Journal of Mathematical Systems, Estimation, and Control
Vol. 4, No. 4, 1994, pp. 1–24
© 1994 Birkhäuser-Boston

Continuous-Time Gauss-Markov Processes with Fixed Reciprocal Dynamics*

Alessandro Beghi †

Abstract

Continuing the work started in [11], in this paper we examine the construction of Gauss-Markov processes with fixed reciprocal dynamics. We show how to construct Gauss-Markov processes, defined on a finite interval, having fixed initial and end-point densities and belonging to a given reciprocal class. The problem of changing the end-point density of a Markov process, while remaining in the same reciprocal class, is also considered. A stochastic interpretation of the results in terms of an optimal control problem is given.

Key words: Gaussian processes, reciprocal processes, stochastic optimal control

AMS Subject Classifications: 60G15

1 Introduction

In this paper we describe a procedure for constructing continuous-time Gauss-Markov processes with fixed reciprocal dynamics. An analogous study for the discrete-time case has been presented in [11]. A $\mathbb{R}^n$ valued stochastic process $x(t)$ defined for $t \in I = [0, T]$ is *reciprocal* if for any $[t_1, t_2] \subset I$ the process in the interior of $[t_1, t_2]$ is conditionally independent of the process in $I - [t_1, t_2]$ given $x(t_1)$ and $x(t_2)$. The time-reversibility of the Markov property implies that Markov processes are necessarily reciprocal, while the converse is false [7]. A motivation for investigating the relations between Markov and reciprocal processes can be found in the work of Schrödinger [17]. In his attempt of giving a probabilistic interpretation to his results in quantum mechanics, Schrödinger had to deal

*Received January 11, 1993; received in final form June 20, 1993; summary appeared in Volume 4, Number 4.

†The research described in this paper was partially performed while visiting the Institute of Theoretical Dynamcis, University of California, Davis CA 95616.

with Markov processes living on a finite interval, with positive transition density $q(s,x;t,y)$ and with the property of having preassigned both the initial probability density $p_0(x)$ and the final one, $p_T(x)$. Clearly, from the knowledge of the transition density and of the initial one we can predict that at the time $t=T$ the final probability density is

$$p(T,x)=\int q(\tau,\xi;T,x)p_0(\xi)d\xi\ . \tag{1.1}$$

An interesting question arises when $p_T(x)$ is different from $p(T,x)$: which is the Markov process $x^*(t)$ that has $p_0(x)$ and $p_T(x)$ as initial and final densities, and is in some way the closest to $x(t)$? Influenced by this problem, Bernstein in 1932 introduced the class of reciprocal processes [1]. In his fundamental contribution to the theory of reciprocal processes, Jamison [7, 8, 9] showed that, similarly to what happens for Markov processes, the finite joint densities of a reciprocal process $x(t)$ can be determined by only two functions, namely the joint density $\mu(x_0,0;x_T,T)$ of the process at two points and the *reciprocal* (or *three-point*) transition density $q(r,x;s,y;t,w)$, where

$$\int_A\int_B q(r,x;s,y;t,w)dxdw=\text{Prob}\left(x(s)=y|x(r)\in A,x(t)\in B\right)\ . \tag{1.2}$$

Two processes having the same three-point transition density are said *locally equivalent* [8], [3], and then the assignment of the function $q(r,x;s,y;t,w)$ defines an equivalence class of processes. Jamison also showed that the reciprocal transition density of a Markov process can be obtained from its Markov transition density via the following factorization:

$$q(s,x;t,y;u,w)=\frac{p(s,x;t,y)p(t,y;u,w)}{p(s,x;u,w)},\qquad 0\le s<t<u\le T\ . \tag{1.3}$$

The factorization (1.3) is not unique. In fact, there are many Markovian transition densities p that give the same reciprocal transition density q, i.e. there are many Markov processes having the same reciprocal description. In this paper we study how these Markov processes are related to each other. We will limit our discussion to the Gaussian context, where an interesting characterization of reciprocal processes is available. In fact, it has been shown by Krener, Levy and Frezza [12], [10] that all the Gaussian reciprocal processes in a given class can be obtained varying the boundary conditions of a self-adjoint, second-order stochastic boundary-value problem (SBVP).

The main goal of this paper is to show how to construct a Markov process belonging to a given reciprocal class with prescribed initial and final densities. We present a method that requires the solution of an algebraic Riccati equation, which is obtained by studying the properties of the set of boundary conditions for the SBVP characterizing the reciprocal class

that yield Markov processes. We consider also the problem of changing the final density of a Markov process while remaining in the same reciprocal class. Recently, different authors [19, 5, 2] have proposed a formulation of Schrödinger's problem in terms of a stochastic optimal control problem. More specifically, the process $x^*(t)$ which has final probability density $p_T(x)$ is considered to be obtained by applying to $x(t)$ a *minimum energy* control. We show that our results are in agreement with these previous ones.

The paper is organized as follows. In Section 2 we review the characterization of Gaussian reciprocal processes proposed by Krener, Frezza and Levy [10], and in Section 3 we introduce an alternative set of boundary conditions for the SBVP considered in [10]. The procedure for constructing a Markov process with given end-point marginal densities and reciprocal dynamics is presented in Section 4. In Section 5 we address the related problem of changing the end-point density of a Markov process while remaining in the same reciprocal class. Finally, we conclude giving an interpretation of the results in terms of a stochastic optimal control problem.

2 Models of Gaussian Reciprocal Processes

Krener, Frezza and Levy showed in [10] that, under suitable assumptions which we will discuss later on, a $\mathbb{R}^n$ valued, zero mean, Gaussian process $x(t)$ defined on the interval $I = [0, T]$ is reciprocal *if and only if* its covariance function $R(t, s)$ solves the linear second-order matrix differential equation

$$\mathcal{L}_t[R(t,s)] = I\delta(t-s) \tag{2.1a}$$

with boundary conditions

$$\begin{bmatrix} R(0,0) & R(0,T) \\ R(T,0) & R(T,T) \end{bmatrix} = P\,, \tag{2.1b}$$

where $\mathcal{L}$ is a self adjoint, positive definite differential operator of second order with no pair of conjugate points on $[0, T]$

$$\mathcal{L} = Q^{-1}(t)(-\frac{d^2}{dt^2} + G(t)\frac{d}{dt} + F(t))\,. \tag{2.2}$$

The $n \times n$ matrices $Q(t)$, $G(t)$ and $F(t)$ are related to $R(t, s)$ as follows

$$Q(t) = \frac{\partial R(t,t^+)}{\partial t} - \frac{\partial R(t,t^-)}{\partial t} \tag{2.3a}$$

$$G(t) = (\frac{\partial^2 R(t,t^+)}{\partial t^2} - \frac{\partial^2 R(t,t^-)}{\partial t^2})Q^{-1}(t) \tag{2.3b}$$

$$F(t) = (\frac{\partial^2 R(t,t^+)}{\partial t^2} - G(t)\frac{\partial R(t,t^+)}{\partial t})R^{-1}(t,t). \tag{2.3c}$$

The assumptions under which this result holds are the following:

A. The covariance function $R(t,s)$ of $x(t)$ is piecewise C^2 on $[0,T]\times[0,T]$.

B. The variance $R(t,t)$ of $x(t)$ is invertible for any $t \in (0,T)$.

C. The matrix $Q(t)$ given by (2.3a) is invertible for any $t \in (0,T)$.

In the sequel we will also assume that

D. The matrix

$$P(t_0,t_1) = \begin{bmatrix} R(t_0,t_0) & R(t_0,t_1) \\ R(t_1,t_0) & R(t_1,t_1) \end{bmatrix} \tag{2.4}$$

is invertible for all $t_0, t_1 \in I$.

The third assumption, which is the most restrictive one, can be relaxed [6] considering higher order models and the concept of higher order reciprocal processes introduced by Miroshin [14].

Since $x(t)$ is Gaussian, it is completely determined by its second-order description. Thus, the knowledge of $F(t), G(t), Q(t), T$ and of the boundary data P determines $R(t,s)$ and hence $x(t)$. In [10] Krener showed that $x(t)$ can also be characterized as the solution of the second-order stochastic differential equation

$$\mathcal{L}[x(t)] = \xi(t) \tag{2.5a}$$

with Dirichlet boundary conditions

$$\begin{bmatrix} x(0) \\ x(T) \end{bmatrix} = b \sim N(0,P) \tag{2.5b}$$

where $\xi(t)$ is a zero mean, generalized, Gaussian process, independent of the boundary conditions b, with covariance

$$E[\xi(t)\xi^T(s)] = \mathcal{L}I\delta(t-s)\,. \tag{2.6}$$

The driving process $\xi(t)$ in (2.5a) has the property that the random variables spanned by $\xi(\cdot)$ and $x(\cdot)$ on disjointed subintervals are orthogonal, and is called the *conjugate process* [16] of $x(t)$. We refer to [10] for a rigorous definition of the meaning of the differential equation (2.5a) in terms of second-order analogs of Feller's postulates for Markov diffusions. The differential equation (2.5a) is a dynamical model for $x(t)$, and we will say that a process that solves (2.5a) has *reciprocal dynamics* (2.5a). It is also possible to express the solution $x(t)$ of (2.5a) as

$$x(t) = \Psi(t)b + \int_0^T , (t,s)\xi(s)\, ds \tag{2.7}$$

where , (t,s) is the Green's function of the operator $\mathcal{L}$ on $[0,T]$ and the boundary transition matrix $\Psi(t) = [\Psi_1(t) \quad \Psi_2(t)]$ solves the deterministic boundary value problem

$$-\frac{d^2\Psi}{dt^2}(t) + F(t)\Psi(t) + G(t)\frac{d\Psi}{dt}(t) = 0 \tag{2.8a}$$

$$\begin{bmatrix} \Psi_1(0) & \Psi_2(0) \\ \Psi_1(T) & \Psi_2(T) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} . \tag{2.8b}$$

Observing that $x(t)$ can be decomposed in the orthogonal sum

$$x(t) = \tilde{x}(t) + \hat{x}(t) \tag{2.9}$$

where

$$\hat{x}(t) = E[x(t)|x(0), x(T)] \tag{2.10}$$

and

$$\tilde{x}(t) = x(t) - \hat{x}(t) \, , \tag{2.11}$$

we see that $R(t,s)$ splits in the sum of two components $\hat{R}(t,s)$ and $\tilde{R}(t,s)$, where $\hat{R}$ and $\tilde{R}$ are respectively the covariances of the processes $\hat{x}(t)$ and $\tilde{x}(t)$ [10]. Expressions for the covariances $\hat{R}(t,s)$ and $\tilde{R}(t,s)$ in terms of the matrix $\Psi(t)$ are given in the following two lemmas.

Lemma 2.1 [10]

$$\hat{R}(t,s) = \begin{bmatrix} \Psi_1(t) & \Psi_2(t) \end{bmatrix} \begin{bmatrix} R(0,0) & R(0,T) \\ R(T,0) & R(T,T) \end{bmatrix} \begin{bmatrix} \Psi_1^T(s) \\ \Psi_2^T(s) \end{bmatrix} \tag{2.12}$$

Lemma 2.2 [4]

$$\tilde{R}(t,s) = \begin{cases} \Psi_2(t) K \Psi_1^T(s) & t \leq s \\ \Psi_1(t) K^T \Psi_2^T(s) & s \leq t \end{cases} \tag{2.13}$$

where the matrix K is constant in t and s but depends in general on F, G, Q and T.

Observe that the matrix K is the same for every process in the reciprocal class, then, in accordance with Jamison [8] and Clark [3], it is a *reciprocal invariant*. K has a variety of representations, that can be obtained using Green's functions construction techniques; in the sequel we will use the following ones [4]

$$K = \dot{\Psi}_2^{-1}(0) Q(0) \tag{2.14a}$$

$$K = -Q(T)\dot{\Psi}_1^{-T}(T). \tag{2.14b}$$

We also notice that $\tilde{R}(t,s)$ is in fact the Green's function , (t,s). The decomposition introduced above is important for the discussion of the following sections since it provides a way to separate the component of $x(t)$ that is common to all the processes in the reciprocal class from the component that is actually determined by the boundary conditions.

As already mentioned in the introduction, Markov processes are a subclass of reciprocal processes. Then, Markov processes whose covariances satisfy assumptions (A)–(C) previously introduced must satisfy also a second-order model like (2.5a). It is well known that a Gauss-Markov process $x(t)$, whose covariance $R(t,s)$ satisfies the assumptions (A)–(B), solves a causal first order model like the following

$$\frac{dx(t)}{dt} = A(t)x(t) + B(t)\nu(t) \tag{2.15a}$$

$$x(0) = x_0 \sim N(0, \Pi(0)) \tag{2.15b}$$

where $\nu(t)$ is Gaussian white noise, i.e. $E[\nu(t)\nu^T(s)] = \mathrm{I}\delta(t-s)$ and the matrices $A(t)$ and $B(t)$ are given by

$$B(t)B^T(t) = \frac{\partial R(t,t^+)}{\partial t} - \frac{\partial R(t,t^-)}{\partial t} \tag{2.16a}$$

$$A(t) = \frac{\partial R(t,t^-)}{\partial t} R^{-1}(t,t). \tag{2.16b}$$

Clearly, if $B(t)B^T(t)$ is invertible then also assumption (C) is satisfied and $x(t)$ must also be the solution of a second order model. As a consequence of (2.3a) and (2.16a), $Q(t) = B(t)B^T(t)$, and it was shown in [10], that (2.3b),(2.3c) and (2.16b) imply that $F(t)$ and $G(t)$ are determined by the following relations

$$G(t)Q(t) = A(t)Q(t) - Q(t)A^T(t) + \tfrac{dQ(t)}{dt} \tag{2.17a}$$

$$F(t) + G(t)A(t) = \tfrac{dA(t)}{dt} + A^2(t)\,. \tag{2.17b}$$

3 Cyclic Boundary Conditions for Second-Order Models of Reciprocal Processes

We have seen in the previous section that Gaussian reciprocal processes can be characterized either specifying the covariance function $R(t,s)$ via (2.1a) or using the stochastic differential equation (2.5a). In both cases, it is necessary to impose some boundary conditions to describe completely the process $x(t)$ over the whole interval I. This can be accomplished satisfactorily in more than one way. In [12], in the discrete-time context, it has been shown that, besides Dirichlet boundary conditions of the kind (2.1b), (2.5b), it is possible to assign *cyclic* boundary conditions to the

second-order model describing the process. This set of conditions can be seen as having the effect of wrapping the interval I onto a circle, extending the second-order description of the process $x(k)$ to the whole interval I. In this way, a coupling between the dynamics of the process $x(k)$ at the boundaries of the interval I is introduced. In this section, we investigate the possibility of formulating a similar concept in the continuous-time context. We consider the covariance function of a reciprocal process and we study how it behaves at the boundaries of I. We show that, similarly to what happens in the discrete-time case, a coupling between the covariances $R(0,s)$ and $R(T,s)$ can be found. As a consequence of this fact, it is possible to introduce an alternative way of assigning the boundary conditions to the differential equation (2.1a), which is presented in the following lemma.

Lemma 3.1 *Let $R(t,s)$ be the covariance function of a zero-mean Gaussian reciprocal process defined on the interval I satisfying assumptions A-D. Then it satisfies the boundary conditions*

$$\begin{bmatrix} \frac{\partial R}{\partial t}(0,s) \\ \frac{\partial R}{\partial t}(T,s) \end{bmatrix} = \begin{bmatrix} M_{00} & M_{0T} \\ M_{T0} & M_{TT} \end{bmatrix} \begin{bmatrix} R(0,s) \\ R(T,s) \end{bmatrix} = M \begin{bmatrix} R(0,s) \\ R(T,s) \end{bmatrix}, \quad (3.1)$$

where the constant matrix M is given by

$$\begin{bmatrix} M_{00} & M_{0T} \\ M_{T0} & M_{TT} \end{bmatrix} = \begin{bmatrix} \left(\frac{\partial R}{\partial t}\right)(0,0) & \left(\frac{\partial R}{\partial t}\right)(0,T) \\ \left(\frac{\partial R}{\partial t}\right)(T,0) & \left(\frac{\partial R}{\partial t}\right)(T,T) \end{bmatrix} P^{-1}. \quad (3.2)$$

Proof: Given $t_0 \in I$ and $t \in [0,t_0], s \in [t_0,T]$, the reciprocal property of $x(t)$ implies that

$$R(t,s) = \begin{bmatrix} R(t,t_0) & R(t,T) \end{bmatrix} P^{-1}(t_0,T) \begin{bmatrix} R(t_0,s) \\ R(T,s) \end{bmatrix}. \quad (3.3)$$

Differentiating (3.3) with respect to t and letting t_0 and t go to 0 we obtain

$$\frac{\partial R}{\partial t}(0,s) = \begin{bmatrix} \left(\frac{\partial R}{\partial t}\right)(0,0) & \left(\frac{\partial R}{\partial t}\right)(0,T) \end{bmatrix} P^{-1} \begin{bmatrix} R(0,s) \\ R(T,s) \end{bmatrix}. \quad (3.4)$$

In a similar way, taking $s \in [0,t_0]$ and $t \in [t_0,T]$, we can write

$$R(t,s) = \begin{bmatrix} R(t,0) & R(t,t_0) \end{bmatrix} P^{-1}(0,t_0) \begin{bmatrix} R(0,s) \\ R(t_0,s) \end{bmatrix}. \quad (3.5)$$

By differentiating with respect to t and letting t_0 and t go to T we obtain

$$\frac{\partial R}{\partial t}(T,s) = \begin{bmatrix} \left(\frac{\partial R}{\partial t}\right)(T,0) & \left(\frac{\partial R}{\partial t}\right)(T,T) \end{bmatrix} P^{-1} \begin{bmatrix} R(0,s) \\ R(T,s) \end{bmatrix}. \quad (3.6)$$

Relations (3.4) and (3.6) can be written more compactly as (3.1), where expression (3.2) for the matrix M is obtained by taking $s = 0$ and $s = T$ in (3.4), (3.6). ■

Thus, the covariances $R(0,s)$ and $R(T,s)$ at the boundaries of I are related to each other by the fact that they each satisfy a first-order non-homogeneous linear differential equation where the other appears as a forcing term. These differential equations can be seen as boundary conditions for the equation (2.1a), which are known once we have assigned the matrix M. Thus, according to the motivation that led us to the coupled differential equations (3.1), we will say that the matrix M specifies the *cyclic* boundary conditions for the equation (2.1a). As we have seen, the matrix M can be computed via (3.2) starting from P and the second order description of $x(t)$ (i.e. the differential equation (2.1a) for the covariance $R(t,s)$). It is also possible to do the converse, that is to obtain P (more precisely, its inverse) from M by only using quantities that are given as data. In fact, let the matrix M be given. Using lemmas 2.1 and 2.2 it is possible to compute the partial derivatives of the covariance $R(t,s)$ of $x(t)$. For $t \leq s$

$$\begin{aligned}\left(\frac{\partial R}{\partial t}\right)(t,s) &= \begin{bmatrix} \dot\Psi_1(t) & \dot\Psi_2(t) \end{bmatrix} \begin{bmatrix} R(0,0) & R(0,T) \\ R(T,0) & R(T,T) \end{bmatrix} \begin{bmatrix} \Psi_1^T(s) \\ \Psi_2^T(s) \end{bmatrix} \\ &\quad + \dot\Psi_2(t) K \Psi_1^T(s) \end{aligned} \tag{3.7}$$

while for $t \geq s$

$$\begin{aligned}\left(\frac{\partial R}{\partial t}\right)(t,s) &= \begin{bmatrix} \dot\Psi_1(t) & \dot\Psi_2(t) \end{bmatrix} \begin{bmatrix} R(0,0) & R(0,T) \\ R(T,0) & R(T,T) \end{bmatrix} \begin{bmatrix} \Psi_1^T(s) \\ \Psi_2^T(s) \end{bmatrix} \\ &\quad + \dot\Psi_1(t) K^T \Psi_2^T(s)\,. \end{aligned} \tag{3.8}$$

Specializing (3.7) and (3.8) for $(t,s) = (0,0),(0,T),(T,0),(T,T)$, we find that

$$\begin{aligned}&\begin{bmatrix} \left(\frac{\partial R}{\partial t}\right)(0,0) & \left(\frac{\partial R}{\partial t}\right)(0,T) \\ \left(\frac{\partial R}{\partial t}\right)(T,0) & \left(\frac{\partial R}{\partial t}\right)(T,T) \end{bmatrix} = \\ &\qquad \begin{bmatrix} \dot\Psi_1(0) & \dot\Psi_2(0) \\ \dot\Psi_1(T) & \dot\Psi_2(T) \end{bmatrix} P + \begin{bmatrix} \dot\Psi_2(0)K & 0 \\ 0 & \dot\Psi_1(T)K^T \end{bmatrix}, \end{aligned} \tag{3.9}$$

where it follows from (2.14a)–(2.14b) that

$$\dot\Psi_2(0)K = Q(0) \qquad\qquad \dot\Psi_1(T)K^T = -Q(T)\,. \tag{3.10}$$

Premultiplying (3.9) by

$$\begin{bmatrix} Q^{-1}(0) & 0 \\ 0 & -Q^{-1}(T) \end{bmatrix} \tag{3.11}$$

and then using (3.2) and again (2.14a)–(2.14b), the following equation for the inverse of the boundary matrix P can be found

$$P^{-1} = \begin{bmatrix} Q^{-1}(0)M_{00} & Q^{-1}(0)M_{0T} \\ -Q^{-1}(T)M_{T0} & -Q^{-1}(T)M_{TT} \end{bmatrix} - \begin{bmatrix} Q^{-1}(0)\dot{\Psi}_1(0) & K^{-1} \\ K^{-T} & -Q^{-1}(T)\dot{\Psi}_2(T) \end{bmatrix} . \tag{3.12}$$

Note that the inverse of a symmetric matrix is itself a symmetric matrix, hence (3.12) yields the following symmetry constraint for M:

$$Q^{-1}(0)M_{0T} = -M_{T0}^T Q^{-1}(T) . \tag{3.13}$$

The symmetry structure of the cyclic boundary conditions can be emphasized normalizing (3.1) by premultiplication by the matrix (3.11). This yields the following expression for the cyclic boundary conditions

$$\begin{bmatrix} Q^{-1}(0)\frac{\partial R}{\partial t}(0,s) \\ -Q^{-1}(T)\frac{\partial R}{\partial t}(T,s) \end{bmatrix} = N \begin{bmatrix} R(0,s) \\ R(T,s) \end{bmatrix} , \tag{3.14}$$

where

$$N = \begin{bmatrix} N_{00} & N_{0T} \\ N_{T0} & N_{TT} \end{bmatrix} = \begin{bmatrix} Q^{-1}(0) & 0 \\ 0 & -Q^{-1}(T) \end{bmatrix} M \tag{3.15}$$

and (3.13) implies that N is symmetric, i.e. $N_{0T} = N_{T0}^T$. Thus, independently of whether we assign the boundary conditions via P or M (or equivalently, N), we have exactly the same number of degrees of freedom in specifying a process $x(t)$ in the reciprocal class defined by the differential equation (2.1a).

Relation (3.2) assumes a particular form when $x(t)$ is a Markov process with state-space model (2.15a)–(2.15b). The covariance $R(t,s)$ of a Markov process satisfies the relation

$$R(t,s) = \begin{cases} \Phi(t,s)\Pi(s), & t \geq s \\ \Pi(t)\Phi^T(s,t), & t \leq s \end{cases} \tag{3.16}$$

where $\Phi(t,s)$ is the state-transition matrix solution of

$$\begin{cases} \frac{\partial \Phi(t,s)}{\partial t} = A(t)\Phi(t,s) \\ \Phi(s,s) = I \end{cases} \tag{3.17}$$

and $\Pi(t)$ is the state variance satisfying the Lyapunov equation

$$\dot{\Pi}(t) = A(t)\Pi(t) + \Pi(t)A^T(t) + Q(t). \tag{3.18}$$

When $t \leq s$, we have that

$$\begin{aligned} \frac{\partial R}{\partial t}(t,s) &= \dot{\Pi}(t)\Phi^T(s,t) + \Pi(t)\frac{\partial \Phi^T}{\partial t}(s,t) \\ &= (A(t)\Pi(t) + Q(t))\Phi^T(s,t)\,, \end{aligned} \tag{3.19}$$

where we have used the fact that

$$\frac{\partial \Phi}{\partial t}(s,t) = -\Phi(s,t)A(t). \tag{3.20}$$

Recalling that for a Markov process

$$P = \begin{bmatrix} \Pi(0) & \Pi(0)\Phi^T(T,0) \\ \Phi(T,0)\Pi(0) & \Pi(T) \end{bmatrix}\,, \tag{3.21}$$

we have from (3.2) that

$$\begin{aligned} &(A(0) + Q(0)\Pi^{-1}(0))\begin{bmatrix} \Pi(0) & \Pi(0)\Phi^T(T,0) \end{bmatrix} = \\ &\qquad \begin{bmatrix} M_{00} & M_{0T} \end{bmatrix}\begin{bmatrix} \Pi(0) & \Pi(0)\Phi^T(T,0) \\ \Phi(T,0)\Pi(0) & \Pi(T) \end{bmatrix}\,. \end{aligned} \tag{3.22}$$

Observing that the $n \times 2n$ matrix $\begin{bmatrix} \Pi(0) & \Pi(0)\Phi^T(T,0) \end{bmatrix}$ on the left hand side of (3.22) is the first row block of P, and since the matrix P is full rank, (3.22) shows that

$$M_{00} = (A(0) + Q(0)\Pi^{-1}(0)), \qquad M_{0T} = 0. \tag{3.23}$$

For $t \geq s$

$$\frac{\partial R}{\partial t}(t,s) = \frac{\partial \Phi}{\partial t}(t,s)\Pi(s) = A(t)\Phi(t,s)\Pi(s) = A(t)R(t,s) \tag{3.24}$$

and using again (3.2) and the full rank property of P, we see that

$$\begin{aligned} &A(T)\begin{bmatrix} \Phi(T,0)\Pi(0) & \Pi(T) \end{bmatrix} = \\ &\qquad \begin{bmatrix} M_{T0} & M_{TT} \end{bmatrix}\begin{bmatrix} \Pi(0) & \Pi(0)\Phi^T(T,0) \\ \Phi(T,0)\Pi(0) & \Pi(T) \end{bmatrix} \end{aligned} \tag{3.25}$$

which implies

$$M_{T0} = 0, \qquad M_{TT} = A(T). \tag{3.26}$$

Relations (3.23) and (3.26) show that for a Markov process the cyclic boundary conditions are *separable*, in the sense that since $M_{0T} = M_{T0} = 0$ the covariances at each end of the interval $I = [0,T]$ are decoupled. The particular expressions of M_{00} and M_{TT} in (3.23) and (3.26) are not unexpected. In fact, taking $t = T$ in (3.24) we immediately obtain

$$\frac{\partial R}{\partial t}(T,s) = A(T)R(T,s)\,. \tag{3.27}$$

Then, we notice that the backward model for a Markov process $x(t)$ with state-space model (2.15a)–(2.15b) is [13],[18]

$$\begin{aligned} -\frac{d}{dt}x(t) &= -[A(t)+Q(t)\Pi^{-1}(t)]x(t) - B(t)\tilde{u}(t) \\ &= -A_b(t)x(t) - B(t)\tilde{u}(t) \end{aligned} \tag{3.28a}$$

$$E[x(T)x^T(T)] = \Pi(T) \qquad E[\tilde{u}(t)\tilde{u}^T(s)] = I\delta(t-s)\,. \tag{3.28b}$$

Since the driving noise $\tilde{u}(t)$ in the backward model is uncorrelated with the "past" states $x(s)$ for $t \le s \le T$, we have that the covariance $R(t,s)$ satisfies for $t \le s$

$$\frac{\partial R}{\partial t}(t,s) = A_b(t)R(t,s)\,. \tag{3.29}$$

Taking $t = 0$ in (3.29) and considering the expression for $A_b(t)$ in (3.28a) we get

$$\frac{\partial R}{\partial t}(0,s) = (A(0)+Q(0)\Pi^{-1}(0))R(0,s)\,. \tag{3.30}$$

Expressions (3.27) and (3.30) correspond exactly to (3.1) considering (3.23) and (3.26).

4 Construction of Markov Processes with Fixed Reciprocal Dynamics

In Section 2 we have seen that the second-order model (2.1a) for the covariance of $x(t)$ needs a set of boundary conditions to be solved. In fact, the assignment of the boundary conditions allow us to select a particular process in the reciprocal family described by the second-order differential operator (2.2). Then a question arises naturally: which are the choices of boundary conditions that are such that the solution of the second-order equation (2.1a) is the covariance of a Markov process? In this section we give an answer to this question. More specifically, we address the problem of building the Markov process belonging to the reciprocal class associated to the operator (2.2) which has given marginal end-point densities

$$x(0) \sim \mathcal{N}(0,\Pi(0)) \qquad x(T) \sim \mathcal{N}(0,\Pi(T))\,. \tag{4.1}$$

It is clear that to satisfy the constraint (4.1) on the variance of $x(t)$ we are no more free to choose as boundary data P an arbitrary symmetric positive definite matrix. In fact, for a Markov process the covariance $R(t,s)$ is determined by the relations (3.16) and (3.18), and in particular the value of $R(0,T) = R^T(T,0)$ in P is fixed by the data. Then, as a starting point we investigate the properties of the end-point covariance matrix P of a

Markov process. The results we obtain yield a procedure for computing Dirichlet and cyclic boundary conditions in function of the given reciprocal dynamics (2.5a) and end-point variances (4.1).

Combining (3.2) with the expressions for M_{00}, M_{TT} in (3.23),(3.26) we obtain that for Markov processes the inverse of the end-point covariance matrix P has the following expression

$$\begin{bmatrix} J_0 & J_1 \\ J_2 & J_T \end{bmatrix} = P^{-1} \tag{4.2}$$

where

$$\begin{aligned} J_0 &= -(Q^{-1}(0)\dot{\Psi}_1(0) - Q^{-1}(0)A(0) - \Pi^{-1}(0)) && \text{(4.3a)} \\ J_1 &= -Q^{-1}(0)\dot{\Psi}_2(0) = (\text{inverting (2.14a)}) = -K^{-1} && \text{(4.3b)} \\ J_2 &= Q^{-1}(T)\dot{\Psi}_1(T) = (\text{inverting (2.14b)}) = -K^{-T} && \text{(4.3c)} \\ J_T &= Q^{-1}(T)\dot{\Psi}_2(T) - Q^{-1}(T)A(T). && \text{(4.3d)} \end{aligned}$$

As we have already remarked, the matrix K is a reciprocal invariant, i.e. it doesn't depend on the boundary conditions. Thus, the problem of finding the required Dirichlet boundary conditions can be formulated as the problem of finding a positive definite matrix P such that

$$P = \begin{bmatrix} \Pi(0) & * \\ * & \Pi(T) \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} * & -K^{-1} \\ -K^{-T} & * \end{bmatrix} \tag{4.4}$$

where the entries $*$ need to be determined. A solution of this problem has been given in [11]. We give a sketch of this method referring to [11] for further details. The procedure is based on the LDU factorization of P

$$P = \begin{bmatrix} I & 0 \\ Z & I \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \begin{bmatrix} I & Z^T \\ 0 & I \end{bmatrix}, \tag{4.5}$$

which implies

$$P^{-1} = \begin{bmatrix} I & -Z^T \\ 0 & I \end{bmatrix} \begin{bmatrix} X^{-1} & 0 \\ 0 & Y^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -Z & I \end{bmatrix}. \tag{4.6}$$

Comparing the expressions (4.4),(4.5) and(4.6) for P and its inverse we can identify

$$\begin{aligned} \Pi(0) &= X && \text{(4.7a)} \\ \Pi(T) &= ZXZ^T + Y && \text{(4.7b)} \\ K^{-1} &= Z^T Y^{-1}. && \text{(4.7c)} \end{aligned}$$

Substituting (4.7a) and (4.7c) inside (4.7b) yields the algebraic Riccati equation (ARE)

$$\Pi(T) = YK^{-T}\Pi(0)K^{-1}Y + Y. \tag{4.8}$$

Once we have found a positive definite solution Y of the ARE (4.8), all the blocks in the LDU factorization of P are known, since Z^T is given by

$$Z^T = K^{-1}Y \ . \tag{4.9}$$

Provided that $\Pi(0) > 0$, the condition $Y > 0$ ensures that the matrix P is positive definite. The existence of a positive definite solution of (4.8) can be established thanks to standard results of linear control theory [20],[21]. In [11] it is also described a numerically stable method for computing a positive definite solution of the ARE (4.8), which is based on the computation of the stable eigenspaces of the Hamiltonian matrix

$$\mathcal{H} = \begin{bmatrix} -I/2 & -K^{-T}\Pi(0)K^{-1} \\ -\Pi(T) & I/2 \end{bmatrix} . \tag{4.10}$$

The knowledge of the solution Y of the ARE (4.8) is sufficient to derive both Dirichlet and cyclic boundary conditions for the model. In fact, the covariance $R(0,T) = R^T(T,0)$ is given by

$$R(0,T) = \Pi(0)K^{-1}Y \ , \tag{4.11}$$

and thus the covariance matrix P in the Dirichlet boundary conditions is completely specified since $R(0,0) = \Pi(0)$ and $R(T,T) = \Pi(T)$ are already given as data. We can also relate Y to the state-space dynamics in terms of the state transition matrix $\Phi(t,s)$. From (4.11) and the expression (3.21) for P we can identify

$$\Phi(T,0) = YK^{-T} \ , \tag{4.12}$$

and from the following closed-form expression for $\Pi(T)$

$$\begin{aligned} \Pi(T) &= \Phi(T,0)\Pi(0)\Phi^T(T,0) + \int_0^T \Phi(T,s)Q(s)\Phi^T(T,s)\,ds \\ &= \Phi(T,0)\Pi(0)\Phi^T(T,0) + \Pi(T|0) \end{aligned} \tag{4.13}$$

and again (4.11) we desume that

$$Y = \Pi(T|0) \ . \tag{4.14}$$

The knowledge of Y is also sufficient to obtain the cyclic boundary conditions. In fact, the blocks J_0 and J_T in P^{-1} can be obtained from Y as

$$\begin{aligned} J_0 &= \Pi^{-1}(0) + K^{-1}YK^{-T} \\ &= \Pi^{-1}(0) + \Phi^T(T,0)\Pi^{-1}(T|0)\Phi(T,0) \qquad (4.15a) \\ J_T &= Y^{-1} = \Pi^{-1}(T|0) \ . \qquad (4.15b) \end{aligned}$$

Since K and $\Psi(t)$ are reciprocal invariants, they can be determined from the data. Thus, combining expressions (4.3a), (4.3d) for J_0 and J_T with expressions (3.23), (3.26) for M_{00}, M_{TT}, it easy to see that the following relations hold:

$$M_{00} = \dot{\Psi}_1(0) + Q(0)J_0 \tag{4.16a}$$
$$M_{TT} = \dot{\Psi}_2(T) - Q(T)J_T \ , \tag{4.16b}$$

or equivalently in terms of the matrix N:

$$N_{00} = J_0 + Q^{-1}(0)\dot{\Psi}_1(0) \tag{4.17a}$$
$$N_{TT} = J_T - Q^{-1}(T)\dot{\Psi}_2(T) \ . \tag{4.17b}$$

A state-space model can also be built taking as $B(t)$ a square root of $Q(t)$ and defining implicitly $A(t)$ by the relations

$$\frac{\partial R}{\partial t}(t,0) = A(t)R(t,0) \tag{4.18a}$$
$$R(t,0) = \Psi_1(t)\Pi(0) + \Psi_2(t)R(T,0) \tag{4.18b}$$

where $R(T,0) = R^T(0,T)$ is given in terms of known quantities in (4.11).

5 Change of End-point Density for a Markov Process: A Stochastic Optimal Control Problem

The procedure described in the previous section provides a way to solve Schrödinger's problem in the Gaussian case. However, we can also use a different perspective and view Schrödinger's problem as the one of changing the final density of a Markov process $x(t)$, with the constraint of having fixed reciprocal dynamics. Clearly, the technique of Section 4 can still be used. In fact, we can obtain the second-order model for $x(t)$ using relations (2.17a)–(2.17b) and then construct the Markov process in the reciprocal class obtained in such a way with the same initial-density of $x(t)$ and the new prescribed end-density. However, using a different approach which is related to the methods originally used by Schrödinger, we will be able to give an interesting interpretation of the results in terms of an optimal control problem.

Let $x(t)$ be a Markov process with state-space model (2.15a)–(2.15b). The variance of the final state $x(T)$ is given by $\Pi(T)$, where $\Pi(T)$ is as in (4.13). We want the process to reach a different final state variance, say $\Pi^*(T)$. We have the following theorem.

Theorem 5.1 *Let $x(t)$ be the Markov process with state-space model (2.15a)–(2.15b). Then the Markov process $x^*(t)$ belongs to the same reciprocal class of $x(t)$ if and only if it admits the state-space representation*

$$\begin{aligned} \frac{dx^*(t)}{dt} &= \bar{A}(t)x^*(t) + \bar{B}(t)\nu(t) && \text{(5.1a)} \\ x^*(0) &\sim \mathcal{N}(0, \Pi(0)) && \text{(5.1b)} \end{aligned}$$

where

$$\begin{aligned} \bar{A}(t) &= A(t) - Q(t)S(t) && \text{(5.2a)} \\ \bar{B}(t) &= B(t)U(t)\,, && \text{(5.2b)} \end{aligned}$$

$U(t)$ is a unitary matrix and $S(t)$ solves the following homogeneous Riccati differential equation

$$\frac{dS(t)}{dt} + A^T(t)S(t) + S(t)A(t) - S(t)Q(t)S(t) = 0. \tag{5.3}$$

Proof: It is clear that if $U(t)$ is unitary we have that $\bar{B}(t)\bar{B}^T(t) = Q(t)$. The relations between the coefficients of the first and second order description of a Markov process are given in (2.17a)–(2.17b). Simple computations show that if $(\bar{A}(t), \bar{B}(t))$ satisfy (5.2a)–(5.2b), then $(\bar{A}(t), \bar{B}(t))$ is a solution of (2.17a)–(2.17b), i.e. $x^*(t)$ has the same reciprocal representation as $x(t)$. Conversely, assume that both $(A(t), B(t))$ and $(\bar{A}(t), \bar{B}(t))$ are solution of (2.17a)–(2.17b). Let us define

$$S(t) \stackrel{\Delta}{=} Q^{-1}(t)(A(t) - \bar{A}(t)). \tag{5.4}$$

Then it is straightforward to verify that the matrix function $S(t)$ solves the differential Riccati equation (5.3). ■

Theorem 5.1 shows that two Markov processes in the same reciprocal class can be obtained one from the other by means of a state feedback. The differential Riccati equation (5.3) needs a terminal condition $S(T)$ to be solved. The assignment of the appropriate terminal condition will allow us to select the Markov process with the prescribed end-state variance $\Pi^*(T)$. The value of $S(T)$ can be obtained considering the algebraic Riccati equation (4.8). Relation (4.3d) and (4.15b) imply that the solution of the ARE (4.8) with coefficients $(\Pi(0), \Pi^*(T))$ is such that

$$\begin{aligned} Y^{*-1} &= Q^{-1}(T)\dot{\Psi}_2(T) - Q^{-1}(T)\bar{A}(T) = \\ &\quad Q^{-1}(T)\dot{\Psi}_2(T) - Q^{-1}(T)(A(T) - Q(T)S(T)) \end{aligned} \tag{5.5}$$

that is

$$S(T) = Y^{*-1} + Q^{-1}(T)(A(T) - \dot{\Psi}_2(T)). \tag{5.6}$$

But since for the solution of the ARE (4.8) for $(\Pi(0), \Pi(T))$ it holds that

$$Y^{-1} = Q^{-1}(T)\dot{\Psi}_2(T) - Q^{-1}(T)A(T)\ , \tag{5.7}$$

we have that

$$S(T) = Y^{*-1} - Y^{-1}. \tag{5.8}$$

It is easy to check that a necessary and sufficient condition for having $S(T) > 0$ is $\Pi^*(T) < \Pi(T)$. In the special case when the system starts from a deterministic initial condition (i.e. $\Pi(0) = 0$), we have that $Y = \Pi(T)\,, Y^* = \Pi^*(T)$ and then $S(T) = \Pi^{*-1}(T) - \Pi^{-1}(T)$. We illustrate the procedure with the following example.

Example: *Change of end-point density for a scalar process.*
Let $x(t)$ be a scalar Gauss–Markov process defined on the interval $I = [0, T]$, with state-space model

$$\frac{dx(t)}{dt} = Ax(t) + \nu(t) \tag{5.9a}$$

$$x(0) \sim \mathcal{N}(0, \sigma_0^2) \tag{5.9b}$$

where A is a positive constant and $\nu(t)$ is white noise with constant intensity $Q(t) \equiv 1$. The variance of $x(t)$ is

$$\Pi(t) = \sigma_0^2 e^{2At} + \frac{1}{2A}e^{2At} - \frac{1}{2A}\ . \tag{5.10}$$

We want to change the variance of $x(T)$ to a value $\sigma_T^2 < \Pi(T)$. The differential Riccati equation (5.3) is in this case

$$\frac{dS(t)}{dt} + 2AS(t) - S^2(t) = 0 \tag{5.11}$$

and can be easily solved by separating the variables. Thus we obtain

$$S(t) = \frac{2A}{e^{2A(t+c)} + 1} \tag{5.12}$$

and the integration constant c can be determined using condition (5.8) for $S(t)$ at $t = T$. We need to compute the coefficient K and then solve the ARE (4.8) with coefficients $(\sigma_0^2, \Pi(T))$ and (σ_0^2, σ_T^2). Relations (2.17a)–(2.17b) give $F(t) = A^2$ and $G(t) = 0$ in the second-order model for $x(t)$, and the boundary transition matrix solution of (2.8a)–(2.8b) is

$$\Psi(t) = \left[\begin{array}{cc} \dfrac{e^{-AT}}{e^{-AT} - e^{AT}}e^{At} + \dfrac{e^{AT}}{e^{AT} - e^{-AT}}e^{-At} & \\ \dfrac{1}{e^{AT} - e^{-AT}}e^{At} + \dfrac{1}{e^{-AT} - e^{AT}}e^{-At} & \end{array}\right]\ . \tag{5.13}$$

From (2.14a) we obtain the value of K

$$K = \dot{\Psi}_2^{-1}(0)Q(0) = \frac{1}{2A}(e^{AT} - e^{-AT}) \,. \tag{5.14}$$

The solutions of the AREs

$$\Pi(T) = \frac{\sigma_0^2}{K^2}y^2 + y \tag{5.15a}$$

$$\sigma_T^2 = \frac{\sigma_0^2}{K^2}y^{*2} + y^* \tag{5.15b}$$

are respectively

$$y = \frac{K}{2\sigma_0^2}\left(\sqrt{K^2 + 4\sigma_0^2\Pi(T)} - K\right) \tag{5.16a}$$

$$y^* = \frac{K}{2\sigma_0^2}\left(\sqrt{K^2 + 4\sigma_0^2\sigma_T^2} - K\right) \,. \tag{5.16b}$$

Then the terminal condition for $S(t)$ is

$$S(T) = y^{*-1} - y^{-1} \triangleq \frac{1}{\delta} \,. \tag{5.17}$$

This fixes the value of c in (5.14) to

$$c = \frac{1}{2A}\ln\,(2A\delta - 1) - T \,, \tag{5.18}$$

and finally $S(t)$ is

$$S(t) = \frac{2A}{e^{2A(t-T)}(2A\delta - 1) + 1} \,. \tag{5.19}$$

Thus, the new Markov process $x^*(t)$ satisfying the condition $E[x^*(T)x^{*T}(T)] = \sigma_T^2$ has the following state-space model

$$\frac{dx^*(t)}{dt} = \bar{A}(t)x^*(t) + \nu(t) \tag{5.20a}$$

$$x^*(0) \sim \mathcal{N}(0, \sigma_0^2) \tag{5.20b}$$

where

$$\bar{A}(t) = A - S(t) = A\frac{e^{2A(t-T)}(2A\delta - 1) - 1}{e^{2A(t-T)}(2A\delta - 1) + 1} \,. \tag{5.21}$$

■

We analyse now with more detail how the feedback matrix $S(t)$ is related to the reciprocal description of the process $x(t)$. To this aim, following the

approach used by Schrödinger [17], Jamison [9], and others [22, 15] for the general case of arbitrary, i.e. not necessarily Gaussian, Markov processes, we study the joint probability density of the process $x^*(t)$. The end-point densities of x^* in $t = 0, T$ satisfy a system of two coupled integral equations (which is often referred to as the *Schrödinger system*) which in the Gaussian case can be reduced to a system of two coupled AREs. The solutions of this system are then used to find an expression for the terminal condition (5.8) for $S(t)$. This enables us to show that the matrix $S(t)$ can be related to the a posteriori distribution of $x(t)$ given the observation of its final state state $x(T)$.

Starting from results of Schrödinger [17] and Jamison [8], in [11] a characterization of the joint densities of the process $x^*(t)$ was discussed. We state without proof the following theorem, which is the continuous-time version of a result in [11].

Theorem 5.2 [11] *Given a Markov process $x(t)$ with state-space model (2.15a)-(2.15b) if $x^*(t)$ is a Markov process in the same reciprocal class, with the same initial density, but with $x^*(T) \sim \mathcal{N}(0, \Pi^*(T))$, the joint probability density of x^* at the times $t_0 = 0, t_1, \ldots, t_k, \ldots, t_N = T$ can be expressed as*

$$p^*(x_0, 0; x_1, t_1; \ldots; x_T, T) = \prod_{k=0}^{N-1} G(x_k, t_k; x_{k+1}, t_{k+1}) q_f(x_0) q_b(x_T) \,, \tag{5.22}$$

where if

$$\begin{aligned} G(x_0, 0; x_T, T) = & \frac{1}{(2\pi)^{n/2} |\Pi(T|0)|^{1/2}} \\ & \exp\{-\frac{1}{2}(x_T - \Phi(T,0)x_0)^T \Pi^{-1}(T|0)(x_T - \Phi(T,0)x_0)\} \end{aligned} \tag{5.23}$$

denotes the transition density from $t = 0$ to $t = T$ of the process $x(t)$, with $\Pi(T|0)$ as in (4.13), the end-point densities $q_f(x_0)$ and $q_b(x_T)$ satisfy the coupled integral equations

$$p(x_0, 0) = q_f(x_0) \int G(x_0, 0; x_T, T) q_b(x_T) dx_T \tag{5.24a}$$

$$p^*(x_T, T) = q_b(x_T) \int G(x_0, 0; x_T, T) q_f(x_0) dx_0 \,. \tag{5.24b}$$

For Gaussian processes, we can assume that q_f and q_b have the form

$$q_f(x_0) = C_f \ \exp\{-\frac{1}{2} x_0^T \Theta_f^{-1} x_0\} \tag{5.25a}$$

$$q_b(x_N) = C_b \ \exp\{-\frac{1}{2} x_T^T \Theta_b^{-1} x_T\} \,, \tag{5.25b}$$

where Θ_f and Θ_b are nonsingular symmetric, not necessarily positive definite, matrices. Then it is possible to show [11] that the integral equations (5.24a)–(5.24b) reduce to the coupled AREs

$$\Pi^{-1}(0) = \Theta_f^{-1} + \Phi^T(T,0)[\Theta_b + \Pi(T|0)]^{-1}\Phi(T,0) \quad (5.26a)$$

$$\Pi^{*-1}(T) = \Theta_b^{-1} + [\Pi(T|0) + \Phi(T,0)\Theta_f\Phi^T(T,0)]^{-1} . \quad (5.26b)$$

From (5.22) in Theorem 5.2, the joint probability density of $x^*(0)$ and $x^*(T)$ is given by

$$p^*(x_0,0;x_T,T) = G(x_0,0;x_T,T)q_f(x_0)q_b(x_T) , \quad (5.27)$$

and (5.27) yields the following expression for the inverse of the covariance P^* of the boundary vector $[x^{*T}(0)\ \ x^{*T}(N)]^T$:

$$P^{*-1} = \begin{bmatrix} \Theta_f^{-1} & 0 \\ 0 & \Theta_b^{-1} \end{bmatrix} + \begin{bmatrix} -\Phi^T(T,0) \\ I \end{bmatrix} \Pi^{-1}(T|0) \begin{bmatrix} -\Phi(T,0) & I \end{bmatrix} . \quad (5.28)$$

Using identities (4.15a)–(4.15b) to express the diagonal blocks of P^{*-1} in terms of Y^*, we find

$$J_0^* = \Pi^{-1}(0) + K^{-1}Y^*K^{-T}$$

$$= \Theta_f^{-1} + \Phi^T(T,0)\Pi^{-1}(T|0)\Phi(T,0) \quad (5.29a)$$

$$J_T^* = Y^{*-1} = \Theta_b^{-1} + \Pi^{-1}(T|0) , \quad (5.29b)$$

so that

$$\Theta_f^{-1} = \Pi^{-1}(0) + K^{-1}(Y^* - \Pi(N|0))K^{-T} \quad (5.30a)$$

$$\Theta_b^{-1} = Y^{*-1} - \Pi^{-1}(T|0) . \quad (5.30b)$$

But since the solution of the ARE (4.8) with coefficients $(\Pi(0),\Pi(T))$ is $Y = \Pi(0|T)$, we conclude from (5.30b) and (5.8) that the initial condition for the differential Riccati equation (5.3) is

$$S(T) = \Theta_b^{-1} . \quad (5.31)$$

Consider now the function

$$q_b(x,t) = \int G(x,t;x_T,T)q_b(x_T)\,dx_T , \quad (5.32a)$$

where

$$G(x,t;x_T,T) = \frac{1}{(2\pi)^{n/2}|\Pi(T|t)|^{1/2}} \exp\{-\frac{1}{2}(x_T - \Phi(T,t)x)^T\Pi^{-1}(T|t)(x_T - \Phi(T,t)x\} \quad (5.32b)$$

is the probability density of $x(T)$ given that $x(t) = x$, with

$$\Pi(T|t) \stackrel{\Delta}{=} \Pi(T) - \Phi(T,t)\Pi(t)\Phi^T(T,t) \,. \tag{5.32c}$$

Since $q_b(x,t)$ is obtained by integrating two Gaussian distributions, it is also Gaussian, i.e.

$$q_b(x,t) = C_b(t) \, \exp\{-\frac{1}{2}x^T\Theta_b^{-1}(t)x\} \,, \tag{5.33a}$$

where some computations show that

$$\Theta_b^{-1}(t) = \Phi^T(T,t)[\Theta_b + \Pi(T|t)]^{-1}\Phi(T,t) \,. \tag{5.33b}$$

We can now prove the following lemma.

Lemma 5.1

$$S(t) = \Theta_b^{-1}(t) \,. \tag{5.34}$$

Proof: Obviously $\Theta_b^{-1}(T) = \Theta_b^{-1}$. We show that $\Theta_b^{-1}(t)$ satisfies the differential Riccati equation (5.3). In fact, using (3.20), we have

$$\begin{aligned}\dot{\Theta}_b^{-1}(t) = &-A^T(t)\Phi^T(T,t)[\Theta_b + \Pi(T|t)]^{-1}\Phi(T,t) \\ &-\Phi^T(T,t)[\Theta_b + \Pi(T|t)]^{-1}\dot{\Pi}(T|t)[\Theta_b + \Pi(T|t)]^{-1}\Phi(T,t) \\ &-\Phi^T(T,t)[\Theta_b + \Pi(T|t)]^{-1}\Phi(T,t)A(t) \,.\end{aligned} \tag{5.35}$$

Using the Lyapunov equation (3.18) for $\Pi(t)$ and again (3.20) we obtain

$$\dot{\Pi}(T|t) = -\Phi(T,t)Q(t)\Phi^T(T,t) \,. \tag{5.36}$$

Plugging (5.36) in (5.35) we see that

$$\dot{\Theta}_b^{-1}(t) = -A^T(t)\Theta_b^{-1}(t) + \Theta_b^{-1}(t)Q(t)\Theta_b^{-1}(t) - \Theta_b^{-1}(t)A(t) \tag{5.37}$$

and then $\Theta_b^{-1}(t)$ is the required solution of (5.3). ■

The differential Riccati equation (5.3) is very well known in optimal control theory. In fact, consider the following problem.

Problem 5.1 *Find a control function $u(t)$ that minimizes the performance index*

$$J = E\left[\frac{1}{2}\int_0^T u^T(s)(B^T(s)B(s))^{-1}u(s)\,ds + x_u^T(T)Wx_u(T)\right] \tag{5.38}$$

subject to the constraint

$$\frac{dx_u(t)}{dt} = A(t)x_u(t) + u(t) + B(t)\nu(t) \tag{5.39a}$$

$$x_u(0) \sim \mathcal{N}(0, \Pi(0)) \,. \tag{5.39b}$$

Optimal control problems with performance index like (5.38) are often referred to as *minimum energy* control problems. The optimal control function that solves Problem 5.1 is the linear state feedback

$$u(t) = -Q(t)S(t)x(t) \tag{5.40}$$

where $S(t)$ solves the homogeneous Riccati differential equation (5.3) with terminal condition $S(T) = W$. Then, it follows from Theorem 5.1 that all the Markov processes in the same reciprocal class of $x(t)$ can be obtained solving the control problem 5.1 for different values of the matrix W. Equivalently, we can say that the subclass of Markov processes in a reciprocal class can be generated by giving one of the Markov processes, $x(t)$, and taking all the processes $x_u(t)$ obtained by solving for different values of $\Pi(0)$ and $\Pi(T)$ the following stochastic optimal control problem

Problem 5.2 *Find a control function $u(t)$ that minimizes the performance index*

$$J = E\left[\frac{1}{2}\int_0^T u^T(s)(B^T(s)B(s))^{-1}u(s)\,ds\right] \tag{5.41}$$

subject to the constraint

$$\frac{dx_u(t)}{dt} = A(t)x_u(t) + u(t) + B(t)\nu(t) \tag{5.42a}$$

$$x_u(0) \sim \mathcal{N}(0, \Pi(0)) \tag{5.42b}$$

$$x_u(T) \sim \mathcal{N}(0, \Pi(T))\,. \tag{5.42c}$$

The connection between reciprocal processes and stochastic optimal control theory in the general non-Gaussian context has recently been the object of the attention of different authors [19, 2, 5], in connection to quantum mechanics. In particular, the stochastic interpretation of the change of end-point density for a Markov process given in this section can be viewed as a special case of the results of Dai Pra, who has solved the analogue of problem 5.2 for a generic diffusion process, deriving the optimal control function starting from the transition density of the Markov process $x(t)$ and the coupled integral equations (5.24a)–(5.24b).

6 Conclusions

In this paper we have shown how to construct Gauss-Markov processes, defined on a finite interval, having fixed initial and end-point densities and reciprocal dynamics. Specifically, we have presented a procedure for choosing the appropriate boundary conditions for the second-order model characterizing the reciprocal class in order to achieve the aforementioned goal. The method requires the solution of an algebraic Riccati equation.

The strictly related problem of changing the end-point density of a Markov process while remaining in the same reciprocal class was also considered. The new process satisfying the constraint on the end-point density can be obtained from the given one by means of a state feedback. This result can also be interpreted in the framework of stochastic optimal control theory: in fact, the state feedback changing in the desired way the variance of the final state corresponds to a minimum energy control law.

Acknowledgments

I would like to thank Prof. Ruggero Frezza and Prof. Bernard C. Levy for their valuable suggestions and comments on the different versions of this paper. I am also grateful to an anonymous referee who pointed out a mistake in the original formulation and proof of Lemma 3.1 and suggested the present ones.

References

[1] S. Bernstein. Sur les liaisons entre les grandeurs aléatoires, In *Proc. Int. Cong. of Math.*, pages 288-309, Zurich, Switzerland, 1932.

[2] A. Blaquière. Controllability of a Fokker-Plank equation, the Schrödinger system, and a related stochastic optimal control, *Dynamics and Control*, 1991.

[3] J. M. C. Clark. A local characterization of reciprocal diffusions, In (M. H. A. Davis and R. J. Elliott), *Applied Stochastic Analysis* volume 5, pages 45–59. New York: Gordon and Breach, 1991.

[4] J. Coleman. *Gaussian Reciprocal Processes and Their Associated Conservation Laws.* Master's thesis, University of California, Davis, 1992.

[5] P. Dai Pra. A stochastic control approach to reciprocal diffusion processes, *Applied Math. Optim.* **23** (1991), 313–329.

[6] R. Frezza. *Models of Higher-order and Mixed-order Gaussian Reciprocal Processes with Application to the Smoothing Problem.* Ph. D. dissertation, Applied Mathematics Program, UC Davis, June 1990.

[7] B. Jamison. Reciprocal processes: The stationary Gaussian case, *Ann. Math. Statist.* **41**(5) (1970), 1624–1630.

[8] B. Jamison. Reciprocal processes, *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **30** (1974), 65–86.

[9] B. Jamison. The Markov processes of Schrödinger, *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **32** (1975), 323–331.

[10] A. J. Krener, R. Frezza, and B. C. Levy. Gaussian reciprocal processes and self-adjoint stochastic differential equations of second order, *Stochastics and Stochastics Reports* **34** (1991), 29–56.

[11] B. C. Levy and A. Beghi. Discrete-time Gauss-Markov processes with fixed reciprocal dynamics, *J. Math. Systems, Estimation, and Control.* To appear.

[12] B. C. Levy, R. Frezza, and A. J. Krener. Modeling and estimation of discrete-time Gaussian reciprocal processes, *IEEE Trans. Automat. Control* **35** (1990), 1013–1023.

[13] A. Lindquist and G. Picci. On the stochastic realization problem, *SIAM J. Control Optim.* **17** (1979), 365–389.

[14] R. N. Miroshin. Second-order Markov and reciprocal Gaussian processes. *Theor. Probab. Appl.* **24** (1979), 845–852.

[15] M. Nagasawa. Stochastic variational principle of Schrödinger processes. In *Seminar on Stochastic Processes.* Basel: Birkhäuser, 1989.

[16] Yu. A. Rozanov. *Stationary Random Processes.* San Francisco: Holden-Day, 1967.

[17] E. Schrödinger. Über die umkehrung der naturgesetze. *Sitzungsberichte der Preussischen Akademie der Wissenschaften Physikalisch-Mathematische Klasse*, pages 144–153, 1931.

[18] G. Verghese and T. Kailath. A further note on backward markovian models, *IEEE Trans. Informat. Theory* **25** (1979), 121–124.

[19] A. Wakolbinger. A simplified variational characterization of Schrödinger processes, *J. Math. Phys.* **30**(12) (1989), 2943–2946.

[20] J. C. Willems. Least squares stationary optimal control and the algebraic Riccati equation, *IEEE Trans. Automat. Control* **16** (1971), 621–634.

[21] W. M. Wonham. *Linear Multivariable Control – A Geometric Approach.* Berlin: Springer Verlag, 1974.

[22] J. C. Zambrini. A new probabilistic approach to the classical heat equation. In (A. Truman and I. M. Davies), *Stochastic Mechanics and*

Stochastic Processes, Lecture Notes in Mathematics, no. 1325, pages 205–220. Berlin: Springer-Verlag, 1988.

Department of Electronics and Computer Science, University of Padova, via Gradenigo 6/A, 35131 Padova, Italy

Communicated by Anders Lindquist