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Singular Perturbation for Controlled Wave Equations*

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Abstract

In this paper we study the approximation of the solutions to an optimal control problem with distributed parameters for the wave equation, let's say $\mathcal{P}$, through solutions of a sequence of regularized problems $\mathcal{P}_\epsilon$. We consider both the finite and infinite time horizon case. We deduce convergence of the optimal pairs of $\mathcal{P}_\epsilon$ to those of $\mathcal{P}$, as ϵ tends to zero, by means of continuous dependence on data theorems for the associated integral/algebraic Riccati equations.

Key words: optimal control, Riccati equation, dynamic programming

AMS Subject Classifications: 49J20, 49L20, 35B25

Introduction

Throughout this paper Ω will be an open, bounded domain in $\mathbf{R}^n$, with smooth boundary $\partial\Omega$. We consider the controlled boundary value problem

$$\begin{cases} y_{tt}(t,x) = \Delta y(t,x) + u(t,x) & (t,x) \in]0,T[\times\Omega \\ y(0,x) = y_0(x),\ y_t(0,x) = y_1(x) & x \in \Omega \\ y(t,x) = 0 & (t,x) \in]0,T[\times\partial\Omega, \end{cases} \tag{1}$$

where $y_0 \in H_0^1(\Omega)$, $y_1 \in L^2(\Omega)$, $T > 0$ is given (possibly $T = +\infty$), and $u \in L^2(0,T;L^2(\Omega))$.

The purpose of the present work is to obtain approximation results for two optimal control problems associated with (1)—both in the finite and

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infinite time horizon case—by using parabolic regularization on one side, and convergence results for Riccati equations on the other side.

Our motivation comes from well-known regularity properties of both the solutions to Riccati equations and the optimal pairs for optimal control problems in the case of parabolic-like dynamics, whose distinctive feature is the analyticity of the underlying semigroup.

At the outset, we fix $T \in (0, +\infty)$ and consider the problem of minimizing the quadratic cost functional

$$J(u) = \int_0^T \int_\Omega (|\nabla y(t,x)|^2 + |y_t(t,x)|^2 + |u(t,x)|^2) dx\, dt + \\ + \int_\Omega (|\nabla y(T,x)|^2 + |y_t(T,x)|^2) dx, \tag{2}$$

overall $u \in L^2([0,T] \times \Omega)$, where y is subject to (1). As a consequence of general theory on minimization of coercive forms it is known that problem $(1) - (2)$ admits a unique optimal control (see [7]).

Following Lions [7], for given $\epsilon > 0$, we consider the natural regularized boundary value problem, namely

$$\begin{cases} y^\epsilon_{tt}(t,x) = \Delta y^\epsilon(t,x) + \epsilon\, \Delta\, y^\epsilon_t(t,x) + u(t,x) & (t,x) \in]0,T[\times\Omega \\ y^\epsilon(0,x) = y_0(x),\; y^\epsilon_t(0,x) = y_1(x) & x \in \Omega \\ y^\epsilon(t,x) = 0 & (t,x) \in]0,T[\times\partial\Omega. \end{cases} \tag{3}$$

With this we associate the cost functional

$$J_\epsilon(u) = \int_0^T \int_\Omega (|\nabla y^\epsilon(t,x)|^2 + |y^\epsilon_t(t,x)|^2 + |u(t,x)|^2) dx\, dt + \\ + \int_\Omega (|\nabla y^\epsilon(T,x)|^2 + |y^\epsilon_t(T,x)|^2) dx. \tag{4}$$

As for the existence and uniqueness of an optimal control for problem $(3) - (4)$, the same comment holds true as in the case $\epsilon = 0$.

Regularization methods were introduced by J.L. Lions as an approach to the study of some boundary value problems and related optimal control problems (see [9] and [7, 8]). In [7] the author obtains convergence of the solutions of problem (3) to those of problem (1), as ϵ tends to zero, and gives applications to different linear quadratic optimal control problems. The arguments used therein are purely variational.

In the present paper, according with the *direct* approach, we shall focus our attention on integral Riccati equations associated with problem $(1)-(2)$ and $(3) - (4)$: our goal is then to prove an approximation result for the related solutions, let say P, P_ϵ respectively. (As for differential Riccati equations in infinite dimensional spaces we refer to [3]; see also [1] for a complete treatment and references).

In fact we shall show, by means of a continuous dependence on data theorem for sequences of Riccati equations, that P_ϵ converges to P—in a sense to be specified below—as ϵ tends to zero (Theorem 2.1). As a corollary we recover the cited result in [7]. As we shall see below, to achieve our goal in the finite time horizon case we use a convergence result which is contained in the paper [4].

This work, although ultimately directed to numerical purposes, deals with the problem of approximating Riccati equations in the case where the control operator is bounded, as in (1), which is a typical feature of distributed parameters systems. A more general approximation theory for Riccati equations, particularly dedicated to the case where the input operator is genuinely unbounded—such as it arises in boundary control and point control for p.d.e.—has been developed by I. Lasiecka and R. Triggiani (see, among all, the review book [6] and the references contained therein).

In the second part of this work we shall treat the more challenging infinite time horizon case. Accordingly, we set $T = +\infty$ and consider the problem of minimizing the quadratic functional

$$J_\infty(u) = \int_0^\infty \int_\Omega \left(|\nabla y(t,x)|^2 + |y_t(t,x)|^2 + |u(t,x)|^2\right) dx\, dt \tag{5}$$

overall $u \in L^2(0,\infty; L^2(\Omega))$, with y subject to (1).

Analogously to the case $T < +\infty$, we take, as regularized parabolic problem, the boundary value problem (3) with corresponding cost functional given by

$$J_{\infty,\epsilon}(u) = \int_0^\infty \int_\Omega \left(|\nabla y^\epsilon(t,x)|^2 + |y^\epsilon_t(t,x)|^2 + |u(t,x)|^2\right) dx\, dt. \tag{6}$$

Thus we consider the algebraic Riccati equations associated with problems $(1)-(5)$, $(3)-(6)$, which formally read, in the space $H^1_0(\Omega) \times L^2(\Omega)$, as

$$A^* X + XA - XBB^* X + I = 0, \tag{7}$$

$$A^*_\epsilon X_\epsilon + X_\epsilon A_\epsilon - X_\epsilon BB^* X_\epsilon + I = 0, \tag{8}$$

respectively, where A, A_ϵ, B are suitable linear operators to be specified in Section 2.

It is known ([12], [1]) that a necessary and sufficient condition for the existence of a minimal nonnegative solution to (7) ((8)) is given by stabilizability of the pairs (A, B), $((A_\epsilon, B))$ with respect to the observation operator (I in this case). In other words it is sufficient that, for any initial data (y_0, y_1), an admissible control does exist. If this happens, the dynamic programming method provides the unique optimal control in feedback form.

In fact it is known that the stabilization property holds true in both cases (see Section 3).

Actually we shall see that in this case, in order to apply a continuous dependence on data theorem for sequences of algebraic Riccati equations, we need additional information, namely stabilizability of the pair (A_ϵ, B) with respect to I which has to be uniform in ϵ (see [4], [5]). Therefore our goal will be to prove that for any initial data $(y_0, y_1) \in H_0^1(\Omega) \times L^2(\Omega)$, there exists a feedback control $u_\epsilon \in L^2(0,\infty; L^2(\Omega))$ such that $\sup_{\epsilon>0} J_{\infty,\epsilon}(u_\epsilon) < +\infty$ (Proposition 3.1).

The basic idea in the proof of Proposition 3.1 is the following: Given the data $(y_0,\ y_1)$, we build up a feedback control u_ϵ such that the closed loop equation resulting from (3) has a "stronger" damping than the one of the free system and therefore we can show—by means of energy estimates techniques—that it has solutions $(y^\epsilon, y^\epsilon_t)$ with a uniform exponential rate of decay (uniform in ϵ, too). We stress that the feedback used above is exactly of the same type as the one we can use to stabilize the wave equation. Finally, in the same framework of Theorem 2.1, we can show an approximation result even in the more delicate case $T = +\infty$, which is not contained in [7].

It should be noted that similar arguments can be applied to the case of other hyperbolic equations, such as for instance the Euler-Bernoulli equation, with natural associated cost functional. More generally, we can take

$$y_{tt}(t,x) = -\mathcal{A}(x)y(t,x) + u(t,x), \quad (t,x) \in]0,T[\times\Omega \tag{9}$$

provided that $\mathcal{A}$ is a strongly elliptic operator of order $2m$, $m \geq 1$, whose realization in $L^2(\Omega)$ - with homogeneous Dirichlet/Neumann/mixed boundary conditions - is a non-negative, self-adjoint operator. The parabolic regularized problem is still obtained by adding in the P.D.E. a strong damping depending on a little parameter $\epsilon > 0$.

The outline of the paper is the following.

In Section 1 we fix the notations and recall some known results on Riccati equations which are needed in the sequel.

In Section 2 we introduce the abstract setting for the concrete problems $(1)-(2)$, $(3)-(4)$, and we present a straightforward proof of the approximation result in the finite time horizon case.

Section 3 is mostly devoted to showing uniform stabilizability for the strongly damped wave equation (with respect to the parameter ϵ). Thus we present the approximation result in the infinite time horizon case.

1 Notations and Preliminaries

Let X and Y be two Hilbert spaces. We denote norms and inner products with $|\cdot|$ and $<\cdot,\cdot>$ respectively. We represent with $\mathcal{L}(X,Y)$ ($\mathcal{L}(X)$ if

$X = Y$), $\Sigma(X)$, $\Sigma^+(X)$ the space of all bounded linear operators from X to Y, the space of all bounded self-adjoint operators in X, and the subset of $\Sigma(X)$ of non-negative definite operators respectively.

We denote by $||\cdot||$ norms in $\mathcal{L}(X, Y)$. For any interval $I \subset \mathbf{R}$, we shall denote by $C(I, \mathcal{L}(X))$ the set of all continuous mappings from I to $\mathcal{L}(X)$. We denote by $C_s(I, \mathcal{L}(X))$ the set of all mappings $F : I \to \mathcal{L}(X)$ such that $F(\cdot)x$ is continuous for any $x \in X$. For more details on the topological structure of $C_s(I, \mathcal{L}(X))$, see [1].

If A is a linear closed operator with dense domain $D(A)$, we denote its adjoint with A^*. We denote by $\varrho(A)$, $\sigma(A)$ and $R(\lambda, A) = (\lambda - A)^{-1}$ the resolvent set, the spectrum and the resolvent operator of A, respectively.

If A is the infinitesimal generator of a strongly continuous semigroup $G(t)$ on X, we set $G(t) = e^{tA}$. Moreover, we will use the notation $A \in \mathcal{G}(M, \omega)$ for an operator A which is the generator of a C_0-semigroup e^{tA} satisfying $||e^{tA}|| \leq M\, e^{\omega t}$, $t \geq 0$, for some $M > 0$, $\omega \in \mathbf{R}$.

We recall some general results on continuous dependence on data for both differential and algebraic Riccati equations.

Let H, Y, U be three Hilbert spaces, $T > 0$. Consider the optimal control problem consisting in minimizing the quadratic functional

$$J(u) = \int_0^T (|Cy(s)|_Y^2 + |u(s)|_U^2)\, ds + < P_0 y(T), y(T) >_H \tag{1.1}$$

overall controls $u \in L^2(0, T; U)$, where y is subject to the differential equation

$$\begin{cases} y'(t) = Ay(t) + Bu(t), & t \in]0, T[, \\ y(0) = y_0 \in H. \end{cases} \tag{1.2}$$

Concerning the operators A, B, C, P_0 we shall assume that

$$\begin{array}{ll} (i) & A \text{ generates a } C_0\text{-semigroup } e^{tA} \text{ in H ;} \\ (ii) & B \in \mathcal{L}(U, H); \\ (iii) & P_0 \in \Sigma^+(H); \\ (iv) & C \in \mathcal{L}(H, Y). \end{array} \tag{1.3}$$

It is well known that, if (1.3i), (1.3ii) are fulfilled, then for any $y_0 \in H$ problem (1.2) has a unique *mild* solution y in $L^2(0, T; H)$, that is y belongs to $C([0, T]; H)$ and is given by the formula

$$y(t) = e^{tA} y_0 + \int_0^t e^{(t-s)A} Bu(s)\, ds.$$

It is also well known ([3], and [1] for complete references as for Riccati equations in infinite dimensional spaces) that under hypotheses (1.3) there exists a unique *mild* solution to the Riccati equation associated with problem $(1.1)-(1.2)$, which reads as follows:

$$\begin{cases} P' = A^*P + PA - PBB^*P + C^*C \\ P(0) = P_0. \end{cases} \tag{1.4}$$

Here, as a mild solution of (1.4), we mean a $P \in C_s([0,T];\Sigma^+(H))$ which solves the following integral equation

$$P(t)x = e^{tA^*}P_0e^{tA}x - \int_0^t e^{(t-s)A^*}[C^*C - P(s)BB^*P(s)]e^{(t-s)A}xds, x \in H.$$

Moreover, if this is the case, the dynamic programming method provides the feedback optimal control by means of the *closed loop equation* (see for instance [7]).

Consider now a sequence of Riccati equations

$$\begin{cases} P_k' = A_k^*P_k + P_kA_k - P_kB_kB_k^*P_k + C_k^*C_k \\ P_k(0) = P_{k,0} \end{cases} \tag{1.5}$$

and suppose that the following hypotheses hold:

$$\begin{array}{ll} (i) & \text{for any } k \in \mathbf{N}\ (A_k, B_k, C_k, P_{k,0}) \text{ fulfill (1.3), with } e^{tA_k} \in \mathcal{G}(M,\omega); \\ (ii) & \lim_{k\to\infty} e^{tA_k}x = e^{tA}x \ \text{ uniformly in } [0,T], \forall T>0, x \in H; \\ (iii) & \lim_{k\to\infty} e^{tA_k^*}x = e^{tA^*}x \ \text{ uniformly in } [0,T], \forall T>0, x \in H; \\ (iv) & \text{the sequences } B_k, B_k^*, C_k, C_k^*, P_{k,0} \text{ are strongly convergent} \\ & \text{respectively to } B, B^*, C, C^*, P_0. \end{array} \tag{1.6}$$

Denote by (u^*, y^*) the optimal pair for problem $(1.1)-(1.2)$, and by (u_k^*, y_k^*) the approximating optimal pair.

Then we have the following [4, Theorem 5.1]:

Theorem 1.1 *Assume* (1.3) *and* (1.6)*. Let P and P_k be the mild solutions to* (1.4) *and* (1.5) *respectively. Let (u^*, y^*) and (u_k^*, y_k^*) the related optimal pairs.*

Then, for any $T > 0$ we have

$$\lim_{k\to\infty} P_k = P \qquad \textit{in } C_s([0,T];\Sigma^+(H)),$$

$$\lim_{k\to\infty} u_k^*(t) = u^*(t) \qquad \textit{strongly and in } L^2(0,T;U),$$

$$\lim_{k\to\infty} y_k^*(t) = y^*(t) \qquad \textit{strongly and in } L^2(0,T;H).$$

In the infinite time horizon case we are concerned with a dynamical system of type (1.2) with $T = \infty$, and we want to minimize the cost functional

$$J_\infty(u) = \int_0^\infty (|Cy(s)|_Y^2 + |u(s)|_U^2)\, ds. \tag{1.7}$$

We recall that (A, B) is said C-stabilizable if, for any $y_0 \in H$, there exists $u \in L^2(0, \infty; U)$ such that the corresponding solution y of system (1.2) is such that $J_\infty(u) < +\infty$.

It is well known ([12]) that if (A, B) is C-stabilizable, then the algebraic Riccati equation

$$A^*X + XA - XBB^*X + C^*C = 0 \tag{1.8}$$

has a minimal nonnegative solution P^∞_{min} which provides the way to solve the above optimal control problem by means of dynamic programming.

Consider now a sequence of algebraic Riccati equations

$$A_k^* X_k + X_k A_k - X_k B_k B_k^* X_k + C_k^* C_k = 0. \tag{1.9}$$

Before stating the corresponding approximation result, we need to introduce the following definitions.

Definition 1.1 *We say that (A_k, B_k) is stabilizable with respect to C_k uniformly in k if for any $y_0 \in H$ there exists $u \in L^2(0, \infty; U)$ such that*

$$\sup_k J_{\infty,k}(u) < +\infty.$$

Remark 1.1 Uniform stabilization trivially implies C_k-stabilization of each pair (A_k, B_k) for k fixed. Under this assumption the feedback operator $F_k = A_k - B_k B_k^* P^\infty_{k,min}$ is obviously well defined, where $P^\infty_{k,min}$ is the minimal nonnegative solution to (1.9).

Definition 1.2 *We say that (A_k, C_k) is detectable uniformly in k if there exist $\mathcal{K}_k \in \mathcal{L}(Y, H)$ and positive constants M, N, a, independent of k, such that $|\mathcal{K}_k x| \le M|x|$ and*

$$||e^{t(A_k - \mathcal{K}_k C_k)}|| \le N e^{-at}, \quad \text{for any } t > 0. \tag{1.10}$$

The following theorem holds (see [5]):

Theorem 1.2 *Assume (1.3), (1.6) and, in addition, that (A_k, B_k) is stabilizable with respect to C_k uniformly in k and the uniform detectability condition holds true. Then, as $k \to \infty$, we have*

$$\begin{array}{ll} |P^\infty_{k,min}x - P^\infty_{min}x| \to 0 & \text{for any } x \in H; \\ |y_k^* - y^*| \to 0 & \text{in } L^2(0,\infty;H) \text{ and in } C(0,\infty;H); \\ |u_k^* - u^*| \to 0 & \text{in } L^2(0,\infty;U). \end{array}$$

2 The Finite Time Horizon Case

Concerning the abstract formulation of problems $(1)-(2)$ and $(3)-(4)$, we set $K = L^2(\Omega)$, $\Lambda: H^2(\Omega) \cap H^1_0(\Omega) = D(\Lambda) \subset K \to K$ the Dirichlet realization of $-\Delta$ in K. Therefore Λ is a strictly positive self-adjoint operator on K with discrete spectrum $\sigma(\Lambda) = \{\lambda_n \mid \quad \lambda_n \geq \lambda_1 > 0, \quad \lambda_n \to \infty\}$.

We introduce the Hilbert space $H = D(\sqrt{\Lambda}) + K$, endowed with the inner product

$$< \begin{pmatrix} v_0 \\ v_1 \end{pmatrix}, \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} >_H = < \sqrt{\Lambda} v_0, \sqrt{\Lambda} z_0 >_K + < v_1, z_1 >_K,$$

and define $A : D(A) \subset H \to H, D(A) = D(\Lambda) + D(\sqrt{\Lambda})$, as follows:

$$A \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} = \begin{pmatrix} 0 & I \\ -\Lambda & 0 \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \end{pmatrix}.$$

For any function $w(t,x)$ we set $w(t) = w(t, \cdot)$. Then, introduced $Y(t) = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}$, problem (1) may be written in the abstract form

$$\begin{cases} Y'(t) = AY(t) + Bu(t) & t \in]0,T[\\ Y(0) = Y_0 \end{cases}, \tag{2.1}$$

where $Y_0 = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix}$ and B is defined by $Bu(t) = \begin{pmatrix} 0 \\ u(t) \end{pmatrix}$, while the functional (2) becomes

$$J(u) = \int_0^T (|Y(t)|_H^2 + |u(t)|_U^2) dt + |Y(T)|_H^2, \tag{2.2}$$

$U = L^2(\Omega)$ being the controls space.

Since $A = -A^*$, it is well known that A is the infinitesimal generator of a C_0-group of contractions e^{tA} in H (see for instance [10]). As (1.3) are fulfilled, [3] applies to problem $(2.1)-(2.2)$ and guarantees the existence of a unique mild solution $P \in C_s([0,T]; \Sigma^+(H))$ to the related Riccati equation.

In a completely similar way the abstract formulation of (3) in H is given by

$$\begin{cases} Y_\epsilon'(t) = A_\epsilon Y_\epsilon(t) + Bu(t) & t \in]0,T[\\ Y(0) = Y_0 \end{cases}, \tag{2.3}$$

where $A_\epsilon : D(A_\epsilon) \subset H \to H$ is defined by

$$A_\epsilon \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} = \begin{pmatrix} 0 & I \\ -\Lambda & -\epsilon\Lambda \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \end{pmatrix}$$

for any $\begin{pmatrix} v_0 \\ v_1 \end{pmatrix} \in D(A_\epsilon) = \left\{ \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} \in H : v_1 \in D(\sqrt{\Lambda}), v_0 + \epsilon v_1 \in D(\Lambda) \right\}$ and B is the same as in (2.1). The corresponding cost functional is given by

$$J_\epsilon(u) = \int_0^T (|Y_\epsilon(t)|_H^2 + |u(t)|_U^2)dt + |Y_\epsilon(T)|_H^2. \tag{2.4}$$

Since A_ϵ and A_ϵ^* are dissipative for any $\epsilon > 0$, it is well known [10] that A_ϵ is the infinitesimal generator of a C_0-semigroup of contractions e^{tA_ϵ} in H, which moreover is *analytic* (see for instance [2]).

Remark 2.1 Note that $A, A_\epsilon, A^*, A_\epsilon^* \in \mathcal{G}(1,0)$. Even in this case, as (1.3) are trivially fulfilled for any $\epsilon > 0$, the Riccati equation associated with problem $(2.3) - (2.4)$ admits a unique solution $P_\epsilon \in C_s([0,T];\Sigma^+(H))$ for any $\epsilon > 0$.

Before coming to the main theorem of this section we prove some preliminary result.

Lemma 2.1 *For any $Y \in H$ and $\lambda \in \mathbf{C}$, with* $\mathrm{Re}\lambda > 0$*, we have*

$$\begin{array}{ll} (i) & R(\lambda; A_\epsilon)Y \to R(\lambda; A)Y \\ (ii) & R(\lambda; A_\epsilon^*)Y \to R(\lambda; A^*)Y \end{array} \tag{2.5}$$

as ϵ tends to zero (in the H norm).

Proof: It is sufficient to write down the expressions of the resolvent operators $R(\lambda, A)$, $R(\lambda, A_\epsilon)$, $R(\lambda, A^*)$, $R(\lambda, A_\epsilon^*)$ in terms of the resolvent of $-\Lambda$.

Easy calculations show that

$$R(\lambda, A) = \begin{pmatrix} \lambda R(\lambda^2; -\Lambda) & R(\lambda^2; -\Lambda) \\ -\Lambda R(\lambda^2; -\Lambda) & \lambda R(\lambda^2; -\Lambda) \end{pmatrix}$$

for any $\lambda \in \varrho(A) = \{\lambda \in \mathbf{C} : \quad \lambda \neq \pm i\sqrt{\lambda_k},\ \lambda_k \in \sigma(\Lambda)\}$, and, respectively,

$$R(\lambda, A_\epsilon) = \frac{1}{\lambda\epsilon + 1} \begin{pmatrix} (\lambda + \epsilon\Lambda) R(\frac{\lambda^2}{\lambda\epsilon+1}; -\Lambda) & R(\frac{\lambda^2}{\lambda\epsilon+1}; -\Lambda) \\ -\Lambda R(\frac{\lambda^2}{\lambda\epsilon+1}; -\Lambda) & \lambda R(\frac{\lambda^2}{\lambda\epsilon+1}, -\Lambda) \end{pmatrix}$$

for $\lambda \in \varrho(A^\epsilon) = \{\lambda \in \mathbf{C} : \lambda \neq -\frac{1}{\epsilon},\ \lambda \neq \frac{-\epsilon\lambda_k \pm \sqrt{\epsilon^2\lambda_k^2 - 4\lambda_k}}{2},\ \lambda_k \in \sigma(\Lambda)\}$.

Hence $2.5(i)$ is a trivial consequence of the continuity of the function $\mu \to R(\mu, -\Lambda)$. $2.5(ii)$ can be showed exactly in the same way. □

Lemma 2.2 *Let $e^{tA}, e^{tA_\epsilon}, e^{tA^*}, e^{tA_\epsilon^*}$ be the semigroups generated by $A, A_\epsilon, A^*, A_\epsilon^*$ respectively. Then, for any $Y \in H$ and $t \geq 0$*

$$\begin{array}{ll} (i) & e^{tA_\epsilon}Y \to e^{tA}Y \\ (ii) & e^{tA_\epsilon^*}Y \to e^{tA^*}Y \end{array} \tag{2.6}$$

as ϵ tends to zero. Moreover the convergence in (2.6) *is uniform on bounded t-intervals.*

Proof: $(2.6(i))$ $((2.6(ii)))$ follows immediately from $(2.5(i))$ $((2.5(ii)))$ of Lemma 2.1 and the Trotter approximation theorem (see [10, p.85]) taking into account Remark 2.1. □

Now, given the sequence of Riccati equations

$$\begin{cases} P_\epsilon{}' = A_\epsilon^* P_\epsilon + P_\epsilon A_\epsilon - P_\epsilon BB^* P_\epsilon + I \\ P_\epsilon(0) = I \end{cases} \tag{2.7}$$

associated with problem $(2.3)-(2.4)$, and the Riccati equation

$$\begin{cases} P' = A^* P + PA - PBB^* P + I \\ P(0) = I \end{cases} \tag{2.8}$$

associated with problem $(2.1)-(2.2)$, we can finally state

Theorem 2.1 *Let P_ϵ, P be the mild solutions to* $(2.7), (2.8)$ *respectively. Let $Y_0 \in H$ and $T > 0$ be given, and let (Y^*, u^*), $(Y_\epsilon^*, u_\epsilon^*)$, be the optimal pairs of the problems* $(2.1)-(2.2)$, $(2.3)-(2.4)$ *respectively.*

Then, as $\epsilon \to 0$, we have

$$P_\epsilon \to P \quad \text{in } C_s([0,T];\Sigma^+(H)),$$

$$J_\epsilon(u_\epsilon^*) \to J(u^*),$$

$$Y_\epsilon^*(t) \to Y^*(t) \quad \text{strongly and in } L^2(0,T;H),$$

$$u_\epsilon^*(t) \to u^*(t) \quad \text{strongly and in } L^2(0,T;U).$$

Proof: It is sufficient to invoke the continuous dependence on data theorem (Section 1, Theorem 1.1) and take into account Lemma 2.2. □

Remark 2.2 As we have already noticed in the introduction, the convergence results obtained in this chapter are stated for problem $(1)-(2)$ mainly for the sake of simplicity. In fact they can be extended to more general situations, at least in the following directions.

(I) If one replaces in (1) the Dirichlet with Neumann boundary condition, all the above considerations still hold true, except for some details in the abstract formulation of the concrete problem, which actually do not change the substance of the proofs.

We just remark that the state space H is given now by the Hilbert space $H^1(\Omega) \times L^2(\Omega)$, endowed with the natural scalar product

$$\begin{aligned} < \begin{pmatrix} v_0 \\ v_1 \end{pmatrix}, \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} >_H \quad &= \int_\Omega (v_0(x) z_0(x) + \nabla v_0(x) \nabla z_0(x)) dx + \\ &+ \int_\Omega v_1(x) z_1(x) dx. \end{aligned}$$

Accordingly, we shall take $y_0 \in H^1(\Omega)$. Moreover we stress that the Neumann realization of $-\Delta$ in $L^2(\Omega)$ is no longer a strictly positive operator. Nevertheless, Lemma 2.1 (and Lemma 2.2) still applies, since the proof is essentially based on the fact that Λ is a self-adjoint nonnegative operator.

(II) We can also consider different boundary value problems for controlled hyperbolic equations of type (9), provided we can still reduce to an abstract problem—in a suitable Hilbert space H—of the form (2.1), with Λ self-adjoint and nonnegative.

3 The Infinite Time Horizon Case

In this section we consider the infinite time horizon case for the boundary value problem (1).

According with the abstract setting introduced in Section 2, the cost functionals (5) and (6) may be expressed as follows:

$$J_\infty(u) = \int_0^\infty (|Y(t)|_H^2 + |u(t)|_U^2)dt, \tag{3.1}$$

$$J_{\infty,\epsilon}(u) = \int_0^\infty (|Y_\epsilon(t)|_H^2 + |u(t)|_U^2)dt, \tag{3.2}$$

where $H = D(\sqrt{\Lambda}) + K$ is the states space, $U = K$ is the controls space, u belongs to $L^2(0,\infty;U)$ and Y, Y_ϵ satisfy (2.1), (2.3) respectively.

Let now $P^\infty_{\epsilon,min}$, P^∞_{min} be the minimal non-negative solutions to the algebraic Riccati equations

$$A_\epsilon^* X_\epsilon + A_\epsilon X_\epsilon - X_\epsilon BB^* X_\epsilon + I = 0 \tag{3.3}$$

$$A^* X + AX - XBB^* X + I = 0 \tag{3.4}$$

associated with problems $(2.3)-(3.2), (2.1)-(3.1)$ respectively, whose existence is guaranteed by the I-stabilizability of the pairs (A_ϵ, B), $\epsilon > 0$ fixed, (A, B) respectively. In fact the first case is trivial, since the free system is exponentially stable; for the case $\epsilon = 0$ see for instance [11].

Now we are interested in proving an approximation result on P^∞_{min} through $P^\infty_{\epsilon,min}$. As we want to apply Theorem 1.2, we will show that (A_ϵ, B) is stabilizable with respect to I uniformly in $\epsilon > 0$. It is clear that this is the most crucial condition to be verified: indeed here the proof of the uniform detectability condition is trivial, since the observation operator is the identity (see (3.2)).

Thus, we fix $Y_0 \in H$, and let

$$u_\epsilon(t) = -\alpha B^* Y_\epsilon(t), \quad t > 0, \tag{3.5}$$

where B^* is the adjoint of B, Y_ϵ is the mild solution of (2.3) with $u = u_\epsilon$ as in (3.5), and α is a real positive number.

We will show that

$$\int_0^\infty |Y_\epsilon(t)|^2 \, dt \; < \; C(Y_0), \qquad t > 0,$$

for any $\epsilon > 0$, and therefore

$$\sup_{\epsilon > 0} J_{\infty,\epsilon}(u_\epsilon) < +\infty.$$

Proposition 3.1 *Let $\alpha > 0$, and let $Y_0 \in H$ be fixed.*

Moreover, let Y_ϵ be the mild solution of (2.3)*, with u given by* (3.5)*. Then we have*

$$\int_0^\infty |Y_\epsilon(t)|^2 \, dt \; < \; C(Y_0), \qquad t > 0. \tag{3.6}$$

Proof: First we assume $Y_0 \in D(\Lambda) \times D(\Lambda)$. Let $Y(t) \equiv Y_\epsilon(t) = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}$, $Y_0 = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix}$ (we suppress the index ϵ to simplify the notation; the dependence of Y on ϵ will be clear from the context).

As $D(\Lambda) \times D(\Lambda) \subset D(A_\epsilon)$ for any $\epsilon > 0$, from (2.3) we obtain that $y(t)$ is a classical solution of

$$\begin{cases} y''(t) + (\epsilon\Lambda + \alpha)y'(t) + \Lambda y(t) = 0, & t > 0 \\ y(0) = y_0 \\ y'(0) = y_1. \end{cases} \tag{3.7}$$

(I step) We first multiply the differential equation in (3.7) by y' and integrate between 0 and t. Then we obtain

$$\int_0^t \left(\frac{1}{2}\frac{d}{ds}|y'(s)|^2 + < (\epsilon\Lambda + \alpha)y'(s), y'(s) > + \frac{1}{2}\frac{d}{ds} < \Lambda y(s), y(s) > \right) ds = 0,$$

that is

$$\begin{aligned} \frac{1}{2}(|y'(t)|^2 \quad &+ \quad < \Lambda y(t), y(t) >) + \int_0^t < (\epsilon\Lambda + \alpha)y'(s), y'(s) > ds = \\ &= \quad \frac{1}{2}(|y_1|^2 + |\sqrt{\Lambda} y_0|^2), \end{aligned}$$

and therefore we deduce that

$$\frac{1}{2}\sup_{t>0}\,(|y'(t)|^2 + |\sqrt{\Lambda} y(t)|^2) + \alpha \int_0^\infty |y'(s)|^2 ds \leq \frac{1}{2}(|y_1|^2 + |\sqrt{\Lambda} y_0|^2). \tag{3.8}$$

This shows that
$$\int_0^\infty |y'(s)|^2 ds < c_1(Y_0). \tag{3.9}$$

(II step) By multiplying by y in (3.7) we have:
$$< y'', y > + < (\epsilon\Lambda + \alpha)y', y > + < \Lambda y, y >= 0,$$

that is
$$\frac{d}{ds} < y'(s), y(s) > -|y'(s)|^2 + \frac{1}{2}\frac{d}{ds} < (\epsilon\Lambda + \alpha)y(s), y(s) > +|\sqrt{\Lambda}y(s)|^2 = 0.$$

We now integrate between 0 and t:
$$\begin{aligned} < y'(t), y(t) > &- < y_1, y_0 > - \int_0^t |y'(s)|^2\, ds + \int_0^t |\sqrt{\Lambda}y(s)|^2\, ds \\ &+ \frac{1}{2}[< (\epsilon\Lambda + \alpha)y(t), y(t) > - < (\epsilon\Lambda + \alpha)y_0, y_0 >] = 0, \end{aligned}$$

or
$$\begin{aligned} &\frac{1}{2} < (\epsilon\Lambda + \alpha)y(t), y(t) > + \int_0^t |\sqrt{\Lambda}y(s)|^2 ds = \\ &\quad =< y_1, y_0 > + \int_0^t |y'(s)|^2 ds + \frac{1}{2} < (\epsilon\Lambda + \alpha)y_0, y_0 > - < y'(t), y(t) > . \end{aligned}$$

Since
$$- < y, y' >\le \frac{\alpha}{4}|y|^2 + \frac{1}{\alpha}|y'|^2,$$

then we finally obtain
$$\frac{\alpha}{4}|y(t)|^2 + \int_0^t |\sqrt{\Lambda}y(s)|^2 ds \le$$
$$\le \frac{1}{\alpha}|y'(t)|^2 + \int_0^\infty |y'(s)|^2 ds + |y_1||y_0| + \frac{1}{2}(\epsilon|\sqrt{\Lambda}y_0|^2 + \alpha|y_0|^2). \tag{3.10}$$

From (3.8) and (3.10) we have $\int_0^\infty |\sqrt{\Lambda}y(s)|^2 ds < c_2(Y_0)$ and this, together with (3.9), yields the estimate (3.6).

Let now $Y_0 \in H$. Since $D(\Lambda) \times D(\Lambda)$ is dense in H, the conclusion easily follows by using regularization arguments. □

We finally state the main result of this section:

Theorem 3.1 *Let $P^\infty_{\epsilon,min}, P^\infty_{min}$ be the minimal solutions to the algebraic Riccati equations* $(3.3) - (3.4)$ *respectively. Let $Y_0 \in H$ be given, and let (Y^*, u^*), $(Y^*_\epsilon, u^*_\epsilon)$, be the optimal pairs of the problems* $(2.3) - (3.2)$, $(2.1) -$ (3.1) *respectively.*

Then, as $\epsilon \to 0$*, we have*

$$\begin{array}{ll} |P^{\infty}_{\epsilon,min} Y - P^{\infty}_{min} Y| \to 0 & \textit{for any } Y \in H; \\ J_{\infty,\epsilon}(u^*_\epsilon) \to J_\infty(u^*); & \\ |Y^*_\epsilon - Y^*| \to 0 & \textit{in } L^2(0,\infty;H) \textit{ and in } C(0,\infty;H); \\ |u^*_\epsilon - u^*| \to 0 & \textit{in } L^2(0,\infty;U). \end{array}$$

Proof: It is sufficient to apply Theorem 1.2. As we already noticed in section 2, the conditions (1.3) and (1.6) are satisfied. The uniform detectability condition is easily verified by simply taking, for instance, $\mathcal{K}_k = I$. Finally, the decisive stabilization property has been showed in Proposition 3.1. □

Remark 3.1 Similar considerations as in Remark 2.2 can be repeated in the infinite time horizon case. We just point out that if we want to show uniform (in ϵ) stabilizability of the strongly damped wave equation with Neumann boundary condition, we have to replace the feedback control u_ϵ in (3.5) by the following

$$u_\epsilon(t) = -KY_\epsilon(t),$$

where $K \in \mathcal{L}(H,U)$ is given by

$$K \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} = \beta v_0 + \alpha v_1,$$

with α and β positive constants.

Then it is easy to follow the scheme of the proof of Proposition 3.1 to obtain a completely similar result. Therefore Theorem 3.1 still holds true.

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