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Well-Posedness, Stabilizability, and Admissibility for Pritchard-Salamon Systems*

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Abstract

The object of this paper is to further develop the theory of Pritchard-Salamon systems, which are abstract infinite-dimensional systems allowing for a certain unboundedness of the control and observation operators. New results are derived on the transfer function and the impulse response of a Pritchard-Salamon system, on the well-posedness of feedback systems, on the invariance properties of the Pritchard-Salamon class under feedback and output injection, on the relation between bounded and admissible stabilizability and on the relationship between exponential and external stability.

Key words: Infinite-dimensional control systems, feedback, stability, stabilizability, transfer functions, C_0-semigroups

AMS Subject Classifications: 93C25, 93D15, 93D20, 93D25, 47D06

Nomenclature

$\mathbb{C}_\lambda := \{s \in \mathbb{C} \mid \mathrm{Re}(s) > \lambda\}, \quad \lambda \in \mathbb{R}.$

Let X and Y be Banach spaces and A a linear operator defined on some subspace of X with values in X. Then:

$$\begin{aligned}
D(A) &:= \text{domain of } A\\
R(A) &:= \text{range of } A\\
\rho(A) &:= \text{resolvent set of } A\\
\sigma(A) &:= \text{spectrum of } A\\
\sigma_p(A) &:= \text{point spectrum of } A \text{ (= set of all eigenvalues of } A)\\
r(A) &:= \text{spectral radius of } A \text{ (if } A \text{ is bounded on } X)
\end{aligned}$$

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Moreover:

$\mathcal{L}(X,Y)$	:=	bounded linear operators from X to Y
$\mathcal{L}(X)$	:=	$\mathcal{L}(X,X)$
$LL^p(0,\infty;X)$	:=	locally p-integrable functions (in the sense of Bochner) defined on $[0,\infty)$ with values in X
$H^\infty(\mathbb{C}_\lambda, X)$	:=	the usual Hardy spaces of bounded holomorphic functions defined on $\mathbb{C}_\lambda$ with values in X

If $S(t)$ is a C_0-semigroup on X, then

$$\omega(S(\cdot)) := \text{ exponential growth constant of } S(t)$$

Finally, $\mathbb{L}$ and the superscript $\hat{}$ are used to denote the Laplace transform.

1 Introduction

Whilst the Pritchard-Salamon class of linear infinite-dimensional systems does include many examples of partial differential systems with boundary control and observation and of delay systems with delayed control and sensing action, it is by no means the largest class of infinite-dimensional systems which has been treated in the literature. For example, it does not include all the examples treated in Lasiecka and Triggiani [15] or in Pedersen [21]. In fact, the Pritchard-Salamon class is a strict subset of the class considered in Salamon [30], [31] and in Weiss [35]. Consequently, this detailed analysis of a special subclass needs some motivation.

First, we recall that the class was first introduced in Pritchard and Salamon [22], [23], to provide a general abstract framework for the linear quadratic control problem. While many other solutions to this problem for even more general classes exist, other proofs tend to be tailored for a specific class, for example, one proof for hyperbolic partial differential equations and another for retarded delay equations. The Pritchard-Salamon class includes both retarded delay systems and many partial differential systems as well, and the one abstract proof applies for all these examples. Later, it was recognized by others that this same class had just the right properties for control synthesis in both time and in frequency domain, and many papers on a wide range of control problems for Pritchard-Salamon systems have appeared: Curtain [3] on the equivalence of exponential and external stability, Curtain and Salamon [9] on stabilization by finite-dimensional output feedback, Logemann [17] on circle criteria and small gain conditions, Pritchard and Townley [24], [25] on the stability radius problem (which is a H^∞-type problem), Logemann and Mårtensson [18] on adaptive stabilization, Curtain [4] on robust stabilization with respect to normalized coprime factorizations, Curtain and Ran [8] on the relaxed optimal Hankel norm problem and in Curtain [6] various robust control problems for Pritchard-Salamon systems have been surveyed. So there already exists an extensive literature on properties of and control problems for the Pritchard-Salamon class of systems. Moreover, examples of Pritchard-Salamon systems have been well-documented in the literature, for example in Pritchard and Salamon [22], [23], Bontsema [1] and Curtain [6]. In spite of this impressive list of publications, there remain several unresolved or only partially resolved fundamental issues concerning Pritchard-Salamon systems, for example,

- The existence and well-posedness of transfer functions and impulse responses (in particular, for systems with infinite-rank inputs and output).
- Perturbation results which cover perturbations induced by output feedback.
- The identification of sufficiently rich classes of feedback and output injection operators such that the closed-loop system is again a Pritchard-Salamon system.
- Stabilizability and detectability concepts which have the property that stabilizability and detectability are retained under feedback and output injection.
- The relation between exponential and external stability for systems with inputs and outputs of infinite rank.

These issues are fundamental to analysis and control synthesis for Pritchard-Salamon systems. Some of these points have received attention in the literature, some properties have been shown to be true under extra assumptions (for example, finite-rank inputs and outputs) and others have been conjectured to be true in general. In this paper we examine these issues in some detail and so lay the necessary basis for continuing research on control design for Pritchard-Salamon systems. For example, the results of this paper form an essential first step in extending the recent results on H^∞-control for infinite-dimensional systems in van Keulen et al. [14], van Keulen [13] and Curtain [5] to the Pritchard-Salamon class. Whilst we accept that it seems artificial to allow for the possibility of infinite-dimensional input and output spaces, we emphasize that this situation arises naturally in perturbation problems for linear systems. In particular, it is useful to interpret infinite-rank weightings attached to a perturbation class as input and output operators for an associated control system. Indeed in Pritchard and Townley [24, 25], the stability radius of a strongly continuous semigroup under structured perturbations is characterized via a transfer function of a control system with infinite-dimensional rank input and output operators.

In more detail, the content of the present work is as follows. In Section 2 we define the concepts of admissible input and admissible output operators, which are due to Salamon [29], Pritchard and Salamon [22, 23] and Weiss [33, 34], introduce the Pritchard-Salamon class and show that any system in this class has a well-defined transfer function. Moreover, we prove some technical results which will be useful in the following sections. In particular, if $S(t)$ is a C_0-semigroup on Hilbert spaces W and V, where $W \subset V$ with continuous dense injection, we give a number of sufficient conditions for the exponential growth constants of $S(t)$ on W and V to coincide.

The main result of Section 3 shows that for any Pritchard-Salamon system

$$C(sI - A^W)^{-1}B = C(sI - A^V)^{-1}B, \tag{1.1}$$

where B is the input operator, C is the output operator and A^W and A^V denote the infinitesimal generators of $S(t)$ on W and V, respectively. In particular, it becomes clear that the additional assumption

$$D(A^V) \hookrightarrow W \quad ^{\dagger} \tag{1.2}$$

originally imposed in [23] is not necessary for (1.1) to hold.

It follows from the results in Section 2 that the impulse response of a Pritchard-Salamon system is in general a (operator-valued) distribution. The main result of Section 3 is then used to prove that the impulse response is a locally square integrable function, provided the input space is finite-dimensional.

In Section 4 we show that state feedback (output injection) applied to a Pritchard-Salamon system produces a well-posed closed-loop system which is again a Pritchard-Salamon system if the feedback operator (output injection operator) is an admissible output operator (admissible input operator). The advantage of taking the feedback and output injection operators to be admissible was suggested by Weiss [37] in a more general context. Morever, we prove that nesting of feedback loops is equivalent to closing the loop for the sum of the feedback operators. Under the extra assumption (1.2) we calculate the infinitesimal generators of the perturbed semigroup on W and V. Furthermore, we give another sufficient condition for the exponential growth constants of $S(t)$ on W and V to coincide.

In Section 5 we introduce the concepts of bounded stabilizability and admissible stabilizability and prove that they are equivalent. Finally, we show that the boundedness of the transfer function of a Pritchard-Salamon

† Here $D(A^V)$ is endowed with the graph norm of A^V and $\hookrightarrow$ means that $D(A^V) \subset W$, $D(A^V)$ is dense in W (with respect to the norm topology of W) and the canonical injection $D(A^V) \to W$, $x \mapsto x$ is continuous.

system in the open right-half plane implies exponential stability of the semigroup on W and V, provided the system is admissibly stabilizable and detectable. A result in a similar vein can be found in Rebarber [28]. However, the admissible stabilizability concepts are different and apply to different classes of systems

Although we work in a Hilbert space context we make clear in a remark placed at the end of Section 5 which results extend to Banach spaces.

2 Pritchard-Salamon Systems

Let W and V be Hilbert spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ satisfying

$$W \hookrightarrow V \ ,$$

i.e. $W \subset V$ and the canonical injection $W \to V$, $x \mapsto x$ is continuous and dense. We consider a C_0-semigroup $S(t)$ on V which restricts to a C_0-semigroup on W. Occasionally we will write $S^W(t)$ or $S^V(t)$ in order to indicate that we consider $S(t)$ as a semigroup on W or on V. The infinitesimal generators of $S(t)$ on W and V will be denoted by A^W and A^V, respectively.

The following example shows that the exponential growth constants $\omega_W := \omega(S^W(\cdot))$ and $\omega_V := \omega(S^V(\cdot))$ of $S(t)$ on W and V may be different and that both of the inequalities $\omega_W < \omega_V$ and $\omega_W > \omega_V$ are possible.

Example 2.1 Define $W := L^2(0,\infty;\mathbb{R})$ and $V = \{f \in LL^2(0,\infty;\mathbb{R}) \mid \exp(-\not\Vdash\cdot)\mho \in \mathbb{W}\}$, where $\|f\|_V := \|\exp(-1\cdot)f(\cdot)\|_W$. Clearly $W \subset V$ with continuous dense injection.

(i) The translation semigroup given by $(S(t)f)(x) = f(x+t)$ is a C_0-semigroup on W and V. It is straightforward to show that $\|S(t)\|_{\mathcal{L}(W)} = 1$ and $\|S(t)\|_{\mathcal{L}(V)} = e^{2t}$ and hence $\omega_W = 0$ and $\omega_V = 2$, and so $\omega_W < \omega_V$.

(ii) The semigroup defined by

$$(S(t)f)(x) = \begin{cases} 0, & 0 \le x < t \\ f(x-t), & x \ge t \end{cases}$$

is strongly continuous on W and V. Now $\|S(t)\|_{\mathcal{L}(W)} = 1$ and $\|S(t)\|_{\mathcal{L}(V)} = e^{-2t}$ and therefore $\omega_W = 0$ and $\omega_V = -2$ showing that for this example $\omega_W > \omega_V$.

The following proposition describes the relationship between ω_W and ω_V in the selfadjoint case.

Proposition 2.2 *Suppose that $S^V(t_0)$ is self-adjoint for some $t_0 > 0$. Then $\omega_W \ge \omega_V$ and under the extra assumption that*

$$\sigma(S^W(t_1)) \subset \overline{\sigma_p(S^W(t_1))} \cup \{0\}$$

cd for some $t_1 > 0$ we have $\omega_W = \omega_V$.

Proof: It follows from Lax [16] that $\sigma(S^V(t_0)) \subset \sigma(S^W(t_0))$ and hence $r(S^V(t_0)) \leq r(S^W(t_0))$. Now it is easy to see that for $X = W, V$

$$r(S^X(t)) = e^{\omega_X t} \text{ for all } t \geq 0 \tag{2.1}$$

(see e.g. Nagel [19], p. 60), which implies $\omega_W \geq \omega_V$. Moreover, it is trivial to see that $\sigma_p(S^W(t)) \subset \sigma_p(S^V(t))$ for all $t \geq 0$. By the closedness of the spectrum it follows that

$$\overline{\sigma_p(S^W(t))} \subset \sigma(S^V(t)) \text{ for all } t \geq 0\,.$$

Under the extra assumption

$$\sigma(S^W(t_1)) \subset \overline{\sigma_p(S^W(t_1))} \cup \{0\}$$

we obtain $r(S^W(t_1)) \leq r(S^V(t_1))$, and hence by (2.1) we see that $\omega_W \leq \omega_V$. $\square$

Remark 2.3 Proposition 2.2 says, in particular, that $\omega_W \geq \omega_V$ if A^V is self-adjoint. Under the extra assumption that $S^W(t_1)$ is compact for some $t_1 > 0$ equality holds.

For the following it is useful to introduce the space $Z := D(A^V)$. Endowed with the inner product

$$\langle x, x\rangle_Z := \langle x, x\rangle_V + \langle A^V x, A^V x\rangle_V$$

Z becomes a Hilbert space. It is clear that the resulting norm $\|x\|_Z = (\langle x, x\rangle_Z)^{\frac{1}{2}}$ is equivalent to the graph norm of A^V.

Remark 2.4 (i) As is well-known, $S^V(t)$ restricts to a C_0-semigroup $S^Z(t)$ on Z and $\omega_Z = \omega_V$, where $\omega_Z := \omega(S^Z(\cdot))$, see e.g. Salamon [29].

(ii) If $Z \subset W$ then it follows that $Z \hookrightarrow W$. Indeed, since $D(A^W)$ is dense in W (with respect to $\|\cdot\|_W$) the same holds for $Z = D(A^V) \supseteq D(A^W)$. Moreover an application of the closed graph theorem shows that the canonical injection $Z \to W$, $x \mapsto x$ is bounded.

On the basis of Remark 2.4 one might conjecture that if $Z \subset W$, then $\omega_W = \omega_V$. However, the following example shows that this is not true in general.

Example 2.5 Set $V := L^2(0,\infty;\mathbb{R})$ and consider again the translation semigroup on V given by $(S(t)f)(x) = f(t+x)$. It is well-known that

$$Z = D(A^V) = \{f \in V \mid f \text{ is a.c. and } f' \in V\}^{\dagger}$$

and

$$A^V f = f' \text{ for all } f \in Z.$$

† The abbreviation "a.c." stands for "absolutely continuous".

Morover, if we define

$$W := \{ f \in V \mid f \text{ is a.c. and } \int_0^\infty |f'(x)|^2 e^{-2x} dx < \infty \}$$

and

$$\langle f, g \rangle_W := \int_0^\infty f(x) g(x) dx + \int_0^\infty f'(x) g'(x) e^{-2x} dx$$

then it is clear that W is a Hilbert space, $S(t)$ restricts to a C_0-semigroup on W and

$$Z \hookrightarrow W \hookrightarrow V \, .$$

Since $\|S(t)\|_{\mathcal{L}(V)} = 1$ for all $t \geq 0$ we see that $\omega_V = 0$. Next we shall calculate $\|S(t)\|_{\mathcal{L}(W)}$. It is easy to see that

$$\|S(t)\|_{\mathcal{L}(W)} \leq e^t \text{ for all } t \geq 0 \, . \tag{2.2}$$

We shall show that equality holds. To this end define functions $f_{t,n} \in W$ $(t \geq 0, n \in \mathbb{N})$ by

$$f_{t,n}(x) = \begin{cases} 0 & , \quad x \in [0, t] \\ n(x-t) & , \quad x \in [t, t + \frac{1}{n}] \\ 1 & , \quad x \in [t + \frac{1}{n}, t + 1 - \frac{1}{n}] \\ n(t+1-x) & , \quad x \in [t + 1 - \frac{1}{n}, t+1] \\ 0 & , \quad x \in [t+1, \infty] \end{cases}$$

Then

$$\begin{aligned} \|f_{t,n}\|_W^2 &< 1 + \int_t^{t+1} \mid f'_{t,n}(x) \mid^2 e^{-2x} dx \\ &= \frac{n^2}{2} e^{-2t} (1 - e^{-\frac{2}{n}} + e^{-2+\frac{2}{n}} - e^{-2}) + 1 \end{aligned}$$

and

$$\begin{aligned} \|S(t) f_{t,n}\|_W^2 &\geq e^{2t} \int_t^\infty \mid f'_{t,n}(x) \mid^2 e^{-2x} dx \\ &= \frac{n^2}{2} (1 - e^{-\frac{2}{n}} + e^{-2+\frac{2}{n}} - e^{-2}) \, . \end{aligned}$$

Defining

$$\psi(n) := \frac{n^2}{2} (1 - e^{-\frac{2}{n}} + e^{-2+\frac{2}{n}} - e^{-2}) \text{ and } \varphi(n) := \frac{\psi(n)}{\psi(n)+1} \, ,$$

we obtain

$$\frac{\|S(t)f_{t,n}\|_W^2}{\|f_{t,n}\|_W^2} \geq \varphi(n)e^{2t}\,. \tag{2.3}$$

Finally, since $\psi(n) \to \infty$ and hence $\varphi(n) \to 1$ as $n \to \infty$, it follows from (2.3) that

$$\|S(t)\|_{\mathcal{L}(W)} \geq e^t \text{ for all } t \geq 0\,. \tag{2.4}$$

Combining (2.2) and (2.4) shows that $\omega_W = 1$, and so $\omega_W > \omega_V$.

In the following we present some sufficient conditions for $\omega_W = \omega_V$ to hold under the extra assumption that $Z \subset W$.

Lemma 2.6 *Suppose $Z \hookrightarrow W$. Then we have:*
(i) $\sigma_p(S^W(t)) = \sigma_p(S^V(t))$ for all $t \geq 0$.
(ii) $\sigma_p(A^W) = \sigma_p(A^V)$ and $\sigma(A^V) = \sigma(A^W)$.

Proof: (i) The inclusion $\sigma_p(S^W(t)) \subset \sigma_p(S^V(t))$ is trivial. In order to verify the converse inclusion let $s \in \sigma_p(S^V(t))$ and let $x \in V$ be a corresponding eigenvector. For $\lambda \in \rho(A^V)$ we obtain

$$(\lambda I - A^V)^{-1}S^V(t)x = s(\lambda I - A^V)^{-1}x\,.$$

Now $(\lambda I - A^V)^{-1}$ and $S^V(t)$ commute and $(\lambda I - A^V)^{-1}x \in W$. Hence

$$S^W(t)(\lambda I - A^V)^{-1}x = s(\lambda I - A^V)^{-1}x\,,$$

which proves that $s \in \sigma_p(S^W(t))$.

(ii) Recall from Pazy [20], p. 123 that A^W and A^V are related as follows

$$\left.\begin{array}{rcl} D(A^W) & = & \{x \in D(A^V) \mid A^V x \in W\} \\ A^W x & = & A^V x \text{ for } x \in D(A^W) \end{array}\right\} \tag{2.5}$$

(2.5) implies in particular that

$$D(A^W) \subset D(A^V) \tag{2.6}$$

and

$$(\lambda I - A^W)^{-1} = (\lambda I - A^V)^{-1} \mid_W \quad \text{for all } \lambda \in \rho(A^W) \cap \rho(A^V)\,. \tag{2.7}$$

The inclusion $\sigma_p(A^W) \subset \sigma_p(A^V)$ now follows trivially from (2.6). In order to prove the converse inclusion let $s \in \sigma_p(A^V)$ and let $x \in D(A^V)$ be a corresponding eigenvector. For $\lambda \in \rho(A^W) \cap \rho(A^V)$ we obtain

$$(\lambda I - A^V)^{-1}A^V x = s(\lambda I - A^V)^{-1}x\,. \tag{2.8}$$

Since $x \in D(A^V) \subset W$ we obtain from (2.5) - (2.8)

$$A^W(\lambda I - A^W)^{-1}x = s(\lambda I - A^W)^{-1}x$$

which shows that $s \in \sigma_p(A^W)$.

Since A^W and A^V are closed operators, it remains to show that $R(sI - A^W) = W$ if and only if $R(sI - A^V) = V$. Suppose first that $(sI - A^V)$ is onto V. So for $w \in W$ there exists $x \in D(A^V) \subset W$ such that $w = (sI - A^V)x$. It follows that $A^V x \in W$ and therefore that $x \in D(A^W)$ and $w = (sI - A^W)x$ by (2.5). This holds for all $w \in W$ and so $R(sI - A^W) = W$. Conversely, suppose now that $(sI - A^W)$ is onto W and consider an arbitrary $v \in V$ and $\lambda \in \rho(A^V)$. Now $(\lambda I - A^V)^{-1} v \in D(A^V) \subset W$ and hence there exists $x \in D(A^W)$ such that

$$(sI - A^W)x = (\lambda I - A^V)^{-1} v\,.$$

It follows that $A^W x = A^V x \in D(A^V)$ and we obtain

$$v = (sI - A^V)(\lambda I - A^V)x\,.$$

Since v was arbitrary we have proved that $(sI - A^V)$ maps onto V. □

Proposition 2.7 *Suppose $Z \subset W$. If any of the conditions*

(i) $S(t)$ satisfies the spectrum determined growth assumption on both spaces W and V

(ii) $\sigma(S^X(t_0) \subset \overline{\sigma_p(S^X(t_0))} \cup \{0\}$ for $X = W, V$ and for some $t_0 > 0$

(iii) $S^W(t_0)$ and $S^V(t_1)$ are self-adjoint for some $t_0 > 0$ and $t_1 > 0$

(iv) $S(t_0)(W) \subset Z$ and $S(t_0)(Z) \subset D(A^W)$ for some $t_0 > 0$

holds, then $\omega_W = \omega_V$.

Remark 2.8 Condition (i) is satisfied if $S(t)$ is a holomorphic semigroup on W and V. Condition (ii) will hold if $S(t_0)$ is compact on W and V. If A^W and A^V are both selfadjoint then (iii) is true. Finally, condition (iv) is satisfied if $S^V(t)x$ is right sided differentiable in t at t_0 for all $x \in W$ and if $S^W(t)x$ is right sided differentiable in t at t_0 for all $x \in Z$.

Proof of Proposition 2.7: (i) This follows from Lemma 2.6 (ii).

(ii) It follows from Lemma 2.6 (i) that $r(S^W(t_0)) = r(S^V(t_0))$, which yields $\omega_W = \omega_V$ by (2.1).

(iii) From Proposition 2.2 we have $\omega_W \geq \omega_V$ and $\omega_Z \geq \omega_W$. Since $\omega_V = \omega_Z$ by Remark 2.4 it follows that $\omega_W = \omega_V$.

(iv) Let $\epsilon > 0$ and set $S_\epsilon(t)x = S(t)e^{-(\omega_V+\epsilon)t}x$ for $x \in V$. Clearly $S_\epsilon(t)$ is a C_0-semigroup on W and V. For any $x \in W$ and any $t \geq t_0$ we obtain using Remark 2.4

$$\begin{aligned} \|S_\epsilon(t)x\|_W &= \|S_\epsilon(t-t_0)S_\epsilon(t_0)x\|_W \\ &\leq \alpha\|S_\epsilon(t-t_0)\|_{\mathcal{L}(Z)}\|S_\epsilon(t_0)x\|_Z\,, \end{aligned}$$

where $\alpha > 0$ is a constant satisfying $\|x\|_W \leq \alpha\|x\|_Z$. Applying again Remark 2.4 the above inequality shows that

$$\int_0^\infty \|S_\epsilon(t)x\|_W\,dt < \infty \text{ for all } x \in W\,.$$

A result by Pazy (see [20], p. 116) implies that $S_\epsilon(t)$ is exponentially stable on W. Since this is true for all $\epsilon > 0$ we get that $\omega_W \leq \omega_V$. Replacing W by Z and V by W in the above argument shows that $\omega_Z \leq \omega_W$. Since $\omega_Z = \omega_V$ by Remark 2.4 it follows that $\omega_W = \omega_V$. □

Next we shall introduce admissible input and output operators for the semigroup $S(t)$ defined on W and V. These concepts are due to Salamon [29], Pritchard and Salamon [22, 23], and Weiss [33, 34].

Definition 2.9 *(i) Let U be a Hilbert space. An operator $B \in \mathcal{L}(U,V)$ is called an admissible input operator for $S(t)$ if there exist numbers $t_1 > 0$ and $\alpha > 0$ such that*

$$\int_0^{t_1} S(t_1 - \tau)Bu(\tau)dt \in W \tag{2.9}$$

and

$$\| \int_0^{t_1} S(t_1 - \tau)Bu(\tau)d\tau \|_W \leq \alpha \|u\|_{L^2(0,t_1)} \tag{2.10}$$

for all $u \in L^2(0,t_1;U)$.

(ii) Let Y be a Hilbert space. An operator $C \in \mathcal{L}(W,Y)$ is called an admissible output operator for $S(t)$ if there exist numbers $t_2 > 0$ and $\beta > 0$ such that

$$\|CS(\cdot)x\|_{L^2(0,t_2)} \leq \beta \|x\|_V \text{ for all } x \in W\,. \tag{2.11}$$

Remark 2.10 (i) If (2.10) holds for one particular t_1, then it can be shown that it holds for all $t_1 > 0$, where α will depend on t_1. In case that $S(t)$ is exponentially stable on W then we can choose a constant α which does not depend on t_1 and moreover, we have that $\| \int_0^\infty S(\tau)Bu(\tau)d\tau\|_W \leq \alpha \|u\|_{L^2(0,\infty)}$.

(ii) The previous statement remains valid if we replace (2.10) by (2.11), t_1 by t_2, α by β and exponential stability on W by exponential stability on V.

(iii) Let $B \in \mathcal{L}(U,V)$ be an admissible input operator for $S(t)$. Then for all $u \in L^2(0,T;U)$ the map $t \mapsto \int_0^t S(t-\tau)Bu(\tau)d\tau$ is continuous on $[0,T]$ with values in W. Moreover, the controllability operator

$$\mathcal{C} : L^2(0,T;U) \to V\,,\ u \mapsto \int_0^T S(t-\tau)Bu(\tau)d\tau$$

satisfies

$$\mathrm{Range}(\mathcal{C}) \subset W \tag{2.12}$$

and

$$\mathcal{C} \in \mathcal{L}(L^2(0,T;U), W)\,. \tag{2.13}$$

It can be shown as in Weiss [34] that (2.13) is implied by (2.12) provided $Z \subset W$.

(iv) Suppose that $C \in \mathcal{L}(W, Y)$ is an admissible output operator for $S(t)$. Then the bounded linear operator $\mathcal{O}_W : W \to L^2(0, T; Y)$, $x \mapsto CS(\cdot)x$ can be extended uniquely to a bounded linear operator $\mathcal{O}_V : V \to L^2(0, T; Y)$. The operators $\mathcal{O}_W$ and $\mathcal{O}_V$ are called the observability operators on W and V, respectively. For $x \in V$ we define $CS(\cdot)x := \mathcal{O}_V x$.

(v) The concepts of an admissble input operator and an admissible output operator are dual to each other, cf. Pritchard and Salamon [23].

Definition 2.11 *A control system of the form*

$$\left.\begin{array}{l} x(t) = S(t)x_0 + \int_0^t S(t-\tau)Bu(\tau)d\tau \\ y(t) = Cx(t)\,, \end{array}\right\} \tag{2.14}$$

where $x_0 \in V$ and $t \geq 0$, is called a Pritchard-Salamon system if $B \in \mathcal{L}(U, V)$ is an admissible input operator for $S(t)$ and $C \in \mathcal{L}(W, Y)$ in an admissible output operator for $S(t)$.

Notice that for every $x_0 \in W$ and every $u \in LL^2(0, \infty; U)$ the output y of a Pritchard-Salamon system is a continuous function on $[0, \infty)$ with values in Y (see Remark 2.10 (iii)). If $x_0 \in V$ we can make sense of y as a function in $LL^2(0, \infty; Y)$ by applying Remark 2.10 (iv).

The following simple result on admissible input and output operators will be useful for the frequency-domain analysis of Pritchard-Salamon systems.

Lemma 2.12 *(i) Let U be a Hilbert space and let $B \in \mathcal{L}(U, V)$ be an admissible input operator for $S(t)$. Then for any $\lambda > \max(\omega_W, \omega_V)$ there exists a constant $L_\lambda > 0$ such that*

$$(sI - A^V)^{-1}B \in \mathcal{L}(U, W) \quad \text{for all } s \in \mathbb{C}_\lambda$$

and

$$\|(sI - A^V)^{-1}B\|_{\mathcal{L}(U,W)} \leq \frac{L_\lambda}{\sqrt{\mathrm{Re}(s) - \lambda}} \quad \text{for all } s \in \mathbb{C}_\lambda .$$

(ii) Let Y be a Hilbert space and let $C \in \mathcal{L}(W, Y)$ be an admissible output operator for $S(t)$. Then for any $\lambda > \max(\omega_W, \omega_V)$ there exists a constant $M_\lambda > 0$ such that

$$\|C(sI - A^W)^{-1}x\|_Y \leq \frac{M_\lambda \|x\|_V}{\sqrt{\mathrm{Re}(s) - \lambda}} \quad \text{for all } x \in W \text{ and } s \in \mathbb{C}_\lambda .$$

Proof: (i) See Weiss [36] and Curtain [3].

(ii) For $\lambda > \max(\omega_W, \omega_V)$ it is clear that $S_\lambda(t) := e^{-\lambda t}S(t)$ is an exponentially stable C_0-semigroup on W and V. Now C is an admissible output operator for $S_\lambda(t)$ and hence it follows from Remark 2.10 (ii) that for some $\beta_\lambda > 0$

$$\|CS_\lambda(\cdot)x\|_{L^2(0,\infty)} \leq \beta_\lambda \|x\|_V \text{ for all } x \in W\,.$$

Hence the following estimate holds for $z \in \mathbb{C}_0$ and $x \in W$

$$\begin{aligned}\|C((z+\lambda)I - A^W)^{-1}\|_Y &= \|\int_0^\infty CS_\lambda(t)xe^{-zt}dt\|_Y \\ &\leq (\int_0^\infty \|CS_\lambda(t)x\|_Y^2 dt)^{\frac{1}{2}}(\int_0^\infty e^{-2Re(z)t}dt)^{\frac{1}{2}} \\ &\leq \frac{\beta_\lambda}{\sqrt{2}}\frac{\|x\|_V}{\sqrt{\mathrm{Re}(z)}}\,.\end{aligned}$$

A change of variables $s = z + \lambda$, $s \in \mathbb{C}_\lambda$, completes the proof. □

Remark 2.13 Lemma 2.12 (ii) shows that if C is an admissible output operator for $S(t)$, then for all s with $\mathrm{Re}(s) > \max(\omega_W, \omega_V)$ the operator $C(sI - A^W)^{-1} \in \mathcal{L}(W,Y)$ can be uniquely extended to an operator $\mathcal{O}(s) \in \mathcal{L}(V,Y)$. We define $C(sI - A^W)^{-1}B := \mathcal{O}(s)B$ and $C(sI - A^W)^{-1}v := \mathcal{O}(s)v$ for all $v \in V$.

We would like to close this section by showing that any system of the form (2.14) has a well-defined transfer function provided $B \in \mathcal{L}(U,V)$ is an admissible input operator for $S(t)$ and $C \in \mathcal{L}(W,Y)$. First we make precise what we mean by a transfer function of (2.14). To this end it is useful to define the space

$$\Omega := \{u \in LL^2(0,\infty;U) \mid \exists\, \gamma = \gamma(u) \in \mathbb{R} \text{ s.t. } u(\cdot)e^{-\gamma\,\cdot} \in L^2(0,\infty;U)\}\,.$$

Furthermore, for $u \in \Omega$ we set

$$\lambda(u) := \inf\{\lambda \in \mathbb{R} \mid u(\cdot)e^{-\lambda\,\cdot} \in L^2(0,\infty;U)\}\,.$$

Definition 2.14 *Suppose that in (2.14) $x_0 = 0$, $B \in \mathcal{L}(U,V)$ is an admissible input operator and $C \in \mathcal{L}(W,Y)$. A holomorphic function $G : \mathbb{C}_\xi \to \mathcal{L}(U,Y)$ is called a transfer function of (2.14) if for any $u \in \Omega$ there holds*

$$\hat{y}(s) = G(s)\hat{u}(s) \quad \text{ for } s \in \mathbb{C}_{\max(\xi,\lambda(u))}.$$

It is clear that if $G_1 : \mathbb{C}_{\xi_1} \to \mathcal{L}(U,Y)$ and $G_2 : \mathbb{C}_{\xi_2} \to \mathcal{L}(U,Y)$ are two transfer functions of (2.14), then $G_1(s) = G_2(s)$ for all $s \in \mathbb{C}_{\max(\xi_1,\xi_2)}$.

Proposition 2.15 *Consider the system (2.14) and suppose that $B \in \mathcal{L}(U,V)$ is an admissible input operator and $C \in \mathcal{L}(W,Y)$. Let $u \in \Omega$ and let η be any number which satisfies $\eta > \max(\omega_W, \omega_V, \lambda(u))$. Then the following statements hold true:*

(i) $y(\cdot)e^{-\eta\cdot} \in L^1(0,\infty;Y) \cap L^2(0,\infty;Y)$

(ii) $\hat{y}(s) = C(sI - A^V)^{-1}B\hat{u}(s)$ for all $s \in \mathbb{C}_\eta$

(iii) $C(\cdot I - A^V)^{-1}B \in H^\infty(\mathbb{C}_\xi, \mathcal{L}(U,Y))$ for all $\xi > \max(\omega_W, \omega_V)$.

It follows in particular that $C(sI-A^V)^{-1}B$ is a transfer function of (2.14).

Proof: (i) Pick $\mu \in (\max(\omega_W, \omega_V, \lambda(u)), \eta)$ and set $\epsilon := \eta - \mu > 0$. Then

$$y(t)e^{-\eta t} = e^{-\epsilon t} C \int_0^t S(t-\tau)e^{-\mu(t-\tau)}Bu(\tau)e^{-\mu\tau}d\tau\,.$$

Now, since $S(t)e^{-\mu t}$ is an exponentially stable C_0-semigroup on W and V and $u(\cdot)e^{-\mu\cdot} \in L^2(0,\infty;U)$, it follows from the admissibility of B via Remark 2.10 (i) that there exists $\alpha > 0$ such that

$$\|y(t)e^{-\eta t}\|_Y \le \alpha \|C\| e^{-\epsilon t} \|u(\cdot)e^{-\mu\cdot}\|_{L^2(0,\infty;U)}\,.$$

(ii) For $s \in \mathbb{C}_\eta$ set

$$z(t) = e^{-st}\int_0^t S(t-\tau)Bu(\tau)d\tau\,.$$

A similar argument as in (i) shows that $z \in L^1(0,\infty;W)$. Hence $z \in L^1(0,\infty;V)$ and ${}_W\!\int_0^\infty z(t)dt = {}_V\!\int_0^\infty z(t)dt$, where ${}_W\!\int$ and ${}_V\!\int$ denote integration in W and V, respectively. It follows that

$$\begin{aligned}
\hat{y}(s) &= \int_0^\infty Cz(t)dt \\
&= C\ {}_W\!\int_0^\infty z(t)dt \\
&= C\ {}_V\!\int_0^\infty z(t)dt \\
&= C(sI - A^V)^{-1}B\hat{u}(s) \qquad \text{for all } s \in \mathbb{C}_\eta\,.
\end{aligned}$$

(iii) Let $u_0 \in U$ and set $u(t) = e^{\omega t}u_0$ for $t \ge 0$, where $\omega := \max(\omega_W, \omega_V)$. By (ii) we have for $s \in \mathbb{C}_\omega$ that

$$\hat{y}(s) = \frac{1}{s-\omega}C(sI - A^V)^{-1}Bu_0\,.$$

Since $(s-\omega)\hat{y}(s)$ is (strongly) holomorphic in $\mathbb{C}_\omega$ the same is true for $C(sI - A^V)^{-1}Bu_0$. Now $u_0 \in U$ was arbitrary and so $C(sI-A^V)^{-1}B$ is holomorphic in $\mathbb{C}_\omega$ with respect to the norm topology of $\mathcal{L}(U,Y)$ (see Hille and Phillips [12], p. 93). The boundedness of $C(sI-A^V)^{-1}B$ on $\mathbb{C}_\xi$ for $\xi>\omega$ follows from Lemma 2.12 (i). □

Remark 2.16 Under the assumptions of Proposition 2.15 it follows from the theory of vector-valued distributions (see e.g. Fattorini [11], pp. 461) that there exists a unique tempered $\mathcal{L}(U,Y)$-valued distribution H with support in $[0,\infty)$ such that $(\mathbb{L}H)(s)=C(sI-A^V)^{-1}B$ for $\mathrm{Re}(s)>\max(\omega_W,\omega_V)$. The distribution H is called the impulse response of the system (2.14).

The reader should notice that for a Pritchard-Salamon system the expressions $C(sI-A^V)^{-1}B$ and $C(sI-A^W)^{-1}B$ both make sense (see Lemma 2.12 and Remark 2.13). In the next section we will show that they are equal for all $s\in\mathbb{C}$ with $\mathrm{Re}(s)>\max(\omega_W,\omega_V)$. This result will be used to express the impulse response H of a Pritchard-Salamon system in terms of $S(t)$, B and C for the special case that $\dim U<\infty$.

3 An Important Property of Pritchard-Salamon Systems

Suppose that (2.14) is a Pritchard-Salamon system. In [23] Pritchard and Salamon introduced the assumption

$$Z=D(A^V)\hookrightarrow W \tag{3.1}$$

in order to ensure that

$$C\int_0^t S(\tau)Bud\tau=\int_0^t CS(\tau)Bud\tau \quad \text{for all } u\in U,\ t\geq 0\,. \tag{3.2}$$

Equation (3.2) seems to be a trivial fact. It should be noticed however that the R.H.S. of (3.2) has to be interpreted via the admissibility of C (cf. Remark 2.10 (iv)), while the L.H.S. makes sense since B is an admissible input operator for $S(t)$ (cf. Remark 2.10 (i)). It is the main goal of this section to show that (3.1) is not required for (3.2) to hold, i.e. (3.2) holds for every Pritchard-Salamon system as defined in Section 2.

Lemma 3.1 *Suppose that in (2.14) the operator B belongs to $\mathcal{L}(U,V)$ and $C\in\mathcal{L}(W,Y)$ is an admissible output operator for $S(t)$. Then*

$$C(sI-A^W)^{-1}Bu=(\mathbb{L}(CS(\cdot)Bu))(s) \tag{3.3}$$

for all $s\in\mathbb{C}$ with $\mathrm{Re}(s)>\max(\omega_W,\omega_V)$ and for all $u\in U$.

The proof of Lemma 3.1 can be found in Logemann [17]. Notice that R.H.S. of (3.3) has to be interpreted in the sense of Remark 2.10 (iv) while the L.H.S. is meaningful in the sense of Remark 2.13.

Lemma 3.2 *Suppose (2.14) is a Pritchard-Salamon system. Then (3.2) is satisfied if and only if*

$$C(sI - A^W)^{-1}B = C(sI - A^V)^{-1}B \tag{3.4}$$

for all $s \in \mathbb{C}$ *with* $\mathrm{Re}(s) > \max(\omega_W, \omega_V)$.

Proof: The necessity of (3.4) for (3.2) to hold has been proved in Logemann [17]. In order to prove sufficiency assume that (3.4) is satisfied. For $u \in U$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > \max(\omega_W, \omega_V, 0)$ it follows from the admissibility of B that the function

$$z(t) := e^{-st} \int_0^t S(\tau)Bud\tau$$

is in $L^1(0, \infty; W)$. Hence

$$\begin{aligned}
[\mathbb{L}(C\int_0^{\cdot} S(\tau)Bud\tau)](s) &= \int_0^\infty Cz(t)dt \\
&= C_W\int_0^\infty z(t)dt \\
&= C_V\int_0^\infty z(t)dt \\
&= C\frac{1}{s}(sI - A^V)^{-1}Bu \\
&= \frac{1}{s}C(sI - A^W)^{-1}Bu \\
&= [\mathbb{L}(\int_0^{\cdot} CS(\tau)Bud\tau)](s),
\end{aligned}$$

where we have made use of Lemma 3.1. It follows that (3.2) holds a.e. on $[0, \infty)$. Now the L.H.S. of (3.2) is continuous in t by Remark (2.10(iii)) and so is the R.H.S. since it is the integral of a LL^2-function. So we see that (3.2) holds for all $t \in [0, \infty)$. □

Theorem 3.3 *If (2.14) is a Pritchard-Salamon system, then*

$$C\int_0^t S(\tau)Bud\tau = \int_0^t CS(\tau)Bud\tau \qquad \text{for all } u \in U, t \geq 0,$$

i.e. (3.2) holds for any Pritchard-Salamon system.

Proof: Set $\omega := \max(\omega_W, \omega_V)$. We shall prove that

$$C(sI - A^W)^{-1}B = C(sI - A^V)^{-1}B \qquad \text{for all } \ s \in \}_\omega.$$

The theorem then follows from Lemma 3.2. The following fact will be useful in the sequel

$$(sI - A^V)^{-1}\mid_W = (sI - A^W)^{-1} \ \text{ for all } \ s \in \mathbb{C}_\omega. \tag{3.5}$$

We define

$$\begin{aligned} T(s) &:= C(sI-A^W)^{-1}B - C(sI-A^V)^{-1}B \\ &= \mathcal{O}(s)(sI-A^V)(sI-A^V)^{-1}B - C(sI-A^V)^{-1}B, \end{aligned}$$

where we have made use of Lemma 2.12 (ii) and Remark 2.13. Let both $u \in U$ and $s \in \mathbb{C}_\omega$ be fixed but arbitrary. We have to show that $T(s)u = 0$. To this end set

$$z(\lambda) := \lambda(\lambda I - A^V)^{-1}(sI-A^V)^{-1}Bu \quad \text{for all } \lambda \in \mathbb{C}_\omega\,. \tag{3.6}$$

Using (3.5) and Lemma 2.12 (i) we obtain

$$z(\lambda) = \lambda(\lambda I - A^W)^{-1}(sI-A^V)^{-1}Bu \text{ for all } \lambda \in \mathbb{C}_\omega\,. \tag{3.7}$$

Since A^W is the generator of a C_0-semigroup on W, we have that

$$\lim_{\lambda\to\infty,\,\lambda\in\mathbb{R}} z(\lambda) = (sI-A^V)^{-1}Bu \quad \text{in } W, \tag{3.8}$$

(see e.g. Curtain and Pritchard [7], p. 19). On the other hand, since A^V generates a C_0-semigroup on V it follows from (3.6) that

$$\lim_{\lambda\to\infty,\,\lambda\in\mathbb{R}} (sI-A^V)z(\lambda) = \lim_{\lambda\to\infty,\,\lambda\in\mathbb{R}} \lambda(\lambda I-A^V)^{-1}Bu = Bu \quad (\text{in } V)\,. \tag{3.9}$$

Setting

$$\begin{aligned} h(\lambda) &:= \mathcal{O}(s)(\lambda I-A^V)^{-1}Bu - C(\lambda I-A^W)^{-1}(sI-A^V)^{-1}Bu \\ &= \mathcal{O}(s)(sI-A^V)(\lambda I-A^V)^{-1}(sI-A^V)^{-1}Bu \\ &\quad -C(\lambda I-A^W)^{-1}(sI-A^V)^{-1}Bu\,, \end{aligned}$$

and using (3.6) and (3.7), we see that

$$\lambda h(\lambda) = \mathcal{O}(s)(sI-A^V)z(\lambda) - Cz(\lambda)\,.$$

Now, by Lemma 2.12 (ii), Remark 2.13, (3.9) and (3.8)

$$\lim_{\lambda\to\infty,\,\lambda\in\mathbb{R}} \lambda h(\lambda) = \mathcal{O}(s)Bu - C(sI-A^V)^{-1}Bu = T(s)u \quad (\text{in } Y)\,. \tag{3.10}$$

Notice that h is a holomorphic function on $\mathbb{C}_\omega$ with values in Y. Hence it follows from (3.10) that it is sufficient to show that $h^{(n)}(\lambda)|_{\lambda=s} = 0$ for $n = 0, 1, 2, \ldots$ in order to prove that $T(s)u = 0$. A straightforward computation shows

$$\begin{aligned} h^{(n)}(\lambda) &= (-1)^n n!(\mathcal{O}(s)(\lambda I-A^V)^{-(n+1)}Bu \\ &\quad -C(\lambda I-A^W)^{(-n+1)}(sI-A^V)^{-1}Bu. \end{aligned} \tag{3.11}$$

Finally, it follows from Lemma 2.12 (i) and (3.5) that

$$\begin{aligned}\mathcal{O}(s)(sI-A^V)^{-(n+1)}Bu &= \mathcal{O}(s)(sI-A^V)^{-n}(sI-A^V)^{-1}Bu \\ &= \mathcal{O}(s)(sI-A^W)^{-n}(sI-A^V)^{-1}Bu \\ &= C(sI-A^W)^{-(n+1)}(sI-A^V)^{-1}Bu\,,\end{aligned}$$

and hence by (3.11) that $h^{(n)}(\lambda)|_{\lambda=s}=0$ for $n=0,1,2,\ldots$. □

In Section 2 we saw that the impulse response (= inverse Laplace transform of the transfer function) of a Pritchard-Salamon system is a tempered $\mathcal{L}(U,Y)$-valued distribution with support in $[0,\infty)$. We shall close this section by showing that if $\dim U<\infty$, then the impulse response is a regular distribution, i.e. a locally integrable $\mathcal{L}(U,Y)$-valued function defined on $[0,\infty)$. This case is the one which has received most attention in the literature up to date.

So let us suppose that $\dim U=n<\infty$ and let $u_1,\cdots,u_n$ denote a basis of U, and denote the coordinates of $u\in U$ with respect to $u_1,\cdots,u_n$ by $\gamma_i(u)$, i.e. $u=\sum_{i=1}^n\gamma_i(u)u_i$. Since C is an admissible output operator for $S(t)$, the expression $CS(\cdot)Bu_i$ has a meaning as an element in $LL^2(0,\infty;Y)$, see Remark 2.10 (ii) and (iv). Recall that $CS(\cdot)Bu_i$ are not functions but equivalence classes of functions which differ only on a set of measure zero. Let the functions f_i be members of the equivalence classes $CS(\cdot)Bu_i$, $i=1,\cdots,n$. Then the $f_i(t)$ are well-defined elements in Y for almost all $t\in[0,\infty)$. Hence, if we define the family of operators $R(t):U\to Y$, $t\geq 0$, by

$$R(t)u=\sum_{i=1}^n\gamma_i(u)f_i(t)$$

we have $R(t)\in\mathcal{L}(U,Y)$ a.e. on $[0,\infty)$. Notice that $R(\cdot)$ is strongly measurable in the uniform operator topology.

Definition 3.4 *Suppose that in (2.14) $B\in\mathcal{L}(U,Y)$, $\dim U<\infty$, and $C\in\mathcal{L}(W,Y)$ is an admissible output operator for $S(t)$. We define the expression $CS(\cdot)B$ as the equivalence class of all functions from $[0,\infty)$ to $\mathcal{L}(U,Y)$ which differ from $R(t)$ only on a set of measure zero.*

Lemma 3.5 *Under the conditions of Definition 3.4 we have:*

(i) The definition of $CS(\cdot)B$ does not depend on the choice of the basis $u_1,\cdots,u_n$ of U.

(ii) $CS(\cdot)B\in LL^2(0,\infty;\mathcal{L}(U,Y))$.

(iii) $(\mathbb{L}(CS(\cdot)B))(s)=C(sI-A^W)^{-1}B$ for all $s\in\mathbb{C}$ with $\mathrm{Re}(s)>\max(\omega_W,\omega_V)$.

Proof: (i) Is a routine exercise and is left to the reader.

(ii) This follows from the definition of $CS(\cdot)B$ and the fact that $CS(\cdot)Bu_i \in LL^2(0,\infty;U)$.

(iii) Apply Lemma 3.1. □

The following corollary shows that for Pritchard-Salamon systems with finite-dimensional input space the impulse response is given by the expression $CS(\cdot)B$ as defined in Definition 3.4.

Corollary 3.6 *If (2.14) is a Pritchard-Salamon system with finite-dimensional input space, then:*

(i) $(\mathbb{L}(CS(\cdot)B))(s) = C(sI - A^V)^{-1}B$ *for all* $s \in \mathbb{C}$ *with* $\mathrm{Re}(s) > \max(\omega_W, \omega_V)$.

(ii) $C\int_0^t S(t-\tau)Bu(\tau)d\tau = \int_0^t CS(t-\tau)Bu(\tau)d\tau$ *for all* $t \geq 0$ *and for all* $u \in LL^2(0,\infty;U)$.

Proof: (i) This follows from Lemma 3.2, Theorem 3.3 and Lemma 3.5 (iii).

(ii) This follows from Theorem 3.3 and the fact that the step-functions are dense in $L^2(0,T;U)$, $T \in (0,\infty)$. The details are left to the reader. □

Remark 3.7 The reader should notice that in the case $\dim U = \infty$ it is (in general) not possible to make sense of $CS(\cdot)B$ as a $\mathcal{L}(U,Y)$-valued function. This implies in particular that if $\dim U = \infty$, then expressions like $CS(\cdot)Bu(\cdot)$ or $\int_0^t CS(t-\tau)Bu(\tau)d\tau$ do not necessarily make sense for arbitrary $u \in LL^2(0,\infty;U)$.

4 Perturbations Induced by Admissible State-feedback and Admissible Output-injection

It is well-known that the Pritchard-Salamon class is invariant under state-feedback with $F \in \mathcal{L}(V,U)$ (see Pritchard and Salamon [23]). However, if $F \in \mathcal{L}(W,U)$ only, then all that can be said, in general, is that there exists a perturbed semigroup on W, which is unsatisfactory for control applications. (The perturbation results in Bontsema and Curtain [2] on Pritchard-Salamon systems assume that the semigroup is smoothing). A common example of a perturbation $F \in \mathcal{L}(W,U)$ arises from output feedback, $u = Ky$, where $K \in \mathcal{L}(Y,U)$, which produces $F = KC$. It is the aim of this section to show that the Pritchard-Salamon class is invariant under such perturbations. More precisely, we show that the Pritchard-Salamon class is invariant under state-feedback with $F \in \mathcal{L}(W,U)$ and output injection with $H \in \mathcal{L}(Y,V)$, provided that F is an admissible output operator and H is an admissible input operator.

Theorem 4.1 *The following statements are valid for a Pritchard-Salamon system (2.14):*

(i) Let $F \in \mathcal{L}(W,U)$ be an admissible output operator for $S(t)$. Then there exists a unique C_0-semigroup $S_{BF}(t)$ on W which is the unique solution of

$$S_{BF}(t)x = S(t)x + \int_0^t S(t-\tau)BFS_{BF}(\tau)x d\tau \tag{4.1}$$

for all $x \in W$. Moreover $S_{BF}(t)$ extends to a C_0-semigroup on V, B is an admissible input operator for $S_{BF}(t)$ and C and F are admissible output operators for $S_{BF}(t)$.

(ii) Let $H \in \mathcal{L}(Y,V)$ be an admissible input operator for $S(t)$. Then there exists a unique C_0-semigroup $\tilde{S}_{HC}(t)$ on V which is the unique solution of

$$\tilde{S}_{HC}(t)x = S(t)x + \int_0^t \tilde{S}_{HC}(t-\tau)HCS(\tau)x d\tau \tag{4.2}$$

for all $x \in V$. Moreover $\tilde{S}_{HC}(t)$ restricts to a C_0-semigroup on W, B and H are admissible input operators for $\tilde{S}_{HC}(t)$ and C is an admissible output operator for $\tilde{S}_{HC}(t)$.

Remark 4.2 As already indicated in the introduction of this section, Theorem 4.1 shows that the Pritchard-Salamon class is invariant under output-feedback of the form $u = Ky$, where $K \in \mathcal{L}(Y,U)$. Just set $F = KC$ in (i) or $H = BK$ in (ii) and notice that F is an admissible output operator and H an admissible input operator.

Proof of Theorem 4.1: (i) Define a sequence $S^n_{BF}(t)$ by

$$S^0_{BF}(t)x := S(t)x\,, \;\; S^{n+1}_{BF}(t)x = \int_0^t S(t-\tau)BFS^n_{BF}(\tau)x d\tau\,,$$

where $x \in W$. Using the admissibility of B we obtain by induction for $t \geq 0$

$$||S^n_{BF}(t)||_{\mathcal{L}(W)} \leq M\alpha^n ||F||^n \sqrt{\frac{t^n}{n!}}, \tag{4.3}$$

where $M = M(t) := \sup_{\tau\in[0,t]} ||S(\tau)||_{\mathcal{L}(W)}$ and $\alpha = \alpha(t)$ is the constant introduced in Definition 2.9 (i) (cf. also Remark 2.10 (i)). It follows from (4.3) that $\sum_{n=0}^{\infty} S^n_{BF}(t)$ converges in the norm topology of $\mathcal{L}(W)$ and hence

$$S_{BF}(t) := \sum_{n=0}^{\infty} S^n_{BF}(t) \in \mathcal{L}(W).$$

It is now easily verified that $S_{BF}(t)x$ solves (4.1) for all $x \in W$. The C_0-semigroup properties and uniqueness of $S_{BF}(t)$ can be shown as in the bounded case (i.e. $W = V$ and $\|\cdot\|_W = \|\cdot\|_V$), see Curtain and Pritchard [7]. In order to make the paper more self-contained we shall prove strong continuity at 0 and uniqueness:

• Strong continuity: By (4.1) and admissibility of B we have for $x \in W$

$$\|S_{BF}(t)x - x\|_W \leq \|S(t)x - x\|_W + \alpha\|F\|\,\|S_{BF}(\cdot)x\|_{L^2(0,t;W)}$$

So using the strong continuity of $S(t)$ on W we see that $\lim_{t\to 0}\|S_{BF}(t)x - x\|_W = 0$.

• Uniqueness: Suppose $T(t) \in \mathcal{L}(W)$ is another family of operators satisfying (4.1). Using the admissibility of B we obtain

$$\|S_{BF}(t)x - T(t)x\|_W^2 \leq \alpha^2\|F\|^2\int_0^t \|S_{BF}(\tau)x - T(\tau)x\|_W^2\,d\tau$$

It follows from Gronwall's lemma that $S_{BF}(t)x = T(t)x$. Since $x \in W$ is arbitrary we have that $S_{BF}(t) = T(t)$ for all $t \geq 0$.

In order to show that $S_{BF}(t)$ extends to a C_0-semigroup on V it is useful to verify the following estimate

$$\|FS_{BF}^n(\cdot)x\|_{L^2(0,t;U)} \leq \beta\alpha^n\|F\|^n\sqrt{\frac{t^n}{n!}}\|x\|_V \text{ for } x \in W,\ n \in \mathbb{N}. \tag{4.4}$$

The estimate (4.4) is easily proved by induction using the admissibility of B and F. The constant $\beta = \beta(t)$ is the one introduced in Definition 2.9 (ii) (cf. also Remark 2.10 (ii)). It follows from the definition of $S_{BF}^n(t)$, the admissibility of B and (4.4) that

$$\begin{aligned}\|S_{BF}^n(t)x\|_W &\leq \alpha\|FS_{BF}^{n-1}(\cdot)x\|_{L^2(0,t;U)}\\ &\leq \beta\alpha^n\|F\|^{n-1}\sqrt{\frac{t^{n-1}}{(n-1)!}}\|x\|_V\end{aligned} \tag{4.5}$$

for $n \geq 1$ and $x \in W$. The estimate (4.5) shows that for $n \geq 1$ the operator $S_{BF}^n(t) \in \mathcal{L}(W)$ can be extended to an element in $\mathcal{L}(V,W) \subset \mathcal{L}(V,V)$. We shall denote this extension by $S_{BF}^n(t)$ as well. It follows from (4.5) that the series $\sum_{n=1}^\infty S_{BF}^n(t)$ converges absolutely in $\mathcal{L}(V,W)$ and hence in $\mathcal{L}(V,V)$. The limit is the same in both spaces and we define

$$R_{BF}(t) = \sum_{n=1}^\infty S_{BF}^n(t)\,. \tag{4.6}$$

Setting $S^V_{BF}(t) := S(t) + R_{BF}(t)$, we obtain an operator in $\mathcal{L}(V,V)$ which extends $S_{BF}(t)$. Let M, λ and γ be real constants such that

$$\|S(t)\|_{\mathcal{L}(V)} \leq Me^{\lambda t} \quad \text{for all } t \geq 0$$

and

$$\|x\|_V \leq \gamma \|x\|_W \text{ for } x \in W.$$

Moreover set

$$f(t) := \beta \sum_{n=1}^{\infty} \alpha^n \|F\|^{n-1} \sqrt{\frac{t^{n-1}}{(n-1)!}}\,. \tag{4.7}$$

It is clear that the function f is continuous and monotonically increasing on $\mathbb{R}_+$. It follows from the definition of $S^V_{BF}(t)$ that

$$\|S^V_{BF}(t)\|_{\mathcal{L}(V)} \leq Me^{\lambda t} + \gamma f(t)\,,$$

which shows that $t \mapsto S^V_{BF}(t)$ is bounded in the norm topology of $\mathcal{L}(V)$ on compact intervals. We claim that $S^V_{BF}(t)$ is a C_0-semigroup on V:

- It is trivial that $S^V_{BF}(0) = I_V$.

- Strong continuity: Pick $t^* > 0$ and set $\kappa := Me^{\lambda t^*} + \gamma f(t^*)$. Let $x \in V$ and choose a sequence $x_n \in W$ such that $x = \lim_{n\to\infty} x_n$ (in V). Then for all $t \in [0,t^*]$

$$\begin{aligned} \|S^V_{BF}(t)x - x\|_V &= \|S^V_{BF}(t)x - S^V_{BF}(t)x_n - x + x_n + S_{BF}(t)x_n - x_n\|_V \\ &\leq [\|S^V_{BF}(t)\|_{\mathcal{L}(V)} + 1]\|x - x_n\|_V \\ &\quad + \gamma\|S_{BF}(t)x_n - x_n\|_W \\ &\leq (\kappa + 1)\|x - x_n\|_V + \gamma\|S_{BF}(t)x_n - x_n\|_W\,. \end{aligned} \tag{4.8}$$

Now for given $\epsilon > 0$ let $N \in \mathbb{N}$ be such that $\|x - x_N\| \leq \frac{\epsilon}{2(\kappa+1)}$. Moreover, by the strong continuity of $S_{BF}(t)$ on W there exists $\delta \in (0,t^*]$ such that $\|S_{BF}(t)x_N - x_N\|_W \leq \frac{\epsilon}{2\gamma}$ for all $t \in [0,\delta]$. Hence, by (4.8)

$$\|S^V_{BF}(t)x - x\|_V \leq \epsilon \text{ for all } t \in [0,\delta].$$

- Semigroup property: Again let $x \in V$ and pick a sequence $x_n \in W$ which converges to x (in V). Using the semigroup property of $S_{BF}(t)$ on W we obtain

$$\begin{aligned} &\|S^V_{BF}(t+s)x - S^V_{BF}(t)S^V_{BF}(s)x\|_V \\ &\quad = \|S^V_{BF}(t+s)x - S^V_{BF}(t+s)x_n \\ &\qquad + S^V_{BF}(t)S^V_{BF}(s)x_n - S^V_{BF}(t)S^V_{BF}(s)x\|_V \\ &\quad \leq \left(\|S^V_{BF}(t+s)\|_{\mathcal{L}(V)} + \|S^V_{BF}(t)S^V_{BF}(s)\|_{\mathcal{L}(V)}\right)\|x - x_n\|_V\,. \end{aligned}$$

Since the R.H.S. converges to 0 as $n \to \infty$, it follows that $S_{BF}^V(t+s) = S_{BF}^V(t)S_{BF}^V(s)$.

It remains to show that B is an admissible input operator and C and F are admissible output operators for $S_{BF}(t)$:

• Admissibility of B: From (4.5)-(4.7) we obtain that $\|R_{BF}(t)\|_{\mathcal{L}(V,W)} \leq f(t)$. Hence it follows for $u \in L^2(0,t_1;U)$:

$$\begin{aligned}
&\| \int_0^{t_1} S_{BF}^V(t_1-\tau)Bu(\tau)d\tau\|_W \\
&\quad = \quad \| \int_0^t S(t_1-\tau)Bu(\tau)d\tau + \int_0^{t_1} R_{BF}(t_1-\tau)Bu(\tau)d\tau\|_W \\
&\quad \leq \quad \alpha\|u\|_{L^2(0,t_1)} + \|B\| \int_0^{t_1} f(t_1-\tau)\|u(\tau)\|_U d\tau \\
&\quad \leq \quad (\alpha + \|B\|\ \|f\|_{L^2(0,t_1)})\|u\|_{L^2(0,t_1)}\,.
\end{aligned}$$

• Admissibility of C and F: For $x \in W$ we have

$$\begin{aligned}
\|CS_{BF}^V(\cdot)x\|_{L^2(0,t_2)} &\leq \|CS(\cdot)x\|_{L^2(0,t_2)} + \|CR_{BF}(\cdot)x\|_{L^2(0,t_2)} \\
&\leq \beta\|x\|_V + \|C\|\ \|f\|_{L^2(0,t_2)}\|x\|_V \\
&= (\beta + \|C\|\ \|f\|_{L^2(0,t_2)})\|x\|_V\,.
\end{aligned}$$

The same estimate holds true if we replace C by F.

(ii) Statement (ii) can be proved in a similar way. □

The following theorem shows that nesting of feedback loops is equivalent to closing the loop for the sum of the feedback operators. Although this seems to be a trivial fact, it requires a proof which is by no means trivial.

Theorem 4.3 *Suppose that (2.14) is a Pritchard-Salamon system and that $F_1, F_2 \in \mathcal{L}(W,U)$ are admissible output operators for $S(t)$. Then using the notation of Theorem 4.1*

$$S_{B(F_1+F_2)}(t)x = S_{BF_1}(t)x + \int_0^t S_{BF_1}(t-\tau)BF_2S_{B(F_1+F_2)}(\tau)xd\tau \quad (4.9)$$

for all $x \in V$.

Remark 4.4 Since the semigroup $(S_{BF_1})_{BF_2}(t)$ gives the unique solution of (4.9) (by Theorem 4.1), it follows that $(S_{BF_1})_{BF_2}(t) = S_{B(F_1+F_2)}(t)$.

In order to prove Theorem 4.3 we need two lemmas.

Lemma 4.5 *Suppose that (2.14) is a Pritchard Salamon system and that $F \in \mathcal{L}(W, U)$ is an admissible output operator for $S(t)$. For all $n \in \mathbb{N}$ with $n > \omega_W$ define the operators $F_n \in \mathcal{L}(W, U)$ by*

$$F_n := nF(nI - A^W)^{-1} . \tag{4.10}$$

Under these conditions the following statements hold true:

(i) F_n can be uniquely extended to an element in $\mathcal{L}(V, U)$. †

(ii) $\lim_{n\to\infty} F_n x = Fx$ for all $x \in W$.

(iii) There exists $L \in \mathbb{R}_+$ such that $\|F_n\|_{\mathcal{L}(W,U)} \le L$ for all $n > \omega_W$.

(iv) For $T > 0$ we have

$$\lim_{n\to\infty} \|F_n S(\cdot)x - FS(\cdot)x\|_{L^2(0,T;U)} = 0 \text{ for all } x \in V .$$

(v) $S_{BF}(t)x - S_{BF_n}(t)x \in W$ for all $x \in V$, all $t \ge 0$ and all $n > \omega_W$ and $\lim_{n\to\infty} \|S_{BF}(t)x - S_{BF_n}(t)x\|_W = 0$ for all $t \ge 0$ and all $x \in V$, where the convergence is uniform in t on compact intervals.

(vi) $\lim_{n\to\infty} \| \int_0^t S_{BF_n}(t-\tau)Bu(\tau)\, d\tau - \int_0^t S_{BF}(t-\tau)Bu(\tau)\, d\tau\|_W = 0$ for all $t \ge 0$ and all $u \in L^2(0, t; U)$.

Proof of Lemma 4.5: (i) This follows from Lemma 2.12 and Remark 2.13.

(ii) The second statement follows from the fact that A^W is the generator of a strongly continuous semigroup on W (see e.g. Curtain and Pritchard [7], p. 19).

(iii) This follows from (ii) and the uniform boundedness principle. Alternatively, statement (iii) follows also from the Hille-Yosida theorem applied to A^W.

(iv) Set $\Phi_n(x) := (F - F_n)S(\cdot)x$ for all $x \in V$. Then, by the admissibility of F, there exists a constant $\beta_F > 0$ such that for all $x \in W$

$$\begin{aligned} \|\Phi_n(x)\|_{L^2(0,T)} &\le \beta_F(\|x\|_V + \|n(nI - A^W)^{-1}x\|_V) \\ &\le \beta_F(1 + \|n(nI - A^V)^{-1}\|_{\mathcal{L}(V)})\|x\|_V \\ &\le \gamma\|x\|_V , \end{aligned}$$

where the existence of the constant γ follows from an application of the Hille-Yosida theorem to A^V. Moreover, by (ii)

$$\lim_{n\to\infty} \|(F_n - F)S(t)x\|_U^2 = 0 \quad \text{for all } t \ge 0, x \in W ,$$

and by (iii)

$$\|(F_n - F)S(t)x\|_U^2 \le (L + \|F\|)^2 \|S(t)x\|_W^2 .$$

† The extension will be denoted by F_n as well.

So by Lebesgue's dominated convergence theorem we have

$$\lim_{n\to\infty} \|\Phi_n(x)\|_{L^2(0,T;U)} = \lim_{n\to\infty} \|(F_n - F)S(\cdot)x\|_{L^2(0,T;U)} = 0 \ \text{ for } \ x \in W\,.$$

It remains to show that $\Phi_n(x) \to 0$ in $L^2(0,T;U)$ for all $x \in V$ as $n \to \infty$. To this end let $x \in V$, $\epsilon > 0$ and choose $y \in W$ so that $\|x - y\| \le \epsilon$. Then

$$\begin{aligned} \|\Phi_n(x)\|_{L^2(0,T)} &\le \|\Phi_n(y)\|_{L^2(0,T)} + \|\Phi_n(x-y)\|_{L^2(0,T)} \\ &\le \epsilon + \gamma\epsilon \end{aligned}$$

for all sufficiently large n.

(v) As in the proof of Theorem 4.1 we define recursively

$$S_0(t) = S(t)\,, \ \ S_k(t)x = \int_0^t S(t-\tau)BFS_{k-1}(\tau)x d\tau, \quad k \ge 1$$

and for $n > \omega_W$

$$S_0^n(t) = S(t)\,, \ \ S_k^n(t)x = \int_0^t S(t-\tau)BF_nS_{k-1}^n(\tau)x d\tau, \quad k \ge 1\,.$$

We know from the proof of Theorem 4.1 that

$$S_k(t), S_k^n(t) \in \mathcal{L}(V) \cap \mathcal{L}(W) \quad \text{ for all } \ k \in \mathbb{N} \text{ and all } n > \omega_W$$

and

$$S_k(t), S_k^n(t) \in \mathcal{L}(V, W) \quad \text{ for all } k \ge 1 \text{ and all } n > \omega_W\,.$$

Moreover

$$S_{BF}(t) = \sum_{k=0}^{\infty} S_k(t), \ \ S_{BF_n}(t) = \sum_{k=0}^{\infty} S_k^n(t)$$

where both series converge in $\mathcal{L}(W)$ and $\mathcal{L}(V)$. We claim that:

a) $\lim_{n\to\infty} \|S_{k+1}^n(t)x - S_{k+1}(t)x\|_W = 0$ uniformly in t on compact intervals for all $x \in V, t \ge 0, k \ge 0$.

b) $\lim_{n\to\infty} \|F_nS_k^n(\cdot)x - FS_k(\cdot)x\|_{L^2(0,T;U)} = 0$ for all $x \in V$, $T > 0$, $k \ge 0$.

We show a) and b) by induction on k. Statement b) is true for $k = 0$ by (iv). Hence, by admissibility of B, statement a) is true for $k = 0$. Assume that a) and b) hold for $k = \ell$. It then follows that for $x \in V$

$$\begin{aligned} &\|F_nS_{\ell+1}^n(t)x - FS_{\ell+1}(t)x\|_U \\ &\qquad \le \|F_n\|_{\mathcal{L}(W,U)}\|S_{\ell+1}^n(t)x - S_{\ell+1}(t)x\|_W \\ &\qquad\quad + \|(F_n - F)S_{\ell+1}(t)x\|_U \\ &\qquad \to 0 \text{ as } n \to \infty\,. \end{aligned}$$

Moreover, using (iii) and the fact that a) holds for $k = \ell$, it is easy to show that $\|F_n S^n_{\ell+1}(t)x\|_U \leq$ const. on $[0,T]$ for all $x \in V$ and for all $n > \omega_W$. It follows from Lebesgue's dominated convergence theorem that b) is true for $k = \ell + 1$. In order to show that a) holds for $k = \ell + 1$ notice that for all $x \in V$

$$\begin{aligned}
&\|S^n_{\ell+2}(t)x - S_{\ell+2}(t)x\|_W \\
&\quad = \|\int_0^t S(t-\tau)B(F_n S^n_{\ell+1}(\tau)x - FS_{\ell+1}(\tau)x)d\tau\|_W \\
&\quad \leq \alpha\|F_n S^n_{\ell+1}(\cdot)x - FS_{\ell+1}(\cdot)x\|_{L^2(0,T)},
\end{aligned}$$

where α is the constant introduced in Definition 2.9 (i) (cf. also Remark 2.10 (i)). Since we have already proved that b) is true for $k = \ell + 1$ it follows that a) holds for $k = \ell + 1$. By the admissibility of F there exists a constant $\beta_F > 0$ such that

$$\|FS(\cdot)x\|_{L^2(0,t)} \leq \beta_F\|x\|_V \text{ for all } x \in W\,.$$

It follows that for all $x \in W$

$$\|F_n S(\cdot)x\|_{L^2(0,t)} \leq n\beta_F\|(nI - A^V)^{-1}\|_{\mathcal{L}(V)}\|x\|_V\,.$$

Applying the Hille-Yosida theorem to A^V shows that there exists a constant $\tilde{\beta} > 0$ such that

$$\|F_n S(\cdot)x\|_{L^2(0,t)} \leq \tilde{\beta}\|x\|_V \quad \text{for all } n > \omega_W \text{ and all } x \in W\,.$$

Recall from the proof of Theorem 4.1 that $\sum_{k=1}^\infty S_k(t)$ and $\sum_{k=1}^\infty S^n_k(t)$ converge in $\mathcal{L}(V,W)$. Therefore

$$S_{BF_n}(t)x - S_{BF}(t)x \in W \qquad \text{for all } x \in V\,.$$

Using the estimate (4.5) and statement (iii) it follows for $x \in V$:

$$\begin{aligned}
\|S_{BF_n}(t)x - S_{BF}(t)x\|_W \quad &\leq \quad \|\sum_{k=1}^{k_0}(S^n_k(t)x - S_k(t)x)\|_W \\
&\quad +\tilde{\beta}\sum_{k=k_0+1}^{\infty}\alpha^k L^{k-1}\sqrt{\frac{t^{k-1}}{(k-1)!}}\|x\|_V \\
&\quad +\beta_F\sum_{k=k_0+1}^{\infty}\alpha^k\|F\|^{k-1}\sqrt{\frac{t^{k-1}}{(k-1)!}}\|x\|_V\,.
\end{aligned}$$

Hence, given $\epsilon > 0$ and $T > 0$, we have for all sufficiently large k_0 that

$$\|S_{BF_n}(t)x - S_{BF}(t)x\|_W \;\leq \|\sum_{k=1}^{k_0}(S^n_k(t)x - S_k(t)x)\|_W + \epsilon$$

for all $x \in V$ and all $t \in [0,T]$. It follows from a) that $\lim_{n\to 0} \|S_{BF_n}(t)x - S_{BF}(t)x\|_W = 0$ for all $x \in V$ uniformly in t on compact intervals, which proves (v).

(vi) Let $t \geq 0$ be fixed but arbitrary and let φ be a step function. Then by (v) $\tau \mapsto (S_{BF}(t-\tau) - S_{BF_n}(t-\tau)B\varphi(\tau)$is a function with values in W which is Bochner integrable in W on $[0,t]$. This follows from the properties of the operators $S_k(t)$ and $S^n_K(t)$. Hence

$$\begin{aligned}
&\|{}_V\!\int_0^t S_{BF}(t-\tau)B\varphi(\tau)d\tau - {}_V\!\int_0^t S_{BF_n}(t-\tau)B\varphi(\tau)d\tau\|_W \\
&\quad= \|{}_W\!\int_0^t (S_{BF}(t-\tau) - S_{BF_n}(t-\tau))B\varphi(\tau)d\tau\|_W \\
&\quad\leq \int_0^t \|(S_{BF}(t-\tau) - S_{BF_n}(t-\tau))B\varphi(\tau)\|_W\, d\tau \\
&\quad\to 0 \text{ as } n \to \infty,
\end{aligned}$$

where we have used (v) and the fact that φ is a step function. In order to complete the proof it is sufficient to show that there exists a constant Λ such that

$$\|\int_0^t S_{BF_n}(t-\tau)Bu(\tau)d\tau\|_W \leq \Lambda\|u\|_{L^2(0,t)} \tag{4.11}$$

for all $n > \omega_W$ and for all $u \in L^2(0,t;U)$. Using (4.5) we obtain

$$\begin{aligned}
&\|\int_0^t S_{BF_n}(t-\tau)Bu(\tau)d\tau\|_W \\
&\quad= \|\int_0^t S(t-\tau)Bu(\tau) + \int_0^t \sum_{k=1}^{\infty} S^n_k(t-\tau)Bu(\tau)d\tau\|_W \\
&\quad\leq \alpha\|u\|_{L^2(0,t)} + \tilde{\beta}\sum_{k=1}^{\infty}\alpha^k\|F_n\|^{k-1} \\
&\qquad\times \int_0^t \sqrt{\frac{(t-\tau)^{k-1}}{(k-1)!}}\|Bu(\tau)\|_V\, d\tau\ ,
\end{aligned}$$

where $\tilde{\beta} > 0$ is a suitable constant which is independent of n. It follows now from (iii) that there exists a constant Λ such that (4.11) is satisfied. □

Lemma 4.6 *Suppose that (2.14) is a Pritchard-Salamon system and that $F, \tilde{F} \in \mathcal{L}(W,U)$ are admissible output operators for $S(t)$. Define F_n and $\tilde{F}_n$ according to (4.10). Then we have for $T > 0$ and $x \in V$ that*

$$\lim_{n\to\infty} \|\tilde{F}_n S_{BF_n}(\cdot)x - \tilde{F}S_{BF}(\cdot)x\|_{L^2(0,T;U)} = 0\,.$$

Proof of Lemma 4.6: First we prove that the claim is true for all $x \in W$:

$$\begin{aligned}
\|\tilde{F}_n S_{BF_n}(\cdot)x - \tilde{F} S_{BF}(\cdot)x\|_{L^2(0,T;U)} \\
&\leq \|\tilde{F}_n(S_{BF_n}(\cdot)x - S_{BF}(\cdot)x)\|_{L^2(0,T;U)} \\
&\quad + \|(\tilde{F}_n - \tilde{F})S_{BF}(\cdot)x\|_{L^2(0,T;U)} \\
&\leq \|\tilde{F}_n\|_{\mathcal{L}(W,U)} \|S_{BF_n}(\cdot)x - S_{BF}(\cdot)x\|_{L^2(0,T;W)} \\
&\quad + \|(\tilde{F}_n - \tilde{F})S_{BF}(\cdot)x\|_{L^2(0,T;U)}
\end{aligned}$$

The first term on the R.H.S. converges to 0 by Lemma 4.5 (iii) and (v). Moreover, by Lemma 4.5 (ii), the sequence $(\tilde{F}_n - F)S_{BF}(t)x$ converges pointwise to zero. Applying Lemma 4.5 (iii) shows that it can be bounded by a function which is integrable on $[0,T]$. Hence the second term on the R.H.S. converges to 0 by Lebesgue's dominated convergence theorem. In order to prove that the claim is true for all $x \in V$ it is sufficient to show that there exists a constant $\Lambda > 0$ such that

$$\|\tilde{F}_n S_{BF_n}(\cdot)x\|_{L^2(0,T;U)} \leq \Lambda \|x\|_V \qquad \text{for all } x \in W \text{ and } n > \omega_W .$$

This follows easily from the definition of $\tilde{F}_n$, Lemma 4.5 (iii) and the integral equation for $S_{BF}(t)$. □

Proof of Theorem 4.3: Define F_{1n} and F_{2n} according to (4.10). Then

$$S_{B(F_{1n}+F_{2n})}(t)x = S_{BF_{1n}}(t)x + \int_0^t S_{BF_{1n}}(t-\tau)BF_{2n}S_{B(F_{1n}+F_{2n})}(\tau)x\,d\tau \tag{4.12}$$

is true for all $x \in V$, since F_{1n}, $F_{2n} \in \mathcal{L}(V,U)$ by Lemma 4.5 (i). Using Lemma 4.5 (v) we see that the L.H.S. of (4.12) converges to $S_{B(F_1+F_2)}(t)x$ while the first term on the R.H.S. of (4.12) goes to $S_{BF_1}(t)x$. Applying (4.11) to $S_{BF_{1n}}(\cdot)B$ instead of $S_{BF_n}(\cdot)B$ and using Lemma 4.5 (vi) and Lemma 4.6 shows that the integral on the R.H.S. of (4.12) converges to $\int_0^t S_{BF_1}(t-\tau)BF_2S_{B(F_1+F_2)}(\tau)x\,d\tau$. □

Corollary 4.7 *Suppose that (2.14) is a Pritchard-Salamon system, that $F \in \mathcal{L}(W,U)$ is an admissible output operator for $S(t)$ and that $H \in \mathcal{L}(Y,V)$ is an admissible input operator for $S(t)$. Using the notation of Theorem 4.1 we have:*

(i) $S_{BF}(t)x = S(t)x + \int_0^t S_{BF}(t-\tau)BFS(\tau)x\,d\tau$ for all $x \in V$.

(ii) If $BF = HC$, then $S_{BF}(t) = \tilde{S}_{HC}(t)$.

Proof: (i) Apply Theorem 4.3 to $F_1 = F$ and $F_2 = -F$.

(ii) This follows from (i), since by Theorem 4.1 (ii) the integral equation has the unique solution $\tilde{S}_{HC}(t)x$. □

The following proposition contains two results on the infinitesimal generator of the perturbed semigroup $S_{BF}(t)$.

Proposition 4.8 *Suppose that (2.14) is a Pritchard-Salamon system and that $F \in \mathcal{L}(W,U)$ is an admissible output operator for $S(t)$. Let A^W_{BF} and A^V_{BF} denote the infinitesimal generators of $S_{BF}(t)$ on W and V, respectively. Then we have:*

(i) $D(A^V_{BF}) = D(A^V)$ and $D(A^W_{BF}) = \{x \in D(A^V) \cap W \mid A^V_{BF}x \in W\}$.

(ii) Under the additional assumption that $D(A^V) \subset W$ we have

$$A^V_{BF}x = Ax + BFx \quad \text{for all } x \in D(A^V)\,.$$

Proof: (i) Let ω be a real number which is larger than the maximum of the exponential growth constants of $S^W(t)$, $S^V(t)$, $S^W_{BF}(t)$ and $S^V_{BF}(t)$. Laplace transformation of (4.1) gives

$$(sI - A^V_{BF})^{-1}x = (sI - A^V)^{-1}x + (sI - A^V)^{-1}BF(sI - A^W_{BF})^{-1}x \quad (4.13)$$

for $s \in \mathbb{C}_\omega$ and $x \in W$. By part (i) of Theorem 4.1 F is an admissible output operator for $S_{BF}(t)$and hence according to Lemma 2.12 (ii) and Remark 2.13 the operator $F(sI - A^W_{BF})^{-1} \in \mathcal{L}(W,U)$ can be extended to an operator in $\mathcal{L}(V,U)$. Therefore (4.13) holds true for all $x \in V$ and, moreover,

$$D(A^V_{BF}) \subset D(A^V)\,, \quad (4.14)$$

where we used the fact that $D(A^V_{BF}) = (sI - A^V_{BF})^{-1}V$. From Corollary 4.7 we obtain that

$$(sI - A^V_{BF})^{-1}x = (sI - A^V)^{-1}x + (sI - A^V_{BF})^{-1}BF(sI - A^W)^{-1}x \quad (4.15)$$

for $s \in \mathbb{C}_\omega$ and $x \in W$. It follows as above that (4.15) extends to V and so

$$D(A^V) \subset D(A^V_{BF}). \quad (4.16)$$

Hence, by (4.14) and (4.16)

$$D(A^V_{BF}) = D(A^V). \quad (4.17)$$

The second claim in statement (i) follows from Pazy [20], p. 123.

(ii) Since $(sI - A^V_{BF})^{-1}x = (sI - A^W_{BF})^{-1}x$ for $x \in W$, we may conclude from (4.13) that

$$(sI - A^V - BF)(sI - A^V_{BF})^{-1}x = x \quad \text{for } x \in W. \quad (4.18)$$

We have already mentioned that $F(sI - A^W_{BF})^{-1}$ extends to an operator in $\mathcal{L}(V,U)$. Taking into account that $(sI - A^V)^{-1} \in \mathcal{L}(V,W)$ (this follows from $D(A^V) \subset W$) we obtain from (4.13) that $(sI - A^V_{BF})^{-1} \in \mathcal{L}(V, W)$. As a consequence there holds

$$BF(sI - A^V_{BF})^{-1} \in \mathcal{L}(V, V)\,. \tag{4.19}$$

Furthermore, with (4.17) and the closedness of A^V, it follows from the closed-graph theorem that

$$(sI - A^V)(sI - A^V_{BF})^{-1} \in \mathcal{L}(V, V)\,. \tag{4.20}$$

So, by (4.19) and (4.20), the operator on the L.H.S. of (4.18) belongs to $\mathcal{L}(V, V)$ and hence (4.18) extends to V. An application of (4.18) to $(sI - A^V_{BF})y$, where $y \in D(A^V_{BF})$, leads to

$$A^V_{BF} y = A^V y + BFy \ \text{ for all } \ y \in D(A^V_{BF})\,.$$

$\square$

In Section 2 we gave a number of sufficient conditions for the exponential growth constants ω_W and ω_V of $S(t)$ to coincide. Theorem 4.1 can be used to derive another one.

Corollary 4.9 *Suppose that $B \in \mathcal{L}(U, V)$ is an admissible input operator for $S(t)$. If there exists an admissible output operator $F \in \mathcal{L}(W, U)$ such that the exponential growth constants ω^{BF}_W and ω^{BF}_V on W and V of the C_0-semigroup $S_{BF}(t)$ given by (4.1) satisfy $\max(\omega^{BF}_W, \omega^{BF}_V) \leq \min(\omega_W, \omega_V)$, then $\omega_W = \omega_V$.*

Proof: From (4.1) we obtain for $\epsilon > 0$

$$\begin{aligned} e^{-(\omega_W+\epsilon)t} S(t)x &= e^{-(\omega_W+\epsilon)t} S_{BF}(t)x \\ &\quad - \int_0^t e^{-(\omega_W+\epsilon)(t-\tau)} S(t-\tau) BF e^{-(\omega_W+\epsilon)\tau} S_{BF}(\tau)x d\tau. \end{aligned}$$

Now $e^{-(\omega_W+\epsilon)t} S_{BF}(t)$ is exponentially stable on V and $e^{-(\omega_W+\epsilon)t} S(t)$ is exponentially stable on W. Using the admissibility of B for $S(t)$ and Remark 2.10 (i) we see that for suitable constants $\alpha > 0$ and $\gamma > 0$

$$\|e^{-(\omega_W+\epsilon)t} S(t)x\|_V \leq \gamma \|x\|_V + \alpha \|F e^{-(\omega_W+\epsilon)\cdot} S_{BF}(\cdot)x\|_{L^2(0,t)}\,.$$

Since $e^{-(\omega_W+\epsilon)t} S_{BF}(t)$ is exponentially stable on V and since F is an admissible output operator for $S_{BF}(t)$ (by Theorem 4.1 (i)) it follows from Remark 2.10 (ii) that

$$\|e^{-(\omega_W+\epsilon)t} S(t)x\|_V \leq (\gamma + \alpha\beta)\|x\|_V \tag{4.21}$$

for some constant $\beta > 0$. The inequality (4.21) is true for any $\epsilon > 0$ and hence $\omega_V \leq \omega_W$. In order to prove the converse inequality, note that by Corollary 4.7

$$\begin{aligned} e^{-(\omega_V+\epsilon)t}S(t)x &= e^{-(\omega_V+\epsilon)t}S_{BF}(t)x \\ &\quad - \int_0^t e^{-(\omega_V+\epsilon)(t-\tau)}S_{BF}(t-\tau)BFe^{-(\omega_V+\epsilon)\tau}S(\tau)x d\tau\,. \end{aligned}$$

Taking norms in W, using the exponential stability of $e^{-(\omega_V+\epsilon)t}S_{BF}(t)$ on W, the exponential stability of $e^{-(\omega_V+\epsilon)t}S(t)$ on V, the admissibility of B for $S_{BF}(t)$ and the admissibility of F for $S(t)$ it follows that

$$\|e^{-(\omega_V+\epsilon)t}S(t)x\|_W \leq \kappa\|x\|_W \quad \text{for all } \epsilon > 0 \text{ and } x \in W.$$

The last estimate shows that $\omega_W \leq \omega_V$. □

Remark 4.10 Although Corollary 4.9 may be difficult to apply in general, it does have one very important consequence. If $S(t)$ is unstable on W and V, and it is admissibly exponentially stabilizable in the sense of Definition 5.1, then $\omega_W = \omega_V$.

5 Stabilizability, Detectability and Equivalence of Exponential and External Stability

We now introduce the concepts of stabilizability and detectability for Pritchard-Salamon systems which are appropriate for 'BKC'-type perturbations, i.e. perturbations induced by output feedback.

Definition 5.1 *(i) Suppose that $B \in \mathcal{L}(U,V)$ is an admissible input operator for $S(t)$. The pair $(S(t),B)$ is called boundedly (admissibly) stabilizable if there exists an operator $F \in \mathcal{L}(V,U)$ (an admissible output operator $F \in \mathcal{L}(W,U)$ for $S(t)$) such that the C_0-semigroup $S_{BF}(t)$ given by (4.1) is exponentially stable on W and V.*

(ii) Suppose that $C \in \mathcal{L}(W,Y)$ is an admissible output operator for $S(t)$. The pair $(C,S(t))$ is called boundedly (admissibly) detectable if there exists an operator $H \in \mathcal{L}(Y,W)$ (an admissible input operator $H \in \mathcal{L}(Y,V)$ for $S(t)$) such that the C_0-semigroup $\tilde{S}_{HC}(t)$ given by (4.2) is exponentially stable on W and V.

Remark 5.2 (i) The above definition makes sense for $F \in \mathcal{L}(V,U)$ ($H \in \mathcal{L}(Y,W)$) since all elements in $\mathcal{L}(V,U) \subset \mathcal{L}(W,U)$ ($\mathcal{L}(Y,W) \subset \mathcal{L}(Y,V)$) are admissible output operators (admissible input operators) and hence, by Theorem 4.1, $S_{BF}(t)$ ($\tilde{S}_{HC}(t)$) is a C_0-semigroup on W and V.

(ii) Admissibly stabilizable Pritchard-Salamon systems have the following nice system theoretic property: If (2.14) is a Pritchard-Salamon system, then by Theorem 4.3 it is admissibly stabilizable (detectable) if and only if $(S_{BKC}(\cdot), B, C)$ is admissibly stabilizable (detectable), where $K \in \mathcal{L}(Y, U)$ and $S_{BKC}(t)$ is the perturbed C_0-semigroup of Theorem 4.1.

The following lemma demonstrates an important connection between exponential stability of the perturbed semigroup on W and V.

Lemma 5.3 *Suppose that $B \in \mathcal{L}(U, V)$ is an admissible input operator for $S(t)$. If $(S(t), B)$ is admissibly stabilizable and there exists $\tilde{F} \in \mathcal{L}(V, U)$ such that $S_{B\tilde{F}}(t)$ is exponentially stable on V, then $S_{B\tilde{F}}(t)$ is also exponentially stable on W.*

Proof: Let $F \in \mathcal{L}(W, U)$ be an admissible output operator for $S(t)$ such that $S_{BF}(t)$ is exponentially stable on W and V. By Theorem 4.1 and Theorem 4.3

$$S_{B\tilde{F}}(t)x = S_{BF}(t)x + \int_0^t S_{BF}(t-\tau)B(\tilde{F} - F)S_{B\tilde{F}}(\tau)x d\tau . \qquad (5.1)$$

Now $(\tilde{F} - F)$ is an admissible output operator for $S_{B\tilde{F}}(t)$ (by Theorem 4.1) and hence, from the exponential stability of $S_{\tilde{F}}(t)$ on V, it follows that for sufficiently small $\epsilon > 0$

$$\int_0^\infty \|e^{\epsilon t}(\tilde{F} - F)S_{B\tilde{F}}(t)x\|_U^2 dt \leq \gamma \|x\|_V^2$$

for all $x \in W$, where γ is some suitable positive constant. Now by the admissibility of B for $S_{BF}(t)$, and the exponential stability of $e^{\epsilon t}S_{BF}(t)$ on W for $\epsilon > 0$ sufficiently small, it follows from multiplying (5.1) by $e^{\epsilon t}$ and taking norms in W that $\|e^{\epsilon t}S_{B\tilde{F}}(t)x\|_W \leq \tilde{\gamma}\|x\|_W$ for some positive constant $\tilde{\gamma}$. This estimate holds for all $x \in W$ and hence we obtain that $\|S_{B\tilde{F}}(t)\|_{\mathcal{L}(W)} \leq \tilde{\gamma}e^{-\epsilon t}$. □

Remark 5.4 Notice that in the above proof we have not made use of the fact that $S_{BF}(t)$ is also exponentially stable on V.

The following result shows that the two concepts of stabilizability introduced in Definition 5.1 coincide.

Theorem 5.5 *Suppose that $B \in \mathcal{L}(U, V)$ is an admissible input operator for $S(t)$. Then the following statements hold:*

(i) The pair $(S(t), B)$ is admissibly stabilizable if and only if $(S(t), B)$ is boundedly stabilizable.

(ii) Suppose that $F \in \mathcal{L}(W, U)$ is an admissible output operator for $S(t)$ such that $S_{BF}(t)$ is exponentially stable on W and V and let $P \in \mathcal{L}(V)$ be the selfadjoint positive semi-definite operator defined by

$$< x, Py >_V := \int_0^\infty < FS_{BF}(t)x, FS_{BF}(t)y >_U \, dt .$$

If $D(A^V) \subset W$, then $\tilde{F} := -B^* P \in \mathcal{L}(V,U)$ stabilizes $(S(t), B)$ on W and V.

Proof: (i) Since for any operator $F \in \mathcal{L}(V,U)$ the restriction $F\mid_W$ is an admissible output operator for $S(t)$, it follows trivially that $(S(t), B)$ is admissibly stabilizable if $(S(t), B)$ is boundedly stabilizable. The following proof of the converse was suggested to us by B. van Keulen: Let $F \in \mathcal{L}(W,U)$ be an admissible output operator $S(t)$ such that $S_{BF}(t)$ is exponentially stable on V. Let $x_0 \in V$ and define $u_{x_0}(\cdot) = F S_{BF}(\cdot)x_0$. Then $u_{x_0}(\cdot) \in L^2(0,\infty);U)$ by Remark 2.10 (ii) and $x(\cdot)$ given by

$$x(t) = S(t)x_0 + \int_0^t S(t-\tau)Bu_{x_0}(\tau)d\tau = S_{BF}(t)x_0$$

is in $L^2(0,\infty;V)$. Hence, using Datko's result [10] on the equivalence between open and closed-loop stabilizability, it follows that there exists an operator $\tilde{F} \in \mathcal{L}(V,U)$ which stabilizes $(S(t), B)$ on V. By Lemma 5.3 the feedback $\tilde{F}$ stabilizes $(S(t), B)$ on W as well.

(ii) Using Proposition 4.8 we have that if $x \in D(A^V)$ then

$$\frac{d}{dt}(S_{B\tilde{F}}(t)x) = A_{BF} S_{B\tilde{F}}(t)x + B(\tilde{F} - F)S_{B\tilde{F}}(t)x\,. \tag{5.2}$$

Using the definition of $\tilde{F}$ and P and (5.2) we can easily show that

$$\begin{aligned}&\langle S_{B\tilde{F}}(t)x, PS_{B\tilde{F}}(t)x\rangle_V - \langle x, Px\rangle_V = \\ &\quad -\int_0^t \|(F-\tilde{F})S_{B\tilde{F}}(\tau)x\|_U^2 d\tau - \int_0^t \|\tilde{F}S_{B\tilde{F}}(\tau)x\|_U^2 d\tau\,.\end{aligned}$$

Since $P \geq 0$, it follows that

$$\int_0^t \|(F-\tilde{F})S_{B\tilde{F}}(\tau)x\|_U^2 \leq \langle x, Px\rangle_V \quad \text{for all } t \geq 0 \text{ and } x \in D(A^V) \tag{5.3}$$

Now, by Theorem (4.1) (i) $(F - \tilde{F})$ is an admissible output operator for $S_{B\tilde{F}}(t)$ and hence (5.3) extends to all $x \in V$. Thus there exists a constant $\gamma > 0$ such that

$$\|(F-\tilde{F})S_{B\tilde{F}}(\cdot)x\|_{L^2(0,\infty;U)} \leq \gamma\|x\|_V \text{ for all } x \in V. \tag{5.4}$$

Choose constants $M \geq 1$ and $\alpha > 0$ such that $\|S_{BF}(t)\|_{\mathcal{L}(V)} \leq Me^{-\alpha t}$. Taking norm estimates in $L^2(0,\infty;V)$ for (5.1) we obtain

$$\begin{aligned}\|S_{B\tilde{F}}(\cdot)x\|_{L^2(0,\infty;V)} \quad &\leq \quad \|S_{BF}(\cdot)x\|_{L^2(0,\infty;V)} + \|\int_0^{\cdot} S_{BF}(\cdot-\tau)B \\ &\qquad \times (\tilde{F}-F)S_{B\tilde{F}}(\tau)x d\tau\|_{L^2(0,\infty;V)}\end{aligned}$$

$$\begin{aligned} &\leq \|S_{BF}(\cdot)x\|_{L^2(0,\infty;V)} + \frac{M}{\alpha}\|B\|_{\mathcal{L}(U,V)} \\ &\qquad \times \|(\tilde{F}-F)S_{B\tilde{F}}(\cdot)x\|_{L^2(0,\infty;U)} \\ &< \infty, \end{aligned}$$

using (5.4) and the exponential stability of $S_{BF}(\cdot)$ on V. The above estimate holds for all $x \in V$ and hence $S_{B\tilde{F}}(\cdot)$ is exponentially stable on V (Theorem 4.1, Pazy [20], p. 116), and by Lemma 5.3 it is also stable on W. □

Remark 5.6 (i) We shall leave the obvious statement and proof of the dual result on detectability to the reader.

(ii) Suppose that (2.14) is a Pritchard-Salamon system. Since it is clear by Theorem 4.3 that stabilizability of (2.14) by admissible feedback is retained under output feedback of the form $u = Ky$, where $K \in \mathcal{L}(Y,U)$ (see Remark 5.2) it follows from Theorem 5.5 that stabilizability by bounded feedback is invariant under output feedback as well.

(iii) The proof of Theorem 5.5 (ii) is similar in many ways to the recursive procedure in Pritchard and Salamon [23] by which the solution of an algebraic Riccati equation is constructed from solutions of a sequence of Lyapunov equations. It is clear from the constructive nature of the proof that similar argumants can be used in order to show that for any Pritchard-Salamon system (satisfying $D(A^V) \subset W$) and any LQ or H^∞-performance index (see e.g. Pritchard and Salamon [22, 23] and Pritchard and Townley [26]) the achievable closed-loop costs are the same with respect to bounded feedbacks and admissible feedbacks. It remains an open problem as to whether the domain condition $D(A^V) \subset W$ is necessary in the constructive argument.

We would like to close this section with a characterization of exponential stability in terms of transfer functions.

Definition 5.7 *Suppose that in (2.14) the operator $B \in \mathcal{L}(U,V)$ is an admissible input operator for $S(t)$ and $C \in \mathcal{L}(W,Y)$. The system (2.14) is called externally stable if its transfer function G is in $H^\infty(\mathbb{C}_0, \mathcal{L}(U,Y))$.*

It follows from Proposition 2.15 that the system (2.14) is externally stable if the semigroup $S(t)$ is exponentially stable on W and V. Under some extra assumptions the converse holds true as well.

Theorem 5.8 *Suppose that (2.14) is a Pritchard-Salamon system. If (2.14) is admissibly stabilizable and admissibly detectable, then the semigroup $S(t)$ is exponentially stable on W and V if and only if (2.14) is externally stable.*

Proof: We only need to show that external stability implies exponential stability of $S(t)$ on W and V. By assumption there exist an admissible output operator $F \in \mathcal{L}(W,U)$ and an admissible input operator $H \in \mathcal{L}(Y,V)$ for $S(t)$ such that the perturbed semigroups $S_{BF}(t)$ and $\tilde{S}_{HC}(t)$ given by (4.1) and (4.2) are exponentially stable on W and V. Using the notation of Proposition 4.8 it follows from (4.1) and (4.2) via Laplace transformation that

$$(sI - A^W_{BF})^{-1}w = (sI - A^W)^{-1}w + (sI - A^V)^{-1}BF(sI - A^W_{BF})^{-1}w \quad (5.5)$$

and

$$(sI - \tilde{A}^V_{HC})^{-1}w = (sI - A^V)^{-1}w + (sI - \tilde{A}^V_{HC})^{-1}HC(sI - A^W)^{-1}w\,, \quad (5.6)$$

for all $w \in W$ and all $s \in \mathbb{C}_\omega$, where $\omega := \max(\omega_W, \omega_V, 0)$ and $\tilde{A}^V_{HC}$ denotes the infinitesimal generator of the semigroup $\tilde{S}_{HC}(t)$ on V. Since B, C, F and H are admissible we obtain from (5.5) and (5.6) using Theorem 4.1, Lemma 2.12 and Remark 2.13 that

$$C(sI - A^W)^{-1}v = C(sI - A^W_{BF})^{-1}v - G(s)BF(sI - A^W_{BF})^{-1}v \quad (5.7)$$

and

$$(sI - A^V)^{-1}v = (sI - \tilde{A}^V_{HC})^{-1}v - (sI - \tilde{A}^V_{HC})^{-1}HC(sI - A^W)^{-1}v \quad (5.8)$$

for all $v \in V$ and $s \in \mathbb{C}_\omega$. In (5.7) we have used that by Proposition 2.15 the transfer function $G(s)$ of (2.14) satisfies $G(s) = C(sI - A^V)^{-1}B$ for all $s \in \mathbb{C}_\omega$.

Now, by assumption and Lemma 2.12, the $\mathcal{L}(V,Y)$-valued functions on the R.H.S. of (5.7) are bounded, and therefore $\|C(sI - A^W)^{-1}\|_{\mathcal{L}(V,Y)}$ is bounded on $\mathbb{C}_\omega$. It follows from (5.8) that

$$(sI - A^V)^{-1} \in H^\infty(\mathbb{C}_\omega, \mathcal{L}(V)). \quad (5.9)$$

We have to show that $\max(\omega_W, \omega_V) < 0$. Assume the contrary, i.e. $\max(\omega_W, \omega_V) \geq 0$. Then, by the definition of ω

$$\omega = \max(\omega_W, \omega_V)\,, \quad (5.10)$$

and (5.9) implies that

$$(sI - (A^V - \omega I))^{-1} \in H^\infty(\mathbb{C}_0, \mathcal{L}(V))\,,$$

which is equivalent to the exponential stability of $S(t)e^{-\omega t}$ on V (this follows from a result by Prüss [27], cf. also Nagel [19], p. 96 or see Weiss [32] for a direct proof). Hence we may conclude that $\omega_V < \omega$. An application of Lemma 5.3 to $(S(t)e^{-\omega t}, B)$ and $\tilde{F} = 0$ shows that $\omega_W < \omega$. Therefore $\max(\omega_W, \omega_V) < \omega$, which is in contradiction to (5.10). □

Remark 5.9 Theorem 5.8 is a nice generalization of the well-known finite-dimensional result. It is important to note that U and Y may be infinite-dimensional. This is in contrast to earlier publications (e.g. Curtain [3]) which proved a similar type of equivalence assuming that U and Y be finite-dimensional. See also Rebarber [28] for a related result.

Remark 5.10 Almost every result in this paper remains valid if we replace the Hilbert spaces W, V, U, and Y by arbitrary Banach spaces. The only exceptions are the following:

- The proof of Proposition 2.2 is based on a result by Lax [16] which requires V to be a Hilbert space. We do not know if Proposition 2.2 extends to Banach spaces. As a consequence it is not clear whether Remark 2.3 and Proposition 2.7 (iii) remain true in the Banach space case.
- In the proof of Theorem 5.5 we have explicitly used the Hilbert space structure of V and U. We do not know whether Theorem 5.5 holds true in the Banach space case or not.
- For arbitrary Banach spaces it is generally not true that C_0-semigroups with H^∞-resolvent are exponentially stable. So, the proof of Theorem 5.8 does not extend to Banach spaces.

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