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The Generalized Solutions of Ordinary Differential Equations in the Impulse Control Problems*

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Abstract

The paper is devoted to the problems of an impulse control for the systems, whose dynamic is described by ordinary nonlinear differential equations. The problem of description for a discontinuous solution, arising from the using of an impulse control, is considered. The representation of the discontinuous (generalized) solutions is obtained on the basis of the method of a discontinuous time change. The problem of an impulse control is also considered. An existence theorem for the generalized optimization problem is proved.

Key words: nonlinear systems, impulse control, generalized solutions, differential equations with a measure

1 Introduction

The purpose of this paper is to obtain the representation of a discontinuous solution for ordinary nonlinear differential equations with an impulse control. The solution of this problem is necessary for investigating different mathematical models which arise in flight dynamics [4], in the control of observations [1], and in the control of radiation and chemical therapy [2].

The main problem for this class of systems is to find the response of the dynamic system to an impulse control of a δ- function type. Some approaches to this problem are known and they are based on using the differential equations with a measure. But using the differential equations with a measure is possible only for the special class of differential equations satisfying conditions of a Frobenious type [11]. Only in this case the response of the dynamic system to an impulse control does not depend on

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the realization of this control. In other cases the way of realization of an impulse control will be an other additional component of the control and will give new possibilities for optimization.

In this paper we have used an approach which is based on the method of a discontinuous time change, which was proposed in the papers of Rishel [10] and Warga [12] for the systems with sublinear dependence upon the unbounded control. This method was extended on a nonlinear system in the author's work [7,8], where the representation of discontinuous (generalized) solutions for the system with an impulse control has been obtained. Here we reduce the initial optimization problem to an auxiliary optimal control problem with bounded controls, which is equivalent to the initial problem. Thus we can obtain existence theorems for the optimal path of a discontinuous type, described by a generalized differential equation with a measure.

2 Statement of the Problem

Let a controllable system be described by the equations

$$\begin{aligned} &\dot{x}(t) = f(x(t), v(t), u(t), t) + B(x(t), v(t), u(t), t)w(t), \\ &\dot{v}(t) = \|w(t)\|, \\ &u(t) \in U, w(t) \in K. \end{aligned} \tag{2.1}$$

Here $x \in R^n$, $v \in R^1$, $u \in R^k$, $w \in R^m$. The vector variable $x(t)$ describes the variation of the phase variables of a controllable system, where the controls $u(t)$ and $w(t)$ respectively denote the ordinary and the generalized control components; the former corresponds to bounded controls, and the latter to controls, which are not bounded in the norm, but bounded in the integral sense. The values of the control $u(t)$ are selected from a closed bounded set $U \subset R^k$, whereas the values of the control $w(t)$ are not bounded in the norm, being selected from a closed cone $K \subset R^m$. A special equation is needed for the variable $v(t) \in R^1$, for taking into account the integral constraints on the control $w(.)$; this can be done with the aid of any of the norms of the vector W in R^m.

We shall assume that the vector function f and the matrix function B are continuous in the totality of variables $(x, v, u, t) \in R^{n+k+2}$ and for any $(u, t) : u \in V$, $t \in [0, T]$, they satisfy Lipschitz's condition in the totality of variables (x, v), i.e.

$$\begin{aligned} &\|f(x_1, v_1, u, t) - f(x_2, v_2, u, t)\| + \\ &\|B(x_1, v_1, u, t) - B(x_2, v_2, u, t)\| \leq \\ &L_1\{\|x_1 - x_2\| + |v_1 - v_2|\}, \end{aligned} \tag{2.2}$$

for any $x_1, x_2 \in R^n$, $v_1, v_2 \in R^1$ with a constant $L_1 > 0$.

The condition of the linear growth in (x, v) also follows from the inequality (2.2) and the continuity of functions f and B, i.e.,

$$\|f(x,v,u,t)\| + \|B(x,v,u,t)\| \leq L_2(1 + \|x\| + |v|), \tag{2.3}$$

for any $(x, v) \in R^{n+1}$ and a constant $L_2 > 0$.

Suppose that the control $w(\cdot)$ has a constraint

$$\int_0^T \|w(t)\| dt = V(T) \leq M < \infty \tag{2.4}$$

i.e., the allowed controls can be taken in the form of functions that are as close as desired to impulse functions of a δ-function type. This, in turn, may result in the presence of a discontinuous solution of system (2.1) under the constraint (2.4), and it requires that the concept of solution be extended. In the paper [8] the following definition of the generalized solution of system (2.1) is proposed.

Definition 2.1 *A pair of functions* $\{x(\cdot), v(\cdot)\}$ *that are continuous from the right and have a bounded variation in the interval* $[0, T]$ *is said to be a generalized solution of the system (2.1) if there exists a sequence of allowed controls* $\{u^k(\cdot), w^k(\cdot)\}$ *that satisfy the constraints*

$$u^k(t) \in U, \quad w^k(t) \in K, \quad \int_0^T \|w^k(t)\| dt \leq M < \infty \tag{2.5}$$

and such that the corresponding sequence $\{x^k(\cdot), v^k(\cdot)\}$ *of solutions of system (2.1) is convergent to the functions* $\{x(\cdot), v(\cdot)\}$ *at all the points of the continuity of the functions* $\{x(\cdot), v(\cdot)\}$*.*

Let us consider the totality of all generalized solutions of the system (2.1) under the constraint (2.4) with the initial conditions $\{x(0), v(0)\} \in A$, where the set A is bounded and closed in R^{n+1}.

The set of generalized solutions of (2.1) under these propositions is compact in the topology of weak-$*$ convergence in the space of the bounded variation function. Compactness in the topology of weak-$*$ convergence signifies that from any set of uniformly bounded functions $f_\alpha(\cdot)$ of the uniformly bounded variation we can select a sequence $f_{\alpha_k}(\cdot)$ that converges in the interval $[0, T]$ to a function $f(\cdot)$ of the bounded variation in the sense of

$$\lim_k f_{\alpha_k}(0) = f(0),$$

$$\lim_k f_{\alpha_k}(T) = f(T),$$

and

$$\lim_k f_{\alpha_k}(t) = f(t),$$

at all the points of continuity of the function $f(\cdot)$.

This result follows from the properties of the functions f and B, and from the Gronwall-Bellman lemma [5], because under the constraints (2.4) and conditions (2.2),(2.3) the set of solutions of the system (2.1) is uniformly bounded and has a uniformly bounded variation. The set of generalized solutions is the closure of the set of solutions of the system (2.1) in the topology of weak-$*$ convergence, and hence is compact in this topology. Our first purpose is to describe the set of generalized solutions in the conventional way.

3 Representation of Generalized Solutions by a Discontinuous Change of Time

Let us consider an auxiliary controllable system of differential equations for the variables $y \in R^n$, $z \in R^1$, $\eta \in R^1$ that is defined in the interval $[0, T], T_1 \leq T + M$:

$$\begin{aligned} \dot{y}(s) &= \alpha(s) f(y(s), z(s), n(s), \eta(s)) + \\ &\quad (1-\alpha(s)) B(y(s), z(s), n(s), \eta(s)) e(s), \\ \dot{z}(s) &= (1-\alpha(s)) \|e(s)\|, \\ \dot{\eta}(s) &= \alpha(s), \end{aligned} \tag{3.1}$$

with initial conditions $y(0) = x(0)$, $z(0) = 0$, $\eta(0) = 0$, and the controls in the form of functions $\alpha(\cdot)$, $n(\cdot)$, $e(\cdot)$, that satisfy the constraints

$$\alpha(s) \in [0,1], \quad n(s) \in U, \quad e(s) \in \{K \cap (\|e\| \leq 1)\}. \tag{3.2}$$

In the description of the auxiliary system (3.1) and constraints (3.2) the functions f and B, sets U and K, and constants T and M, are the same as for the initial system (2.1). Between the systems (2.1) and (3.1) there exists a correspondence specified by the two following theorems, which are proved in [8].

Theorem 3.1 *Let the functions $\{x(\cdot), v(\cdot), u(\cdot), w(\cdot)\}$ satisfy the system (2.1), and let the functions $\{u(\cdot), w(\cdot)\}$ be measurable and satisfy the constraints*

$$u(t) \in U, \quad w(t) \in K, \quad \int_0^T \|w(t)\| dt \leq M.$$

Then there exist functions $\{y(\cdot), z(\cdot), \eta(\cdot), \alpha(\cdot), n(\cdot), e(\cdot)\}$ defined in the interval $[0, T + v(T)]$, that satisfy the system (3.1) and the constraints (3.2), and such that for any $t \in [0, T]$ we have

$$\begin{aligned} x(0) &= y(0), \qquad v(0) = z(0) = 0, \\ x(t) &= y(,(t)), \quad v(t) = z(,(t)) \end{aligned} \tag{3.3}$$

where

$$, (t) = t + v(t). \tag{3.4}$$

Theorem 3.2 *Let the functions* $\{y(\cdot), z(\cdot), \eta(\cdot), \alpha(\cdot), n(\cdot), e(\cdot)\}$*, which are defined in the interval* $[0, T_1]$*,* $T_1 \leq T + M$*, satisfy the system (3.1) and the constraints (3.2), and let also*

$$\eta(T_1) = T. \tag{3.5}$$

Then there exists a sequence of controls $\{u^k(\cdot), w^k(\cdot)\}$*, which satisfy the constraints*

$$u^k(t) \in U, \quad w^k(t) \in K, \quad \int_0^T \|w^k(t)\|dt \leq M, \tag{3.6}$$

such that the corresponding sequence of solutions $\{x^k(\cdot), v^k(\cdot)\}$ *of the system (2.1) converges to the functions*

$$x(t) = y(, (t)), \quad v(t) = z(, (t))$$

at all points of continuity of the function $, (\cdot)$ *defined by the relation*

$$, (t) = \inf\{s : \eta(s) > t\}. \tag{3.7}$$

The following theorem gives us the total description of the set of generalized solutions of the system (2.1).

Theorem 3.3 *Let the pair of functions* $\{x(\cdot), v(\cdot)\}$ *be the generalized solution of the system (2.1), which is defined in the interval* $[0, T]$*. Then there exists a sequence of measurable functions* $\{\alpha^k(\cdot), n^k(\cdot), e^k(\cdot)\}$*, which are defined in the interval* $[0, T_1]$*,* $T_1 \geq T$*, that satisfy the constraints (3.2) everywhere in* $[0, T_1]$ *and such that the corresponding sequence of solution of the system (3.1)* $\{y^k(\cdot), z^k(\cdot), \eta^k(\cdot)\}$ *converges to the functions* $\{y(\cdot), z(\cdot), \eta(\cdot)\}$ *uniformly in* $[0, T_1]$*. The functions* $\{y(\cdot), z(\cdot), \eta(\cdot)\}$ *satisfy the relations*

$$\begin{array}{ll} x(t) = y(, (t)), & v(t) = z(, (t)), \\ , (t) = \inf\{s \in [0, T_1] : \eta(s) > t\}, & \end{array} \tag{3.8}$$

where $, (T) = \eta(T_1)$ *by definition.*

3.1 Proof of Theorem 3.3.

If the pair $\{x(\cdot), v(\cdot)\}$ is the generalized solution of the system (2.1), then by the definition there exists a sequence of functions $\{x^k(\cdot), v^k(\cdot), u^k(\cdot), w^k(\cdot)\}$, that satisfy the system (2.1) with measurable functions $\{u^k(\cdot), w^k(\cdot)\}$, satisfying the constraints (2.5), such that the sequence $\{x^k(\cdot), v^k(\cdot)\}$ converges to $\{x(\cdot), v(\cdot)\}$ at all points of continuity. For every k, by virtue of Theorem

3.1, there exists a totality of functions $\{y^k(\cdot), z^k(\cdot), \eta^k(\cdot), \alpha^k(\cdot), n^k(\cdot), e^k(\cdot)\}$ that satisfy the system (3.1) and the constraints (3.2) in an interval $[0, T_1]$, where $T_1 = T + v(T)$, such that

$$\begin{aligned} &x^k(t) = y^k(\Gamma^k(t)), \qquad\qquad v^k(t) = z^k(\Gamma^k(t)), \\ &\Gamma^k(t) = \inf\{s : \eta^k(s) > t\}. \end{aligned} \tag{3.9}$$

The sequence $v^k(T) = \int_0^T \|w^k(t)\|dt$ is uniformly bounded by the constant M, hence, there exists the constant T_1, such that $T_1 \geq T_1^k$ for every k. Let us complete the definition of functions $\alpha^k(\cdot), e^k(\cdot)$ by zero values in the half-interval $(T_1^k, T_1]$ and leave the same designation for them.

Let us consider now the sequence of the functions $\{y^k(\cdot), z^k(\cdot), \eta^k(\cdot)\}$ in the interval $[0, T]$; because of Lipschitz's and linear growth conditions of functions f and B (2.2),(2.3), the sequence $\{y^k(\cdot), z^k(\cdot), \eta^k(\cdot)\}$ is uniformly bounded and equicontinuous [5]. Then, by virtue of Artsella's theorem [3], we can select from the sequence $\{y^k(\cdot), z^k(\cdot), \eta^k(\cdot)\}$ a subsequence, which converges to the functions $\{y(\cdot), z(\cdot), \eta(\cdot)\}$ uniformly in the interval $[0, T]$. Let us take for this subsequence the same designation $\{y^k(\cdot), z^k(\cdot), \eta^k(\cdot)\}$ and prove that this subsequence is the same that is needed by the theorem. For proving the theorem it suffices to show that the sequence $\{y^k(\Gamma^k(t), z^k(\Gamma^k(t))\}$ converges to $\{y(\Gamma(t), z(\Gamma(t))\}$ at all points of continuity of the function $\Gamma(\cdot)$. As a first step let us prove that the sequence $\Gamma^k(t)$ converges to the value of the function $\Gamma(t) = inf\{s : \eta(s) > t\}$ at all points of continuity.

Indeed, let t be a point of continuity of the function $\Gamma(\cdot)$. Then we specify the sequence $\{s^k\}$, $s^k \in [0, T]$ that satisfies the relation $\eta^k(s^k) = t$ (such a point s^k is unique for any fixed t by virtue of the monotonicity of the function $\eta^k(\cdot)$ for any k). Let us show that s^k converges to a point $s^\star$ such that $\eta(s^\star) = t$. For any k we have the relation

$$\eta(s^k) - \eta(s^\star) = \eta(s^k) - \eta^k(s^k)$$

whose right-hand side tends to zero by virtue of the uniform convergence of η^k to η, and hence $(\lim_k \eta(s^k) = t)$.

The sequence $\{s^k\}$ is bounded, and for any of its partial limits $\bar{s}$ we have $\eta(\bar{s}) = t$, and by virtue of the continuity of the function $\Gamma(\cdot)$ at the point t, a point $s^\star$ such that $\eta(s^\star) = t$ will be unique.

This signifies that

$$\lim_k \Gamma^k(t) = \bar{s} = s^\star = \Gamma(t),$$

and hence we have established that $\Gamma^k(t)$ converges to $\Gamma^k(t)$ at all the points of continuity of the function $\Gamma(\cdot)$. Then by virtue of the uniform convergence of $\{y^k, z^k\}$ to $\{y, z\}$ we have the relations

$$y^k(, {}^k(t)) - y(, (t)) = y^k(, {}^k(t)) - y(, {}^k(t)) + y(, {}^k(t)) - y(, (t))$$
$$z^k(, {}^k(t)) - z(, (t)) = z^k(, {}^k(t)) - z(, {}^k(t)) + z(, {}^k(t)) - z(, (t))$$

and

$$\lim_k y^k(, {}^k(t)) = y(, (t))$$

$$\lim_k z^k(, {}^k(t)) = z(, (t))$$

at all points of continuity of the function , $(\cdot)$; and by virtue of the relations (3.9)

$$\lim_k x^k(t) = x(t) =$$

$$\lim_k y^k(, {}^k(t)) = y(, (t)),$$

$$\lim_k v^k(t) = v(t) =$$

$$\lim_k z^k(, {}^k(t)) = z(, (t))$$

at all points of continuity of the functions $x(\cdot), v(\cdot)$.

Hence the relation (3.8) is valid at all points of continuity; but the functions $x(\cdot), v(\cdot)$ and , $(\cdot)$ are continuous from the right, and , $(\cdot)$ is monotonically increasing, hence for every t in the interval $[0, T]$

$$\lim_{\tau^n \downarrow t} x(\tau^n) = x(t) =$$

$$\lim_{\tau^n \downarrow t} y(, (\tau^n) = y(, (t)),$$

$$\tag{3.10}$$

$$\lim_{\tau^n \downarrow t} v(\tau^n) = v(t) =$$

$$\lim_{\tau^n \downarrow t} z(, (\tau^n) = z(, (t))$$

where $\tau^n \downarrow t$ is monotonically decreasing sequence of the points of continuity. Theorem 3.3 is proved.

Let us consider the set of vectors $L \in R^{n+2}$, denoted by $L(y, z, \eta)$, such that

$$L(y, z, \eta) = \tag{3.11}$$

$$\left\{ \begin{array}{l|l} \alpha f(y, z, n, \eta) + & \\ (1-\alpha) B(y, z, n, \eta) e & \alpha \in [0, 1] \\ (1-\alpha)\|e\| & n \in U \\ \alpha & e \in \{K \cap (\|e\| \leq 1)\} \end{array} \right\}$$

Theorem 3.4 *Let us suppose that sets $L(y, z, \eta)$ be convex for every (y, z, η). Then for every generalized solution $\{x(\cdot), v(\cdot)\}$ defined in the interval $[0, T]$ there exists a totality of functions $\{y(\cdot), z(\cdot), \eta(\cdot), \alpha(\cdot), n(\cdot), e(\cdot)\}$, that satisfy the system (3.1) under the constraints (3.2) almost everywhere in the interval $[0, T_1]$, with $T_1 \geq T$; such that at all points of interval $[0, T]$ the equality (3.8) takes place.*

3.2 Proof of Theorem 3.4

By virtue of the convexity condition of the set (3.11) the right-hand side of the system (3.1) is the convex set for every (y, z, η). The existence of the uniformly convergent sequence (y^n, z^n, η^n) follows from the Theorem 3.3 and the limit functions (y, z, η) satisfy the equality (3.8). Then from the convexity condition and by virtue of Fillipov lemma [13] there exists the totality of functions (α, n, e), such that the totality of functions $(y, z, \eta, \alpha, n, e)$ satisfies the system (3.1) under the constraints (3.2) almost everywhere in the interval $[0, T_1]$. Theorem 3.4 is proved.

Now we shall consider some examples of systems which satisfy the condition of convexity.

3.3 Example 1

Let the sets

$$f(y, z, U, \eta) = \{L \in R^n : L = f(y, z, n, \eta) \mid n \in U\} \tag{3.12}$$

be convex for every (y, z, η); the function B does not depend on the control n and the sets

$$B(y, z, \eta)K = \tag{3.13}$$

$$\left\{ \frac{B(y, z, \eta)e}{||e||} \;\middle|\; e \in K \cap \{||e|| \leq 1\} \right\}$$

be convex for every (y, z, η). Then the sets $L(y, z, \eta)$ are convex for every (y, z, η) and conditions of Theorem 3.4 are satisfied.

3.4 Example 2

Let the cone K be the set of vectors with non-negative component and the norm of the vector is defined as a sum of the absolute values of components. Then for every $e \in K$ the norm of e is equal to

$$||e|| = \sum_{i=1}^{m} e_i,$$

and the set $B(y,z,\eta)K$ in above example can be represented in the following form

$$B(y,z,\eta)K = \tag{3.14}$$

$$\left\{ \begin{array}{ll|l} L_1 \in R^n = \sum_{i=1}^m B_i(y,z,\eta)e_i & & e_i \geq 0, i = 1,...,m \\ L_2 \in R^1 = \sum_{i=1}^m e_i & & \sum_{i=1}^m e_i \leq 1 \end{array} \right\}$$

It is obvious that the sets (3.14) are convex for every (y,z,η).

If conditions of Theorem 3.4 are held, then the generalized solution of the system (2.1) may be represented by a differential equation with a measure.

4 Representation of Generalized Solutions via Differential Equations with Measure

Let us consider $\{x(\cdot), v(\cdot)\}$ which is a generalized solution of a system (2.1) in the interval $[0,T]$. The pair of functions $\{x(\cdot), v(\cdot)\}$ has a bounded variation, and the function $v(\cdot)$ is monotonically non-decreasing in the interval $[0,T]$, and bounded, hence it defines in the interval $[0,T]$ a scalar nonnegative measure $V(dt)$. The set of points of discontinuity of $v(t) = V\{[0,t]\}$

$$D = \{\tau : \Delta v(\tau) = v(\tau) - v(\tau-) > 0\} \tag{4.1}$$

is countable, and the function $v(\cdot)$ can be represented in the form

$$v(t) = v^c(t) + \sum_{\tau \in D \cap \{\tau \leq t\}} \Delta v(\tau) \tag{4.2}$$

where $v^c(t)$ is a continuous function.

The generalized solution $\{x(\cdot), v(\cdot)\}$ can be represented in the following way.

Theorem 4.1 *Let the sets (3.11) be convex for every (y,z,η). Then for every $\{x(\cdot), v(\cdot)\}$ which is a generalized solution of the system (2.1) in the interval $[0,T]$ there exists:*

(i) a vector-measure $a(dt)$ in the interval $[0,T]$, such that

$$a(A) \in K \tag{4.3}$$

for all V-measurable sets A;

(ii) both V and Lebesgue measurable function $u(\cdot)$, such that

$$u(t) \in U \tag{4.4}$$

almost everywhere with respect to both V and Lebesgue measure;

(iii) the totality of Lebesgue measurable functions $\{n_\tau(\cdot), e_\tau(\cdot)\}$ *which are defined for every* $\tau \in D$ *in the interval* $[0, \Delta v(\tau)]$*; every pair* $\{n_\tau(\cdot), e_\tau(\cdot)\}$ *satisfies the constraints*

$$n_\tau(s) \in U, e_\tau(s) \in K \cap \{\|e\| \leq 1\} \tag{4.5}$$

almost everywhere in the interval $[0, \Delta v(\tau)]$ *with respect to Lebesgue measure, and satisfies the condition*

$$\int_0^{\Delta v(\tau)} e_\tau(s) ds = \Delta a(\tau) = a(\{\tau\});$$

such that generalized solution $\{x(\cdot), v(\cdot)\}$ *satisfies the differential equation with a measure*

$$\begin{aligned} dx(t) &= f(x(t), v(t), u(t), t)dt+ \\ & B(x(t), v(t), u(t), t)da^c(t)+ \\ & \sum_{\tau \in D \cap (\tau \leq t)} \Delta x(\tau)\delta(t-\tau)dt, \end{aligned} \tag{4.6}$$

$$v(t) = Var_{[0,t]} a^c(s) + \sum_{\tau \in D \cap (\tau \leq t)} \Delta v(\tau),$$

and values of functions $x(\cdot), v(\cdot)$ *in the points* $\tau \in D$ *are defined by relations*

$$\begin{aligned} x(\tau) &= y_\tau(\Delta v(\tau)), \\ v(\tau) &= z_\tau(\Delta v(\tau)), \end{aligned}$$

where functions $\{y_\tau(\cdot), z_\tau(\cdot)\}$ *satisfy the system of the differential equations*

$$\begin{aligned} \dot{y}(s) &= B(y_\tau(s), z_\tau(s), n_\tau(s), \tau)e(s) \\ \dot{z}(s) &= 1. \end{aligned} \tag{4.7}$$

with the initial conditions

$$y(0) = x(\tau-), z(0) = v(\tau-).$$

4.1 Proof of Theorem 4.1

By virtue of the Theorem 3.4 there exists a totality of functions $\{y(\cdot), z(\cdot), \eta(\cdot), \alpha(\cdot), n(\cdot), e(\cdot)\}$, such that $\{y(\cdot), z(\cdot), \eta(\cdot)\}$ satisfies the system of differential equation (3.1) under the controls $\{\alpha(\cdot), n(\cdot), (\cdot)\}$ that satisfy the constraints (3.2), and equalities (3.8) are valid everywhere in $[0,T]$. Let us define the vector-function $a(\cdot)$ by the relation

$$a(t) = \int_0^{,\,(t)} (1-\alpha(s))e(s)ds, \tag{4.8}$$

and the vector-function $u(\cdot)$ by the relation

$$u(t) = n(,\,(t)). \tag{4.9}$$

Let us prove that the function $u(\cdot)$ is both $, -$ and Lebesgue measurable in the interval $[0,T]$, where measure $,\,(dt)$ is defined in half-intervals $(a,b]$ by the relation

$$,\,\{(a,b]\} = ,\,(b) - ,\,(a).$$

The measurability of a function with respect to Lebesgue's measure follows from the equation

$$\eta(N_c) = L_c \cup D_c \tag{4.10}$$

which holds for any constant C and any component n_i of the vector function $n(s)$. Here

$$L_c = \{t : n_i(,\,(t)) \le c\}, \quad N_c = \{s : n_i(s) \le c\}$$

and D_c is a subset of set $D^, = \{t : \Delta,\,(t) > 0\}$. The subset D_c is at most countable, and therefore its Lebesgue measure is zero, whereas the set N_c is Lebesgue measurable by virtue of the measurability of $n(\cdot)$ in the interval $[0,T]$, and the function $\eta(\cdot)$ is absolutely continuous; therefore the set $\eta(N_c)$ is Lebesgue measurable, and hence the set L_c is also measurable [9].

For proving that the function $u(t) = n(,\,(t))$ is $, -$ measurable let us consider once again the set L and show that its indicator function

$$I\{L_c\} = \begin{cases} 1 & \text{for } t \in L_c \\ 0 & \text{otherwise} \end{cases}$$

is $, -$ integrable for any c. It follows from the properties of the function that

$$,\,(\eta(s)) = \begin{cases} s & \text{for } \eta(s) \in [0,T] \setminus D \\ ,\,(\tau) & \text{for } \eta(s) = \tau \in D \end{cases}$$

and therefore

$$\begin{aligned} &I\{s : ,\,(\eta(s)) \in N_c\} I\{s : \eta(s) \in [0,T] \setminus D\} = \\ &I\{s : s \in N_c\} I\{s : \eta(s) \in [0,T] \setminus D\}. \end{aligned} \tag{4.11}$$

The function in the right-hand side of (4.11) is Lebesgue measurable; according to the formula for a change of variables in the Lebesgue-Stielties integral. We therefore obtain

$$\begin{array}{l}\int_0^{,\,(t)} I\{s:,\,(\eta(s))\in N_c\}I\{s:\eta(s)\in[0,T]\setminus D\}ds= \\ \int_0^{t} I\{t:,\,(t)\in N_c\}I\{t:t\in[0,T]\setminus D\}d,\,(t).\end{array} \tag{4.12}$$

Now let us note that the set L_c can be represented by a union of two disjoint sets

$$\begin{array}{l}L_c=\{\{t:,\,(t)\in N_c\}\cap\{t:t\in[0,T]\setminus D\}\}\cup \\ \{\{t:,\,(t)\in N_c\}\cap\{t:t\in D\}\},\end{array}$$

where the second set is at most countable and is a subset of the set D; hence it is $,-$ measurable. With regard to the first set, its indicator function is $,-$ integrable by virtue of (4.12), and therefore the set itself is $,-$ measurable.

Thus, the indicator function of the set L_c is $,-$ integrable for any c; by virtue of the absolute continuity of the measure $V(dt)$ with respect to the measure $,(dt)$ (that is result of equality (3.8) $v(t)=z(,\,(t))=V\{[0,t]\}$ and of the absolute continuity of the function $z(\cdot)$ with respect to Lebesgue measure), it hence follows that the indicator function of the set L_c is integrable with respect to the measure $V(dt)$. Thus we have proved that the function $n(,\,(t))$ is both $,-$ and $V-$ measurable.

The generalized solution $\{x(t),v(t)\}$ by virtue of Theorem 3.4 can be represented in form

$$\begin{aligned}
x(t) &= y(,\,(t)) = x(0-) \\
&\quad + \int_0^{,\,(t)} \alpha(s)f(y(s),z(s),n(s),\eta(s))ds \\
&\quad + \int_0^{,\,(t)} (1-\alpha(s))B(y(s),z(s),n(s),\eta(s))e(s)ds \qquad (4.13)\\
&= x(0-) + \int_0^{,\,(t)} \alpha(s)f(y(s),z(s),n(s),\eta(s))ds + \\
&\quad + \int_0^{,\,(t)} (1-\alpha(s))I\{s:\eta(s)\in[0,T]\setminus D\}\times \\
&\qquad B(y(s),z(s),n(s),\eta(s))e(s)ds \\
&\quad + \sum_{\tau\in D\cup(\tau\le t)} \int_{,\,(\tau-)}^{,\,(\tau)} B(y(s),z(s),n(s),\tau)e(s)ds \\
&= x(0-) + x^a(t) + x^c(t) + x^d(t),
\end{aligned}$$

$$v(t) = z(,\,(t))$$

$$
\begin{aligned}
&= \int_0^{,\,(t)} (1-\alpha(s))\|e(s)\|ds \\
&= \int_0^{,\,(t)} (1-\alpha(s))I\{s : \eta(s) \in [0,T] \setminus D\}\|e(s)\|ds \qquad (4.14) \\
&+ \sum_{\tau \in D \cup (\tau \le t)} \int_{,\,(\tau-)}^{,\,(\tau)} \|e(s)\|ds = v^c(t) + v^d(t).
\end{aligned}
$$

In accordance with the formula for the change of variable in the Lebesgue-Stielties integral [6] for $\tau = \eta(s)$, and $\alpha(s) = \dot{\eta}(s)$ we can obtain the following relations

$$
\begin{aligned}
x^a(t) &= \int_0^{,\,(t)} \alpha(s) f(y(s), z(s), n(s), \eta(s))ds \\
&= \int_0^{,\,(t)} f(y(s), z(s), n(s), \eta(s))d\eta(s) \\
&= \int_0^{\eta(,\,(t))} f(y(,\,(\tau)), z(,\,(\tau)), n(,\,(\tau)), \tau)d\tau \qquad (4.15) \\
&= \int_0^{t} f(x(\tau), v(\tau), u(\tau), \tau)d\tau,
\end{aligned}
$$

$$
\begin{aligned}
x^c(t) &= \int_0^{,\,(t)} (1-\alpha(s))I\{s : \eta(s) \in [0,T] \setminus D\} \times \\
&\quad B(y(s), z(s), n(s), \eta(s))e(s)ds \\
&= \int_0^{,\,(t)} B(y(s), z(s), n(s), \eta(s)) \times \\
&\quad d\{\int_0^{s} (1-]alpha(\omega : \eta(\omega) \in [0,T] \setminus D\}e(\omega)d\omega\} \qquad (4.16) \\
&= \int_0^{,\,(t)} B(y(,\,(\tau)), n(,\,(\tau)), \tau)da^c(\tau) \\
&= \int_0^{t} B(x(\tau), v(\tau), u(\tau), \tau)da^c(\tau),
\end{aligned}
$$

where

$$
\begin{aligned}
a^c(t) &= \int_0^{,\,(t)} (1-\alpha(s))I\{s : \eta(s) \in [0,T] \setminus D\}e(s)ds \\
&= \int_0^{t} I\{\tau : \tau \in [0,T] \setminus D\}da(\tau), \qquad (4.17)
\end{aligned}
$$

and

$$\begin{aligned} v^c(t) &= \int_0^{,(t)} (1-\alpha(s))Is : \eta(s) \in [0,T] \setminus D\} \|e(s)\| ds = \\ &= Var_{[0,t]} a^c(\tau). \end{aligned} \tag{4.18}$$

Then, for $\Delta x(\tau)$, $\Delta v(\tau)$ we can obtain relations

$$\Delta x(\tau) = y(,(\tau)) - y(,(\tau-)),$$
$$\Delta v(\tau) = z(,(\tau)) - z(,(\tau-)),$$

in interval $[,(\tau-),,(\tau)]$ functions $\{y(\cdot), z(\cdot)\}$ satisfy the system of differential equations

$$\dot{y}(s) = B(y(s), z(s), n(s), \tau)e(s),$$
$$\dot{z}(s) = \|e(s)$$

with the initial condition $y(,(\tau-)) = x(\tau-)$, $z(,(\tau-)) = v(\tau-)$.

Let us map the interval $[,(\tau-),,(\tau)]$ into the interval $[0, \Delta v(\tau)]$ by the change of variable

$$\omega = z(s) - z(,(\tau-)).$$

Then by the change of variable $p_\tau(\omega) = inf\{s : z(s) - z(,(\tau-)) > \omega\}$ we can obtain the system of equations for functions $y_\tau(\omega) = y(p_\tau(\omega))$, $z_\tau(\omega) = z(p_\tau(\omega))$,

$$\begin{aligned} &\dot{y}_\tau(\omega) = B(y_\tau(\omega), z_\tau(\omega), n(p_\tau(\omega)), \tau)e(p_\tau(\omega), \\ &\dot{z}_\tau(\omega) = 1 \end{aligned} \tag{4.19}$$

with the initial condition $y_\tau(0) = y(,(\tau-)) = x(\tau-)$, $z_\tau(0) = z(,(\tau-)) = v(\tau-)$. For every τ functions $n_\tau(\cdot), e_\tau(\cdot)$ can be defined by relations

$$\begin{aligned} n_\tau(\omega) &= n(p_\tau(\omega)), \\ e_\tau(\omega) &= e(p_\tau(\omega)), \end{aligned} \tag{4.20}$$

and hence functions $n_\tau(\cdot), e_\tau(\cdot)$ are defined in the interval $[0, \Delta v(\tau)]$, and they are measurable with respect to Lebesgue measure. The measurability of functions $n_\tau(\cdot), e_\tau(\cdot)$ can be proved as above for the function $u(t) = n(,(t))$.

The combination of relations (4.16)-(4.21) proves the theorem.

5 Impulse Control Problem

Let us consider the problem of control of system (2.1) with the following performance criterion which must be minimized

$$J[x(\cdot), v(\cdot), u(\cdot), w(\cdot)] = \varphi(x(0), x(T), v(T)), \tag{5.1}$$

under terminal and phase constraints

$$\begin{array}{l} h(x(0), x(T), v(T)) = 0, \\ S(x(0), x(T), v(T)) \leq 0 \\ g(x(t), v(t), t) \leq 0 \text{ for any } t \in [0, T], \end{array} \tag{5.2}$$

where φ, h, S and g are continuous (in the totality of variables) vector functions of a correspondent dimension, with (5.2) being understood as componentwise relation.

A solution of the problem of control of system (2.1) under the constraints (2.4) and (5.2) will be sought in the class of generalized solutions. We shall require that the constraints (5.2) should hold for a generalized solution, whereas (5.2) holds only in the limit for the sequence of ordinary solutions that approximates the former, i.e. the approximating sequence $\{x^n(\cdot), v^n(\cdot)\}$ must satisfy the equations

$$\begin{array}{l} \lim_n h(x^n(0), x^n(T), v^n(T)) = 0, \\ \lim_n S(x^n(0), x^n(T), v^n(T)) \leq 0, \\ \lim_n Sup_{[0,T]} g(x^n(t), v^n(t), t) \leq 0 \;\text{ for any }\; t \in [0, T]. \end{array} \tag{5.3}$$

Now, in order to solve this problem, we consider an auxiliary control problem, which will be formulated as a problem of control of system (3.1) under the constraints on the control (3.2), with a performance criterion

$$J'[y(\cdot), z(\cdot), \eta(\cdot), \alpha(\cdot), n(\cdot), e(\cdot), T] = \varphi(y(0), y(T), z(T)) \tag{5.4}$$

under terminal and phase constraints

$$\begin{array}{ll} h(y(0), y(T), z(T)) = 0, & \eta(T_1) = T, \\ S(y(0), y(T), z(T)) = 0, & z(T) \leq M, \\ g(y(s), z(s), \eta(s)) \leq 0 & \text{for any } s \in [0, T]. \end{array} \tag{5.5}$$

By virtue of compactness of the totality of generalized solutions and continuity of performance criterion the solution of the primary optimization problem exists, if the totality of admissible solutions non empty. But this solution can be expected to have points where the equations (5.3) are not satisfied. This can be illustrated by the following example.

5.1 Example 3

Let the system be described by the differential equations:

$$\begin{array}{l} \dot{x_1}(t) = x_2(t) w(t), \\ \dot{x_2}(t) = -x_1(t) w(t), \end{array} \tag{5.6}$$

with the initial condition $x(0) = 0, x(t) = 1$; under constraints

$$w(t) \geq 0, \quad \int_0^1 w(t)dt \leq \pi, \quad x_1(t) \leq 0, \tag{5.7}$$

with a performance criterion

$$J[x_1(\cdot), x_2(\cdot)] = (x_2(1) + 1)^2 \to \min. \tag{5.8}$$

The solution of the system (5.6) can be represented in form

$$x(t) = \sin a(t), \quad x(t) = \cos a(t), \tag{5.9}$$

where $a(t) = \int_0^t w(\tau)d\tau$. The generalized solution of the system (5.6) has the same form, where $a(t)$ is the nondecreasing function which satisfies the constraints $a(0) = 0$, $a(1) \leq \pi$, and the optimal generalized solution are the functions

$$x_1(t) \equiv 0,$$

$$x_2(t) = \begin{cases} 1 & \text{at } t < \tau, \\ -1 & \text{at } t \geq \tau, \end{cases}$$

where $\tau \in (0, 1)$, and

$$a(t) = \begin{cases} 0 & \text{at } t < \tau, \\ \pi & \text{at } t \geq \tau. \end{cases}$$

The performance criterion is zero. But this solution does not satisfy the constraint

$$\limsup_{n} \sup_{[0,1]} x^n(t) \leq 0$$

because this limit is equal to 1 for every sequence that approximates the generalized solution $(x_1(\cdot), x_2(\cdot))$.

Definition 5.1 *A pair of functions $\{x(\cdot), v(\cdot)\}$ that are continuous from the right and have a bounded variation in the interval $[0, T]$ is said to be an admissible generalized solution of system (2.1) under constraints (2.4),(5.2) if:*

(i) functions $\{x(\cdot), v(\cdot)\}$ satisfy the constraints (5.2) in the following sense

$$h(x(0-), x(T), v(T)) = 0, \qquad S(x(0-), x(T), v(T)) \leq 0,$$
$$g(x(t), v(t), t) \leq 0 \textit{for any } t \in [0, T];$$

(ii) there exists a sequence of admissible controls $\{u^n(\cdot), w^n(\cdot)\}$ *that satisfy the constraints (2.5), and such that the corresponding sequence* $\{x^n(\cdot), v^n(\cdot)\}$ *of solutions of the system (2.1) converges to the functions* $\{x(\cdot), v(\cdot)\}$ *at all points of continuity, and in addition*

$$\lim_n x_n(0) = x(0-), \quad \lim_n x_n(T) = x(T), \quad \lim_n v_n(T) = v(T),$$

$$\limsup_{n \ [0,T]} g_k(x^n(t), v^n(t), t) \le 0 \text{ for any } k = 1, ..., N_3.$$

Definition 5.2 *An admissible generalized solution* $\{x^0(\cdot), v^0(\cdot)\}$ *is said to be an optimal generalized solution if the following inequality takes place*

$$\varphi(x^0(0-), x^0(T), v^0(T)) \le \varphi(x^{'}(0-), x^{'}(T), v^{'}(T)). \qquad (5.10)$$

for any admissible generalized solution $\{x^{'}(\cdot), v^{'}(\cdot)\}$.

An approach based on the using the auxiliary control problem (5.4),(5.5) permits us to obtain the theorem of the existence of the optimal generalized solution.

Theorem 5.1 *Let us suppose that sets* $L(y, z, \eta)$ *be convex for every* (y, z, η) *and let the totality of solutions of the system (2.1) under constraints (2.4), (5.2) is nonempty. Then there exists the optimal generalized solution.*

5.2 Proof of Theorem 5.1

Let us consider the auxiliary control problem for the system (3.1) under constraints (3.2), (5.5) with a performance criterion (5.4) in the interval $[0, T+M]$. By virtue of the existence of an admissible solution the totality of admissible controls in the problem (3.1), (3.2), (5.4), (5.5) is nonempty, and hence, by virtue of convexity of sets $L(y, z, \eta)$, there exists the optimal control $\{\alpha^0(\cdot), e^0(\cdot), n^0(\cdot)\}$ that is defined in the interval $[0, T_1]$, where $T_1 \le T+M$ [5]. The optimal solution of the system (3.1) $\{y^0(\cdot), z^0(\cdot), \eta^0(\cdot)\}$ corresponding to the optimal control defines a generalized solution of a system (2.1) $\{x^0\cdot), v^0(.\cdot)\}$ by the relations $x^0(t) = y^0(, {}^0(t))$, $v^0(t) = z^0(, {}^0(t))$, where $, {}^0(t) = \inf\{s : \eta^0(s) > t\}$ and $, {}^0(T) = T_1^0$ by virtue of the constraints $\eta^0(T_1^0) = T$.

Let us prove that $\{x^0(\cdot), v^0(\cdot)\}$ is the optimal generalized solution. Indeed this solution satisfies the constraints (5.2), besides there exists the sequence of functions $\{\alpha^n(\cdot)\}$ which is uniformly convergent to $\alpha^0(\cdot)$ in the interval $[0, T]$ and satisfies the constraints

$$0 < \alpha(s) \le 1, \quad \int_0^{T_1^0} \alpha^n(s) ds = T \qquad (5.11)$$

(see the proof of Theorem 2 in [8]). Then the sequence of solutions of the system (3.1) $\{y^n(\cdot), z^n(\cdot), \eta^n(\cdot)\}$ with controls $\{\alpha^n(\cdot), n^0(\cdot), e^0(\cdot)\}$ converges to $\{y^0(\cdot), z^0(\cdot), \eta^0(\cdot)\}$ uniformly in the interval $[0,T]$. By virtue of (5.11) the function $,^n(\cdot)$ which is the inverse of $\eta^n(\cdot)$ (with $\eta^n(s) = \dot{\alpha}(s) > 0$) and is defined in the interval $[0,T]$ will be absolutely continuous, and we can specify a sequence of controls

$$\begin{aligned} u^n(t) &= n(,^n(t)), \\ w^n(t) &= \tfrac{(1-\alpha^n(,^n(t))}{\alpha^n(,^n(t))} e(,^n(t)) \end{aligned} \tag{5.12}$$

that generates a sequence of solutions $\{x^n(\cdot), v^n(\cdot)\}$ of system (2.1) i.e.

$$x^n(t) = y^n(,^n(t)), \quad v^n(t) = z^n(,^n(t)). \tag{5.13}$$

By virtue of the absolute continuity of the functions $,^n(\cdot)$ we can prove the measurability of controls $\{u^n(\cdot), w^n(\cdot)\}$ with respect to the Lebesgue measure in exactly the same way as in the proof of Theorem 1 in [8]. Besides by definition $x^n(0) = y^n(0) = y^0(0) = x^0(0-)$, $v^n(0) = z^n(0) = z^0(0) = v^0(0-) = 0$, and $x^n(T) = y^n(,^0(T)) = y^n(T_1^0)$, $v^n(T) = z^n(,^0(T)) = z^n(T_1^0)$, where the sequence $\{y^n(T_1^0), z^n(T_1^0)\}$ converges to $\{y^0(T_1^0), z^0(T_1^0)\}$. Hence

$$\lim_n x^n(T) = y^0(T_1^0) = x^0(T), \quad \lim_n v^n(T) = z^0(T_1^0) = v^0(T).$$

By virtue of the uniform convergence of sequence $\{y^n(\cdot), z^n(\cdot), \eta^n(\cdot)\}$ to $\{y^0(\cdot), z^0(\cdot), \eta^0(\cdot)\}$ and of the continuity of functions $g(x,v,t)$ $k = 1, ..., N_3$,

$$\lim_n \sup_{[0,T]} g_k(x^n(t), v^n(t), t) =$$

$$\lim_n \sup_{[0,T]} g_k(y^n(,^n(t)), z^n(,^n(t)), \eta^n(,^n(t))) \le$$

$$\lim_n \sup_{[0,T_1^0]} g_k(y^n(s), z^n(s), \eta^n(s)) \le 0.$$

Hence we have proved that $\{x^0(\cdot), v^0(\cdot)\}$ is admissible generalized solution.

Let us take any admissible generalized solution $\{x'(\cdot), v'(\cdot)\}$. By virtue of the Definition 5.1 there exists a sequence of controls $\{u^n(\cdot), w^n(\cdot)\}$ and a corresponding pair of functions that satisfies the constraints (2.4), and the system of equation (2.1), such that by virtue of the Theorem 3.1 there exists a totality of functions $\{y^n(\cdot), z^n(\cdot), \eta^n(\cdot), \alpha^n(\cdot), n^n(\cdot), e^n(\cdot)\}$ that satisfy the system (3.1) and the constraints (3.2), and such that

$$\begin{aligned} &y^n(s) = x^n(\eta^n(s)), \qquad\qquad z^n(s) = v^n(\eta^n(s)), \\ &\eta^n(s) = \inf\{t : t + v^n(t) > s\}. \end{aligned} \tag{5.14}$$

Functions $\{y^n(\cdot), z^n(\cdot), \eta^n(\cdot)\}$ are defined in the interval $[0, T_1^n] = [0, T + v^n(T)]$ where $T + v^n(T) \leq T + M$ and are uniformly bounded and equicontinuous. Let us complete these functions by values $y^n(T_1^n)$, $z^n(T_1^n)$, $\eta^n(T_1^n)$ in the half-intervals $(T_1^0, T + M]$, then there exists a totality of functions $\{y(\cdot), z(\cdot), \eta(\cdot)\}$ and a subsequence of functions $\{y^{n_k}(\cdot), z^{n_k}(\cdot), \eta^{n_k}(\cdot)\}$ that converges to $\{y(\cdot), z(\cdot), \eta(\cdot)\}$ uniformly in the some interval $[0, T_1]$, $T_1 \leq T + M$. By virtue of the convexity of the sets $L(y, z, \eta)$ there exists the totality of admissible controls $\{\alpha(\cdot), n(\cdot), e(\cdot)\}$ such that functions $\{y(\cdot), z(\cdot), \eta(\cdot)\}$ satisfy the system (3.1) with this controls. The functions $\{y(\cdot), z(\cdot), \eta(\cdot)\}$ and the generalized solution $\{x'(\cdot), v'(\cdot)\}$ are connected by relations

$$\begin{aligned} &x'(t) = y(,\ (t)), \qquad\qquad v'(t) = z(,\ (t)), \\ &\text{where }\ ,\ (t) = \inf\{s : \eta(s) > t\}, \quad ,\ (T) = T_1. \end{aligned} \tag{5.15}$$

By virtue of the uniform convergence of $\{y^n(\cdot), z^n(\cdot), \eta^n(\cdot)\}$ to $\{y(\cdot), z(\cdot), \eta(\cdot)\}$, and by virtue of the relations (5.14), (5.15), the relations

$$\begin{aligned} y(0) &= \lim_n y^n(0) = \lim_n x^n(0), \\ y(T_1) &= \lim_n y^n(T_1) = \lim_n y^n(T_1^n) = x^n(T), \\ z(T_1) &= \lim_n z^n(T_1) = \lim_n z^n(T_1^n) = v^n(T), \end{aligned} \tag{5.16}$$

are valid, and hence $\{y(0), y(T_1), z(T_1)\}$ satisfy the constraints (5.5) at terminal points. In addition for any $\epsilon > 0$ and $k = 1, ..., N_3$, the function $g_k(y^n(s), z^n(s), \eta^n(s)) = g_k(x^n(\eta^n(s)), v^n(\eta^n(s)), \eta^n(s)) \leq \epsilon$ at $n \geq N(\epsilon)$ at any $s \in [0, T + M]$. Hence

$$\lim_n \sup_{[0,T_1]} g_k(y^n(s), z^n(s), \eta^n(s)) \leq 0 \tag{5.17}$$

and by virtue of the continuity of g and uniform convergence of $\{y^n, z^n, \eta^n\}$ to $\{y, z, \eta\}$ from (5.17) follows the inequality

$$\lim_n \sup_{[0,T]} g_k(y^n(s), z^n(s), \eta^n(s)) \leq 0.$$

Then it is proved that to any admissible generalized solution $\{x'(\cdot), v'(\cdot)\}$ there corresponds an admissible solution $\{y(\cdot), z(\cdot), \eta(\cdot), \alpha(\cdot), n(\cdot), e(\cdot)\}$ of system (3.1) in any interval $[0, T_1]$ with the same value of the performance criterion. It follows from the relations (5.16). Hence from the optimality of solution $\{y^0(\cdot), z^0(\cdot), \eta^0(\cdot)\}$ in the problem (3.1),(3.2),(5.4),(5.5) the following inequality takes place

$$\begin{aligned} &\varphi(x^0(0-), x^0(T), v^0(T)) = \\ &\varphi(y^0(0), y^0(T_1^0), z^0(T_1^0 \leq \\ &\varphi(y(0), y(T_1), z(T_1)) = \\ &\varphi(x'(0-), x'(T), v'(T)). \end{aligned} \tag{5.18}$$

The inequality (5.18) proves the theorem.

6 Conclusion

In this paper we have studied a new class of problems of generalized control of dynamic systems that can be transformed to conventional control problems by the method of discontinuous time change. This transformation makes it possible to find a generalized solution, to establish a theorem of existence of an optimal generalized solution, to construct an approximating sequence of ordinary controls, and to derive an equation for the generalized solution.

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