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Necessary and Sufficient Conditions for Nonlinear Worst Case (H_∞) Control and Estimation*

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Abstract

We present necessary and sufficient conditions for the existence of worst case controllers and estimators for nonlinear systems. These are also called H_∞ suboptimal controllers and estimators. We consider affine and more general nonlinear systems, both time varying and autonomous over finite, semi-infinite and infinite intervals. In particular, we give necessary and sufficient conditions for the solvability of a standard H_∞ suboptimal control problem by measurement feedback that involve the solvability of a pair of partial differential equations of the Hamilton-Jacobi type. The first is the one associated with the problem of H_∞ suboptimal control by state feedback that has appeared previously in the work of several authors. The second is a new Hamilton-Jacobi equation associated with H_∞ suboptimal estimation.

Key words: nonlinear H_∞ control, nonlinear H_∞ estimation, nonlinear worst case control, nonlinear worst case estimation

AMS Subject Classifications: 93C10, 49A40

1 Introduction

Over the past decade, two of the most active areas of system theory have been linear H_∞ control and nonlinear control. Recently several groups of authors including van der Schaft [14], [15], Başar-Bernhard [2], Ball-Helton-Walker [1], Isidori-Astolfi [6] and James-Baras-Elliott [8], [9] have addressed the nonlinear H_∞ control problem and made significant progress.

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This paper builds on their work and presents new necessary and sufficient conditions for the solvability of the nonlinear H_∞ control problem by measurement feedback (Theorem 2.1).

The linear H_∞ control problem in state space form is known to be equivalent to the solvability of a pair of Riccati inequalities either in uncoupled form with a compatibility condition [3] or in coupled form [13]. The nonlinear generalization of the first of the Riccati inequalities is an integral dissipation inequality in the sense of Willems [17]. Assuming differentiability, this reduces to a Hamilton-Jacobi partial differential inequality [14], [6], [2], [1]. Several different nonlinear generalizations of the second Riccati inequality have been proposed, including a linear-quadratic approximation [6] and a partial differential inequality [2], [15] which generalizes the second Riccati inequality of [3]. In this paper, we present a new generalization of the second Riccati inequality in the coupled form of Tadmor [13]. It is a conditional integral dissipation inequality for a nonlinear estimator. It is called conditional because it is conditioned on the measurements. Along with a solution of the first Hamilton-Jacobi PDE, it yields necessary and sufficient conditions for H_∞ suboptimal control (Theorem 2.4). Under certain assumptions, the conditional integral dissipation inequality becomes a partial differential inequality of Hamilton-Jacobi type and the solvability of this and the first Hamilton-Jacobi inequality lead to sufficient conditions for H_∞ suboptimal control (Theorem 2.5).

The rest of the paper is organized as follows. In Section 2 we present five theorems which give necessary and sufficient conditions for nonlinear H^∞ control of a standard system. Two of these are well-known, theorems 2.2 and 2.3, and three are new. In Section 3, we specialize these theorems to linear systems. In Section 4, we treat various generalizations and extensions of these theorems. Section 5 concludes the paper and discusses future directions of research.

2 A Standard Worst Case Control Problem

Consider a nonlinear time varying system of the form

$$\dot{x} = a(x,t) + b\,(x,t)\,u + g\,(x,t)\,w \tag{2.1}$$

$$y = c(x,t) + v \tag{2.2}$$

$$z = \begin{bmatrix} h\,(x,t) \\ u \end{bmatrix} \tag{2.3}$$

$$x\,(t_0) = x^0. \tag{2.4}$$

The input $u\,(t)$ is a control while the inputs $w(t)$ and $v\,(t)$ are noises. The output $y\,(t)$ is measurable and the other output $z\,(t)$ is to be regulated, i.e., kept close to zero. The goal is to design a compensator which processes the past measurements, $y(\tau)$, $t_0 \le \tau \le t$, and the estimate $\widehat{x}^0$ of the initial state, x^0, to obtain a control action, $u(t)$. The compensator is chosen to minimize the worst case effect of the initial state, x^0, driving noise, $w(t)$ and observation noise, $v(t)$, on the regulated output, $z(t)$, in an L^2 sense. Notice that $z(t)$ contains the control, $u(t)$, and this discourages the use of large control actions.

More precisely, assume that $x^0, \widetilde{x}^0 = x^0 - \widehat{x}^0$, $w(t)$ and $v(t)$ are bounded in the standard L^2 norm, e.g.

$$\left|x^0\right|^2 + \left|\widetilde{x}^0\right|^2 + \int_{t_0}^{t_f} |w(t)|^2 \, dt + |v(t)|^2 \, dt \le 1. \tag{2.5}$$

We seek a compensator that infimizes the supremum of

$$\int_{t_0}^{t_f} |z(t)|^2 \, dt \tag{2.6}$$

over all $t_f \ge t_0$ and all $x^0, \widetilde{x}^0, w(t), v(t)$ satisfying (2.5).

The problem as stated is in a standard form, later in Section 4, we shall discuss various generalizations. The standard problem is not easy to solve even if the system is linear. The usual approach is to seek a suboptimal compensator. Given a $\gamma > 0$, we seek compensator such that for all $t_f \ge t_0$, and all $x^0, \widetilde{x}^0, w(t)$, and $v(t)$

$$\int_{t_0}^{t_f} |z(t)|^2 \, dt \le \gamma^2 \left[\left|x^0\right|^2 + \left|\widetilde{x}^0\right|^2 + \int_{t_0}^{t_f} |w(t)|^2 \, dt + |v(t)|^2 \, dt \right]. \tag{2.7}$$

A compensator which satisfies (2.7) is said to achieve an L^2 gain less than or equal to γ. By iteration on γ, one hopes to converge to an optimal compensator.

We have not specified the form of the compensator except to require that it be a causal mapping

$$\begin{bmatrix} \widehat{x}^0 \\ y(\tau) \end{bmatrix} \longmapsto u(t), \qquad t_0 \le \tau \le t \tag{2.8}$$

from the initial state estimate $\widehat{x}^0$ and the past observations, $y(t)$, to the current control, $u(t)$. More precisely, for each $\widehat{x}^0$ and $y(t)$ there is a $u(t)$ with the following property. If $\widehat{x}^0, y(t)$ and $\widehat{x}^0, \overline{y}(t)$ lead to controls $u(t)$ and $\overline{u}(t)$ respectively and $y(\tau) = \overline{y}(\tau)$ for $t_0 \le \tau \le t$ then $u(\tau) = \overline{u}(\tau)$ for $t_0 \le \tau \le t$.

Theorem 2.1 *A compensator (2.8) achieves an L^2 gain $\le \gamma$ iff there exists a causal mapping*

$$\begin{bmatrix} \widehat{x}^0 \\ y(\tau) \end{bmatrix} \longmapsto S(x,t), \qquad t_0 \le \tau \le t \tag{2.9}$$

such that for all $t_0 \le t_1 \le t_2$ *and all* x^0, x^1

$$S\left(x^0, t_0\right) \le \frac{\gamma^2}{2}\left(\left|x^0\right|^2 + \left|\widetilde{x}^0\right|^2\right) \tag{2.10}$$

$$S\left(x^1, t_1\right) \ge 0 \tag{2.11}$$

$$S\left(x\left(t\right), t\right)\big]_{t_1}^{t_2} \le \int_{t_1}^{t_2} \frac{\gamma^2}{2}\left|\begin{array}{c} w\left(t\right) \\ v\left(t\right) \end{array}\right|^2 - \frac{1}{2}\left|z\left(t\right)\right|^2 dt. \tag{2.12}$$

Remarks:

(i) $S(x,t)$ is causal in the same sense as $u(t)$; for each $\widehat{x}^0$ and $y(\tau)$ there exists $S(x,t)$. If $\widehat{x}^0, y(\tau)$ and $\widehat{x}^0, \overline{y}(\tau)$ lead to $S(x,t)$ and $\overline{S}(x,t)$ respectively and $y(\tau) = \overline{y}(\tau)$ for $t_0 \le \tau \le t$ then $S(x,\tau) = \overline{S}(x,\tau)$ for all x and for $t_0 \le \tau \le t$.

(ii) We have not made precise the spaces where x, u, w, v, y, z live, for convenience, we assume that they are Euclidean spaces of varying dimensions. The inequality (2.12) should hold along any trajectory of the closed loop system. If the actual inputs are known to be bounded as in (2.5) then S need only be defined and satisfy (2.10 to 2.12) where such trajectories are possible.

(iii) The proof of this theorem is similar to those found in Willems [17], where functions such as $S(x,t)$ are called storage functions. Because $S(x,t)$ depends on the initial estimate and past observations we call it a conditional storage function. In a loose sense, $S(x,t)$ measures the "energy" stored in the closed loop system when $x(t) = x$ assuming the initial estimate $\widehat{x}^0$ and observation $y(\tau)$. The initial "energy" $S\left(x^0, t_0\right)$ is bounded above by (2.10). The integrand on the right side of (2.12) is called the supply rate and can be thought of as the net "power" supplied to the system. The noises, $w(t)$ and $v(t)$, supply "power" to the system and the regulated output, $z(t)$, extracts "power" from the system. The inequality (2.12) is called the integral dissipation inequality and postulates that the system always dissipates "energy".

(iv) James-Baras-Elliott [8] and James-Baras [9] have introduced the concept of the information state which is closely relaed to the concept of a conditional storage function. The conditional storage function defined by (2.13) below is essentially the negative of their information state. In [9] they give a similar result for discrete time systems.

Proof: Suppose a compensator (2.8) and a conditional supply function (2.9) satisfy (2.10-2.12) for the closed loop system, then by (2.12) for any $t_f \ge t_0$ and $x^0, \widehat{x}^0, w(t), v(t)$

$$S\left(x\left(t_f\right), t_f\right) + \frac{1}{2}\int_{t_0}^{t_f} \left|z\left(t\right)\right|^2 dt \le S\left(x\left(t_0\right), t_0\right) + \frac{\gamma^2}{2}\int_{t_0}^{t_f} \left|w\left(t\right)\right|^2 + \left|v\left(t\right)\right|^2 dt.$$

Using (2.10, 2.11) this implies (2.7).

On the other hand suppose (2.7) holds for all $t_f \geq t_0$ and $x^0, \widehat{x}^0, w(t)$, $v(t)$ for the closed loop system with compensator (2.8). For each $\widehat{x}^0$ $y(\tau)$,
$t_0 \leq \tau \leq t$ we define the conditional required supply

$$S\left(x^1, t_1\right) = \inf \left[\frac{\gamma^2}{2}\left(\left|x^0\right|^2 + \left|\widetilde{x}^0\right|^2\right)\right.$$

$$\left. + \int_{t_0}^{t_1} \frac{\gamma^2}{2}\left(\left|w(t)\right|^2 + \left|v(t)\right|^2\right) - \frac{1}{2}\left|z(t)\right|^2 dt\right] \tag{2.13}$$

where the infimum is over all $x^0, w(t), v(t)$ consistent with the observations, $y(t)$, and such that $x(t_1) = x^1$. Clearly S satisfies (2.10) with equality and (2.11) follows from (2.7). To verify (2.12) suppose $t_0 \leq t_1 \leq t_2$, then

$$S\left(x^2, t_2\right) = \inf_{x(t_2)=x^2} \left[\frac{\gamma^2}{2}\left(\left|x^0\right|^2 + \left|\widetilde{x}^0\right|^2\right)\right.$$

$$\left. + \int_{t_0}^{t_2} \frac{\gamma^2}{2}\left(\left|w(t)\right|^2 + \left|v(t)\right|^2\right) - \frac{1}{2}\left|z(t)\right|^2 dt\right]$$

$$\leq \inf_{\substack{x(t_1)=x^1 \\ x(t_2)=x^2}} \left[\frac{\gamma^2}{2}\left(|x|^0 + \left|\widetilde{x}^0\right|^2\right) + \int_{t_0}^{t_1} \frac{\gamma^2}{2}\left(\left|w(t)\right|^2 + \left|v(t)\right|^2\right) - \frac{1}{2}\left|z(t)\right|^2 dt\right.$$

$$\left. + \int_{t_1}^{t_2} \frac{\gamma^2}{2}\left(\left|w(t)\right|^2 + \left|v(t)\right|^2\right) - \frac{1}{2}\left|z(t)\right|^2 dt\right].$$

Hence for any trajectory consistent with the observation and satisfying $x(t_1) = x^1$ and $x(t_2) = x^2$

$$S\left(x^2, t_2\right) \leq S\left(x^1, t_1\right) + \int_{t_1}^{t_2} \frac{\gamma^2}{2}\left(\left|w(t)\right|^2 + \left|v(t)\right|^2\right) - \frac{1}{2}\left|z(t)\right|^2 dt.$$

Q.E.D.

The next two theorems are slight variations of those found in Willems [17], Başar-Bernhard [2], van der Schaft [14] and Isidori-Astolfi [6]. We assume that the state is directly measurable with no observation noise and we seek to find a state feedback that achieves an L^2 gain $\leq \gamma$, i.e., for all $t_f \geq t_0$ and all $x^0, w(t)$

$$\int_{t_0}^{t_f} \left|z(t)\right|^2 dt \leq \gamma^2 \left(\left|x^0\right|^2 + \int_{t_0}^{t_f} \left|w(t)\right|^2 dt\right). \tag{2.14}$$

Theorem 2.2 *The state feedback $u = k(x,t)$ achieves an L^2 gain $\leq \gamma$ in the sense of (2.14) iff there exists a storage function $P(x,t)$ such that for all $t_0 \leq t_1 \leq t_2$ and all x^0, x^1*

$$P\left(x^0, t_0\right) \leq \frac{\gamma^2}{2} \left|x^0\right|^2 \tag{2.15}$$

$$P\left(x^1, t_1\right) \geq 0 \tag{2.16}$$

$$P\left(x(t), t\right)\Big]_{t_1}^{t_2} \leq \int_{t_1}^{t_2} \frac{\gamma^2}{2} \left|w(t)\right|^2 - \frac{1}{2} \left|z(t)\right|^2 dt. \tag{2.17}$$

Remarks: Notice that $P(x,t)$ does not depend on $\hat{x}^0$ or $y(\tau)$. The proof is omitted as it is essentially the same as Theorem 2.1. In particular, one such $P(x,t)$ is defined by

$$P\left(x^1, t_1\right) = \inf \left[\frac{\gamma^2}{2} \left|x^0\right|^2 + \int_{t_0}^{t_1} \frac{\gamma^2}{2} \left|w(t)\right|^2 - \frac{1}{2} \left|z(t)\right|^2 dt \right] \tag{2.18}$$

where the infimum is over all x^0 and $w(t)$ such that $x(t_1) = x^1$. This particularly $P(x,t)$ is called the required supply by Willems [17]. In most other treatments of nonlinear H^∞ control, a different storage function, called the available storage, is used instead. However the required supply seems more natural as it satisfies an initial condition (2.15 with equality). The available storage satisfies a terminal condition and therefore requires a fixed terminal time.

Theorem 2.3 *Suppose $P(x,t)$ is C^1 and satisfies (2.15, 2.16) and (2.19)*

$$P_t + P_x a + \frac{1}{2} P_x \left(\frac{1}{\gamma^2} g g' - b b' \right) P_x' + \frac{1}{2} \left|h\right|^2 \leq 0 \tag{2.19}$$

then the state feedback

$$u = k(x,t) = -b'(x,t) P_x'(x,t) \tag{2.20}$$

achieves L^2 gain $\leq \gamma$. The "worst case" driving noise is

$$w = d(x,t) = \frac{1}{\gamma^2} g'(x,t) P_x'(x,t). \tag{2.21}$$

Proof: If P satisfies (2.19) then adding

$$-\frac{\gamma^2}{2} \left|w - d\right|^2 + \frac{1}{2} \left|u - k\right|^2$$

to both sides yields

$$P_t + P_x (a + bu + gw)$$

$$\leq \frac{\gamma^2}{2}|w|^2 - \frac{1}{2}|h|^2 - \frac{1}{2}|u|^2 - \frac{\gamma^2}{2}|w-d|^2 + \frac{1}{2}|u-k|^2. \tag{2.22}$$

Therefore if $u(t) = k(x(t), t)$ then $P(x(t), t)$ satisfies the integral dissipation inequality (2.17) and the result follows from Theorem 2.1.

Q.E.D.

Remarks:

(i) The equivalent partial differential inequalities (2.22) and (2.19) are called the differential dissipation inequality and the Hamilton-Jacobi-Issacs partial differential inequality.

(ii) If $P(x,t)$ is called a C^1 function satisfying (2.15) and (2.19) both with equality then $P\left(x^1, t_1\right)$ is the value function of the differential game whose payoff is

$$\sup_k \inf_{x^0, w(t)} \left[\frac{\gamma^2}{2}\left|x^0\right|^2 + \int_{t_0}^{t_1} \frac{\gamma^2}{2}|w(t)|^2 - \frac{1}{2}|z(t)|^2\, dt\right]$$

$$= \inf_{x^0, w(t)} \sup_k \left[\frac{\gamma^2}{2}\left|x^0\right|^2 + \int_{t_0}^{t_1} \frac{\gamma^2}{2}|w(t)|^2 - \frac{1}{2}|z(t)|^2\, dt\right]. \tag{2.23}$$

The value of the game given that $x(t_1) = x_1$ is $P\left(x^1, t_1\right)$ and the saddle point solution is $u(t), w(t)$ given by (2.20, 2.21). The saddle point initial condition x^0 is obtained by integrating the closed loop system backwards in time from $x(t_1) = x^1$. For more details, see Başar-Bernhard [2]. The next theorem is a reformulation of necessary and sufficient conditions of Theorem 2.1 given that the differential game (2.23) admits a saddle point solution with smooth value function.

Theorem 2.4 *Suppose there exists a smooth $P(x,t)$ satisfying (2.15) and (2.19) both with equality and let $k(x,t), d(x,t)$ be defined by (2.20,2.21).*

A measurement feedback compensator (2.8) achieves an L^2 gain $\leq \gamma$ iff there exists a causal mapping

$$\left|\begin{array}{c} \widehat{x}^0 \\ y(\tau) \end{array}\right| \longmapsto Q(x,t) \qquad t_0 \leq \tau \leq t \tag{2.24}$$

such that for $t_0 \leq t_1 \leq t_2$ and all x^0, x^1,

$$Q\left(x^0, t_0\right) \leq \frac{\gamma^2}{2}\left|\widetilde{x}^0\right|^2 \tag{2.25}$$

$$P\left(x^1, t_1\right) + Q\left(x^1, t_1\right) \geq 0 \tag{2.26}$$

$$Q(x(t), t)]_{t_1}^{t_2} \leq \int_{t_1}^{t_2} \frac{\gamma^2}{2}|w-d|^2 + \frac{\gamma^2}{2}|v|^2 - \frac{1}{2}|u-k|^2\, dt \tag{2.27}$$

for all $t_0 \leq t_1 \leq t_2$ and $x^0, w(t), v(t)$ consistent with $y(t)$.

(Note $P(x,t)$ need not satisfy (2.16).)

Proof: Since $P(x,t)$ satisfies (2.15) with equality, it satisfies (2.22) with equality and therefore for any $x^0, w(t), u(t)$

$$P(x(t),t)]_{t_1}^{t_2}$$

$$= \int_{t_1}^{t_2} \frac{\gamma^2}{2}|w|^2 - \frac{1}{2}|h|^2 - \frac{1}{2}|u|^2 - \frac{\gamma^2}{2}|w-d|^2 + \frac{1}{2}|u-k|^2\, dt. \qquad (2.28)$$

In particular (2.28) holds for $u(t)$ given by the compensator (2.8).

If there exists $Q(x,t)$ satisfying (2.25, 2.26, 2.27) then it is straightforward to verify that $S(x,t) = P(x,t) + Q(x,t)$ satisfies (2.10, 2.11, 2.12) and so (2.7) follows from Theorem 2.1.

On the other hand if (2.7) holds then, by Theorem 2.1, there exists $S(x,t)$ satisfying (2.10, 2.11, 2.12). Define $Q(x,t) = S(x,t) - P(x,t)$ then (2.25, 2.26, 2.27) hold.

Q.E.D.

The next theorem gives sufficient conditions for existence of a measurement feedback compensator achieving a L^2 gain $\leq \gamma$ and suggests an explicit method for constructing an infinite dimensional compensator.

Theorem 2.5 *Suppose*

(i) there exists a smooth $P(x,t)$ satisfying (2.15) and ($k(x,t), d(x,t)$ are defined by (2.20,2.21),

(ii) there exists a smooth $Q(x,t)$ satisfying (2.25) with unique minimum at $\widehat{x}(t)$

$$\widehat{x}(t) = \arg\min_x Q(x,t) \qquad (2.29)$$

and satisfying the partial differential inequality

$$Q_t + Q_x(a + b\widehat{u} + gd) + \frac{1}{2\gamma^2} Q_x g g' Q_x'$$

$$- \frac{\gamma^2}{2}|y-c|^2 + \frac{1}{2}|k-\widehat{u}|^2 \leq 0 \qquad (2.30)$$

where $y = y(t)$ is the observation and $\widehat{u} = \widehat{u}(t)$ is given by

$$\widehat{u}(t) = k(\widehat{x}(t), t), \qquad (2.31)$$

(iii)

$$P(x,t) + Q(x,t) \geq 0.$$

Then the compensator defined by (2.29-2.31) achieves an L^2 gain $\leq \gamma$ in the sense of (2.7).

Proof: Define $S(x,t) = P(x,t) + Q(x,t)$ then $S(x,t)$ clearly satisfies (2.10, 2.11). The integral forms of (2.19) (more precisely (2.22)) and (2.30) yield (2.12).

Q.E.D.

Remarks:

(i) While $P(x,t)$ may be computed off-line, the computation of $Q(x,t)$ requires $y(t)$ and so must be done on-line. Hence the compensator is infinite dimensional with state $Q(\cdot,t)$ at time t.

(ii) As we shall discuss in Section 4.5, $\widehat{x}(t)$ can be thought of as a worst case (H_∞) estimate of $x(t)$ generalizing the maximum likelihood and minimum energy estimates of Mortenson [12] and Hijab [4]. Following [4] we can derive a differential equation for $\widehat{x}(t)$ when $Q(x,t)$ satisfies (2.30) with equality. By differentiating the relation

$$Q_x(\widehat{x}(t),t) = 0$$

with respect to t and differentiating (2.30) with respect to x and evaluating at $\widehat{x}(t)$ we obtain

$$\begin{aligned}\dot{\widehat{x}} = {} & a(\widehat{x},t) + b(\widehat{x},t)\,k(\widehat{x},t) + g(\widehat{x},t)\,d(\widehat{x},t) \\ & + \gamma^2 Q_{xx}^{-1}(\widehat{x}(t),t)\,c_x'(\widehat{x},t)\,(y - c(\widehat{x},t))\end{aligned} \tag{2.32}$$

which is similar to (108) of [15]. By differentiating (2.30) twice with respect to x and evaluating at $\widehat{x}(t)$ we obtain an ordinary differential equation for $Q_{xx}^{-1}(\widehat{x}(t),t)$ that is driven by $y(t)$ and $Q_{xxx}^{-1}(\widehat{x}(t),t)$. By continuing to differentiate (2.30)with respect to x and evaluating at $\widehat{x}(t)$, one obtains an infinte sequence of coupled ODE's driven by $y(t)$ that is formally equivalent to the PDE (2.30). As in extended Kalman filtering, these ODE's can be truncated at degree two by assuming that $Q(x,t)$ is approximately of the form

$$q(t) + \frac{1}{2}(x - \widehat{x}(t))'\,Q(t)\,(x - \widehat{x}(t))$$

where $\widehat{x}(t)$ satisfies (2.32) and $Q(t)$ is a matrix approximating $Q_{xx}^{-1}(\widehat{x}(t),t)$ obtained by linearizing the system around the trajectory $\widehat{x}(t)$. Then $Q(t)$ satisfies the Riccati differential equation

$$\dot{Q} + \overline{A}'Q + Q\overline{A} + \frac{1}{\gamma^2}QGG'Q - \gamma^2 C'C + K'K = 0$$

where A, C, K, D are the Jacobians with respect to x of a, c, k, d evaluated at $\widehat{x}(t)$, $\overline{A} = A + GD$ and $G(t) = g(\widehat{x}(t),t)$

(iii) Formula (2.31) is called a certainty equivalence controller because it is the controller we would use if we were certain that $\widehat{x}(t) = x(t)$ asuming that the state feedback (2.20) achieves an L^2 gain $\leq \gamma$.

The certainty equivalence principle asserts that if there exists a measurement feedback controller achieving an L^2 gain $\leq \gamma$ then there exist a state feedback controller and a corresponding certainty equivalence controller that also achieve an L^2 gain $\leq \gamma$. But Theorem 2.5 only give sufficient conditions for the existence of a measurement feedback, certainty equivalence controller achieving an L^2 gain $\leq \gamma$.

The certainty equivalence principle holds in the linear case [2], [13] but it is not known whether the certainty equivalence principle holds in the general nonlinear setting. But the nonlinear certainty equivalence control (2.31) is motivated by the fact that it maximizes

$$\frac{d}{dt}Q\left(\widehat{x}\left(t\right),t\right)=\frac{\gamma^2}{2}\left|y\left(t\right)-c\left(\widehat{x}\left(t\right),t\right)\right|^2-\frac{1}{2}\left|k\left(\widehat{x}\left(t\right),t\right)-u\left(t\right)\right|^2. \quad (2.33)$$

(iv) A function $P(x,t)$ satisfying (2.19) is a potential Lyapunov function for the system (2.1) under the state feedback (2.20) if the driving noise $w(t)=0$ because

$$\frac{d}{dt}P\left(x\left(t\right),t\right)\leq-\frac{\gamma^2}{2}\left|d\left(x\left(t\right),t\right)\right|^2$$

$$-\frac{1}{2}\left|\begin{array}{c} h\left(x\left(t\right),t\right) \\ k\left(x\left(t\right),t\right)\end{array}\right|^2.$$

If $a(0,t)=0$, $h(0,t)=0$,

$$\alpha_1\left(|x|\right)\leq P\left(x,t\right)\leq\alpha_2\left(|x|\right) \quad (2.34)$$

where α_i are functions of class K_∞ and the system is zero state detectable through the output $h(x(t),t)$ with zero inputs then the system under state feedback is asymptotically stable to $x=0$. Recall a real valued function is of class K_∞ if it continuous, monotone increasing, $\alpha_i(0)=0$ and $\alpha_i(s)\rightarrow\infty$ as $s\rightarrow\infty$. A state x^0 is detectable with respect to the output $h(x(t),t)$ with zero inputs if $\left|x^0(t)-x^1(t)\right|\rightarrow 0$ whenever $h\left(x^0(t),t\right)=h\left(x^1(t),t\right)$ where $x^i(t)$ is the trajectory satisfying $x^i(t_0)=x^i$ with zero inputs.

If $Q(x,t)$ satisfies the inequality (2.30) with equality at $x=\widehat{x}(t)$ then

$$\widetilde{Q}\left(x,t\right)=Q\left(x,t\right)-Q\left(\widehat{x}\left(t\right),t\right)$$

is a potential Lyapunov function for state estimation problem under the measurement feedback $\widehat{u}(t)=k(\widehat{x}(t),t)$, the worst case driving noise $w(t)=d(x(t),t)$ and zero observation noise $v(t)=0$ because

$$\frac{d}{dt}\widetilde{Q}\left(x\left(t\right),t\right)\leq-\frac{1}{2\gamma^2}Q_xgg'Q_x'$$

$$-\frac{1}{2}\left|\widehat{u}(t)-k(x(t),t)\right|^2-\frac{\gamma^2}{2}\left|y(t)-c(\widehat{x}(t),t)\right|^2.$$

If $\widetilde{Q}(x,t)$ is uniformly bounded below and above by functions of class K_∞ as in (2.34) then $\left|\widehat{u}(t)-k(x(t),t)\right|\rightarrow 0$. If every state is detectable through $c(x(t),t)$ with inputs (2.20) and (2.21) then $\left|\widehat{x}(t)-x(t)\right|\rightarrow 0$.
Finally we note that

$$\widetilde{S}(x,t)=P(x,t)+\widetilde{Q}(x,t)$$

is a potential Lyapunov function for the system under measurement feedback $\widehat{u}(t)=k(\widehat{x}(t),t)$ with $w(t)=0$ and $v(t)=0$ because

$$\frac{d}{dt}\widetilde{S}(x(t),t)\leq-\frac{1}{2}\left|\begin{array}{c} h(x(t),t) \\ k(\widehat{x}(t),t)\end{array}\right|^2.$$

$$-\frac{\gamma^2}{2}\left|y(t)-c(\widehat{x}(t),t)\right|^2.$$

If $a(0,t)=0$, $h(0,t)=0$, $\widetilde{S}(x,t)$ is uniformly bounded below and above by functions of class K_∞ as in (2.34) and the system is zero state detectable through the output $h(x(t),t)$ with zero inputs then $|x(t)|\rightarrow 0$. If the system is also zero state detectable through the output $c(x(t),t)$ with zero inputs then $|\widehat{x}(t)|\rightarrow 0$.

3 Linear Worst Case Control

We study the implications of Theorem 2.5 for linear time varying systems of the form

$$\dot{x}=A(t)x+B(t)u+G(t)w \tag{3.1}$$

$$y=C(t)x+v \tag{3.2}$$

$$z=\left[\begin{array}{c} H(t)x \\ u\end{array}\right] \tag{3.3}$$

$$x(t_0)=x^0. \tag{3.4}$$

As before the goal is to find a measurement feedback compensator that achieves an L^2 gain less than or equal to γ in the sense of (2.7) for the closed loop system. Because (3.1-3.4) is linear and (2.7) is quadratic we expect that, when it exists, the compensator will be linear and the storage functions, S, P and Q will be quadratic in x. Also because of the linear-quadratic structure, the compensator will be finite dimensional, optimal in a game theoretic sense and each of the supply functions $P(x,t), Q(x,t)$ will be nonnegative. We shall derive only sufficient conditions based on Theorem 2.5 but similar conditions are already known to be both necessary and sufficient. See for example, Başar-Bernhard [2] and Tadmor [13].

Theorem 3.1 *Suppose*

(i) there exists a smooth

$$P(x,t) = \frac{1}{2} x' P(t) x$$

(please excuse the abuse of notation, $P(x,t)$ is a scalar function, while $P(t)$ is $n \times n$ matrix) such that

$$P(t_0) \leq \gamma^2 I \tag{3.5}$$

$$P(t) \geq 0 \qquad \text{for } t \geq t_0 \tag{3.6}$$

$$\dot{P} + A'P + PA + P\left(\frac{1}{\gamma^2} GG' - BB'\right) P + H'H \leq 0 \tag{3.7}$$

$$K = -B'P \tag{3.8}$$

$$D = \frac{1}{\gamma^2} G'P \tag{3.9}$$

(ii) there exists a smooth

$$Q(x,t) = q(t) + \frac{1}{2}(x - \widehat{x}(t))' Q(t)(x - \widehat{x}(t))$$

($Q(x,t)$ is a scalar function while $Q(t)$ is a $n \times n$ matrix function) such that

$$q(t_0) = 0, \quad Q(t_0) \leq \gamma^2 I \tag{3.10}$$

$$q(t) \geq 0, \quad Q(t) > 0 \qquad \text{for } t \geq t_0 \tag{3.11}$$

$$\dot{q} = \frac{\gamma^2}{2} |y - C\widehat{x}|^2 \tag{3.12}$$

$$\dot{\widehat{x}} = \left(\overline{A} + BK\right) \widehat{x} + \gamma^2 Q^{-1} C' (y - C\widehat{x}) \tag{3.13}$$

$$\dot{Q} + \overline{A}' Q + Q\overline{A} + \frac{1}{\gamma^2} QGG'Q - \gamma^2 C'C + K'K \leq 0 \tag{3.14}$$

where

$$\overline{A} = A + GD \tag{3.15}$$

then the finite dimensional compensator defined by (3.13, 3.14) and

$$\widehat{u} = K\widehat{x} \tag{3.16}$$

achieves an L^2 gain less than or equal to γ.

The proof is a straightforward specialization of Theorem 2.5 and is omitted.

Remarks:

(i) Notice that $Q(t)$ does not depend on the observation, $y(t)$, and can be computed off-line as in Kalman filtering. The other parts of $Q(x,t)$, namely $q(t)$ and $\widehat{x}(t)$ do depend on the observation $y(t)$. In the linear case, the certainty equivalence control (3.16) is optimal because it maximizes $\dot{q}(t)$ and hence $q(t)$, see (2.33). Since $Q(t)$ does not depend on $y(t)$ and is assumed to be positive definite, the certainty equivalence control maximizes (over $u(t)$) the minimum (over x) of $Q(x,t)$.

(ii) In Theorem 3.1, $P(t)$ is assumed to be nonnegative definite and $Q(t)$ is assumed to be positive definite while in Theorem 2.5, only the sum $S(x,t) = P(x,t) + Q(x,t)$ is required to be nonnegative. The discrepancy is explained as follows. In (2.9) we assumed that $Q(x,t)$ had a unique minimum in x for each t. In the linear quadratic context of Theorem 3.1 this implies that $Q(t)$ must be positive definite. Moreover in this context we have

$$S(x,t) = \frac{1}{2}x'P(t)\,x + q(t) + \frac{1}{2}\left(x - \widehat{x}(t)\right)' Q(t)\left(x - \widehat{x}(t)\right).$$

Now suppose $y(t) = C(t)\,\widehat{x}(t)$ then $q(t) = 0$ and $\widehat{x}(t)$ satisfies

$$\dot{\widehat{x}} = \left(\overline{A} + BK\right)\widehat{x}$$

$$\widehat{x}(0) = \widehat{x}^0.$$

It follows immediately that if $S(x,t) \geq 0$ for all x,t and $\widehat{x}^0$ then both $P(t)$ and $Q(t)$ must be nonnegative definite. Neither depend on $y(t)$ or $\widehat{x}^0$.

4 Extensions and Generalizations

EXTENSIONS AND GENERALIZATIONS OF THE STANDARD WORST CASE CONTROL PROBLEM

In this section we shall discuss several extensions and modifications of the standard worst case control problem treated in Section 2. Some of these can be handled by more or less straightforward modifications of the previous results while others lead to open research topics.

4.1 Finite time interval

We consider a system on a finite time interval $[t_0, t_f]$ and seek a causal compensator that achieves an L^2 gain less than or equal to γ for the mapping

$$\begin{bmatrix} x^0 \\ \widetilde{x}^0 \\ w(t) \\ v(t) \end{bmatrix} \longmapsto \begin{bmatrix} x(t_f) \\ \widetilde{x}(t_f) \\ z(t) \end{bmatrix}. \tag{4.1.1}$$

In other words we seek a compensator that processes the initial state estimate $\widehat{x}^0$ and the past observations $y(\tau)$, $t_0 \leq \tau \leq t$ to obtain a control action $u(t)$ and final state estimate $\widehat{x}(t_f)$ so that for all $x^0, w(t), v(t)$

$$\begin{aligned} &|x(t_f)|^2 + |\widetilde{x}(t_f)|^2 + \int_{t_0}^{t_f} |z(t)|^2 \, dt \\ \leq \gamma^2 &\left[\left|x^0\right|^2 + \left|\widetilde{x}^0\right|^2 + \int_{t_0}^{t_f} |w(t)|^2 + |v(t)|^2 \, dt \right]. \end{aligned} \tag{4.1.2}$$

We omit the proofs of the following theorems as they are slight modifications of those in Section 2. This problem has also been considered by Lu [11].

Theorem 4.1 *A causal compensator*

$$\begin{bmatrix} \widehat{x}^0 \\ y(\tau) \end{bmatrix} \longmapsto \begin{bmatrix} \widehat{x}(t_f) \\ u(t) \end{bmatrix} \qquad t_0 \leq \tau \leq t \leq t_f \tag{4.1.3}$$

achieves an L^2 gain $\leq \gamma$ on $[t_0, t_f]$ iff there exists a causal mapping

$$\begin{bmatrix} \widehat{x}^0 \\ y(\tau) \end{bmatrix} \longmapsto S(x, t) \qquad t_0 \leq \tau \leq t \leq t_f \tag{4.1.4}$$

such that for $t_0 \leq t_1 \leq t_2 \leq t_f$ and all x^0, x^f

$$S\left(x^0, t_0\right) \leq \frac{\gamma^2}{2}\left(\left|x^0\right|^2 + \left|\widetilde{x}^0\right|^2\right) \tag{4.1.5}$$

$$S\left(x^f, t_f\right) \geq \frac{1}{2}\left(\left|x^f\right|^2 + \left|\widetilde{x}^f\right|^2\right) \tag{4.1.6}$$

$$S(x(t), t)]_{t_1}^{t_2} \leq \int_{t_1}^{t_2} \frac{\gamma^2}{2} \left| \begin{array}{c} w(t) \\ v(t) \end{array} \right|^2 - \frac{1}{2}|z(t)|^2 \, dt. \tag{4.1.7}$$

If exact state measurements are possible then we seek a state feedback $u = k(x(t), t)$ so that the closed loop system has L^2 gain less than or equal to γ for the mapping

$$\begin{bmatrix} x^0 \\ w(t) \end{bmatrix} \longmapsto \begin{bmatrix} x(t_f) \\ z(t) \end{bmatrix}. \tag{4.1.8}$$

Theorem 4.2 *The state feedback* $u = k(x,t)$ *achieves an* L^2 *gain* $\leq \gamma$ *on* $[t_0, t_f]$ *iff there exists a* $P(x,t)$ *such that for all* $t_0 \leq t_1 \leq t_2 \leq t_f$ *and all* x^0, x^f

$$P\left(x^0, t_0\right) \leq \frac{\gamma^2}{2}\left|x^0\right|^2 \tag{4.1.9}$$

$$P\left(x^f, t_f\right) \geq \frac{1}{2}\left|x^f\right|^2 \tag{4.1.10}$$

$$P\left(x(t), t\right)\Big]_{t_1}^{t_2} \leq \int_{t_1}^{t_2} \frac{\gamma^2}{2}\left|w(t)\right|^2 - \frac{1}{2}\left|z(t)\right|^2 dt. \tag{4.1.11}$$

Theorem 4.3 *Suppose* $P(x,t)$ *is* C^1 *and satisfies (4.1.9) and (4.1.10) and (2.19), then the state feedback (2.20) achieves an* L^2 *gain* $\leq \gamma$ *on* $[t_0, t_f]$ *for the mapping (4.1.8).*

Theorem 4.4 *Suppose there exists a smooth* $P(x,t)$ *satisfying (4.1.9) and (2.19) both with equality and let* $k(x,t), d(x,t)$ *be as in (2.20,2.21). A compensator (4.1.3) achieves an* L^2 *gain* $\leq \gamma$ *for the measurement feedback problem on* $[t_0, t_f]$ *in the sense of (4.1.2) iff there exists a causal conditional storage function (2.24) such that*

$$Q\left(x^0, t_0\right) \leq \frac{\gamma^2}{2}\left|\widetilde{x}^0\right|^2 \tag{4.1.12}$$

$$P\left(x^f, t_f\right) + Q\left(x^f, t_f\right) \geq \frac{1}{2}\left(\left|x^f\right|^2 + \left|\widetilde{x}^f\right|^2\right) \tag{4.1.13}$$

$$Q\left(x(t), t\right)\Big]_{t_1}^{t_2} \leq \int_{t_1}^{t_2} \frac{\gamma^2}{2}\left|w-d\right|^2 + \frac{\gamma^2}{2}\left|v\right|^2 - \frac{1}{2}\left|u-k\right|^2 dt. \tag{4.1.14}$$

Note: In (4.1.13) $\widetilde{x}^f = x(t_f) - \widehat{x}(t_f)$ where $\widehat{x}(t_f)$ is given by (4.1.3) and need not be the arg min of $Q\left(x^f, t_f\right)$.

Theorem 4.5 *Suppose for* $t \in [t_0, t_f]$

(i) there exists a smooth $P(x,t)$ *satisfying (4.1.9) and (2.19) and* $k(x,t), d(x,t)$ *are defined by (2.20,2.21),*

(ii) there exists a smooth $Q(x,t)$ *satisfying (4.1.12) with unique minimum at* $\widehat{x}(t)$ *(2.29) and satisfying (2.30) with* $\widehat{u}(t)$ *given by (2.31),*

(iii) P *and* Q *satisfy (4.1.13).*

Then the compensator (2.29-2.31) achieves an L^2 *gain* $\leq \gamma$ *on* $[t_0, t_f]$ *in the sense of (4.1.2).*

4.2 Autonomous systems on infinite time interval

Consider an autonomous version of the nonlinear system (2.1-2.4) such that $a(x), c(x)$ and $h(x)$ are all zero at $x = 0$. Assume that for each pair of noises $w(t), v(t)$, there exists a t_0 such that $w(t) = 0, v(t) = 0, x(t) = 0, \widehat{x}(t) = 0$ for $t \leq t_0$, and so $y(t) = 0, z(t) = 0$ for $t \leq t_0$.

We seek a causal compensator

$$y(\tau) \mapsto u(t), \qquad t_0 \leq \tau \leq t \tag{4.2.1}$$

which achieves an L^2 gain $\leq \gamma$

$$\int_{t_0}^{t_1} |z(t)|^2 \, dt \leq \gamma^2 \int_{t_0}^{t_1} \left| \begin{array}{c} w(t) \\ v(t) \end{array} \right|^2 dt \tag{4.2.2}$$

on any interval $[t_0, t_1]$ over all pairs of noises $w(t), v(t)$ with support bounded below by t_0.

Theorem 4.6 *A causal compensator (4.2.1) achieves an L^2 gain $\leq \gamma$ for an autonomous version of (2.1-2.4) on $(-\infty, \infty)$ iff there exists a causal conditional storage*

$$y(\tau) \mapsto S(x, t), \qquad t_0 \leq \tau \leq t \tag{4.2.3}$$

such that for all $t_0 \leq t_1 \leq t_2$ and all x^1

$$S(0, t_0) = 0 \tag{4.2.4}$$

$$S\left(x^1, t_1\right) \geq 0 \tag{4.2.5}$$

$$S(x(t), t)]_{t_1}^{t_2} \leq \int_{t_1}^{t_2} \frac{\gamma^2}{2} \left| \begin{array}{c} w(t) \\ v(t) \end{array} \right|^2 - \frac{1}{2} |z(t)|^2 \, dt. \tag{4.2.6}$$

Proof: Suppose there exists S satisfying (4.2.4,4.2.5) then for any noises $w(t), v(t)$ and trajectory $x(t)$, whose support is bounded below by t_0 we have

$$\begin{aligned} \frac{1}{2} \int_{t_0}^{t_1} |z(t)|^2 \, dt &\leq S(x(t_1), t_1) + \frac{\gamma^2}{2} \int_{t_0}^{t_1} \left| \begin{array}{c} w(t) \\ v(t) \end{array} \right|^2 dt \\ &\leq S(x(t_0), t_0) + \frac{\gamma^2}{2} \int_{t_0}^{t_1} \left| \begin{array}{c} w(t) \\ v(t) \end{array} \right|^2 dt \\ &= \frac{\gamma^2}{2} \int_{0}^{t_1} \left| \begin{array}{c} w(t) \\ v(t) \end{array} \right|^2 dt. \end{aligned}$$

On the other hand suppose, there exists a causal compensator (4.2.1) which achieves an L^2 gain $\leq \gamma$. That is, for each $y(t)$ with support bounded below by some t_0 there exists a causal $u(t)$ also with support bounded below by t_0 such that for any $t_1 \geq t_0$ and any $w(t), v(t)$ with support bounded below by t_0 and compatible with $y(t)$

$$0 \leq \int_{t_0}^{t_1} \frac{\gamma^2}{2} \left| \begin{array}{c} w(t) \\ v(t) \end{array} \right|^2 - \frac{1}{2} |z(t)|^2 \, dt \tag{4.2.7}$$

along the trajectories of the autonomous system and compensator starting from $x(t_0) = \widehat{x}(t_0) = 0$. For each measurement history $y(t)$, define $S(x^1, t_1)$ to be the infimum of (4.2.7) over all $w(t), v(t)$ compatible with $y(t)$, etc. It is straightforward to verify that $S(x,t)$ satisfies (4.2.4, 4.2.5, 4.2.6).

Q.E.D.

If the state of the system is exactly measurable then it may be possible to achieve an L^2 gain for the mapping

$$w(t) \longmapsto z(t) \tag{4.2.8}$$

by state feedback. Necessary and sufficient conditions for these to be possible follow immediately from the work of Willems [17] as extended by van der Schaft [14],-[15] and Isidori Astolfi [6].

Theorem 4.7 *(Willems [17]) The autonomous state feedback $u = k(x)$ achieves an L^2 gain $\leq \gamma$ for an autonomous version of (2.1-2.4) on $(-\infty, \infty)$ iff there exists a $P(x)$ such that*

$$P(0) = 0 \tag{4.2.9}$$

$$P(x) \geq 0 \tag{4.2.10}$$

$$P(x(t))]_{t_1}^{t_2} \leq \int_{t_1}^{t_2} \frac{\gamma^2}{2} |w(t)|^2 - \frac{1}{2} |z(t)|^2 \, dt. \tag{4.2.11}$$

Proof: If there exists a $P(x)$ satisfying (4.2.9-4.2.11) then clearly $u = k(x)$ achieves an L^2 gain $\leq \gamma$. Define

$$P(x) = \inf \int_{t_0}^{t_1} \frac{\gamma^2}{2} |w(t)|^2 - \frac{1}{2} |z(t)|^2 \, dt. \tag{4.2.12}$$

where the infimum is taken over all t_0 and all $w(t)$ with support bounded below by t_0 and all $t_1 \geq t_0$ such that $x(t_1) = x$. If no such $w(t), t_0, t_1$ exist then $P(x) = \infty$. It is straightforward to verify that P satisfies (4.2.9-4.2.11).

Q.E.D.

Remarks: Willems calls the function $P(x)$ defined by (4.2.12) the required supply. An autonomous version of (2.1-2.4) is said to be reachable from zero if for every x there exists a $w(t)$ and t_1 such that $x(t_0) = 0$ and $x(t_1) = x$. The function $P(x)$ defined by (4.2.12) is reachable iff $P(x) < \infty$, [17].

Theorem 4.8 *[14], [6]. Suppose $P(x)$ is C^1 and satisfies (4.2.9, 4.2.10) and*

$$P_x a + \frac{1}{2} P_x \left(\frac{1}{\gamma^2} g g' - b b' \right) P_x' + \frac{1}{2} |h|^2 \leq 0 \tag{4.2.13}$$

then

$$u = k(x) = -b'(x) P_x'(x) \tag{4.2.14}$$

achieves L^2 gain $\leq \gamma$ for the autonomous version of (2.1-2.4) on $(-\infty, \infty)$. The "worst case" driving noise is

$$w = d(x) = \frac{1}{\gamma^2} g'(x) P_x'(x). \tag{4.2.15}$$

The proof is omitted as it is very similar to that of Theorem 2.3. The solvability of the partial differential inequality (4.2.13) and the corresponding equality are discussed by van der Schaft [15].

Theorem 4.9 *Suppose there exists a smooth $P(x)$ satisfying (4.2.9, 4.2.13) both with equality and let $k(x), d(x)$ be defined by (4.2.14-4.2.15). A measurement feedback compensator (4.2.1) achieves an L^2 gain $\leq \gamma$ for an autonomous version of (2.1-2.2) on $(-\infty, \infty)$ iff there exists a causal mapping*

$$y(\tau) \mapsto Q(x,t) \qquad t_0 \leq \tau \leq t \tag{4.2.16}$$

such that for all $t_0 \leq t_1 \leq t_2$, all $xw(t), v(t)$ with support bounded below by t_0

$$Q(0, t_0) = 0 \tag{4.2.17}$$

$$P(x) + Q(x,t) \geq 0 \tag{4.2.18}$$

$$Q(x(t), t)]_{t_1}^{t_2} \leq \int_{t_1}^{t_2} \frac{\gamma^2}{2} |w - d|^2 + \frac{\gamma^2}{2} |v|^2 - \frac{1}{2} |u - k|^2 \, dt. \tag{4.2.19}$$

Again the proof is omitted as it follows closely the proof of Theorem 2.4.

Theorem 4.10 *Suppose*

(i) there exists a smooth storage function $P(x)$ satisfying (4.2.9, 4.2.13) and let $k(x)$ and $d(x)$ be defined by (4.2.14,4.2.15),

(ii) there exists a smooth conditional storage function $Q(x,t)$ satisfying (4.2.17) for all $w(t), v(t)$ with support bounded below by t_0 with a unique minimum at $\widehat{x}(t)$ (2.29) and satisfying the partial differential inequality (2.30) with control given by (2.31),

(iii)

$$P(x) + Q(x,t) \geq 0.$$

Then the compensator defined by (2.29-2.31) achieves an L^2 gain $\leq \gamma$ for an autonomous version of (2.1-2.4) on $(-\infty, \infty)$.

The proof is omitted as it follows closely the proof of Theorem 2.5.

Remarks: The compensator defined by (2.29-2.30) is infinite dimensional and autonomous. Since the system (2.1-2.4be autonomous and k, d are autonomous, the partial differential inequality (2.30) is autonomous except for $Q(x,t)$, $y(t)$ and $\widehat{u}(t)$. It is an autonomous infinite dimensional compensator. Further research is needed on finite dimensional compensators which approximate this infinite dimensional compensator.

4.3 More general systems

Consider a nonlinear system of the form

$$\dot{x} = a(x, t, u, w) \tag{4.3.1}$$

$$y = c(x, t, u, w) \tag{4.3.2}$$

$$z = h(x, t, u, w) \tag{4.3.3}$$

$$x(t_0) = x^0. \tag{4.3.4}$$

We present the generalization of the theorems of Section 2, drawing on work of Başar-Bernhard [2] and Isidori-Kang [7]. Previously we considered quadratic supply rates of the form

$$s(x, t, u, w, v) = \frac{\gamma^2}{2} \left| \begin{array}{c} w \\ v \end{array} \right|^2 - \frac{1}{2} \left| \begin{array}{c} h(x,t) \\ u \end{array} \right|^2$$

but in this more general context we allow any function $s(x, t, u, w)$. We are primarily interested in supply rates which are concave in u, convex in w and satisfy several additional conditions which shall be apparent in the later development.

Following Willems [17], a state feedback

$$u = k(x, t) \tag{4.3.5}$$

is dissipative on $[t_0, \infty)$ with respect to supply rate $s(x, t, u, w)$ and initial storage $P^0(x^0)$ if for every $t_1 \geq t_0$ and every $x^0, u(t), w(t)$

$$0 \leq P^0(x^0) + \int_{t_0}^{t_1} s(x(t), t, u(t), w(t))\, dt. \tag{4.3.6}$$

A causal, measurement feedback compensator

$$\left[\begin{array}{c} \widehat{x}^0 \\ y(\tau) \end{array} \right] \mapsto u(t) \qquad t_0 \leq \tau \leq t \tag{4.3.7}$$

is dissipative on $[t_0,\infty]$ with respect to supply rate $s(x,t,u,w)$ and initial conditional storage $S^0\left(x^0,\widehat{x}^0\right)$ if for every $t_1\ge t_0$ and every $x^0,u(t),w(t)$

$$0\le S^0\left(x^0,\widehat{x}^0\right)+\int_{t_0}^{t_1}s\left(x(t),t,u(t),w(t)\right)dt. \tag{4.3.8}$$

The following theorems are presented without proof as they are very similar to those of Section 2.

Theorem 4.11 *A measurement feedback compensator (4.3.7) is dissipative on $[t_0,\infty)$ with respect to supply rate $s(x,t,u,w)$ and initial conditional storage $S^0\left(x^0,\widehat{x}^0\right)$ iff there exists a causal conditional storage*

$$\begin{bmatrix}\widehat{x}^0\\ y(\tau)\end{bmatrix}\mapsto S(x,t)\qquad t_0\le\tau\le t \tag{4.3.9}$$

such that for all $t_0\le t_1\le t_2$, and $x^0,x^1,w(t)$ consistent with the observations $y(t)$

$$S\left(x^0,t_0\right)\le S^0\left(x^0,\widehat{x}^0\right) \tag{4.3.10}$$

$$S\left(x^1,t_1\right)\ge 0 \tag{4.3.11}$$

$$S\left(x(t),t\right)\Big]_{t_1}^{t_2}\le\int_{t_1}^{t_2}s\left(x(t),t,u(t),w(t)\right)dt. \tag{4.3.12}$$

Theorem 4.12 *The state feedback (4.3.5) is dissipative on $[t_0,\infty)$ with respect to the supply rate $s(x,t,u,w)$ and the initial supply $P^0\left(x^0\right)$ iff there exists a storage function $P(x,t)$ such that for all $t_0\le t_1\le t_2$, all $x^0,x^1,w(t)$*

$$P\left(x^0,t_0\right)\le P^0\left(x^0\right) \tag{4.3.13}$$

$$P\left(x^1,t_1\right)\ge 0 \tag{4.3.14}$$

$$P\left(x(t),t\right)\Big]_{t_1}^{t_2}\le\int_{t_1}^{t_2}s\left(x(t),t,u(t),w(t)\right)dt. \tag{4.3.15}$$

Theorem 4.13 *Suppose $P(x,t)$ is C^1 and satisfies (4.3.13, 4.3.14) and suppose*

$$u=k(x,t) \tag{4.3.16}$$

$$w=d(x,t) \tag{4.3.17}$$

satisfy for each $x,t\ge t_0$

$$0\ge\inf_u\sup_w\left(P_t+P_xa-s\right)=\left(P_t+P_xa-s\right)\Big|_{\substack{u=k(x,t)\\ w=d(x,t)}}. \tag{4.3.18}$$

Then the state feedback (4.3.16) is dissipative on $[t_0,\infty)$ with respect to supply rate $s(x,t,u,w)$ and initial storage $P^0\left(x^0\right)$.

Theorem 4.14 *Suppose there exists a smooth $P(x,t)$ satisfying (4.3.13) and (4.3.18) with equality for $k(x,t), d(x,t)$ defined by (4.3.16-4.3.17). A measurement feedback compensator (4.3.7) is dissipative on $[t_0,\infty)$ with respect to supply rate $s(x,t,u,w)$ and initial conditional storage*

$$S^0\left(x^0,\widehat{x}^0\right) = P^0\left(x^0\right) + Q^0\left(\widetilde{x}^0\right) \tag{4.3.19}$$

iff there exists a causal conditional storage

$$\begin{bmatrix} \widehat{x}^0 \\ y(\tau) \end{bmatrix} \mapsto Q(x,t) \qquad t_0 \le \tau \le t$$

such that for and $t_0 \le t_1 \le t_2$, all $x^0, x^1, w(t)$ consistent with the observations $y(t)$

$$Q\left(x^0,t_0\right) \le Q^0\left(\widetilde{x}^0\right) \tag{4.3.20}$$

$$P\left(x^1,t_1\right) + Q\left(x^1,t_1\right) \ge 0 \tag{4.3.21}$$

$$\left. Q(x(t),t)\right]_{t_1}^{t_2} \le \int_{t_1}^{t_2} \widetilde{s}(x(t),t,u(t),w(t))\,dt \tag{4.3.22}$$

where

$$\widetilde{s}(x,t,u,w) = s(x,t,u,w) - s(x,t,k(x,t),d(x,t)). \tag{4.3.23}$$

Theorem 4.15 *Suppose*

(i) there exists smooth $P(x,t)$ and $k(x,t),\ d(x,t)$ satisfying (4.3.13) and (4.3.18),

(ii) there exists a smooth $Q(x,t)$ satisfying (4.3.20) with unique minimum at $\widehat{x}(t)$

$$\widehat{x}(t) = \arg\min_x Q(x,t) \tag{4.3.24}$$

and satisfying the partial differential inequality

$$\inf_w \left(Q_t + Q_x a - \widetilde{s}\right)\Big|_{u=\widehat{u}(t)} \le 0 \tag{4.3.25}$$

where the infimum is overall $w(t)$ consistent with the observations $y(t)$ and

$$\widehat{u}(t) = k(\widehat{x}(t),t), \tag{4.3.26}$$

(iii)

$$P(x,t) + Q(x,t) \ge 0.$$

Then the infinite dimensional compensator (4.3.24-4.3.26) is dissipative on $[t_0,\infty)$ with respect to the supply rate $s(x,t,u,w)$ initial conditional storage function $S^0\left(x^0,\widehat{x}^0\right)$, in the sense of (4.3.8).

4.4 Worst case estimation

Embedded in the measurement feedback compensators of the previous sections are worst case estimators similar to those of Mortensen [12] and Hijab [4]. Consider the system

$$\dot{x} = a(x,t) + g(x,t)\,w \tag{4.4.1}$$

$$y = c(x,t) + v \tag{4.4.2}$$

$$u = k(x(t),t) \tag{4.4.3}$$

$$x(t_0) = x^0. \tag{4.4.4}$$

As before $w(t), v(t)$ are driving and observation noises. But $u(t)$ is no longer an input but rather it is an output that is to be estimated from the past measurements, $y(\tau), t_0 \le \tau \le t$ and the initial state estimate $\widehat{x}^0$. Linear versions of this problem have been treated by Khargonekar-Nagpal [10] and Başar-Bernhard [2]. In particular we seek a causal estimator

$$\begin{bmatrix} \widehat{x}^0 \\ y(\tau) \end{bmatrix} \mapsto \widehat{u}(t) \qquad t_0 \le \tau \le t \tag{4.4.5}$$

with an L^2 error gain $\le \gamma$, i.e., for any $t_0 \le t_1$ and any $x^0, w(t), v(t)$ consistent with the observations, $y(t)$

$$\int_{t_0}^{t_1} |k(x(t),t) - \widehat{u}(t)|^2 \le \gamma^2 \left(\left|\widetilde{x}^0\right|^2 + \int_{t_0}^{t_1} \left| \begin{array}{c} w(t) \\ v(t) \end{array} \right|^2 dt \right). \tag{4.4.6}$$

The following two theorems are essentially specializations of Theorems 2.4 and 2.5 and proven in a similar fashion.

Theorem 4.16 *A causal estimator (4.4.5) has an L^2 error gain $\le \gamma$ on $[t_0, \infty)$ iff there exists a causal conditional storage function*

$$\begin{bmatrix} \widehat{x}^0 \\ y(\tau) \end{bmatrix} \mapsto Q(x,t) \qquad t_0 \le \tau \le t \tag{4.4.7}$$

such that for any $t_0 \le t_1 \le t_2$ and any $x^0, w(t), v(t)$ consistent with the observation $y(t)$

$$Q\left(x^0, t_0\right) \le \frac{\gamma^2}{2} \left|\widetilde{x}^0\right|^2 \tag{4.4.8}$$

$$Q\left(x^1, t_1\right) \ge 0 \tag{4.4.9}$$

$$Q(x(t),t)]_{t_1}^{t_2} \le \int_{t_1}^{t_2} \frac{\gamma^2}{2} \left| \begin{array}{c} w(t) \\ v(t) \end{array} \right|^2 - \frac{1}{2} |k(x(t),t) - \widehat{u}(t)|^2 \, dt. \tag{4.4.10}$$

Theorem 4.17 *Suppose there exists a smooth $Q(x,t)$ satisfying (4.4.8,4.4.9) with unique minimum at $\widehat{x}(t)$*

$$\widehat{x}(t) = \arg\min Q(x,t) \tag{4.4.11}$$

and satisfying the partial differential inequality

$$Q_t + Q_x a + \frac{1}{2\gamma^2} Q_x g g' Q_x' - \frac{\gamma^2}{2} |y-c|^2 + \frac{1}{2} |k - \widehat{u}|^2 \leq 0 \tag{4.4.12}$$

where $y = y(t)$ is the observation and $\widehat{u} = \widehat{u}(t)$ is given by

$$\widehat{u}(t) = k(\widehat{x}(t), t) \tag{4.4.13}$$

then the infinite dimensional estimator (4.4.11-4.4.13) achieves an L^2 error gain $\leq \gamma$.

Remarks: As before

$$\widetilde{Q}(x,t) = Q(x,t) - Q(\widehat{x}(t), t)$$

is a potential Lyapunov function for state estimation problem. See Remark (iv) following Theorem (2.5).

5 Conclusions and Questions

We have presented necessary and sufficient conditions for worst case ($H-$infinity suboptimal) compensators and estimators in a variety of settings. The compensators and estimators are generally infinite dimensional. There are several open questions.

Are there nonlinear systems that admit finite dimensional compensators, [16]? Are there effective finite dimensional approximations? The compensators are based on certainty equivalence, are there other kinds of compensators?

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