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Design of Finite-dimensional Controllers for Infinite-dimensional Systems by Approximation*

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Abstract

Several difficulties in controller design for infinite-dimensional systems arise from using an approximation for the state of the system. In this paper it is demonstrated that the graph topology is an appropriate framework in which to discuss convergence of approximations used for controller design. It is also shown that Galerkin type approximations to a large class of problems possess the required convergence properties and can be used to design controllers which will perform as designed when implemented on the original infinite-dimensional system. An $\mathcal{H}_\infty$–controller design problem is used to illustrate this approach.

Key words: infinite-dimensional systems, Galerkin approximations, coprime factorizations, control theory, graph topology

1 Introduction

There are computational difficulties, apart from the theoretical problems, to designing controllers for systems whose dynamics are described by partial differential equations or integral-differential equations. Consider the following on a Hilbert space $\mathcal{X}$:

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0, \tag{1}$$

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$$y(t) = Cx(t).$$

The operator A generates a strongly continuous semigroup of operators $T(t)$ on $\mathcal{X}$, so A is closed with domain dom$A$ dense in $\mathcal{X}$. This paper is concerned solely with the case where the input and output spaces are finite-dimensional and $B \in \mathcal{L}(R^m, \mathcal{X})$, $C \in \mathcal{L}(\mathcal{X}, R^p)$. Systems (1) which satisfy the above assumptions will be referred to as *bounded control systems* and (1) will often be abbreviated (A, B, C). For control functions in $L_2(0, \infty; R^m)$, and initial conditions in $\mathcal{X}$, the equations (1) have a mild solution:

$$y(t) = CT(t)x_0 + C \int_0^t T(t-\tau)Bu(\tau)d\tau.$$

Further details can be found in, for instance [30].

The equations (1) are a model for a number of control problems, including those where the system dynamics are described by a partial differential equation and hereditary differential systems. A closed form solution can be computed only in the simplest of situations. In general, it is necessary to use an numerical approximation to (1) in order to simulate the response of the system. This approximation will typically be a system of n ordinary differential equations:

$$\begin{aligned} \dot{x}(t) &= A_n x(t) + B_n u(t), \qquad x(0) = x_{0n}, \\ y(t) &= C_n x(t). \end{aligned} \tag{2}$$

Further details on approximation schemes will be presented in subsequent sections. The above finite-dimensional system will often be abbreviated as (A_n, B_n, C_n). This approximation is used not only to simulate the system response, but also to compute controllers for the system whose dynamics are modelled by (1).

There are a number of convergence questions associated with this approach. Is the use of finite-dimensional approximations a valid technique when designing controllers for a given infinite-dimensional system? If so, which approximation methods can be used to design controllers which will perform as designed when implemented on the actual system, and how high an order is required? It is known that convergence of the open loop response, *i.e.,* of the approximating semigroups on bounded intervals, is not sufficient to ensure affirmative answers to these questions (eg. [6]). A scheme which yields good results when used for simulation or identification may be inappropriate for controller design.

Most research in this area has been concerned with state feedback: *i.e.,* the case where the output operator C is the identity. In particular, many researchers (eg. [5, 14, 18, 26]) have studied convergence of solutions to a sequence of Riccati equations. Gibson [14] showed uniform convergence of the solutions, under assumptions of uniform stabilizability and strong convergence of the adjoint semigroups. Subsequently, Banks and Kunisch [5] proved uniform stabilizability for symmetric parabolic equations, and then showed convergence of solutions to the Riccati equation. Convergence of the optimal state feedback operators is then shown by using convergence of solutions to the associated Riccati equations.

Ito [17] showed strong convergence of controllers based on state feedback and estimation, for approximation schemes which are uniformly stabilizable/detectable and for which the approximating resolvants converge uniformly. Gibson and Adamian [15] studied a similar problem for flexible structures and presented some numerical results. These papers are all concerned with controllers obtained using linear-quadratic regulator design.

In this paper the problem of closed loop stability and closed loop convergence for controllers obtained using a arbitrary design method is discussed. It is shown that the graph topology is an appropriate topology in which to study convergence of approximations used in controller design. Simultaneous stabilization of the system and all approximations of high enough order, and convergence of the closed loop systems, is possible if and only if the approximations converge in the graph topology. This is independent of the technique used for controller design. The approximation scheme studied by Kappel and Salamon [21, 22] is used as an illustration.

It is proven that uniform stabilizability or detectability, plus the usual conditions required in simulation, imply convergence in the graph topology. It is proven that Galerkin approximations to a large class of sectorial operators converge in the graph topology.

These ideas are applied to a standard $\mathcal{H}_\infty$ control problem. It is shown that if the approximations converge in the graph topology, performance arbitrarily close to the optimal performance may be achieved by solving a sequence of finite-dimensional $\mathcal{H}_\infty$ problems. Not only is the proof of this result very simple in the context of the graph topology, it has not been obtained by any other approach.

2 Stability

For a given linear time-invariant system, the Laplace transform of the map from inputs u to outputs y is the system transfer function. For bounded control systems (1), the transfer function is $CR(s;A)B$, where $R(s;A)$ indicates the inverse of $s-A$ and s is the Laplace transform variable.

Suppose a system maps inputs in $L_2(0,\infty;R^m)$ to outputs in $L_2(0,\infty;R^p)$, and that furthermore, there is a maximum ratio, γ, called the L_2–gain between the norm of the output and the norm of the input:

$$\| y\|_{L_2(0,\infty;R^p)} \leq \gamma \| u\|_{L_2(0,\infty;R^m)}.$$

Such a system is said to be L_2–*stable*. In other words, L_2–stable systems are those whose input/output map is a bounded operator from $L_2(0,\infty;R^m)$ to $L_2(0,\infty;R^p)$.

The notation $\mathcal{H}_\infty$ indicates the Hardy space of functions $G(s)$ which are analytic in the right-half plane $Re(s) > 0$ and for which

$$\sup_\omega \lim_{x\downarrow 0} |G(x+j\omega)| < \infty.$$

The norm of a function in $\mathcal{H}_\infty$ is

$$\| G\|_\infty = \sup_\omega \lim_{x\downarrow 0} |G(x+j\omega)|.$$

Matrices with entries in $\mathcal{H}_\infty$ are indicated by $M(\mathcal{H}_\infty)$. The norm of a function in $M(\mathcal{H}_\infty)$ is the induced matrix norm

$$\| G\|_\infty = \sup_\omega \lim_{x\downarrow 0} \bar{\sigma}\left[G(x+j\omega)\right]$$

where $\bar{\sigma}$ denotes the largest singular value. By the Paley-Weiner Theorem, a system is L_2–stable if and only if the system transfer function $G \in M(\mathcal{H}_\infty)$.

In the interest of brevity the statement "a linear system with transfer function G" will often be abbreviated to "the system G". The term *plant* is used to indicate a system for which a controller is to be designed.

Now, let G be the transfer function of a given plant and let H be the transfer function of a controller, of compatible dimensions, arranged in the feedback configuration shown in Figure 1. This framework is general enough to include most common control problems. For instance, in tracking, r_1 would be the reference signal to be tracked by the plant output y_1. Since r_1 can also be regarded as modelling sensor noise and r_2 as modelling actuator noise, it is reasonable to regard the control system in Figure 1 as externally stable if the four maps from r_1, r_2 to e_1, e_2 are in $M(\mathcal{H}_\infty)$. (Stability could also be defined in terms of the transfer matrix from (r_1, r_2) to (y_1, y_2): both notions of stability are equivalent [36].)

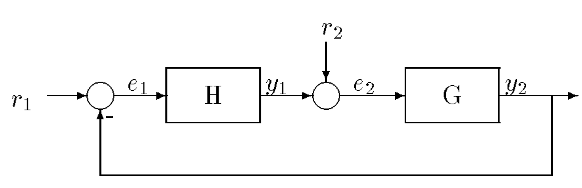

Figure 1. Feedback System

If $\det(I+GH)$ is not identically zero, then the 2×2 transfer matrix $\Delta(G,H)$ which maps the pair (r_1, r_2) into the pair (e_1, e_2) is given by

$$\Delta(G,H) = \begin{bmatrix} (I+GH)^{-1} & -G(I+HG)^{-1} \\ H(I+GH)^{-1} & (I+HG)^{-1} \end{bmatrix}.$$

Definition 2.1 *The feedback system (Figure 1), or alternatively the pair (G,H), is said to be* externally stable *if $\det(I+GH)\neq 0$, and each of the four elements in the above matrix belongs to $M(\mathcal{H}_\infty)$.*

The above definition of external stability is sufficient to ensure that all maps from uncontrolled inputs to outputs are bounded. Furthermore, under an additional assumption of stabilizability/detectability, external stability and internal stability are equivalent. First, some definitions are given.

Definition 2.2 *The C_o-semigroup $T(t)$ is stable if there exist constants M and $\alpha > 0$ such that $\|T(t)\| \leq Me^{-\alpha t}$ for all $t \geq 0$.*

Definition 2.3 *A bounded control system $(A,B,C)$is said to be* internally stable *if the semigroup generated by A is stable according to Definition (2.2).*

Definition 2.4 *The pair (A,B) is* stabilizable *if there exists a bounded linear operator $K : \mathcal{X} \to R^m$ such that $A - BK$ generates a stable semigroup.*

Definition 2.5 *The pair (A,C) is* detectable *if there exists a bounded linear operator $F : R^p \to X$ such that $A - FC$ generates a stable semigroup.*

Definition 2.6 *The system (A,B,C) is* jointly stabilizable/detectable *if (A,B) is stabilizable and (A,C) is detectable.*

It is easy to show, using the Hille-Yosida Theorem, that internally stable systems (1) are externally stable. Also, external stability implies internal exponential stability:

Theorem 2.1 *([20], Theorem 19) A jointly stabilizable/detectable bounded control system is internally stable if and only if it is externally stable.*

Theorem 2.2 *([20], Theorem 35) Assume that (A,B,C) is a jointly stabilizable/detectable bounded control system and that a controller with realization (A_c, B_c, C_c) is also a jointly stabilizable/detectable bounded control system. The closed loop system is externally stable if and only if it is internally stable.*

This equivalence between internal and external stability justifies the use of controller design techniques based on system input/output behaviour for infinite-dimensional systems of the form (1).

Much of modern control theory is concerned with *coprime factorizations* of systems. For the case of L_2-stability, the transfer function of a system is written as $G = ND^{-1}$ where $N, D \in M(\mathcal{H}_\infty)$ and there exists $X, Y \in M(\mathcal{H}_\infty)$ with

$$X(s)N(s) + Y(s)D(s) = I, \qquad \mathrm{Re}(s) \geq 0. \tag{3}$$

The pair (N, D) is called a *right coprime factorization* (r.c.f.) for G. *Left coprime factorizations* (l.c.f.'s) are defined similarly. The pair $(\tilde{N}, \tilde{D})$ is a l.c.f. for G if $G = \tilde{D}^{-1}\tilde{N}$ where $\tilde{N}, \tilde{D} \in M(\mathcal{H}_\infty)$ and there exists $\tilde{X}, \tilde{Y} \in M(\mathcal{H}_\infty)$ with

$$\tilde{N}(s)\tilde{X}(s) + \tilde{D}(s)\tilde{Y}(s) = I, \qquad \mathrm{Re}(s) \geq 0. \tag{4}$$

The statement that a system has a r.c.f. (N, D) will be understood to mean that the transfer function of the system has this r.c.f.; and similarly for l.c.f.'s. The importance of coprime factorizations for controller design is explained by the following theorem.

Theorem 2.3 *([36] Lemma 8.3.2) Suppose a system G has a right coprime factorization (N, D) and a controller H, of compatible dimensions, has a left coprime factorization (X, Y). The closed loop system (G, H) is externally stable if and only if the matrix $XN + YD$ has an inverse in $M(\mathcal{H}_\infty)$. Similarly if G has a left coprime factorization $(\tilde{N}, \tilde{D})$ and H has a right coprime factorization $(\tilde{X}, \tilde{Y})$, (G, H) is externally stable if and only if $\tilde{N}\tilde{X} + \tilde{D}\tilde{Y}$ has an inverse in $M(\mathcal{H}_\infty)$.*

Every system which is described by a linear time-invariant ordinary differential equation has both a left- and a right-coprime factorization. This is a consequence of the fact that the transfer functions of such systems are composed of rational functions. More general systems, which do not have rational transfer functions, do not necessarily have either a left- or a right-coprime factorization. In [28] this is used to show that a undamped Euler-Bernoulli beam with bounded control and sensing operators cannot be externally stabilized by any controller with a coprime factorization. In fact, any system with a transfer function which can be written as a fraction AB^{-1}, where $A, B \in M(\mathcal{H}_\infty)$, is externally stabilizable if and only if it has a right coprime factorization [34].

Bounded control systems which are either stabilizable or detectable possess only a finite number of right-half plane eigenvalues [20]. Such control systems possess both right and left coprime factorizations [8]. If a bounded control system (A, B, C) is jointly stabilizable/detectable, both left and right coprime factorizations can be explicitly written [19]. Let $K \in \mathcal{L}(\mathcal{X}, R^m)$ be such that $A - BK$ generates a stable semigroup and $F \in \mathcal{L}(R^p, \mathcal{X})$ be such that $A - FC$ generates a stable semigroup. Defining

$$N(s) := CR(s; A - BK)B, \quad D(s) := I - KR(s; A - BK)B, \tag{5}$$

$$X(s) := KR(s; A - FC)F, \qquad Y(s) := I + KR(s; A - FC)B,$$

it can be shown [19] that $XN + YD = I$. The pair (N, D) is a r.c.f. for the system *i.e.* $G(s) := CR(s; A)B = ND^{-1}$ [19]. A l.c.f. can be defined similarly. Define

$$\tilde{N}(s) := CR(s; A - FC)B, \quad \tilde{D}(s) := I - CR(s; A - FC)F. \tag{6}$$

Then, $G(s) = \tilde{D}^{-1}(s)\tilde{N}(s)$ and $\tilde{N}\tilde{X} + \tilde{D}\tilde{Y} = I$ for some $\tilde{X}, \tilde{Y} \in M(\mathcal{H}_\infty)$.

The following is now well-known. For completeness, a proof is given.

Theorem 2.4 *Every jointly stabilizable/detectable bounded control system is externally, and internally, stabilizable by a finite-dimensional controller.*

Proof: Let (N, D) be a r.c.f. for the system as defined in (5). Since the time-domain realization of N is bounded by $Me^{-\alpha t}$ for some $M, \alpha > 0$, the Riemann-Lebesgue Lemma implies that

$$\lim_{\substack{|s| \to \infty \\ Re(s) \geq 0}} \overline{\sigma}(N(s)) = 0.$$

Similarly,

$$\lim_{\substack{|s| \to \infty \\ Re(s) \geq 0}} \overline{\sigma}(D(s)) = I.$$

It follows (Mergelyan's Theorem) that each factor can be approximated by a rational element of $M(\mathcal{H}_\infty)$: for sufficiently small $\varepsilon > 0$ there exists $N_n, D_n \in M(\mathcal{H}_\infty)$, coprime and rational with

$$\| N(s) - N_n(s)\|_\infty < \varepsilon, \; \| D(s) - D_n(s)\|_\infty < \varepsilon.$$

Let (X, Y) be a l.c.f. of any finite-dimensional controller which (externally) stabilizes $N_n {D_n}^{-1}$ with $XN_n + YD_n = I$. Then, if ε is small enough, $XN + YD = U$ where U has an inverse in $M(\mathcal{H}_\infty)$. Rewriting, $X(NU^{-1}) + Y(DU^{-1}) = I$. Sijce (NU^{-1}, DU^{-1}) is a r.c.f. for the system (1), it follows that every jointly stabilizable/detectable control system (1) is externally stabilizable by a finite-dimensional controller. If a jointly stabilizable/detectable realization of $Y^{-1}X$ is used, the closed loop is also internally stable. □

Since every bounded control system can be stabilized by a finite-dimensional controller, this suggests an indirect method of controller design for such systems. The infinite-dimensional system (1) is first approximated by a finite-dimensional system (2). Some controller design method is then used to design a controller for this finite-dimensional system. This approach is valid if for sufficiently large approximation order, the designed controller also stabilizes the original system (1), and furthermore, the closed loop responses are close in some sense. In the next two sections conditions under which these criteria are satisfied are established.

For the common situation where the system is already externally stable, and an aim of controller design is to improve the settling time of the system, a real number $\sigma > 0$ is specified. A system is said to be externally σ–stable if its shifted transfer function $G(s-\sigma)$ is in $M(\mathcal{H}_\infty)$. Equivalently, replace $\mathcal{H}_\infty$ by the algebra $\mathcal{H}_{\infty\sigma}$ of functions which are analytic in the right half plane $Re(s) > -\sigma$, and for which

$$\sup_{\omega} \lim_{x \downarrow -\sigma} |G(x+j\omega)| < \infty$$

with corresponding norm.

Definition 2.7 *The feedback system (Figure 1), or alternatively the pair (G, H), is said to be* externally σ–stable *if $det(I + GH) \neq 0$, and each of the four elements in the matrix $\Delta(G, H)$ belongs to $M(\mathcal{H}_{\infty\sigma})$.*

Definition 2.8 *The C_o-semigroup $T(t)$ is σ–stable if there exist constants M and $\alpha > \sigma$ such that $\|T(t)\| \leq Me^{-\alpha t}$ for all $t \geq 0$.*

Definitions (2.3–2.6) also extend in a straightforward manner to σ–internal stability, σ–stabilizability etc. Also, define (N, D) to be a r.c.f. over $\mathcal{H}_{\infty\sigma}$ for G if $G = ND^{-1}$, $N, D \in M(\mathcal{H}_{\infty\sigma})$ and for some $X, Y \in M(\mathcal{H}_{\infty\sigma})$,

$$X(s)N(s) + Y(s)D(s) = I, \qquad \text{Re}(s) \geq -\sigma.$$

Left coprime factorizations over $M(\mathcal{H}_{\infty\sigma})$ are defined similarly. Theorems 2.1–2.4 also extend in a straightforward way to σ–external and σ–internal stability.

3 The Graph Topology

Assume a sequence of finite-dimensional subspaces $\mathcal{X}_n \subset \mathcal{X}$. Define $P_n x$ to be the orthogonal projection of $x \in \mathcal{X}$ onto the finite-dimensional subspace $\mathcal{X}_n$. The norm on $\mathcal{X}_n$ is that inherited from $\mathcal{X}$ and the natural injection $\mathcal{X}_n \rightarrow \mathcal{X}$ will not be explicitly indicated.

For each $\mathcal{X}_n$, the approximating system is (A_n, B_n, C_n) where $B_n := P_n B$, $C_n := C|_{\mathcal{X}_n}$, and A_n is an approximation to A which satisfies the two assumptions (A2) and (A3) below. Note that the operators A_n, B_n, C_n and the semigroup $T_n(t)$ generated by A_n are operators on $\mathcal{X}_n$. The following is assumed:

(A1) For all $x \in \mathcal{X}$, $\lim_{n\to\infty} \| P_n x - x\| = 0$.

(A2) For some $s \in \rho(A)$ and for all $x \in \mathcal{X}$,

$$\lim_{n\to\infty} \| P_n R(s;A)x - R(s;A_n)P_n x\| = 0.$$

(A3) The semigroups $T_n(t)$ generated by A_n are uniformly bounded. That is, there exist real numbers M, k and an integer N such that

$$\|T_n(t)\| \le Me^{kt} \text{ for all } n \ge N. \tag{7}$$

Uniform boundedness of the approximate semigroups is generally referred to as "stability" in the numerical analysis literature. Assumption (A2) is usually referred to as "consistency" of the approximations.

Assumptions (A1)-(A3) are satisfied by typical approximation methods. They are sufficient to ensure that the *open loop* response of the systems (A_n, B_n, C_n) approximate the response of (A, B, C). They are not sufficient to ensure convergence of the closed loop response. In a series of papers which include[21, 22], Kappel and Salamon examine an approximation scheme for a class of delay systems. This approximation scheme satisfies convergence properties stronger than (A1)-(A3) above, and solutions of a sequence of approximating algebraic Riccati equations converge strongly. However, as illustrated below, it is not suitable for use in controller design.

Example: This example is taken from [22]. Let $\mathcal{X} = l_2$ and choose some $b \in l_2$ with $\| b\| = 1$ and with each component b_i real and non-negative. Consider

$$\dot{x}(t) = -x(t) + bu(t) \tag{8}$$

$$y(t) = \langle b, x(t)\rangle \tag{8b}$$

where $\langle , \rangle$ here indicates the l_2 inner product. The solution semigroup of the homogeneous problem is $S(t) = e^{-t}I$ which is clearly exponentially stable, and so (8) is a jointly stabilizable/detectable bounded control system. Note also that the infinitesimal generator, $-I$, is self-adjoint.

For $n = 1, 2, \ldots$ define

$$b_n := col(b_1, b_2, \ldots b_{n+1}) \in R^{(n+1)},$$

$$A_n := diag(-1, -1, \ldots, -b_{n+1}^2) \in R^{(n+1)\times(n+1)}$$

and consider the approximating systems

$$\dot{x}_n(t) = A_n x_n(t) + b_n u(t), \tag{9}$$

$$y_n(t) = b_n^* x_n(t). \tag{9b}$$

The systems (9) satisfy assumptions (A1)-(A3). Since the infinitesimal generators A and A_n are self-adjoint, it is trivial that the adjoint semigroups $S_n^*(t)$ generated by A_n^* converge strongly, uniformly on bounded time intervals, to the semigroup $S^*(t)$ generated by A^*. The transfer function of (8) is $G(s) = \frac{1}{s+1}$ and the transfer functions $G_n(s)$ of the approximations (9) are

$$G_n(s) := \frac{1}{s+1}\sum_{i=1}^{n} b_i^2 \ + \ \frac{b_{n+1}^2}{s+b_{n+1}^2}.$$

Defining $j := \sqrt{-1}$, $|G_n(j\omega)| \leq 2$ for all ω. Thus, the approximations are uniformly externally stable. It is shown in [22] that for each $u \in L_2(0,\infty)$, $y_n(t)$ converges in $L_2(0,\infty)$ to $y(t)$. Also, note that, for $Re(s) \geq 0$,

$$\lim_{n\to\infty} G_n(s) = \begin{cases} G(s) & \text{for } s \neq 0, \\ G(s)+1 & \text{for } s = 0. \end{cases}$$

It follows that the transfer functions converge strongly everywhere in the the right half plane except 0. It is also shown in [22] that the approximations satisfy

$$\int_0^\infty |b_n^* e^{A_n t} P_n x|^2 dt \leq (\| b_n\|^2 + 1)\| x\|^2 \leq 2\| x\|^2.$$

That is, they are *uniformly output stable.*

For each n, let Π_n indicate the unique non-negative definite self-adjoint solution of the approximating Riccati equation

$$A_n^* \Pi_n + \Pi_n A_n - \Pi_n b_n b_n^* \Pi_n + b_n b_n^* = 0.$$

The above assumptions are sufficient to guarantee strong convergence of Π_n to the unique non-negative self-adjoint solution Π to the infinite-dimensional Riccati equation

$$(A^* \Pi + \Pi A) - \Pi b\, b^* \Pi + b\, b^* = 0.$$

The behaviour of the approximations when placed with a controller in feedback is not so well-behaved. Consider first a controller with transfer function $H_1 := -2\frac{s+1}{s+3}$. This controller externally stabilizes the infinite-dimensional system (8). Thus, if an internally stable realization of this controller is used, the closed loop will be internally stable. Now,

$$(1+GH_1)^{-1} = \frac{s+3}{s+1}$$

and

$$(1+G_nH_1)^{-1} = \frac{(s+b_{n+1}^2)(s+3)}{s^2+(3-2\sum_{i=1}^n b_i^2 - b_{n+1}^2)s + b_{n+1}^2(1-2\sum_{i=1}^n b_i^2)}.$$

This controller does not stabilize the approximating systems (9) for large values of n. Simulation results would not predict that this controller stabilizes the original system (8).

As another illustration, consider the controller transfer function $H_2 := -\frac{s+1}{s+2}$. This controller also externally stabilizes the infinite-dimensional system (8). We have

$$(1+GH_2)^{-1} = \frac{s+2}{s+1}$$

and

$$(1+G_nH_2)^{-1} = \frac{(s+2)(s+b_{n+1}^2)}{s^2+(2-\sum_{i=1}^n b_i^2)s + b_{n+1}^2(1-\sum_{i=1}^n b_i^2)}$$

which is a sequence of stable systems. However,

$$\| (1+G_nH_2)^{-1}\|_\infty \geq (1+G_nH_2)^{-1}(0) = \frac{2}{1-\sum_{i=1}^n b_i^2}$$

which is an unbounded sequence. Even if this controller optimized some performance measure for (8), simulation results would indicate that this controller did not stabilize (8). □

It is clear that convergence stronger than that which exists in the above example is required. Let (G, H) be some stable plant-controller pair. A neighbourhood of G should be defined so that, for all sufficiently small neighbourhoods B of G, $G_n \in B$ implies that

1. (G_n, H) is externally stable, and
2. the closed loop response $\Delta(G_n, H)$ is close (in $M(\mathcal{H}_\infty)$) to $\Delta(G, H)$.

If $G, G_n \in M(\mathcal{H}_\infty)$, then for sufficiently small $\varepsilon > 0$, $\| G - G_n\|_\infty < \varepsilon$ implies (1) and (2) above. However, to be applicable to feedback control of possibly unstable systems, a topology must be more general than the operator norm topology for L_2–stable systems.

The graph topology [37, 38] (also referred to as the gap topology) arose from a need to define distance between possibly unbounded operators. The basic idea is outlined in Kato [23]. If E, F are closed linear subspaces of a Banach space then $\delta(E, F)$ is the smallest number δ such that for all $u \in E$,

$$\delta\| u\| \geq \inf_{v\in F} \|u-v\|. \tag{10}$$

The gap between E and F is defined as

$$\hat{\delta}(E,F) := \max(\delta(E,F), \delta(F,E)).$$

The graph of a closed operator S between Banach spaces $\mathcal{U}$ and $\mathcal{Y}$ is the set

$$\{(u, Su),\ u \in \mathrm{dom} S\}$$

where $\mathrm{dom}S$ indicates the domain of S. If S and T are closed operators from $\mathcal{U}$ to $\mathcal{Y}$ then the gap between their graphs as closed subspaces of $\mathcal{U}\times\mathcal{Y}$ is well defined. Define a distance function $d(S,T)$ on closed operators to be the gap between their graphs. Generalized convergence of a sequence $S_n \to S$ is defined by $d(S_n, S) \to 0$. If a sequence of operators $\{S_n\}$ are bounded, then convergence in norm to S is equivalent to generalized convergence. Generalized convergence is thus an extension of the concept of uniform convergence to closed operators.

Now, let $R(\mathcal{H}_\infty)$ indicate transfer functions with both right and left coprime factorizations over $M(\mathcal{H}_\infty)$. Let $G \in R(\mathcal{H}_\infty)$ denote the transfer function of a system with m inputs and p outputs. Define $U^m = L_2(0,\infty; R^m)$. Even if the system is unstable, some inputs $u \in U^m$ will be mapped to outputs $y \in U^p$. Consider these inputs to be the "domain" of the system as an operator between two Banach spaces. The graph of a system is defined as (the Laplace transforms of) this set of input-output pairs, *i.e.*,

$$\mathcal{G}(G) := \{(\hat{u}, \hat{y}), (u, y) \in U^{m+p} | \hat{y} = G\hat{u}\}$$

where $\hat{u}, \hat{y}$ indicate the Laplace transforms of u and y respectively. Letting (N, D) be a r.c.f. for G,

$$\mathcal{G}(G) = \{(Dz, Nz), z \in \mathcal{H}_2^m\}$$

where $z \in \mathcal{H}_2^m$ indicates a vector with entries in the Hardy space $\mathcal{H}_2$ over the right-half plane. It is easy to show that $\mathcal{G}(G)$ is a closed set.

Let G_1, G_2 be any two systems in $R(\mathcal{H}_\infty)$. We say that G_1 is "close" to G_2 in the graph topology if the gap between $\mathcal{G}(G_1)$ and $\mathcal{G}(G_2)$ is small. More formally, let G be a system with a right coprime factorization (N, D) and let $\mu(N, D)$ be a number so that (N_1, D_1) is right coprime for all N_1, D_1 with

$$\left\| \begin{matrix} N_1 - N \\ D_1 - D \end{matrix} \right\|_\infty < \mu(N, D).$$

A basic neighborhood is defined as follows. Let G be some element of $R(\mathcal{H}_\infty)$ with a r.c.f. (N, D) and let ε be any positive number less than $\mu(N, D)$. The set

$$\left\{ H_1 = N_1 D_1^{-1} \mid \quad \left\| \begin{matrix} N_1 - N \\ D_1 - D \end{matrix} \right\|_\infty < \varepsilon \right\}$$

is a basic neighbourhood of G. By varying ε over all $0 < \varepsilon < \mu(N, D)$, (N, D) over all r.c.f's of G and G over all elements of $R(\mathcal{H}_\infty)$ we obtain a collection of sets which is a base for a topology on $R(\mathcal{H}_\infty)$, the graph topology. Further details can be found in [37, 39].

The next theorem will be of use in a subsequent section.

Theorem 3.1 *[37, 39] Suppose G_n is a sequence in $R(\mathcal{H}_\infty)$, and that $G \in R(\mathcal{H}_\infty)$. Then the following statements are equivalent.*

1. *$\{G_n\}$ converges to G in the graph topology.*
2. *There exists a r.c.f. (N, D) of G, and a sequence of r.c.f.'s (N_n, D_n) of G_n such that $N_n \to N$ and $D_n \to D$ in $M(\mathcal{H}_\infty)$.*
3. *There exists a l.c.f. (N, D) of G, and a sequence of l.c.f.'s (N_n, D_n) of G_n such that $N_n \to N$ and $D_n \to D$ in $M(\mathcal{H}_\infty)$.*

For a sequence of stable systems, such as in the example discussed above, (A1)-(A3) imply uniform convergence of $C_n S_n(t) B_n$ on bounded intervals of time, or pointwise convergence of the transfer functions in s. Uniform convergence of the transfer functions is required in order to obtain convergence in the graph topology. Furthermore, convergence in the graph topology of possibly unstable systems can be established by examining convergence *in norm* of stable coprime factorizations.

The importance of the graph topology in controller design is due to the following result: A family of plants G_n can be robustly stabilized by a compensator H which stabilizes some nominal plant G if and only if G_n converges to G in the graph topology. Furthermore, if this is the case, the closed loop response of the feedback pair (G_n, H) converges to that of (G, H). This is stated more precisely below.

Theorem 3.2 *[37, 39] Let G_n be a sequence of plants in $R(\mathcal{H}_\infty)$.*

1. *Suppose G_n converges to $G \in R(\mathcal{H}_\infty)$ in the graph topology. Let $H \in R(\mathcal{H}_\infty)$ stabilize G. Then there exists an N such that H stabilizes G_n, for all $n \geq N$, and moreover, the closed loop transfer matrix $\Delta(G_n, H)$ converges to $\Delta(G, H)$ in $M(\mathcal{H}_\infty)$.*
2. *Conversely, suppose there exists a $H \in R(\mathcal{H}_\infty)$ which stabilizes G_n for all $n \geq N$, and that $\Delta(G_n, H)$ converges to $\Delta(G, H)$. Then G_n converges to G in the graph topology.*

This has obvious implications for controller design using approximations. Failure of a sequence of approximations to converge in the graph topology implies that for each controller H, at least one of the following must occur:

1. (G_n, H) is not stable for all n sufficiently large;
2. the closed loop response $\Delta(G_n, H)$ does not converge uniformly to $\Delta(G, H)$.

The approximating transfer functions in the example discussed earlier in this section are in $\mathcal{H}_\infty$, but do not converge uniformly, and hence do not converge in the graph topology. The controllers discussed illustrate the problems which occur when an approximation scheme does not converge in the graph topology. The first controller does not even stabilize the approximations for large approximation order (item 1). The second controller leads to closed loop responses which do not converge (item 2).

Returning to the case of a general bounded control system (A, B, C), if the transfer functions of the approximations (A_n, B_n, C_n) do not converge in the graph topology to the transfer function of the original system, use of the approximations to design a controller for (A, B, C), is guaranteed to lead to incorrect conclusions about the behaviour of the closed loop. If the approximations do converge in the graph topology, Theorem 3.2 states that use of the approximations in controller design is valid.

The problem where σ–stability is desired is handled entirely analogously, by defining the graph topology with coprime factorizations over $\mathcal{H}_{\infty\sigma}$ instead of $\mathcal{H}_\infty$.

The above example showed that assumptions (A1)-(A3) are not sufficient for convergence in the graph topology. This example also demonstrates that strong convergence of the adjoint semigroups, even when the approximations are uniformly externally stable, is not sufficient. In the next section sufficient conditions for convergence in the graph topology of finite-dimensional approximations are presented.

4 Convergence of Approximation Schemes

Suppose that an approximation scheme for (1) satisfies, in addition to (A1)-(A3), an assumption of uniform $\sigma-$stabilizability.

(A4) If the original system is σ-stabilizable, then the approximations are uniformly σ-stabilizable. That is, there exists a sequence of operators $\{K_n\}$ with $K_n \in \mathcal{L}(\mathcal{X}_n, R^m)$ and some $K \in \mathcal{L}(\mathcal{X}, R^m)$ such that for all $x \in \mathcal{X}$, $\lim_{n\to\infty} K_n P_n x = Kx$. Furthermore, for sufficiently large N the semigroups generated by $A_n - B_n K_n$ are uniformly bounded by $Me^{-\alpha t}$ for some $M > 0, \alpha > \sigma$ and all $n > N$.

Lemma 4.1 *Let (A, B, C) be a σ-stabilizable bounded control system and assume (A_n, B_n, C_n) is a sequence of approximations which satisfy assumptions (A1)-(A4). Let $S(t)$ be the semigroup generated by $A - BK$ and $S_n(t)$*

be the semigroup generated by $A_n - B_nK_n$ *where* K, K_n *are as in assumption (A4) . Then* $\| S(t)\| \le Me^{-\alpha t}$ *and for all* $\tau > 0, x \in \mathcal{X}$,

$$\lim_{n\to\infty} \sup_{0\le t\le\tau} \|S_n(t)P_nx - P_nS(t)x\| = 0.$$

Proof: Define $A_o := A - BK$ and $A_{no} := A_n - B_nK_n$. Since (A2) is satisfied and $\lim_{n\to\infty} B_nK_nP_nx - P_nBKx = 0$ for all $x \in \mathcal{X}$, it follows that for some $s \in \rho(A_o)$,

$$\lim_{n\to\infty} \| P_nR(s; A_o)x - R(s; A_{no})P_nx\| = 0.$$

Using (A4), the conclusions follow from the Trotter-Kato Theorem [25] [Thm. 2.1]. □

Theorem 4.2 *Let* (A, B, C) *be a* σ*-stabilizable/detectable bounded control system, and assume* (A_n, B_n, C_n) *is a sequence of approximations which satisfy assumptions (A1)-(A4). Then the approximating systems with transfer functions* $G_n(s) := C_nR(s; A_n)B_n$ *converge to the original system in the graph topology on* $M(\mathcal{H}_{\infty\sigma})$.

Proof: The theorem will be proven by showing that a sequence of right coprime factorizations for the approximate systems (A_n, B_n, C_n) converge to a right coprime factorization for (A, B, C). The result will then follow from Theorem 3.1. Let feedback operators K and K_n be as defined above (A4) so that $A_{no} := A_n - B_nK_n$ generates a σ–stable semigroup $S_n(t)$ for sufficiently large n and $A_o := A - BK$ generates a σ–stable semigroup $S(t)$. Referring to (5), for sufficiently large n time domain representations of right σ–coprime factors for the approximating systems are given by

$$\mathcal{N}_n(t) = C_nS_n(t)B_n, \qquad \mathcal{D}_n(t) = I - K_nS_n(t)B_n.$$

Similarly, a time domain representation of a right coprime factorization for the original system is

$$\mathcal{N}(t) = CS(t)B, \qquad \mathcal{D}(t) = I - KS(t)B.$$

Convergence of the numerators is proven first. Since the semigroups $S_n(t)$ and $S(t)$ are uniformly $\sigma-$ stable, for any $\varepsilon > 0$ there exists τ such that

$$\int_\tau^\infty \exp(\sigma t) \| \mathcal{N}(t) - \mathcal{N}_n(t) \| \, dt < \frac{\varepsilon}{2} \tag{11}$$

where $\| \cdot \|$ here indicates a norm on $R^{p\times m}$. Also,

$$\| \mathcal{N}(t) - \mathcal{N}_n(t) \| \le \|C\| \, \|P_nS(t)B - S_n(t)P_nB\| + \|CS(t)B - CP_nS(t)B\|.$$

It follows from Lemma 4.1 that the first term above approaches zero, uniformly on bounded intervals $[0,\tau]$. The second term is uniformly bounded, equicontinuous and pointwise convergent to zero. Thus, by Ascoli's Theorem, it also converges to zero uniformly on $[0,\tau]$. Therefore, for any $\varepsilon > 0$, $\tau > 0$ there exists M such that

$$\sup_{0\leq t\leq\tau} \| \mathcal{N}(t) - \mathcal{N}_n(t) \| \leq \frac{\varepsilon}{2e^{\sigma\tau}\tau} \quad \text{for all } n > M. \tag{12}$$

Combining statements (11) and (12) it follows that, since ε was arbitrary,

$$\lim_{n\to\infty} \int_0^\infty \exp(\sigma t)\|\mathcal{N}(t) - \mathcal{N}_n(t)\|dt = 0.$$

Therefore $(\mathcal{N}_n(t) - \mathcal{N}(t))e^{\sigma t}$ converges to zero in $\mathcal{L}_1(0,\infty;R^{p\times m})$. Since convergence in $\mathcal{L}_1$ of a function to zero implies convergence of its Laplace transform to zero in $M(\mathcal{H}_\infty)$, it follows that the numerators converge in $M(\mathcal{H}_{\infty\sigma})$.

We now show convergence of $\mathcal{D}_n(t)$ to $\mathcal{D}(t)$. Defining

$$\mathcal{F}_n(t) := [\ K_nP_n - K\ ]\ S(t)B, \qquad \mathcal{E}_n(t) := K_n\ [\ S_n(t)P_nB - P_nS(t)B\ ],$$

we have

$$\mathcal{D}(t) - \mathcal{D}_n(t) = \mathcal{F}_n(t) + \mathcal{E}_n(t). \tag{13}$$

Since the sequence $\{K_n\}$ is uniformly bounded, a proof identical to that used for the numerators shows that $\{\mathcal{E}_n(t)e^{\sigma t}\}$ converges to zero in the $\mathcal{L}_1$ norm .

Since $S(t)$ is σ–stable, for any ε there exists τ so that

$$\int_\tau^\infty \exp(\sigma t)\|\mathcal{F}_n(t)\|dt \leq \frac{\varepsilon}{2}.$$

The sequence $\mathcal{F}_n(t)$ is clearly uniformly bounded on $[0,\tau]$ and the sequence is also equicontinuous. Since $\{\mathcal{F}_n(t)\}$ is pointwise convergent to zero, it follows from Ascoli's Theorem that

$$\lim_{n\to\infty} \sup_{0\leq t\leq\tau} \|\mathcal{F}_n(t)\| = 0,$$

and so

$$\lim_{n\to\infty} \int_0^\infty \exp(\sigma t)\|\mathcal{F}_n(t)\|dt = 0.$$

Therefore, $(\mathcal{D}_n(t) - \mathcal{D}(t))e^{\sigma t}$ also converges to zero in $\mathcal{L}_1(0,\infty;R^{m\times m})$. Hence, the coprime factors converge in $M(\mathcal{H}_{\infty\sigma})$. □

Thus, if H is *any* stabilizing controller for the original system G, then for sufficiently large N, (G_n, H) is σ–externally stable for all $n > N$. If H

is implemented as a σ–stabilizable/detectable bounded control system, as is common, Theorem 2.2 implies that the closed loop is also internally σ–stable. That is, the closed loop is a bounded control system (A_t, B_t, C_t) and A_t generates an semigroup with bound Me^{-at} where $a > \sigma$. Furthermore, the closed loop response of (G_n, H) converges, uniformly in the norm on $M(\mathcal{H}_{\infty\sigma})$, to that of (G, H).

If (A4) is replaced by a similar assumption of uniform detectability, then similiar conclusions may be drawn by showing convergence of the left coprime factors.

(A5) If the original system is σ-detectable, then the approximations are uniformly σ-detectable. That is, there exists a sequence of operators $\{F_n\}$ with $F_n \in \mathcal{L}(Y, \mathcal{X}_n)$ and some $F \in \mathcal{L}(Y, \mathcal{X})$, such that for all $y \in R^p$, $\lim_{n\to\infty} F_n y - P_n F y = 0$. Furthermore, for sufficiently large N, the semigroups generated by $A_n - F_n C_n$ are uniformly bounded by $Me^{-\alpha t}$ for some $M > 0, \alpha > \sigma$ and all $n > N$.

Theorem 4.3 *Let (A, B, C) be a σ-stabilizable/detectable bounded control system, and assume (A_n, B_n, C_n) is a sequence of approximations which satisfy assumptions (A1)-(A3) and (A5). Then the approximating systems with transfer functions $G_n(s) = C_n R(s; A_n) B_n$ converge to the original system in the graph topology on $M(\mathcal{H}_{\infty\sigma})$.*

Proof: As above, it is sufficient to show convergence of the time domain representations of a sequence of left coprime factors. Let $S_{Fn}(t)$ denote the semigroup of operators generated by $A_n - F_n C_n$ and $S_F(t)$ the semigroup generated by $A - FC$. Define

$$\tilde{N}_n(s) = C_n R(s; A_n - F_n C_n) B_n, \qquad \tilde{D}_n(s) = I - C_n R(s; A_n - F_n C_n) F_n.$$

$(\tilde{N}_n, \tilde{D}_n)$ is a sequence of l.c.f's for the approximating systems The pair

$$\tilde{N}(s) = CR(s; A - FC)B, \qquad \tilde{D}(s) = I - CR(s; A - FC)F$$

is a l.c.f. for the original system.

The proof of convergence of $\tilde{N}_n$ to $\tilde{N}$ is exactly the same as that for the case of right coprime factorizations. To show that the denominator functions converge, define

$$Q_n(t) = C_n S_{Fn}(t)\left[F_n - P_n F\right], \qquad R_n(t) = C_n S_{Fn}(t) P_n F - C S_F(t) F$$

and note that

$$\tilde{D}(s) - \tilde{D}_n(s) = \mathcal{L}\left[Q_n(t)\right] + \mathcal{L}\left[R_n(t)\right]$$

where $\mathcal{L}(Q)$ here indicates the Laplace transform of Q. By assumption (A5) and the fact that the output space Y is finite-dimensional, $Q_n(t)e^{\sigma t}$

converges to zero in $\mathcal{L}_1(0,\infty;R^{p\times p})$. The same approach used for the numerators can be used to show that $R_n(t)e^{\sigma t}$ also converges to zero in the L_1-norm.

Hence, the sequence of l.c.f.'s for (A_n, B_n, C_n) converge to a l.c.f. for (A, B, C) in $M(\mathcal{H}_{\infty\sigma})$. □

It is not clear to what extent the additional assumption (A4) or (A5) is necessary for convergence in the graph topology. Some insight into the requirements for convergence in the graph topology can be obtained by considering stable systems. Suppose that the semigroup $T(t)$ generated by A is bounded by $Me^{-\alpha t}, \alpha > 0$. It is reasonable to demand that the closed loop system have at least this margin of stability. If this is so, then the sequence of approximating transfer functions G_n must converge in the graph topology on $M(H_{\infty\,\sigma})$, $\alpha > \sigma \geq 0$. This implies that $G_n(\cdot - \sigma) \in M(\mathcal{H}_\infty)$ for large n [37, 39] and so for some sequence of constants M_n

$$\| G_n(t)\| \leq M_n e^{-\sigma t}. \tag{14}$$

Thus, for large enough model order, any unstable modes must be both uncontrollable and unobservable. If the approximation scheme is valid for arbitrary choices of sensing and actuating, then for some B, C and some n, (A_n, B_n, C_n) will be σ-stabilizable and σ-detectable. Thus, if a scheme is satisfactory for an arbitrary choice of B, C, then for some M_n, large enough n, [29]

$$\| T_n(t)\| \leq M_n e^{-\sigma t}, \qquad \alpha > \sigma > 0. \tag{15}$$

This heuristic argument suggests that it is necessary that the eigenvalues of the approximating systems are bounded away from the imaginary axis, which is a weaker form of uniform exponential stability. However, further research into these questions is needed.

In the next section common approximation schemes for several important classes of bounded control systems are shown to satisfy assumptions (A1)-(A4), and hence the corresponding sequences of transfer functions converge in the graph topology.

5 Uniform Stabilizability of Approximation Schemes

A number of researchers (eg. [5, 14, 18, 21, 26]) have studied the problem of which approximating schemes provide uniform exponential stabilizability (A4) in addition to the usual (A1)-(A3). The following theorem is immediate from results on approximations to infinite-dimensional Riccati equations.

Theorem 5.1 *Assume that*

1. *(A1)-(A3) hold for the system* (A, B, C)*,*
2. *there exist uniformly bounded* $\{L_n\}$ *such that, for some* $M > 0, \alpha > \sigma$*, the semigroups generated by* $A_n - B_n L_n$ *are uniformly bounded by* $Me^{-\alpha t}$*, and,*
3. *the adjoint semigroup operators converge strongly:*

$$\lim_{n\to\infty} \sup_{0\le t\le\tau} \|T_n^*(t)P_n x - P_n T^*(t)x\| = 0.$$

Then (A, B) *is* $\sigma-$*stabilizable, and the conclusions of Theorem 4.2 hold.*

Proof: By assumption, (A, B) with observation operator I, where I is the identity operator on $\mathcal{X}$ satisfy hypotheses (H1)-(H3) in [18]. The result follows from [18, Theorem 2.1, Theorem 2.3]. □

It is well-known that (A1)-(A4) are true if $(A,\ B)$ is stabilizable, the set of eigenfunctions $\{\phi_i\}$ of A is complete in $\mathcal{X}$, and the approximating subspaces are chosen to be $\mathcal{X}_n := \text{span}\{\phi_i\}_{i=1,n}$. Therefore, the corresponding sequence of finite-dimensional approximations converges in the graph topology.

Several other classes of bounded control systems and common approximation schemes which satisfy assumptions (A1)-(A5) are discussed below.

5.1 Hereditary systems

Consider the delay functional differential equation

$$\begin{aligned} \dot{x}(t) &= A_o x(t) + A_1 x(t-h) + \int_{-h}^{0} D(r)x(t+r)dr + Bu(t), \qquad t \ge 0 \\ x(0) &= x_0, \quad x(\tau) = \phi(\tau), \qquad -h \le \tau < 0, \qquad (16) \\ y(t) &= Cx(t), \end{aligned}$$

where $x(t) \in R^n, y(t) \in R^p, u(t) \in R^m$ and A_o, A_1, B and C are matrices of appropriate dimension. Also, $D \in L_2(-h, 0; R^{n\times n})$. Defining the state-space to be $\mathcal{X} = R^n \times L_2(-h, 0; R^n)$, the hereditary system (16) can be formulated as a bounded control system [3].

Define the finite-dimensional subspaces, $\mathcal{X}_N$, of $\mathcal{X}$ as follows:

$$\left\{ (\phi_1, \phi_2) \in \mathcal{X} |\ \phi_2(\tau) = z_j,\ \frac{-j}{N}h \le \tau < \frac{-(j-1)}{N}h,\ j = 1, ...N \text{ where } z_j \in R^n \right\}.$$

The finite-dimensional Galerkin approximation to (16) derived using these subspaces satisfies assumptions (A1)-(A3) [3, 31]. This scheme is known as the averaging approximation to (16).

In [17] it is shown that this scheme is uniformly stabilizable and detectable. In fact, for a bounded operator K such that $A-BK$ generates a stable semigroup, KP_n satisfies (A4). Similarly, if (16) is detectable, there exists a sequence $F_n \to F$ where $A - FC$ generates a stable semigroup and F_n is a sequence which satisfies (A5).

5.2 Sectorial operators

Banks and Kunisch [5] prove that (A1)-(A4) hold for general Galerkin approximations to symmetric parabolic problems with bounded control. In this section these results are extended to a larger class of systems.

Let V be a Hilbert space, densely and continuously imbedded in $\mathcal{X}$. The notation $\langle \cdot, \cdot \rangle$ indicates the inner product on $\mathcal{X}$, and $\langle \cdot, \cdot \rangle_V$ indicates the inner product on V. In order to avoid confusion with the norm on $\mathcal{X}$, the norm on V will be indicated by $\| \cdot \|_V$. Identify $\mathcal{X}$ with its dual so that $V \hookrightarrow \mathcal{X} = \mathcal{X}' \hookrightarrow V'$.

Let $a : V \times V \mapsto \mathcal{C}$ be a continuous sesquilinear form *i.e.* there exists c_1 such that

$$|a(\phi, \psi)| \leq c_1 \| \phi \|_V \| \psi \|_V \tag{17}$$

for all $\phi, \psi \in V$. Define a closed operator A through this form by

$$\langle -A\phi, \psi \rangle = a(\phi, \psi), \qquad \forall \psi \in V$$

where $\mathrm{dom} A = \{\phi \in V | a(\phi, \cdot) \in \mathcal{X}\}$. We assume that in addition to (17), $a(\cdot, \cdot)$ satisfies Garding's inequality: there exists $k \geq 0$, such that for all $\phi \in V$

$$\mathrm{Re} a(\phi, \phi) + k \langle \phi, \phi \rangle \geq c \| \phi \|_V^{\,2}. \tag{18}$$

The inequalities (17) and (18) guarantee that A generates an analytic semigroup with bound $\| T(t) \| \leq e^{kt}$ and that the spectrum of A is contained in some sector of the complex plane[33]. Further details may be found in [33].

The assumption that Garding's inequality holds is not trivial. For instance, structures with purely viscous damping do not satisfy this condition [4]. However, this classification does include a large number of problems. In particular, the symmetric parabolic systems studied by Banks and Kunisch [5] are a special case of this class of systems. Also, many damping models for structural vibrations lead to problems of this type. Details can be found in [4]. A particular example is described below.

Example: Consider a Euler-Bernoulli beam of unit length rotating about a fixed hub, and let w denote the deflection of the beam from its rigid body motion. Denote the torque applied at the hub by $u(t)$, and assume that the hub inertia I_h is much larger than the beam inertia, so that, letting $\theta(t)$ indicate the rotation angle, $u(t) = I_h \ddot{\theta}(t)$ is a reasonable approximation to the applied torque. Use of the Kelvin-Voigt damping model leads to the following description of the beam vibrations:

$$\frac{\partial^2 w}{\partial t^2} + \frac{\partial^2}{\partial r^2} \left[EI \frac{\partial^2 w}{\partial r^2} + c_D I \frac{\partial^3 w}{\partial r^2 \partial t} \right] = \frac{r}{I_h} u(t), \qquad 0 < r < 1.$$

The appropriate boundary conditions are

$$w(0,t)=0, \qquad \frac{\partial w}{\partial r}_{r=0}=0,$$
$$\left[EI\frac{\partial^2 w}{\partial r^2}+c_D I\frac{\partial^3 w}{\partial r^2 \partial t}\right]_{r=1}=0, \qquad \left[EI\frac{\partial^3 w}{\partial r^3}+c_D I\frac{\partial^4 w}{\partial r^3 \partial t}\right]_{r=1}=0. \tag{19}$$

Let $x := (w, \dot{w})$. If the position is measured at the tip of the beam, a state-space formulation of the above partial differential equation problem is

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) := w(1,t),$$

where

$$A := \begin{bmatrix} 0 & I \\ -EI\frac{\partial^4}{\partial r^4} & -c_D I\frac{\partial^4}{\partial r^4} \end{bmatrix}, \qquad B := \begin{bmatrix} 0 \\ r/I_h \end{bmatrix}.$$

Define

$$H := \left\{ w(r) \in H^2[0,1], w(0) = \frac{dw}{dr}(0) = 0 \right\}.$$

It is known that A can be associated with a sesquilinear form $a(\cdot,\cdot)$ which satisfies (17) and (18) with $V := H \times H$ and $\mathcal{X} := H \times L_2[0,1]$(See [4]). The operator B is clearly a bounded operator from R to $\mathcal{X}$. Since point sensing is a bounded operator on H (Sobolev's Inequality [11]), the output operator C is bounded from $\mathcal{X}$ to R . □

Let $\mathcal{X}_n \subset V$ be a sequence of finite-dimensional subspaces. For the class of bounded control systems (1) where the operator A satisfies the inequalities (17) and (18) we assume only that the approximating subspaces $\mathcal{X}_n$ satisfy a V-approximation property: for all $x \in V$ there exists a sequence $x_n \in \mathcal{X}_n$ with

$$\text{(H1)} \qquad \lim_{n\to\infty} \| x_n - x\|_V = 0.$$

Note that it is only required that projections onto $\mathcal{X}_n$ converge in the V-norm. An earlier result [17] requires an inverse approximation property:

$$\inf_{x\in\mathcal{X}_n} \| R(s;A)z - x\|_V \le \varepsilon_1(n)\| z\|_{\mathcal{X}}$$
$$\inf_{x\in\mathcal{X}_n} \| R(s;\ A^*\ )z - x\|_V \le \varepsilon_2(n)\| z\|_{\mathcal{X}}$$

where $\varepsilon_1(N), \varepsilon_2(N) \to 0$ as $N \to \infty$. These conditions are stronger, and more difficult to verify than (H1). Furthermore, the above conditions can only be satisfied if $R(s;A)$ is compact. Systems such as the Euler-Bernoulli beam with Kelvin-Voigt damping discussed above are excluded.

The approximating generator A_n is defined by

$$\langle -A_n x_n, v_n\rangle = a(x_n, v_n), \quad \forall x_n, v_n \in \mathcal{X}_n, \tag{20}$$

and P_n, B_n, C_n are as defined at the beginning of Section 3. This type of approximation is generally referred to as a *Galerkin* approximation.

Assumption (H1) clearly implies (A1). We now prove that, subject to hypothesis (H1) the approximations also satisfy assumptions (A2)-(A5).

Theorem 5.2 *If (H1) holds, then assumptions (A2) and (A3) are satisfied and so the semigroups $T_n(t)$ generated by A_n converge strongly, uniformly on bounded intervals $0 \leq t \leq \tau$, to the semigroup $T(t)$ generated by A.*

Proof: Since A_n are defined through the sesquilinear form $a(\cdot,\cdot)$ we obtain $\| T_n(t)\| \leq e^{kt}$ from the Lumer-Phillips Theorem [30]. The result will follow from the Trotter-Kato Approximation Theorem if it is proven that, for some λ and for all x,

$$\lim_{n\to\infty} R(\lambda; A_n)P_n x = R(\lambda; A)x.$$

Choose $\lambda > k, x \in \mathcal{X}$, and define $w_n := R(\lambda; A_n)P_n x,\ w := R(\lambda; A)x$. The proof that $w_n \to w$ is identical to that in [4, Theorem 2.2]. □

Theorem 5.3 *Suppose (A, B) is a stabilizable pair and that A satisfies (17) and (18). Let $K \in \mathcal{L}(\mathcal{X}, R^m)$ be such that $A - BK$ generates a stable semigroup. If (H1) is true, there exists N such that for some $M, \alpha > 0$ and all $n > N$, the semigroups $S_n(t)$ generated by $A_{no} := A_n - B_n K P_n$ are uniformly stable* i.e.

$$\| S_n(t)\| \leq M e^{-\alpha t} \qquad \forall n > N. \tag{21}$$

Proof: The technique used in [5, Lemma 3.3] is generalized. Define a sesquilinear form $a_B : V \times V \to \mathcal{C}$ by $a_B(z, v) = a(z, v) + (BKz, v) + k_2(z, v)$ where k_2 is chosen so that there exists $c_4 > 0, 0 < c_3 \leq c_4$ with

$$\text{Re } a_B(z, z) \geq c_3 \| z\|_V^{\ 2}, \qquad |a_B(z, v)| \leq c_4 \| z\|_V \| v\|_V.$$

Therefore,

$$\frac{|\text{Im } a_B(z, z)|}{|\text{Re } a_B(z, z)|} \leq \frac{c_4}{c_3}.$$

Thus, defining $\theta := \tan^{-1} \frac{c_4}{c_3}$, the numerical range of a_B is contained in the sector $\Sigma := \{ \lambda \in \mathcal{C},\ |arg\lambda| \leq \theta \}$. It follows that the spectrum of $A - BK - k_2 I$ is contained in the left sector $\Sigma_- := \{ \lambda,\ |arg(-\lambda)| \leq \theta \}$ [23, pg. 268]. Similarly, for all n, the spectrum of $A_n - B_n K P_n - k_2 I$ is contained in Σ_- . The remainder of the proof is identical to that in [5, Lemma 3.3]. It follows that there is an $\varepsilon > 0$ and an integer N such that, for $n > N$, the spectrum of A_{no} is contained in a left sector $\tilde{\Sigma}$ with vertex $-\varepsilon/2$.

For some C dependent only on c_3, c_4 , $\| R(\lambda; A_{no} - k_2 I)\| \leq C/|\lambda|$ or,

$$\| R(\lambda; A_{no})\| \leq C/|\lambda - k_2|$$

[33, Theorem IV.6.A]. The uniform exponential bound (21) for $S_n(t)$ now follows from the representation

$$S_n(t) = \int_\Gamma e^{\lambda t} R(\lambda; A_{no})\ d\lambda$$

where , is the positively bounded boundary of the sector $\tilde{\Sigma}$, and from the observation that for all $t \leq 1$, $\| S_n(t)\| \leq e^{k_2}$. □

Corollary 5.4 *Suppose (A, B) is a σ-stabilizable pair and A satisfies (17) and (18). If (H1) is true, then there exists an N such that for all $n > N$, the semigroups $S_n(t)$ generated by $A_{no} := A_n - B_n K P_n$ are uniformly σ–stable,* i.e., *there exists $N, M, \alpha > \sigma$ such that*

$$\| S_n(t)\| \leq Me^{-\alpha t} \qquad \forall n > N. \tag{22}$$

Proof: Replace A by $A + \sigma I$ and $a(u, v)$ by $a(u, v) - \sigma\langle u, v\rangle$, in the proof of the previous theorem. □

It follows that if the operator A of a stabilizable/detectable bounded control system (A, B, C) is associated with a continuous sesquilinear form $a(\cdot, \cdot)$ which satisfies Garding's inequality (18), then any sequence of Galerkin approximations (20) satisfying assumption (H1) converge in the graph topology.

The example of an Euler-Bernoulli beam with Kelvin-Voigt damping discussed above does not have a compact resolvant, and satisfies assumptions (A1)-(A4), so it is clear that uniform convergence of the resolvants is not necessary for uniform stabilizability or convergence in the graph topology.

6 Application: $\mathcal{H}_\infty$ Controller Design

It has been demonstrated above that the graph topology is an appropriate topology in which to establish convergence of approximations used for controller design, and also that such convergence exists for several important classes of systems and approximation schemes. In this section these ideas are applied to a problem of controller design.

Recall that G is the transfer function of the system (1) and that G_n is the transfer function of the system (2). Denote the set of all stabilizing controllers for a system G by $S(G)$. Convergence in the graph topology of a sequence of approximating systems G_n to the original system G implies that for sufficiently large n, the sets $S(G_n)$ and $S(G)$ of stabilizing controllers have a non-empty intersection. Furthermore, if H is any stabilizing controller for G, then for large n, $H \in S(G_n)$, and the closed loop response $\Delta(G_n, H)$ converges to the exact closed loop response.

Suppose that a sequence of controllers H_n for the approximating systems G_n is obtained using some design technique. If H_n also converges to $H \in S(G)$ in the graph topology then the closed loop response $\Delta(G_n, H_n)$ converges to $\Delta(G, H)$. Recall that $R(\mathcal{H}_\infty)$ indicates systems with transfer functions which have right- and left- coprime factorizations over $\mathcal{H}_\infty$.

Theorem 6.1 *[37, 39] Let $\{G_n\}, \{H_n\}$ be sequences in $R(\mathcal{H}_\infty)$ and $G, H \in R(\mathcal{H}_\infty)$. Then $\Delta(G_n, H_n) \to \Delta(G, H)$ in $M(\mathcal{H}_\infty)$ if and only if $G_n \to G$ and $H_n \to H$ in the graph topology.*

As an illustration, consider $\mathcal{H}_\infty$ optimal control methods. Let G be a stable plant, and $\{G_n\}$ a sequence of approximations in $M(\mathcal{H}_\infty)$ which converge to G, *i.e.*,

$$\lim_{n\to\infty} \| G - G_n\|_\infty = 0. \tag{23}$$

Such convergence is guaranteed by Theorem 4.2 if the approximations satisfy assumptions (A1)-(A3), and in addition, the constant k in (7) can be chosen less than zero. The most basic $\mathcal{H}_\infty$-control problem is weighted sensitivity minimization:

$$\mu(G) := \inf_{H \in S(G)} m(G, H) \tag{24}$$

where

$$m(G, H) := \| W_1(I + GH)^{-1}\|_\infty \tag{25}$$

and $W_1 \in M(\mathcal{H}_\infty)$. This problem arises in the context of minimizing the steady-state error e_1 (see Figure 1) with the error weighted by W_1. Typically, W_1 is a function which is large near the frequencies where the disturbance r_1 is expected to be significant. Further details can be found, for instance, in [12].

The set of all stabilizing controllers for a system with a right (or left) coprime factorization may be described in terms of the *Youla parametrization*: a controller H externally stabilizes G if and only if it can be written

$$H = (Y - Q\tilde{N})^{-1}(X + Q\tilde{D}), \quad |Y - Q\tilde{N}| \neq 0, \quad Q \in M(\mathcal{H}_\infty) \tag{26}$$

where $X, Y, \tilde{N}, \tilde{D}$ are as defined in (3), (4). The set of all stabilizing controllers for a given system G are parametrized by Q , as Q ranges over all stable systems. In other words,

$$S(G) = \{\ (Y - Q\tilde{N})^{-1}(X + Q\tilde{D}), \quad |Y - Q\tilde{N}| \neq 0, \quad Q \in M(\mathcal{H}_\infty)\ \}.$$

Using the Youla parametrization (26) of all stabilizing controllers, any stabilizing controller for G must be of the form

$$H = Q(I - GQ)^{-1}, \qquad Q \in M(H_\infty), \tag{27}$$

and so (24, 25) can be rewritten

$$\mu(G) := \inf_{Q \in M(\mathcal{H}_\infty)} m(G, H)$$

where

$$m(G, H) = \| W_1(I - GQ)\|_\infty .$$

As shown in [35], this problem is not in general continuous in the argument G. The assumption that $G_n \to G$ in the graph topology, does not imply that $\mu(G_n) \to \mu(G)$. Even if it is known that

$$\lim_{n\to\infty} \mu(G_n) = \mu(G),$$

and ${W_1}^{-1} \in M(\mathcal{H}_\infty)$, the corresponding controllers may not converge. To see this, suppose $\{H_n\}$ is such that $m(G_n, H_n) \to m(G, H)$ and let Q_n, Q be the corresponding Youla parameters. This implies only that $G_nQ_n \to GQ$, so $\{Q_n\}$ may be unbounded. It is not possible prove in general that H_n stabilizes G for all n sufficiently large.

Consider now the mixed sensitivity problem

$$\mu(G) := \inf_{H \in S(G)} m(G, H) \tag{28}$$

where

$$m(G, H) := \left\| \begin{array}{c} W_1(I+GH)^{-1} \\ W_2 H(I+GH)^{-1} \end{array} \right\|_\infty \tag{29}$$

and $W_1, W_2, W_2{}^{-1} \in M(\mathcal{H}_\infty)$. This problem arises when sensitivity is being reduced in conjunction with a robustness constraint (eg. [12]). Using the Youla parametrization to rewrite (29),

$$m(G, H) = \left\| \begin{array}{c} W_1(I-GQ) \\ W_2 Q \end{array} \right\|_\infty \tag{30}$$

where Q is the Youla parameter for $H \in S(G)$. The optimal value for this problem is continuous in G [35]. That is, uniform convergence of G_n to G (23) implies that

$$\lim_{n\to\infty} \mu(G_n) = \mu(G).$$

Since the mixed sensitivity problem is continuous in G, it is possible to construct a sequence of finite-dimensional controllers, H_n, which satisfy the inequality $m(G_n, H_n) \leq p$ where $p > \mu(G)$. For large enough n, H_n stabilizes G, and a level of performance $m(G, H_n)$ arbitrarily close to p can be achieved when the finite-dimensional controller H_n is implemented with the original model G. This is stated precisely and proven below.

Theorem 6.2 *Let G_n be a sequence of transfer functions in $M(\mathcal{H}_\infty)$ which converge to G in norm on $M(\mathcal{H}_\infty)$. Choose some $p > \mu(G)$ where $\mu(G)$ is defined in (28, 29). It is possible to choose a sequence of controllers which satisfy*

$$m(G_n, H_n) \leq p \tag{31}$$

for sufficiently large n, where $m(\cdot,\cdot)$ is given by (29). For sufficiently large N, H_n stabilizes G for $n > N$, and furthermore, for any $\varepsilon > 0$, we can choose N large enough so that

$$m(G, H_n) \leq p + \varepsilon, \quad n > N. \tag{32}$$

If the sequence of H_n are optimal to within a tolerance γ i.e. *for some N,*

$$m(G_n, H_n) \leq \mu(G_n) + \gamma, \quad n > N; \tag{33}$$

then,

$$\text{limsup}_{n\to\infty} m(G, H_n) \leq \mu(G) + \gamma.$$

That is, performance arbitrarily close to optimal can be obtained with a finite-dimensional controller H_n.

Proof: Since $\mu(G_n) \to \mu(G)$, $\mu(G_n) < p$ for sufficiently large n. This proves (31).

The boundedness of the performance measure $m(G_n, H_n)$ guarantees, using (30), that the Youla parameters $\| Q_n\|_\infty \leq M$ for some constant M. For sufficiently large n,

$$H_n(I+GH_n)^{-1} = Q_n(I+(G-G_n)Q_n)^{-1},$$

$$(I+GH_n)^{-1} = (I-G_nQ_n)(I+(G-G_n)Q_n)^{-1}$$

are in $M(\mathcal{H}_\infty)$. Therefore, from the definition of external stability, $H_n = Q_n(I-G_nQ_n)^{-1}$ stabilizes G for large n.

Furthermore,

$$m(G,H_n) \leq \left\| \begin{bmatrix} W_1(I-G_nQ_n) \\ W_2Q_n \end{bmatrix} \right\| \left\|(I+(G-G_n)Q_n)^{-1}\right\|.$$

Thus,

$$m(G,H_n) \leq m(G_n,H_n)\left\|(I+(G-G_n)Q_n)^{-1}\right\|. \tag{34}$$

By choosing N large enough, (32) can be satisfied for any $\varepsilon > 0$.

If , for some γ, N, (33) is satisfied, then since $\lim_{n\to\infty}\mu(G_n) = \mu(G)$, $\| Q_n\|_\infty$ is again bounded and H_n stabilizes G for sufficiently large n. Taking limits of both sides of (34),

$$\text{limsup}_{n\to\infty} m(G,H_n) \leq \mu(G)+\gamma$$

as required. □

Note that not only have these results not been previously obtained, use of the graph topology leads to a short and simple proof.

7 Conclusions

For most applications, finite-dimensional approximations must be used in designing a controller for an infinite-dimensional system. The graph topology has been used here to discuss convergence of approximations to bounded semigroup control systems. Convergence in this topology is stronger than the convergence required for simulation of the original system. However, convergence in the graph topology is a necessary condition for any scheme to be satisfactory for controller design.

It has been shown that the standard assumptions used in numerical analysis, plus an additional condition, uniform stabilizability (or detectability), are sufficient for the convergence in the graph topology. This work is currently being extended to problems where control and/or observation may be unbounded.

Several examples have been given of common problems, and common approximation schemes for these problems, which converge in the graph topology. Banks' and Kunisch's results [5] have been extended to show that Galerkin type approximations to sectorial operators are uniformly stabilizable, and converge in the graph topology.

At the present time, necessary and sufficient conditions for an approximation scheme to converge in the graph topology are not known. The standard convergence conditions (A1)-(A3) with uniform stabilizability (A4) of the approximations are sufficient. Uniform convergence of the resolvants is not necessary for this condition to be satisfied. Furthermore, strong convergence of the adjoint semigroups, even with uniform external stability is not sufficient. However, the questions of whether uniform stabilizability is necessary and whether it can be obtained without strong convergence of the adjoint semigroups remain unanswered.

Once a scheme has been shown to converge in the graph topology, we would also like to be assured that a given design technique will yield reliable results. We need to establish that the sequence of finite-dimensional controllers at least stabilizes the original system for high enough model order. Convergence of controllers designed using finite-dimensional approximations and $\mathcal{H}_\infty$ techniques has been discussed. Future work will demonstrate convergence of linear quadratic regulator type controllers.

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