Journal of Mathematical Systems, Estimation, and Control
Vol. 4, No. 3, 1994, pp. 1–25

Discrete-time Gauss-Markov Processes with Fixed Reciprocal Dynamics*

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Abstract

Motivated by a problem considered earlier by Schrödinger [1]–[2], Jamison [3]–[4] and others, we examine in this paper the construction of Gauss-Markov processes with fixed reciprocal dynamics. Given the class of reciprocal processes specified by a second-order model, a procedure is described for constructing a Markov process in the class with preassigned marginal probability densities at the end points. The problem of changing the final density of a Gauss-Markov process while remaining in the same reciprocal class is also examined, and is interpreted in terms of an estimation problem.

1 Introduction

The goal of this paper is to develop techniques for constructing discrete-time Gauss-Markov processes with given reciprocal dynamics. We recall that a $\mathbb{R}^n$ valued discrete-time stochastic process $x(k)$ defined over the interval $I = [0, N]$ is said to be *reciprocal* if for any subinterval $[K, L] \subset I$, the process in the interior of $[K, L]$ is conditionally independent of the process in $I - [K, L]$ given $x(K)$ and $x(L)$. From this definition, we can immediately conclude that Markov processes are necessarily reciprocal, but the converse is not true [5]. Reciprocal processes were introduced in 1932 by Bernstein [6] who was influenced by an attempt of Schrödinger [1], [2] at giving a stochastic interpretation of quantum mechanics. The stochastic processes considered by Schrödinger were Markov processes, but they had the interesting feature that the marginal probability densities of the process at both

*Received October 7, 1992; received in final form January 23, 1993. Summary appeared in Volume 4, Number 3, 1994.
†The research described in this paper was supported by the Office of Naval Research under Grant USN-N00014-89-J-3153.
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ends of the interval of interest were preassigned. Schrödinger showed that such Markov processes could be constructed by solving two coupled nonlinear integral equations. After their introduction by Bernstein, reciprocal processes were studied in detail by Jamison [5], [3], [4], who showed that they could be constructed by pinning a Markov process at both ends of a fixed time interval, and then assigning an arbitrary probability density to the end points of the process. From this perspective, two reciprocal processes obtained by pinning the same Markov process, but assigning different end-point densities, can be viewed as equivalent since they have the same dynamics. We refer the reader to [7] for a discussion of the concept of equivalence of reciprocal processes.

In the context of Jamison's work, Schrödinger's problem can be viewed as one where given a Markov process, we seek to construct another Markov process in the same reciprocal class, with the same initial density, but a different end-point density. Refining earlier proofs of Fortet [8] and Beurling [9], Jamison showed that the two coupled integral equations of Schrödinger admit a unique solution. More recently, it was shown [10], [11] that the problem of changing the final density of a Markov process can be formulated as a stochastic optimal control problem.

Starting with Krener's work [12], a significant amount of attention has focused on developing dynamical models for reciprocal processes. It was shown in [13], [14] that Gaussian reciprocal processes whose covariance is uniformly positive definite over the interval of interest admit self-adjoint second-order models driven by locally correlated noise, where the noise correlation structure is totally determined by the model dynamics.

In this paper, given the equivalence class of discrete-time Gaussian reciprocal processes specified by a second-order model, we develop a method for constructing a Gauss-Markov process in the class with preassigned marginal end-point probability densities. The construction procedure relies on a characterization of the class of boundary conditions of second-order models corresponding to Markov processes, and requires finding the positive definite solution of a standard algebraic Riccati equation. The problem of changing the end-point density of a discrete-time Gauss-Markov process while remaining in the same reciprocal class is also discussed. Unlike the continuous-time case, it is shown that this problem does not admit a stochastic optimal control interpretation, but an alternative, more general, interpretation is given in terms of an estimation problem.

The paper is organized as follows. In Section 2 we briefly review the properties of the second-order models of Gaussian reciprocal processes introduced in [13]. In Section 3 it is shown how to construct a Markov process with given second-order model and end-point marginal densities. In Section 4 we consider the problem of changing the end-point density of a Markov process, while remaining in the same reciprocal class. A stochastic interpretation of the solution of this problem is provided in Section 5.

2 Second-order Models of Reciprocal Processes

We start by reviewing the properties of the second-order models of discrete-time Gaussian reciprocal processes introduced in [13]. Let $x(k)$ be a zero-mean Gaussian reciprocal process taking values in $\mathbb{R}^n$, defined over the finite interval $I = [0, N]$ and with covariance $R(k,l) = E[x(k)x^T(l)]$. When the process $x(k)$ is nonsingular over I, i.e. when its covariance matrix $\mathbf{R} = (R(k,l))_{0 \leq k,l \leq N}$ is positive definite, $x(k)$ admits a second-order model of the form

$$-M_-(k)x(k-1) + M_0(k)x(k) - M_+(k)x(k+1) = e(k) , \tag{2.1}$$

for $1 \leq k \leq N-1$. This model has the following features:

(i) If Z denotes the forward shift operator $Zf(k) = f(k+1)$, the operator

$$\Lambda \triangleq M_0(k)I - M_+(k)Z - M_-(k)Z^{-1} \tag{2.2a}$$

is self-adjoint, so that

$$M_0(k) = M_0^T(k) , \qquad M_+(k) = M_-^T(k+1) . \tag{2.2b}$$

(ii) The driving noise $e(k)$ is the *conjugate process* of $x(k)$, and has the property

$$E[e(k)x^T(l)] = I\delta(k-l) , \tag{2.3}$$

which implies

$$\Lambda R(k,l) = I . \tag{2.4}$$

(iii) $e(k)$ is locally correlated with covariance

$$E(k,l) = 0 \quad \text{for} \quad |k-l| > 1 \tag{2.5a}$$

$$E(k,k) = M_0(k) , \qquad E(k,k+1) = -M_+(k) . \tag{2.5b}$$

In order to specify completely the reciprocal process $x(k)$ in terms of the model (2.1), some boundary conditions must be provided. These can take either the form of Dirichlet conditions

$$\begin{bmatrix} x(0) \\ x(N) \end{bmatrix} = b \sim \mathcal{N}(0, P) \tag{2.6a}$$

$$P = \begin{bmatrix} R(0,0) & R(0,N) \\ R(N,0) & R(N,N) \end{bmatrix} , \tag{2.6b}$$

where b is independent of the driving noise $e(k)$, or of cyclic boundary conditions

$$-M_-(0)x(N) + M_0(0)x(0) - M_+(0)x(1) = e(0) \quad (2.7a)$$

$$-M_-(N)x(N-1) + M_0(N)x(N) - M_+(N)x(0) = e(N) \quad (2.7b)$$

where it is now assumed that the self-adjointness relations (2.2b) hold for $0 \leq k \leq N$ with $k+1$ defined modulo $N+1$, and $e(0)$ and $e(N)$ satisfy (2.3).

Following Jamison [3] and Clark [7], the concept of local equivalence of reciprocal processes can be formulated as follows.

Definition 2.1 *Two reciprocal processes $x(k)$ and $x'(k)$ are said to be locally equivalent if the three point transition density $r(x_{k-1}, k-1; x_k, k; x_{k+1}, k+1)$ of $x(k)$ given $x(k-1)$ and $x(k+1)$ is the same for both processes, or equivalently, if they admit the same dynamics (2.1) for $1 \leq k \leq N-1$. Quantities which are preserved under local equivalence are called reciprocal invariants.*

Thus, the dynamics (2.1) define an equivalence class of reciprocal processes, where two processes in the same class differ only by their boundary conditions. To specify a reciprocal process $x(k)$ within the equivalence class, we must select covariance matrices

$$\Pi(0) = R(0,0) \qquad \Pi(N) = R(N,N) \qquad R(0,N) \qquad (2.8)$$

which then yield a Dirichlet condition of the form (2.6a)–(2.6b). Equivalently, if we consider the cyclic conditions (2.7a)–(2.7b), we see from the self-adjointness conditions (2.2b) that

$$M_+(0) = M_-^T(1) \qquad M_-(N) = M_+^T(N-1) \qquad (2.9)$$

so that $M_+(0)$ and $M_-(N)$ are specified by the dynamics (2.1), and the only matrices we are free to choose are

$$M_0(0) \qquad M_0(N) \qquad M_+(N) = M_-^T(0)\,, \qquad (2.10)$$

where $M_0(0)$ and $M_0(N)$ must be symmetric positive definite. Thus, independently of whether we consider Dirichlet or cyclic boundary conditions, we have exactly the same number of degrees of degree of freedom in specifying a process $x(k)$ in the reciprocal class defined by (2.1).

Since Markov processes constitute a subclass of reciprocal processes, there are some processes in the class specified by (2.1) which are Markov. In addition to a second-order description, these processes admit a first-order state-space model of the form

$$x(k+1) = A(k)x(k) + w(k) \qquad 0 \leq k \leq N-1\,, \qquad (2.11)$$

where $w(k)$ is a zero-mean white Gaussian noise (WGN) process independent of $x(0)$ with intensity $Q(k)$, i.e. $w(k) \sim \mathcal{N}(0, Q(k))$. If the process $x(k)$ is nonsingular over $[0, N]$, i.e. if the covariance matrix $\mathbf{R}$ is positive definite, it is easy to verify that the noise $w(k)$ has full rank, so that $Q(k) > 0$ for all k. In [13], it is shown that there exists the following relations between the matrices $M_0(k), M_-(k), M_+(k)$ of the second order model (2.1) and the matrices $A(k), Q(k)$ of the state-space model (2.11):

$$M_0(k) = Q^{-1}(k-1) + A^T(k)Q^{-1}(k)A(k) \tag{2.12a}$$

$$M_+(k) = A^T(k)Q^{-1}(k) \tag{2.12b}$$

$$M_-(k) = Q^{-1}(k-1)A(k-1)\,. \tag{2.12c}$$

Furthermore, the conjugate process driving the second-order model (2.1) can be expressed in terms of the white noise $w(k)$ driving the state-space model (2.11) as

$$e(k) = Q^{-1}(k-1)w(k-1) - A^T(k)Q^{-1}(k)w(k)\,. \tag{2.13}$$

3 Construction of Markov Processes with Fixed Reciprocal Dynamics

In [13] the following characterization of nonsingular reciprocal processes defined over a finite interval was obtained.

Theorem 3.1 $\mathbf{R} > 0$ *is the covariance matrix of a reciprocal process defined over* I *if and only if its inverse covariance* $\mathbf{R}^{-1}$ *has a cyclic block tridiagonal structure, i.e.*

$$\mathbf{R}^{-1} = \tag{3.1}$$

$$\begin{bmatrix} M_0(0) & -M_+(0) & 0 & \cdots & 0 & -M_-(0) \\ -M_-(1) & M_0(1) & -M_-(1) & 0 & \cdots & 0 \\ \cdots & & & & & \cdots \\ 0 & \cdots & 0 & -M_-(N-1) & M_0(N-1) & -M_+(N-1) \\ -M_+(N) & 0 & \cdots & 0 & -M_-(N) & M_0(N) \end{bmatrix}.$$

In contrast, it was shown by Ackner and Kailath [15] that $\mathbf{R} > 0$ is the covariance of a Markov process if and only if $\mathbf{R}^{-1}$ is *block tridiagonal*, so that within the equivalence class of reciprocal processes with dynamics (2.1), the subclass of Markov processes has the feature that the corner matrices of $\mathbf{R}^{-1}$ are zero, i.e.

$$M_+(N) = M_-^T(0) = 0\,. \tag{3.2}$$

The identity (3.2) provides a simple characterization of the subclass of Markov processes with reciprocal dynamics (2.1), but it does not tell us how to construct a Markov process $x(k)$ with reciprocal characteristics (2.1) and marginal end-point densities

$$x(0) \sim \mathcal{N}(0, \Pi(0)) \qquad\qquad x(N) \sim \mathcal{N}(0, \Pi(N))\,. \tag{3.3}$$

As was already indicated, once the reciprocal dynamics (2.1) are specified, the selection of a particular process within the reciprocal class is accomplished by imposing a set of boundary conditions. In order to ensure that the resulting model is well posed, the only constraint that needs to be imposed on Dirichlet boundary conditions is that the matrix P in (2.6b) should be positive definite. However, if we want the solution of the model to be a Markov process with marginal end-point densities (3.3), the covariance matrix P cannot be selected arbitrarily. Specifically, the Markov property introduces additional constraints which have the effect of fixing completely the correlation matrix $R(0, N)$ in terms of the given data. Similarly, if we consider cyclic boundary conditions of the form (2.7a)–(2.7b), a consequence of the constraint (3.3) is that in the Markov case, the cyclic conditions are completely specified by $M_0(0)$ and $M_0(N)$, which are also determined by the given data.

The objective of this section is to develop a procedure for computing the correlation matrix $R(0, N)$ and the cyclic boundary matrices $M_0(0)$ and $M_0(N)$ in function of the reciprocal dynamics (2.1) and end-point covariances $\Pi(0)$ and $\Pi(N)$. A method will also be presented for constructing a first-order state-space model for the Markov process $x(k)$.

As starting point, we introduce the Green's function , (k, l) associated to the operator Λ. It satisfies

$$\Lambda,\,(k, l) = I\delta(k - l) \tag{3.4a}$$

$$,\,(0, l) = \,,\,(N, l) = 0 \tag{3.4b}$$

with $1 \leq k, l \leq N-1$, where the self-adjointness property of Λ implies that , (k, l) is self-adjoint, i.e.

$$,\,(k, l) = \,,\,^{T}(l, k)\,. \tag{3.5}$$

Note that , (k, l) is independent of the boundary conditions, and depends only on the dynamics (2.1), so that it is a reciprocal invariant. Then, the solution of the model (2.1) with Dirichlet conditions (2.6a)–(2.6b) is given by

$$\begin{aligned} x(k) \;\; = \;\; & \sum_{l=1}^{N-1} ,\,(k, l)e(l) \\ & +,\,(k, 1)M_{-}(1)x(0) + \,,\,(k, N-1)M_{+}(N-1)x(N) \end{aligned} \tag{3.6}$$

for $1 \leq k \leq N-1$. When $x(k)$ is a Markov process, the property (3.2) implies that the cyclic boundary conditions (2.7a)-(2.7b) take the form

$$
\begin{aligned}
M_0(0)x(0) - M_-^T(1)x(1) &= e(0) && (3.7a) \\
-M_+^T(N-1)x(N-1) + M_0(N)x(N) &= e(N)\,, && (3.7b)
\end{aligned}
$$

which after substitution of the values of $x(1)$, $x(N-1)$ given by (3.6) yield

$$
\begin{aligned}
T_0 x(0) - Sx(N) &= f(0) && (3.8a) \\
-S^T x(0) + T_N x(N) &= f(N) && (3.8b)
\end{aligned}
$$

with

$$
\begin{aligned}
T_0 &= M_0(0) - M_-^T(1),\ (1,1)M_-(1) && (3.9a) \\
T_N &= M_0(N) - M_+^T(N-1),\ (N-1,N-1)M_+(N-1) && (3.9b) \\
S &= M_-^T(1),\ (1,N-1)M_+(N-1) && (3.9c) \\
f(0) &= e(0) + M_-^T(1)\sum_{l=1}^{N-1},\ (1,l)e(l) && (3.9d) \\
f(N) &= e(N) + M_+^T(N-1)\sum_{l=1}^{N-1},\ (N-1,l)e(l)\,. && (3.9e)
\end{aligned}
$$

Taking into account the orthogonality property (2.3) of the conjugate process gives

$$
\begin{aligned}
E[f(0)x^T(0)] &= E[f(N)x^T(N)] = I && (3.10a) \\
E[f(0)x^T(N)] &= E[f(N)x^T(0)] = 0\,. && (3.10b)
\end{aligned}
$$

From (3.8a)–(3.8b) and (3.10a)–(3.10b), we deduce that

$$
P^{-1} = \begin{bmatrix} T_0 & -S \\ -S^T & T_N \end{bmatrix}. \tag{3.11}
$$

The relations (3.9a)–(3.9b) indicate that T_0 and T_N depend on $M_0(0)$ and $M_0(N)$, and thus on the boundary conditions. But the matrix S given by (3.9c) is a *reciprocal invariant*, since it is completely determined by the dynamics (2.1). Thus the problem of finding a Dirichlet condition for a Markov process with dynamics (2.1) and marginal densities (3.3) can be formulated as the problem of finding a positive definite matrix P such that

$$
P = \begin{bmatrix} \Pi(0) & * \\ * & \Pi(N) \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} * & -S \\ -S^T & * \end{bmatrix}, \tag{3.12}
$$

where the entries denoted by a $*$ need to be determined.

To solve this problem, consider the LDU factorization

$$P = \begin{bmatrix} I & 0 \\ Z & I \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \begin{bmatrix} I & Z^T \\ 0 & I \end{bmatrix} , \tag{3.13}$$

which implies

$$P^{-1} = \begin{bmatrix} I & -Z^T \\ 0 & I \end{bmatrix} \begin{bmatrix} X^{-1} & 0 \\ 0 & Y^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -Z & I \end{bmatrix} . \tag{3.14}$$

From

$$P = \begin{bmatrix} X & XZ^T \\ ZX & ZXZ^T + Y \end{bmatrix} \tag{3.15a}$$

$$P^{-1} = \begin{bmatrix} X^{-1} + Z^T Y^{-1} Z & -Z^T Y^{-1} \\ -Y^{-1} Z & Y^{-1} , \end{bmatrix} \tag{3.15b}$$

we can identify

$$\Pi(0) = X \tag{3.16a}$$

$$\Pi(N) = ZXZ^T + Y \tag{3.16b}$$

$$S = Z^T Y^{-1} . \tag{3.16c}$$

Substituting (3.16a) and (3.16c) inside (3.16b) yields the algebraic Riccati equation (ARE)

$$\Pi(N) = YS^T \Pi(0) SY + Y . \tag{3.17}$$

Thus, we have transformed the original problem of finding boundary conditions for the Markov process with dynamics (2.1) and marginal densities (3.3) into the equivalent one of finding a positive definite solution Y to the ARE (3.17). Given such a solution, Z^T is given by

$$Z^T = SY , \tag{3.18}$$

so that all blocks appearing in the LDU factorization of P are known. Also, provided that $\Pi(0) > 0$, the condition $Y > 0$ ensures that the matrix P is positive definite. But the ARE (3.17) can also be written as

$$\left(-\frac{I}{2}\right) Y + Y \left(-\frac{I}{2}\right) + \Pi(N) - YS^T \Pi(0) SY = 0 \tag{3.19}$$

which is in the form of the ARE of continuous-time linear quadratic control (or Kalman filtering). In this equation, the continuous-time state matrix $F = -I/2$ is stable, the pair $(F = -I/2, G = \Pi^{1/2}(N))$ is reachable provided that $\Pi(N) > 0$, and the pair $(H = S, F = -I/2)$ is detectable since F is stable. This ensures [16], [17] that the ARE (3.17) admits a unique positive definite solution.

There are several ways of computing the positive definite solution of the ARE (3.17). A numerically reliable method [18] is based on the computation of the stable eigenspace of the Hamiltonian matrix

$$\mathcal{H} \triangleq \begin{bmatrix} -I/2 & -S^T \Pi(0) S \\ -\Pi(N) & I/2 \end{bmatrix} . \tag{3.20}$$

To describe this method, we first state without proof several properties of the matrix $\mathcal{H}$.

Property 1: *If λ is an eigenvalue of $\mathcal{H}$, so is $-\lambda$.*

Property 2: *$\mathcal{H}$ does not have any eigenvalue on the imaginary axis.*

Since $\mathcal{H}$ has real entries, the Property 1 implies that the complex eigenvalues of $\mathcal{H}$ occur in groups of four: $(\lambda, \lambda^*, -\lambda, -\lambda^*)$, and its real eigenvalues in groups of two: $(\lambda, -\lambda)$. From Property 2 we conclude that $\mathcal{H}$ has n eigenvalues strictly in the left half of the complex plane, and n eigenvalues strictly in the right half-plane. The stable eigenspace of $\mathcal{H}$ is then given by

$$E = \begin{bmatrix} U \\ V \end{bmatrix} \in \mathbb{R}^{2n \times n}, \qquad J \in \mathbb{R}^{n \times n} \tag{3.21a}$$

with

$$\mathcal{H}E = EJ \tag{3.21b}$$

where E has full column rank and J has all its eigenvalues in the left half-plane. The matrices E and J can be obtained by computing the Schur decomposition of $\mathcal{H}$, which takes the form

$$\mathcal{Q}^T \mathcal{H} \mathcal{Q} = \mathcal{R} \tag{3.22}$$

with $\mathcal{Q}$ orthonormal, where

$$\mathcal{R} = \begin{bmatrix} R_{11} & \dots & \dots & R_{1q} \\ 0 & R_{22} & \dots & \vdots \\ & 0 & \ddots & \vdots \\ & & 0 & R_{qq} \end{bmatrix} \tag{3.23}$$

is a block upper triangular matrix such that the diagonal blocks R_{ii} are either 1×1 matrices or 2×2 matrices with complex conjugate eigenvalues. In this decomposition, it is always possible to ensure that the $n \times n$ block R_S in the partition

$$\mathcal{R} = \begin{bmatrix} R_S & R_{SU} \\ 0 & R_U \end{bmatrix} \tag{3.24}$$

corresponds to the stable eigenvalues of $\mathcal{H}$. We can then identify $J = R_S$, and E is formed by the first n columns of $\mathcal{Q}$. It can be shown that the stable eigenspace of $\mathcal{H}$ has the following property:

Property 3: *The matrix $U^T V$ is symmetric positive definite.*

We are now in position to construct the solution of the ARE (3.17). As a consequence of Property 3, U must be invertible and

$$VU^{-1} = U^{-T}(U^T V)U^{-1} \tag{3.25}$$

is symmetric positive definite. We can rewrite (3.21b) as

$$\begin{aligned} -\frac{U}{2} - S^T \Pi(0) S V &= UJ && (3.26a) \\ -\Pi(N)U + \frac{V}{2} &= VJ \,. && (3.26b) \end{aligned}$$

Premultiplying and postmultiplying (3.26a) by VU^{-1} and U^{-1}, respectively, and postmultiplying (3.26b) by U^{-1} gives

$$\begin{aligned} -\frac{VU^{-1}}{2} - UV^{-1} S^T \Pi(0) S U V^{-1} &= VJU^{-1} && (3.27a) \\ -\Pi(N) + \frac{VU^{-1}}{2} &= VJU^{-1} \,. && (3.27b) \end{aligned}$$

Then, subtracting (3.27a) from (3.27b) we see that $Y = VU^{-1}$ satisfies the ARE (3.17), so that we have found a positive definite solution of (3.17).

Given this solution, we can now construct boundary conditions for the Markov process $x(k)$ with reciprocal dynamics (2.1) and end-point covariances $\Pi(0) > 0$ and $\Pi(N) > 0$. From (3.15a) we see that

$$R(0, N) = \Pi(0) S Y \tag{3.28}$$

specifies a Dirichlet boundary condition for $x(k)$.

Combining the expressions (3.9a)–(3.9b) for T_0 and T_N with relation (3.15b) for P^{-1} yields the identities

$$\begin{aligned} T_0 &= M_0(0) - M_-^T(1), (1,1) M_-(1) = \Pi^{-1}(0) + SYS^T && (3.29a) \\ T_N &= M_0(N) - M_+^T(N-1), (N-1, N-1) M_+(N-1) \\ &= Y^{-1} && (3.29b) \end{aligned}$$

which show that once Y has been computed, the boundary matrices $M_0(0)$ and $M_0(N)$ are known. Furthermore, since $\Pi(0) > 0$ and $Y > 0$, we have $M_0(0) > 0$ and $M_0(N) > 0$ as desired. Thus the knowledge of Y yields also cyclic boundary conditions for the Markov process $x(k)$.

Finally, to construct a state-space model for the Markov process $x(k)$, we note from (2.12b) that

$$A(k) = Q(k) M_+^T(k) \,, \tag{3.30}$$

and by combining (2.12a) and (2.12b), we find that $Q^{-1}(k)$ satisfies the backward Riccati equation

$$M_0(k) = Q^{-1}(k-1) + M_+(k)Q(k)M_+^T(k), \qquad 1 \le k \le N-1 \quad (3.31)$$

with initial condition

$$M_0(N) = Q^{-1}(N-1)\,. \quad (3.32)$$

As we just saw, the matrix $M_0(N)$ can be computed from the solution Y of the ARE (3.17), and equations (3.31) and (3.30) can be used to compute $Q(k)$ and $A(k)$, respectively, for $0 \le k \le N-1$.

The entries of P, P^{-1} and their block LDU factorization can be expressed in terms of the state-space dynamics. Since the state variance $\Pi(k)$ of the Markov process $x(k)$ satisfies the Lyapunov equation

$$\Pi(k+1) = A(k)\Pi(k) + \Pi(k)A^T(k) + Q(k) \qquad 0 \le k \le N-1 \quad (3.33)$$

with initial condition $\Pi(0)$, the state variance at $k = N$ can be expressed in closed-form as

$$\Pi(N) = \Phi(N,0)\Pi(0)\Phi^T(N,0) + \sum_{k=0}^{N-1} \Phi(N,k+1)Q(k)\Phi^T(N,k+1) \quad (3.34)$$

where the state transition matrix

$$\Phi(t,s) \triangleq \prod_{k=s}^{t-1} A(k) \quad (3.35)$$

satisfies the recursion $\Phi(t+1,s) = A(t)\Phi(t,s)$, with $\Phi(s,s) = I$. The state covariance matrix $R(k,s)$ is given by

$$R(k,s) = \begin{cases} \Phi(k,s)\Pi(s) & \text{for} \quad k \ge s \\ \Pi(k)\Phi^T(s,k) & \text{for} \quad k \le s\,, \end{cases} \quad (3.36)$$

so that the Dirichlet boundary conditions have the form

$$\begin{bmatrix} x(0) \\ x(N) \end{bmatrix} = b \sim \mathcal{N}(0,P) \quad (3.37a)$$

$$P = \begin{bmatrix} \Pi(0) & \Pi(0)\Phi^T(N,0) \\ \Phi(N,0)\Pi(0) & \Pi(N) \end{bmatrix} \quad (3.37b)$$

with $\Pi(N)$ as in (3.34). Then

$$R(0,N) = \Pi(0)\Phi^T(N,0) \quad (3.38a)$$
$$Z = \Phi(N,0) \quad (3.38b)$$

$$Y = \Pi(N|0) \triangleq \Pi(N) - \Phi(N,0)\Pi(0)\Phi^T(N,0)$$
$$= \sum_{s=0}^{N-1} \Phi(N,s+1)Q(s)\Phi^T(N,s+1) \tag{3.38c}$$
$$S = \Phi^T(N,0)\Pi^{-1}(N|0) \tag{3.38d}$$
$$T_0 = \Pi^{-1}(0) + \Phi^T(N,0)\Pi^{-1}(N|0)\Phi(N,0) \tag{3.38e}$$
$$T_N = \Pi^{-1}(N|0). \tag{3.38f}$$

4 Change of End-point Density

Let us consider now the problem of changing the end-point density of a discrete-time Gauss-Markov process while remaining in the same reciprocal class. This problem can be formulated follows: given a Markov process with state dynamics $(A(k), Q(k))$ and end-point covariances $(\Pi(0), \Pi(N))$, we seek to find a new Gauss-Markov process $x^*(k)$ in the same reciprocal class, but with end-point covariances $(\Pi(0), \Pi^*(N))$.

Of course, this problem can be solved by using the results of the previous section. This would involve using identities (2.12a)–(2.12c) to compute the second-order model of $x(k)$, and then solving the algebraic Riccati equation (3.17) with $\Pi(N)$ replaced by $\Pi^*(N)$. However it is of interest to contrast this approach with the one followed by Schrödinger [1], Jamison [4], and others [19], [20] for the general case of arbitrary, i.e. not necessarily Gaussian, Markov processes. The Schrödinger-Jamison construction of the process $x^*(k)$ proceeds in two steps. First a characterization of the joint probability density of $x^*(0), \ldots, x^*(N)$ is derived. Then, in a second stage, the joint density is used to construct a state-space model for $x^*(k)$.

A Joint Density Characterization

Consider a Gauss-Markov process with state-space model (2.11) and initial density

$$p(x_0, 0) = \frac{1}{(2\pi)^{n/2}|\Pi(0)|^{1/2}} \exp\{-\frac{1}{2}x_0^T\Pi^{-1}(0)x_0\}. \tag{4.1}$$

Then, the joint density of $x(0), \ldots, x(N)$ can be expressed as

$$p(x_0, 0; x_1, 1; \ldots; x_N, N) = p(x_0, 0)\prod_{k=0}^{N-1} G(x_k, k; x_{k+1}, k+1), \tag{4.2}$$

where

$$G(x_k, k; x_{k+1}, k+1) = \frac{1}{(2\pi)^{n/2}|Q(k)|^{1/2}}$$

$$\exp\{-\frac{1}{2}(x_{k+1}-A(k)x_k)^T Q^{-1}(k)(x_{k+1}-A(k)x_k)\} \tag{4.3}$$

is the one-step transition density of the state-space model (2.11). This can be rewritten more compactly as

$$p(x_0,0;x_1,1;\ldots;x_N,N)=\frac{1}{Z}\exp\{-J\} \tag{4.4a}$$

where J is the quadratic form

$$J \triangleq \frac{1}{2}\left[\| x_0\|^2_{\Pi^{-1}(0)}+\sum_{k=0}^{N-1}||x_{k+1}-A(k)x_k||^2_{Q^{-1}(k)}\right] \tag{4.4b}$$

and the partition function Z is selected such that the joint probability density integrates to one. The marginal probability density of the final state is then given by

$$p(x_N,N)=\frac{1}{(2\pi)^{n/2}|\Pi(N)|^{1/2}}\exp\{-\frac{1}{2}x_N^T\Pi^{-1}(N)x_N\} \tag{4.5}$$

where $\Pi(N)$ satisfies (3.34).

The following procedure can be employed to change the end-point density of $x(k)$ while remaining in the same reciprocal class.

Theorem 4.1 *Given a Markov process $x(k)$ with state-space model (2.11) and initial density $p(x_0,0)$, if $x^*(k)$ is a Markov process in the same reciprocal class, with the same initial density, but with final density $p^*(x_N,N)$, where $p^*(x_N,N)$ is obtained by replacing $\Pi(N)$ by $\Pi^*(N)$ in (4.5), the joint probability density of x^* can be expressed as*

$$p^*(x_0,0;x_1,1;\ldots;x_N,N)=\prod_{k=0}^{N-1}G(x_k,k;x_{k+1},k+1)q_f(x_0)q_b(x_N)\,, \tag{4.6}$$

where if

$$G(x_0,0;x_N,N)=\frac{1}{(2\pi)^{n/2}|\Pi(N|0)|^{1/2}}$$
$$\exp\{-\frac{1}{2}(x_N-\Phi(N,0)x_0)^T\Pi^{-1}(N|0)(x_N-\Phi(N,0)x_0)\} \tag{4.7}$$

denotes the N-step transition density of the process $x(k)$, with $\Pi(N|0)$ given by (3.38c), the end-point densities $q_f(x_0)$ and $q_b(x_N)$ satisfy the coupled integral equations

$$p(x_0,0) = q_f(x_0)\int G(x_0,0;x_N,N)q_b(x_N)dx_N \tag{4.8a}$$
$$p^*(x_N,N) = q_b(x_N)\int G(x_0,0;x_N,N)q_f(x_0)dx_0\,. \tag{4.8b}$$

Proof: Jamison showed [3] that two reciprocal processes in the same class differ only by the choice of an end-point density $q(x_0, x_N)$ for the trajectories of a Markov process pinned at x_0 and x_N. Furthermore, he proved that the subclass of Markov processes is characterized by the fact that the end-point density $q(x_0, x_N)$ can be factored as

$$q(x_0, x_N) = q_f(x_0) q_b(x_N) \,. \tag{4.9}$$

Combining these two observations with the expression (4.2) for the joint density of x, we can conclude that the joint density of x^* takes the form (4.6). Also, equating the marginal densities for $x(0)$ and $x(N)$ obtained from the joint density (4.6) to the preassigned densities yields the coupled integral equations (4.8a)–(4.8b). □

Multiplying $q_f(x_0)$ by an arbitrary constant, and dividing $q_b(x_N)$ by the same constant leaves equations (4.8a)–(4.8b) unchanged, so that the solution to these equations is only fixed up to an arbitrary scaling. When such a scaling is provided, the existence and uniqueness of solutions was established in [8], [9], [3].

For Gaussian processes, the structure of equations (4.8a)–(4.8b) can be simplified considerably by assuming that q_f and q_b have the form

$$\begin{aligned} q_f(x_0) &= C_f \, \exp\{-\frac{1}{2} x_0^T \Theta_f^{-1} x_0\} && (4.10a) \\ q_b(x_N) &= C_b \, \exp\{-\frac{1}{2} x_N^T \Theta_b^{-1} x_N\} \,, && (4.10b) \end{aligned}$$

where Θ_f and Θ_b are nonsingular symmetric, not necessarily positive definite, matrices. In this case, we have

$$G(x_0, 0; x_N, N) q_b(x_N) = \frac{1}{Z_b} \, \exp\{-J_b\} \tag{4.11a}$$

with

$$J_b \triangleq \frac{1}{2} \left[(x_N - L_b x_0)^T \Sigma_b^{-1} (x_N - L_b x_0) + x_0^T \Lambda_b x_0 \right] \,, \tag{4.11b}$$

where

$$\begin{aligned} \Sigma_b^{-1} &= \Pi^{-1}(N|0) + \Theta_b^{-1} && (4.12a) \\ L_b &= \Sigma_b \Pi^{-1}(N|0) \Phi(N, 0) && (4.12b) \\ \Lambda_b &= \Phi^T(N, 0) \Pi^{-1}(N|0) \Phi(N, 0) - L_b^T \Sigma_b^{-1} L_b && \\ &= \Phi^T(N, 0) [\Theta_b + \Pi(N|0)]^{-1} \Phi(N, 0) && (4.12c) \end{aligned}$$

where the last equality follows from the Sherman-Morrison-Woodbury matrix inversion identity (see [21], p. 51). This implies

$$\int G(x_0, 0; x_N, N) q_b(x_N) dx_N = \frac{1}{K_b} \, \exp\{-\frac{1}{2} x_0^T \Lambda_b x_0\}, \tag{4.13}$$

so that by equating the coefficients of the quadratic exponents on both sides of (4.8a) we obtain

$$\Pi^{-1}(0) = \Theta_f^{-1} + \Phi^T(N,0)[\Theta_b + \Pi(N|0)]^{-1}\Phi(N,0)\,. \tag{4.14}$$

Similarly we have

$$G(x_0,0;x_N,N)q_f(x_0) = \frac{1}{Z_f}\ \exp\{-J_f\} \tag{4.15a}$$

with

$$J_f = \frac{1}{2}\left[(x_0 - L_f x_N)^T \Sigma_f^{-1}(x_0 - L_f x_N) + x_N^T \Lambda_f x_N\right] \tag{4.15b}$$

where

$$\Sigma_f^{-1} = \Phi^T(N,0)\Pi^{-1}(N|0)\Phi(N,0) + \Theta_f^{-1} \tag{4.16a}$$

$$L_f = \Sigma_f \Phi^T(N,0)\Pi^{-1}(N|0) \tag{4.16b}$$

$$\begin{aligned}\Lambda_f &= \Pi^{-1}(N|0) - L_f^T \Sigma_f^{-1} L_f \\ &= [\Pi(N|0) + \Phi(N,0)\Theta_f \Phi^T(N,0)]^{-1}.\end{aligned} \tag{4.16c}$$

This implies

$$\int G(x_0,0;x_N,N)q_f(x_0)dx_0 = \frac{1}{K_f}\ \exp\{-\frac{1}{2}x_N^T \Lambda_f x_N\} \tag{4.17}$$

so that

$$\Pi^{*-1}(N) = \Theta_b^{-1} + [\Pi(N|0) + \Phi(N,0)\Theta_f \Phi^T(N,0)]^{-1}\,. \tag{4.18}$$

Thus, in the Gaussian case, the coupled integral equations (4.8a)–(4.8b) reduce to the coupled algebraic Riccati equations (4.14) and (4.18).

Remark: In the special case when the end-point density remains the same, i.e. $\Pi^*(N) = \Pi(N)$, it is natural to expect that $x^*(k) = x(k)$. To verify this, note that the solutions of the AREs (4.14) and (4.18) are given by

$$\Theta_f = \Pi(0) \qquad \Theta_b^{-1} = 0 \tag{4.19}$$

so that

$$q_f(x_0) = p(x_0,0) \qquad q_b(x_N) \equiv 1\,, \tag{4.20}$$

which implies that the joint density of x^* is the same as that of x.

Since the coupled AREs (4.14) and (4.18) have a relatively complex form, it is not easy to demonstrate the existence of solutions directly from these two equations. Instead, it is more convenient to relate the solutions of these two equations to the positive definite solution Y^* of the ARE (3.17) with $\Pi(N)$ replaced by $\Pi^*(N)$ and S given by (3.38d). To do so, note that according to (4.6), the joint probability density of $x^*(0)$ and $x^*(N)$ is given by

$$p^*(x_0, 0; x_N, N) = G(x_0, 0; x_N, N) q_f(x_0) q_b(x_N) \,, \tag{4.21}$$

so that if P^* denotes the covariance of the vector $[x^{*T}(0) x^{*T}(N)]^T$, we have

$$P^{*-1} = \begin{bmatrix} \Theta_f^{-1} & 0 \\ 0 & \Theta_b^{-1} \end{bmatrix} + \begin{bmatrix} -\Phi^T(N,0) \\ I \end{bmatrix} \Pi^{-1}(N|0) \begin{bmatrix} -\Phi(N,0) & I \end{bmatrix} . \tag{4.22}$$

Using identities (3.29a)–(3.29b) to express the diagonal blocks of P^{*-1} in terms of Y^*, we find

$$\begin{aligned} T_0^* &= \Pi^{-1}(0) + SY^*S^T = \Theta_f^{-1} + \Phi^T(N,0)\Pi^{-1}(N|0)\Phi(N,0) && (4.23a) \\ T_N^* &= Y^{*-1} = \Theta_b^{-1} + \Pi^{-1}(N|0) \,, && (4.23b) \end{aligned}$$

so that

$$\begin{aligned} \Theta_f^{-1} &= \Pi^{-1}(0) + S(Y^* - \Pi(N|0))S^T && (4.24a) \\ \Theta_b^{-1} &= Y^{*-1} - \Pi^{-1}(N|0) \,. && (4.24b) \end{aligned}$$

This shows that solutions to the AREs (4.14) and (4.18) can be obtained from the positive definite solution Y^* of the ARE (3.17). Note that the matrices Θ_f and Θ_b are not necessarily positive definite. Observing that Y^* and $Y = \Pi(N|0)$ both solve the ARE (3.17), but for different values $\Pi^*(N)$ and $\Pi(N)$ of the end-point covariance, it is easy to check that a necessary and sufficient condition to have $\Theta_b^{-1} > 0$ is $\Pi^*(N) < \Pi(N)$. In other words, Θ_b is positive only if the uncertainty of the final state $x^*(N)$ is less than that of $x(N)$.

B Model Construction

To construct the state-space model of $x^*(k)$, we can proceed as follows. Consider the function

$$q_b(x_k, k) = \int G(x_k, k; x_N, N) q_b(x_N) \, dx_N \,, \tag{4.25a}$$

where

$$G(x_k, k; x_N, N) = \frac{1}{(2\pi)^{n/2} |\Pi(N|k)|^{1/2}} \exp\{-\frac{1}{2}(x_N - \Phi(N,k)x_k)^T \Pi^{-1}(N|k)(x_N - \Phi(N,k)x_k\} \tag{4.25b}$$

is the probability density of $x(N)$ given that $x(k) = x_k$, with

$$\Pi(N|k) \triangleq \Pi(N) - \Phi(N,k)\Pi(k)\Phi^T(N,k) \,. \tag{4.25c}$$

Substituting (4.25a) for $k = 0$ inside the integral equation (4.8a) yields

$$p(x_0, 0) = q_f(x_0) q_b(x_0, 0) \,. \tag{4.26}$$

Also, since $q_b(x_k, k)$ is obtained by integrating two Gaussian distributions, it is also Gaussian, i.e.

$$q_b(x_k, k) = C_b(k) \ \exp\left\{-\frac{1}{2} x_k^T \Theta_b^{-1}(k) x_k\right\} , \tag{4.27a}$$

where we find

$$\Theta_b^{-1}(k) = \Phi^T(N,k)[\Theta_b + \Pi(N|k)]^{-1}\Phi(N,k) \,. \tag{4.27b}$$

Consider now the modified one-step transition density

$$\begin{aligned} G^*(x_k, k; x_{k+1}, k+1) &\triangleq G(x_k, k; x_{k+1}, k+1)\frac{q_b(x_{k+1}, k+1)}{q_b(x_k, k)} && (4.28a) \\ &= C_k^* \ \exp\{-J_k^*\} && (4.28b) \end{aligned}$$

with

$$J_k^* = \frac{1}{2}\left[\| \, x_{k+1} - A(k)x_k\|^2_{Q^{-1}(k)} + \| \, x_{k+1}\|^2_{\Theta_b^{-1}(k+1)} - \| \, x_k\|^2_{\Theta_b^{-1}(k)}\right] . \tag{4.28c}$$

Substituting it inside the expression (4.6) for the joint density of $x^*(0), \ldots, x^*(N)$, and taking into account (4.26), we find that the joint density can be written as

$$p^*(x_0, 0; x_1, 1; \ldots; x_N, N) = p(x_0, 0) \prod_{k=0}^{N-1} G^*(x_k, k; x_{k+1}, k+1) \tag{4.29}$$

which is the usual expression the joint density of a Markov process, so that the function $G^*(x_k, k; x_{k+1}, k+1)$ given by (4.28a) is actually the one-step transition density of $x^*(k)$.

This observation leads to the following characterization of the process $x^*(k)$.

Theorem 4.2 *Let $x(k)$ be a Markov process with state-space model*

$$\begin{aligned} x(k+1) &= A(k)x(k) + w(k) && (4.30a) \\ x(0) &\sim \mathcal{N}(0, \Pi(0)) && (4.30b) \end{aligned}$$

where $w(k)$ is a zero-mean WGN process independent of $x(0)$ and with intensity $Q(k)$. Then the Markov process $x^*(k)$ with the same reciprocal dynamics as $x(k)$ and end-point covariances $(\Pi(0), \Pi^*(N))$ admits the state-space model

$$\begin{aligned} x^*(k+1) &= A^*(k)x^*(k) + w^*(k) && (4.31a) \\ x^*(0) &\sim \mathcal{N}(0, \Pi(0)) && (4.31b) \end{aligned}$$

where $w^*(k)$ is a WGN independent of $x^*(0)$ with intensity

$$Q^*(k) = [Q^{-1}(k) + \Theta_b^{-1}(k+1)]^{-1} , \tag{4.32a}$$

where $\Theta_b(k)$ is given by (4.27b) and

$$A^*(k) = Q^*(k)Q^{-1}(k)A(k) . \tag{4.32b}$$

Proof: We need only to show that the quadratic form J_k^* in (4.28c) can be expressed as

$$J_k^* = \frac{1}{2} \parallel x_{k+1} - A^*(k)x_k\|^2_{Q^{*-1}(k)}. \tag{4.33}$$

Expanding (4.33), it is easy to check that this is equivalent to verifying that

$$A^{*T}(k)Q^{*-1}(k)A^*(k) = A^T(k)Q^{-1}(k)A(k) - \Theta_b^{-1}(k) . \tag{4.34}$$

Substituting (4.32a) and (4.32b) inside (4.34) yields

$$\begin{aligned} &\Theta_b^{-1}(k) \\ &= A^T(k)[Q^{-1}(k) - Q^{-1}(k)[Q^{-1}(k) + \Theta_b^{-1}(k+1)]^{-1}Q^{-1}(k)]A(k) \\ &= A^T(k)[Q(k) + \Theta_b(k+1)]^{-1}A(k) && (4.35) \end{aligned}$$

so that in order to prove (4.34) we must check that the matrix function $\Theta_b(k)$ given by (4.27b) satisfies the backward equation (4.35) with final condition $\Theta_b(N) = \Theta_b$. But (4.35) can be rewritten as the forward Lyapunov equation

$$\Theta_b(k+1) = A(k)\Theta_b(k)A^T(k) - Q(k) \tag{4.36}$$

which admits the solution

$$\Theta_b = \Phi(N,k)\Theta_b(k)\Phi^T(N,k) - \Pi(N|k) \tag{4.37a}$$

with

$$\Pi(N|k) = \sum_{s=k}^{N-1} \Phi(N, s+1)Q(s)\Phi^T(N, s+1) , \tag{4.37b}$$

which is equivalent to (4.27b). □

An alternative method of computing ${Q^*}^{-1}(k), A^*(k)$, relies on the observation that $x(k)$ and $x^*(k)$ admit the same second-order model. From (2.12b), this implies

$$ {Q^*}^{-1}(k)A^*(k) = M_+^T(k) = Q^{-1}(k)A(k) \tag{4.38} $$

which gives (4.32b). Also, both $Q^*(k)$ and $Q(k)$ satisfy the Riccati equation (3.31), but with different final conditions. The final condition for Q^* is given by

$$ {Q^*}^{-1}(N-1) = Q^{-1}(N-1) + \Theta_b^{-1} , \tag{4.39} $$

so that $Q^*(k)$ could be found by propagating (3.31) backward in time. In contrast, the technique described earlier relies on the little-known fact [22] that given a solution $Q^{-1}(k)$ of a Riccati equation, any other solution ${Q^*}^{-1}(k)$ can be obtained from it by adding a correction term, where the correction term

$$ \Theta_b^{-1}(k+1) = {Q^*}^{-1}(k) - Q^{-1}(k) \tag{4.40} $$

admits a closed-form expression, which is given here by (4.27b).

An important feature of the above state-space model for $x^*(k)$ is that the intensity $Q^*(k)$ of the driving noise differs from the noise intensity $Q(k)$ of the model satisfied by $x(k)$. This difference is important, since it precludes the existence of a stochastic control interpretation of the change of end-point density problem of the type discussed in [11] for the continuous-time case. To see this, consider a dynamic system

$$ x(k+1) = A(k)x(k) + u(k) + w(k) , \tag{4.41} $$

where $w(k)$ is a WGN sequence of intensity $Q(k)$ and $u(k)$ is an input function. Then, the solution of a linear quadratic stochastic control problem for this system takes the form of a linear feedback law $u(k) = -L(k)x(k)$ which leaves invariant the the intensity $Q(k)$ of the noise $w(k)$. Thus, the state-space model for $x^*(k)$ cannot be obtained by applying a linear control law to the model satisfied by $x(k)$.

5 Stochastic Interpretation

When the matrices Θ_f and Θ_b obtained by solving the algebraic Riccati equations (4.14) and (4.18) are positive definite, the results of the previous section admit a simple stochastic interpretation.

As starting point, we observe that the joint probability density (4.6) of $x^*(0), \ldots, x^*(N)$ can be expressed as

$$ p^*(x_0, 0; x_1, 1; \ldots; x_N, N) = \frac{1}{Z^*} \exp\left\{-J^*\right\} , \tag{5.1} $$

where

$$J^* = \frac{1}{2}\left[\sum_{k=0}^{N-1} \| x_{k+1} - A(k)x_k\|^2_{Q^{-1}(k)} + \| x_0\|^2_{\Theta_f^{-1}} + \| x_N\|^2_{\Theta_b^{-1}}\right] \qquad (5.2)$$

and Z^* is the corresponding partition function. This means that $x^*(k)$ admits the stochastic model

$$x^*(k+1) = A(k)x^*(k) + n(k) \qquad (5.3a)$$
$$x^*(0) = v_0 \qquad (5.3b)$$
$$x^*(N) = v_N \,, \qquad (5.3c)$$

where $n(k)$ is a zero-mean WGN process with intensity $Q(k)$ and

$$v_0 \sim \mathcal{N}(0, \Theta_f) \qquad v_N \sim \mathcal{N}(0, \Theta_b) \qquad (5.4)$$

where v_0, v_N and $n(k)$ are independent of each other. However, (5.3a) is not a Gauss-Markov state-space model for $x^*(k)$. This is due to the fact that the model (5.3a)–(5.3b) is *overspecified*, in the sense that (5.3a) and (5.3b) alone are sufficient to specify the process $x^*(k)$ for $0 \leq k \leq N$, and (5.3c) can be viewed as an observation for this process. We are interested in obtaining the Gauss-Markov model describing the *a posteriori* distribution of $x^*(k)$ given the observation o_N corresponding to (5.3c). To do so, following an approach similar to the one employed to construct a backward Markovian model from a forward one [23], [24], we construct a quasi-martingale decomposition of the noise process $n(k)$ with respect to the sigma field $\mathcal{F}_k$ generated by $x(s)$ for $0 \leq s \leq k$ and the observation o_N.

Noting that (5.3c) can be rewritten as

$$v_N = x^*(N) = \Phi(N,k)x^*(k) + \sum_{s=k}^{N-1} \Phi(N, s+1)n(s) \,, \qquad (5.5)$$

or equivalently as

$$z(k) = \Phi(N,k)x^*(k) = -\sum_{s=k}^{N-1} \Phi(N, s+1)n(s) + v_N \,, \qquad (5.6)$$

we see immediately that through the observation (5.6), the knowledge of the state $x^*(k)$ provides some information about the driving noise $n(k)$. The part of the noise $n(k)$ which is predictable from the sigma field $\mathcal{F}_k$ is given by

$$\begin{aligned}\hat{n}(k) &= E[n(k)|\mathcal{F}_k] = E[n(k)|z(k)] \\ &= E[n(k)z^T(k)]E[z(k)z^T(k)]^{-1}z(k) \,, \end{aligned} \qquad (5.7a)$$

with

$$E[n(k)z^T(k)] = -Q(k)\Phi^T(N,k+1) \tag{5.7b}$$

$$E[z(k)z^T(k)] = \Pi(N|k) + \Theta_b \,. \tag{5.7c}$$

The residual process

$$\tilde{n}(k) = n(k) - \hat{n}(k) \tag{5.8a}$$

is then a WGN sequence with respect to the increasing family of sigma fields $\mathcal{F}_k$, with intensity

$$\begin{aligned} E[\tilde{n}(k)\tilde{n}^T(k)] & \\ = \; & Q(k) - Q(k)\Phi^T(N,k+1)(\Pi(N|k)+\Theta_b)^{-1}\Phi(N,k+1)Q(k) \\ = \; & Q(k) - Q(k)A^{-T}(k)\Theta_b^{-1}(k)A^{-1}(k)Q(k) \,. \end{aligned} \tag{5.8b}$$

Substituting the Lyapunov equation (4.36) gives

$$\begin{aligned} E[\tilde{n}(k)\tilde{n}^T(k)] &= Q(k) - Q(k)[\Theta_b(k+1) + Q(k)]^{-1}Q(k) \\ &= [Q^{-1}(k) + \Theta_b^{-1}(k+1)]^{-1} = Q^*(k) \,. \end{aligned} \tag{5.9}$$

The state-space model for $x^*(k)$ takes therefore the form

$$x^*(k+1) = A(k)x^*(k) + \hat{n}(k) + \tilde{n}(k) = A^*(k)x^*(k) + \tilde{n}(k) \tag{5.10}$$

where

$$\begin{aligned} A^*(k) &= A(k) - Q(k)\Phi^T(N,k+1)(\Pi(N|k)+\Theta_b)^{-1}\Phi(N,k) \\ &= [I - Q(k)\Phi^T(N,k+1)(\Pi(N|k)+\Theta_b)^{-1}\Phi(N,k+1)]A(k) \\ &= [I - Q(k)(Q(k)+\Theta_b(k+1))^{-1}]A(k) \\ &= Q^*(k)Q^{-1}(k)A(k) \end{aligned} \tag{5.11}$$

as expected.

Thus, the state-space model for the Gauss-Markov process $x^*(k)$ can be viewed as describing the a posteriori distribution of the model (5.3a)–(5.3b) given the observation (5.3c). An important feature of the decomposition of the noise $n(k)$ in its predictable part $\hat{n}(k)$ and residual $\tilde{n}(k)$ is that the intensity $Q^*(k)$ of $\tilde{n}(k)$ is *different* from the intensity $Q(k)$ of $n(k)$. As was noted earlier, this difference precludes the development of a stochastic control interpretation of the transformation leading from the model ($A(k)$, $Q(k)$) of $x(k)$ to the model ($A^*(k)$, $Q^*(k)$) of $x^*(k)$, since the state-feedback law arising in such interpretations leaves the noise intensity invariant. On the other hand, in the continuous time case, the decomposition of the driving noise employed above takes the form of a semi-martingale decomposition [25] into the sum of a predictable component and a martingale with respect to the sigma-field sequence considered. A key feature of such decompositions is that the martingale parts of a sample-path continuous semi-martingale decomposed with respect to different sigma-fields have the *same quadratic variation*. In other words, when the noise decomposition technique applied above is adapted to the continuous-time case, the driving noise of the state-space model for x^* will have the same intensity as the noise of the x-model, which explains why in the continuous-time case it is possible [10], [11] to derive a stochastic control interpretation of the change of end-point density for a Markov process.

We can also give a stochastic interpretation to the coupled Riccati equations (4.14) and (4.18) for Θ_f and Θ_b. To do so, note that by eliminating the vectors $x(s)$ with $1 \leq s \leq k-1$ from (5.3a)–(5.3c), we obtain

$$\begin{aligned} 0 &= -x^*(N) + \Phi(N,0)x^*(0) + w && (5.12a) \\ 0 &= -x^*(0) + v_0 && (5.12b) \\ 0 &= -x^*(N) + v_N \, , && (5.12c) \end{aligned}$$

where

$$w = \sum_{s=0}^{N-1} \Phi(N, s+1) n(s) \;\sim\; \mathcal{N}(0, \Pi(N|0)) \, . \tag{5.13}$$

The equations (5.12a)–(5.12c) can be viewed as observations where $x^*(0)$ and $x^*(N)$ are unknown vectors to be estimated. To compute their ML estimates and associated error covariance matrices, it is convenient to rewrite (5.12a)–(5.12c) as a single vector observation of the form

$$0 = H \begin{bmatrix} x^*(0) \\ x^*(N) \end{bmatrix} + r \tag{5.14a}$$

with

$$H \triangleq \begin{bmatrix} -I & \Phi(N,0) \\ -I & 0 \\ 0 & -I \end{bmatrix} , \tag{5.14b}$$

where $r = [w^T \, v_0^T \, v_N^T]^T \sim \mathcal{N}(0, R)$ with

$$R = \begin{bmatrix} \Pi(N|0) & 0 & 0 \\ 0 & \Theta_b & 0 \\ 0 & 0 & \Theta_f \end{bmatrix} . \tag{5.14c}$$

The ML estimates and error covariance matrix corresponding to the observation (5.14a) are given by

$$\begin{bmatrix} \hat{x}^*(N) \\ \hat{x}^*(0) \end{bmatrix} = H^T R^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 \tag{5.15}$$

$$P^* = \left[\begin{array}{cc} \Pi^*(N) & \Phi^*(N,0)\Pi(0) \\ \Pi(0)\Phi^{*T}(N,0) & \Pi(0) \end{array} \right] = (H^T R^{-1} H)^{-1} . \qquad (5.16)$$

From expressions (5.14b)–(5.14c), we find

$$\begin{aligned} \mathcal{I} &\triangleq H^T R^{-1} H \qquad (5.17) \\ &= \left[\begin{array}{cc} \Theta_b^{-1} + \Pi^{-1}(N|0) & -\Pi^{-1}(N|0)\Phi(N,0) \\ -\Phi^T(N,0)\Pi^{-1}(N|0) & \Theta_f^{-1} + \Phi^T(N,0)\Pi^{-1}(N|0)\Phi(N,0) \end{array} \right] . \end{aligned}$$

Comparing (5.16) and (5.17) we see that Θ_f and Θ_b must satisfy

$$\Pi^*(N) = S_0^{-1} \qquad \Pi(0) = S_N^{-1} \qquad (5.18)$$

where S_0 and S_N denote respectively the Schur complements of the blocks $\Theta_f^{-1} + \Phi^T(N,0)\Pi^{-1}(N|0)\Phi(N,0)$ and $\Theta_b^{-1} + \Pi^{-1}(N|0)$ inside the matrix $\mathcal{I}$. After some algebra, these Schur complements can be expressed as

$$\begin{aligned} S_0 &= \Theta_b^{-1} + [\Pi(N|0) + \Phi(N,0)\Theta_f \Phi^T(N,0)]^{-1} \qquad (5.19a) \\ S_N &= \Theta_f^{-1} + \Phi^T(N,0)[\Theta_b + \Pi(N|0)]^{-1}\Phi(N,0) , \qquad (5.19b) \end{aligned}$$

so that the identities (5.18) reduce to the coupled algebraic Riccati equations (4.14) and (4.18) for Θ_f and Θ_b.

Thus, we have shown that when the matrices Θ_f and Θ_b are positive definite, they can be viewed as the *a priori* covariances that need to imposed on the vectors $x^*(0)$ and $x^*(N)$, so that their *a posteriori* covariances after incorporating the dynamics (5.3a) are $(\Pi(0), \Pi^*(N))$.

6 Conclusions

In this paper we have studied the subclass of Markov processes contained in the class of discrete-time Gaussian reciprocal processes specified by a second-order model defined over a finite interval. It was shown how to choose the boundary conditions to ensure that the solution of the model is a Markov process with given initial and final marginal probability densities. The specification of boundary conditions requires the solution of an algebraic Riccati equation. We then considered the problem of changing the end-point density of a Markov process while remaining in the same reciprocal class as the original Markov process. This problem was solved by using a characterization of the joint density of the transformed process in terms of two Gaussian end-point densities q_f and q_b, which were obtained by solving two coupled algebraic Riccati equations. It was also shown that the transformed process admits a stochastic interpretation as the process corresponding to the a-posteriori density of a Markov process given a measurement of its final state.

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Communicated by Anders Lindquist