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The Spaces of Improper Rational Matrices and ARMA-Systems of fixed McMillan Degree**

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Abstract

The space of rational matrices with fixed size and degree is shown to have a manifold structure with fibers over a Grassmannian. The fibers are homeomorphic to a suitable space of strictly proper rational matrices. This structure is compatible with Willems' partition of external variables into inputs and outputs.

Key words: rational matrix, Mcmillan degree, polynomial coprime factorization, realization theory

AMS Subject Classifications: 93B27, 93B25, 93B20, 93C35

1 Introduction

In this paper we study the space of (not necessarily proper) rational matrices with fixed degree. They occur in a variety of contexts in the control theory literature, primarily as transfer matrices of polynomial systems as introduced by Rosenbrock [15] and of singular systems.

It is well-known that an arbitrary rational matrix G has a (McMillan) degree which, roughly speaking, counts the poles of G in the extended complex plane and that each such matrix can be thought of as the transfer function $G(s) = C(sE - A)^{-1}B + D$ of an irreducible singular system of the form $E\dot{x} = Ax + Bu,\ y = Cx + Du$ (see e. g. [17, p. 822], [4, 8, 11, 12]). Thus, one is lead to consider on the one hand a space of rational matrices of fixed degree r, on the other hand a space of realizations whose "size" depends on r, and to investigate relationships among these two spaces. For state space systems, Byrnes/Duncan [3] showed that the realization map establishes a homeomorphism between similarity classes of triples of minimal systems of order r and the space $\mathrm{Rat}^0_{m,r,p}$ of all strictly proper

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matrices of McMillan degree r over the reals or complexes, provided both sets are suitable topologized.

In this work we analyze a few basic properties of the (differential) space of improper rational matrices and, more generally, of ARMA-systems of fixed degree. In a subsequent paper, the second listed author will study a corresponding quotient space of irreducible system realizations and establish a homeomorphism between the two spaces, in analogy to the state space situation. This result, in turn, sets the problem of finding continuous realizations.

We stress that, in the existing literature, spaces of improper rational matrices occur, to the best of our knowledge, as subspaces (or subvarieties) of compactifications of the space $\mathrm{Rat}^0_{m,r,p} \times \mathbb{K}^{p\times m}$ of proper matrices (see e. g. [2, 16]). Such compactifications are obtained by embedding $\mathrm{Rat}^0_{m,r,p}$ in a high dimensional Grassmannian [2] or projective space [16] and then taking Zariski closures. They have been considered in the geometric framework for studying dynamic feedback compensation.

In the third section we will use the space $\mathrm{Rat}^0_{m,r,p}$ to describe the larger space $\mathrm{Rat}_{m,r,p}$ of rational matrices $G \in \mathbb{K}(s)^{p\times m}$ with McMillan degree (i. e. number of poles of G in $\mathbb{C}\cup\{\infty\}$) $d(G) = r$.
On the space

$$\tilde{\Sigma}_{m,r,p} = \{(A,B,C) \in \mathbb{K}^{r^2+rm+pr} \mid (A,B,C)\,\text{minimal}\} \tag{1.1}$$

of minimal state space systems the similarity action $\overset{s}{\sim}$, which is given by $(A,B,C)\overset{s}{\sim}(A',B',C')$ iff $(A',B',C') = (TAT^{-1}, TB, CT^{-1})$ for some $T \in Gl_r(\mathbb{K})$, leads to a quotient space $\Sigma_{m,r,p} = \tilde{\Sigma}_{m,r,p}/_{\overset{s}{\sim}}$, which is bijective to $\mathrm{Rat}^0_{m,r,p}$, the bijection being given by the realization map: $(A,B,C) \mapsto C(sI-A)^{-1}B$. Hazewinkel [9] showed that $\Sigma_{m,r,p}$ can be given the structure of an $r(p+m)$-dimensional manifold (over $\mathbb{K}$), whose underlying topology is just the quotient Euclidian one.

On the other hand Hermann/Martin [13] viewed $\mathrm{Rat}^0_{m,r,p}$ as a space of rational curves on the Grassmannian $\mathcal{G}_m(\mathbb{C}^{p+m})$ of m-dimensional subspaces of $\mathbb{C}^{p\times m}$: via a polynomial coprime factorization $G = PQ^{-1} \in \mathrm{Rat}^0_{m,r,p}$, we associate with $X = \begin{bmatrix} P \\ Q \end{bmatrix}$ the rational map

$$\begin{aligned} \Phi_X : S^2 &\longrightarrow \mathcal{G}_m(\mathbb{C}^{p+m}) \\ s &\longmapsto \begin{cases} \operatorname{span} X(s) & , s \in \mathbb{C} \\ \operatorname{span} \begin{bmatrix} 0 \\ I_m \end{bmatrix} & , s = \infty. \end{cases} \end{aligned} \tag{1.2}$$

Interpreting this way, $\mathrm{Rat}^0_{m,r,p}$ is just the space of all rational maps from the Riemann sphere S^2 to the Grassmannian of degree r (in the algebraic sense) and with the base point condition $\infty \mapsto \operatorname{span}\left[\begin{smallmatrix}0\\I\end{smallmatrix}\right]$. It is shown by Byrnes/Duncan [3], that the realization map induces an homeomorphism from $\Sigma_{m,r,p}$ to $\mathrm{Rat}^0_{m,r,p}$, endowed with the compact-open topology (or topology of uniform convergence, if we look at $\mathcal{G}_m(\mathbb{C}^{p+m})$ as a metric space).

We will follow the Hermann/Martin approach for the analysis of the space $\mathrm{Rat}_{m,r,p}$. The last part of the introduction is devoted to a brief sketch of the paper.

The properties of coprime factorizations for rational matrices show that $\mathrm{Rat}_{m,r,p}$ is included in the space $\mathcal{I}_{m,r,p} = \{\langle X\rangle \mid \mathrm{rk}X(s) = m \text{ for } s \in \mathbb{C},\, \delta(X) = r\}$, where, for $X \in \mathbb{K}[s]^{(p+m)\times m}$, $\langle X\rangle$ denotes the equivalence class $\{XU \mid U \in \mathbb{K}[s]^{m\times m} \text{ unimodular}\}$ and $\delta(X) = \max\{\deg\det\Delta \mid \Delta\ m\times m\text{-submatrix of } X\}$ is a generalization of the McMillan degree. $\mathcal{I}_{m,r,p}$ (or, better, its dual version using $X \in \mathbb{K}[s]^{p\times(p+m)}$) can then be interpreted as a set of equivalence classes of ARMA-systems.

Using a Hermann/Martin map analogous to (1.2), $\mathcal{I}_{m,r,p}$ can be identified with the space of all rational maps with degree r from S^2 to the Grassmannian, the only difference being the base point condition: for $\langle X\rangle \in \mathcal{I}_{m,r,p}$

$$\begin{aligned} \Phi_X(\infty) \quad &= \quad \operatorname{span} \lim\nolimits_{t\to\infty} X\Delta^{-1}(t), \text{ where} \\ &\qquad \Delta \text{ is an } m\times m\text{-submatrix of } X \text{ with } \deg\det\Delta = r. \end{aligned} \tag{1.3}$$

In the case of a strictly proper matrix G this leads to the definition (1.2).

While $\mathrm{Rat}^0_{m,r,p}$ is the subspace of $\mathcal{I}_{m,r,p}$ of all rational maps from S^2 to $\mathcal{G}_m(\mathbb{C}^{p+m})$ of degree r which satisfy the condition $\infty \mapsto \operatorname{span}\left[\begin{smallmatrix}0\\I\end{smallmatrix}\right]$, each curve corresponding to a non-strictly proper element of $\mathrm{Rat}_{m,r,p}$ will attain at ∞ a different value on $\mathcal{G}_m(\mathbb{C}^{p+m})$. In other words, the non-singularity of Q in $\left[\begin{smallmatrix}P\\Q\end{smallmatrix}\right]$ cannot be detected from the point at ∞ on the associated curve. On the other hand, if we fix a point $\omega \in \mathcal{G}_m(\mathbb{C}^{p+m})$, the set of curves Φ_X for which $\Phi_X(\infty) = \omega$ is in bijective correspondence with $\mathrm{Rat}^0_{m,r,p}$. This suggests that $\mathcal{I}_{m,r,p}$ has a fiber structure with fibers homeomorphic to $\mathrm{Rat}^0_{m,r,p}$ (the fiber over $\operatorname{span}\left[\begin{smallmatrix}0\\I\end{smallmatrix}\right] \in \mathcal{G}_m(\mathbb{C}^{p+m})$) and base space the Grassmannian of "values at infinity". This is in fact true if $\mathcal{I}_{m,r,p}$ is endowed with the compact-open topology.

More precisely, if $\langle X\rangle \in \mathcal{I}_{m,r,p}$ is such that $\Phi_X(\infty) = \operatorname{span}\sigma\left[\begin{smallmatrix}Z\\I\end{smallmatrix}\right]$, where σ is a permutation matrix and $Z \in \mathbb{K}^{p\times m}$, it then follows $X = \sigma\left[\begin{smallmatrix}P\\Q\end{smallmatrix}\right]$ with $\deg\det Q = r$, and hence that PQ^{-1} is proper. Thus there exists a permutation of the data (the rows of X) which leads to a proper matrix. This permutation depends on the standard chart of the Grassmannian in which $\Phi_X(\infty)$ can be located. The constant part of the obtained proper matrix describes the σ-coordinates of $\Phi_X(\infty) \in \mathcal{G}_m(\mathbb{C}^{p+m})$, while the strictly proper part selects the point on the fiber. In this sense the space $\mathcal{I}_{m,r,p}$ is the local product of a Grassmannian with $\mathrm{Rat}^0_{m,r,p}$, and thus an analytic manifold.

2 Preliminaries

We begin with some notations: let $\mathbb{K}$ denote the field of real or complex numbers. For $m \in \mathbb{N}$ put
$\underline{m} = \{1, \ldots, m\}$,
$\mathcal{P}(m) = \{P \in \mathbb{K}^{m \times m} \mid P \text{ permutation matrix}\}$,
$\mathcal{U}_m(\mathbb{K}) = \{U \in \mathbb{K}[s]^{m \times m} \mid \det U \equiv c \in \mathbb{K}^*\}$, the unimodular matrices,
$\mathcal{R}_m(\mathbb{K}) = \{Q \in \mathbb{K}[s]^{m \times m} \mid \det Q \not\equiv 0\}$, the non-singular matrices in $\mathbb{K}[s]^{m \times m}$.

For a proper rational matrix $G \in \mathbb{K}(s)^{p \times m}$ the McMillan degree $d(G)$ is defined to be the number of poles of G counting multiplicities in the complex plane $\mathbb{C}$. It can be computed via the Smith-McMillan form of G, which gives even more information, namely the structure at the various poles. In a simpler way the degree can be expressed by $\deg \det Q$, where $PQ^{-1} = G$ is a polynomial coprime factorization of G, $P \in \mathbb{K}[s]^{p \times m}$, $Q \in \mathcal{R}_m(\mathbb{K})$.

For an improper rational matrix G the McMillan degree $d(G)$ is also defined to be the number of poles of G, which, in this case, has to take into account the poles at ∞. It can be computed by partitioning $G = G_- + G_+$ into its strictly proper part G_- and its polynomial part $G_+ \in \mathbb{K}[s]^{p \times m}$ and counting the finite and infinite poles separately. Thus

$$d(G) = d(G_-) + d(\hat{G}_+),$$

where $\hat{G}_+(s) = G_+(s^{-1}) \in \mathbb{K}(s)^{p \times m}$ is proper with poles only at $s = 0$.

The main results on polynomial coprime factorizations can be summarized in the following

Theorem 2.1 *(Polynomial coprime factorizations)*
a) Let $G \in \mathbb{K}(s)^{p \times m}$. Then there exists a polynomial coprime factorization of G, i. e. there are polynomial matrices $P \in \mathbb{K}[s]^{p \times m}$, $Q \in \mathcal{R}_m(\mathbb{K})$ satisfying $G = PQ^{-1}$ and $\mathrm{rk}[P(s)^{\mathrm{t}}, Q(s)^{\mathrm{t}}]^{\mathrm{t}} = m$ for all $s \in \mathbb{C}$.
b) If $G = PQ^{-1} = \hat{P}\hat{Q}^{-1}$ are two coprime factorizations, then there exists $U \in \mathcal{U}_m(\mathbb{K})$ such that $\hat{P} = PU, \hat{Q} = QU$.

For a proof see e.g. [6, Section 2.1] or [5, Theorem 22.11].

As in the proper case it is possible to compute the McMillan degree of G via polynomial coprime factorizations. This can be done in a more general setting.

Definition 2.2 *For* $X \in \mathbb{K}[s]^{(p+m)\times m}$ *with* $\mathrm{rk} X(s) = m$ *for all* $s \in \mathbb{C}$ *put*

$$\delta(X) = \max\{\deg\det\Delta \mid \Delta\ m \times m\text{-submatrix of } X\}.$$

$\delta(X)$ *is called the McMillan degree of* X. *Each* $m \times m$*-submatrix* Δ *of* X *which fulfills* $\deg\det\Delta = \delta(X)$, *is said to be a maximal* m*-submatrix.*

The following lemma establishes the coincidence of the two different notions of McMillan degree. A proof can be found in Janssen [10, Cor. 3.1].

Lemma 2.3 *Let* $\begin{bmatrix} P \\ Q \end{bmatrix} \in \mathbb{K}[s]^{(p+m)\times m}$ *with* $\mathrm{rk}\begin{bmatrix} P(s) \\ Q(s) \end{bmatrix} = m$ *for all* $s \in \mathbb{C}$ *and* $Q \in \mathcal{R}_m(\mathbb{K})$. *Then*

a) PQ^{-1} *is (strictly) proper* $\Longleftrightarrow$ Q *is (the only) maximal* m*-submatrix of* $[P^{\mathrm{t}}, Q^{\mathrm{t}}]^{\mathrm{t}}$.

b) $\delta([P^{\mathrm{t}}, Q^{\mathrm{t}}]^{\mathrm{t}}) = d(PQ^{-1})$, *the McMillan degree of* PQ^{-1}.

In view of the preceding discussion we define

$$\mathrm{Rat}_{m,r,p}(\mathbb{K}) = \left\{ \begin{bmatrix} P \\ Q \end{bmatrix} \in \mathbb{K}[s]^{(p+m)\times m} \mid \mathrm{rk}\begin{bmatrix} P(s) \\ Q(s) \end{bmatrix} = m \text{ for } s \in \mathbb{C}, \right.$$
$$\left. Q \in \mathcal{R}_m(\mathbb{K}),\ \delta\begin{bmatrix} P \\ Q \end{bmatrix} = r \right\} \Big/ \mathcal{U}_m(\mathbb{K}) \tag{2.1}$$

where $\mathcal{U}_m(\mathbb{K})$ acts by multiplication from the right. In the following we will use the name "rational matrix" not only for $G = PQ^{-1} \in \mathbb{K}(s)^{p\times m}$ but also for the equivalence classes $\langle\begin{bmatrix} P \\ Q \end{bmatrix}\rangle$ satisfying $\mathrm{rk}\begin{bmatrix} P(s) \\ Q(s) \end{bmatrix} = m$ for all $s \in \mathbb{C}$ with $Q \in \mathcal{R}_m(\mathbb{K})$, where the lower $m \times m$-block stands for the denominator. In this way we identify the space $\mathrm{Rat}^0_{m,r,p}$ of strictly proper matrices with the subspace $\{\langle\begin{bmatrix} P \\ Q \end{bmatrix}\rangle \in \mathrm{Rat}_{m,r,p} \mid PQ^{-1} \text{ strictly proper}\}$.

Remark 2.4 From [17] it is known, that the McMillan degree of a rational matrix G is the order of a strong irreducible realization of G, i. e. if $G(s) = C(sE - A)^{-1}B + D$ and the singular system (E, A, B, C, D) is strong irreducible (see [17, p. 822]), then $\mathrm{rk}E = d(G)$.

In a future paper the second listed author will construct a quotient space of strong irreducible singular systems (E, A, B, C, D) corresponding bijectively to the space $\mathrm{Rat}_{m,r,p}(\mathbb{K})$, so that the realization map

$$(E, A, B, C, D) \mapsto C(sE - A)^{-1}B + D$$

induces a homeomorphism (the topology of $\mathrm{Rat}_{m,r,p}(\mathbb{K})$ will be defined in the next section). This result generalizes the well-known homeomorphism between $\mathrm{Rat}^0_{m,r,p}(\mathbb{C})$ and the quotient space of minimal state space systems of order r modulo similarity induced by the realization map. Using the local structure of $\mathrm{Rat}_{m,r,p}$, and more generally of ARMA-systems (in the dual version), which we will derive in Theorem 3.5, the study of the generalized transfer function can then be reduced to the state space situation.

The last lemma of this section will be used later. It shows how a rational matrix $G = PQ^{-1}$ is transformed if the rows of $\begin{bmatrix} P \\ Q \end{bmatrix}$ are partitioned in a different way in numerator and denominator. Note that, by Lemma 2.3, G can be transformed to a proper matrix by appropriate choice of the denominator matrix.

Lemma 2.5 *Suppose $G = PQ^{-1} \in \mathbb{K}(s)^{p\times m}$ with $P \in \mathbb{K}[s]^{p\times m}$, $Q \in \mathcal{R}_m(\mathbb{K})$ is a coprime factorization. Let the matrices be partitioned as*

$$P = \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix},\ Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{bmatrix},\ G = \begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix}$$

where P_4, Q_2 and G_3 are $k \times k$-blocks with $1 \le k \le \min\{m,p\}$. Then

$$\det G_3 \not\equiv 0 \Longleftrightarrow \det \begin{bmatrix} P_3 & P_4 \\ Q_3 & Q_4 \end{bmatrix} \not\equiv 0$$

and in this case $\begin{bmatrix} P_1 & P_2 \\ Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} P_3 & P_4 \\ Q_3 & Q_4 \end{bmatrix}^{-1} = \begin{bmatrix} G_1 G_3^{-1} & G_2 - G_1 G_3^{-1} G_4 \\ G_3^{-1} & -G_3^{-1} G_4 \end{bmatrix}$.

Proof: follows directly from $\begin{bmatrix} P_3 & P_4 \\ Q_3 & Q_4 \end{bmatrix} \begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{bmatrix}^{-1} = \begin{bmatrix} G_3 & G_4 \\ 0 & I_{m-k} \end{bmatrix}$ and $\begin{bmatrix} P_1 & P_2 \\ Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{bmatrix}^{-1} = \begin{bmatrix} G_1 & G_2 \\ I_k & 0 \end{bmatrix}$. □

3 The Space of Rational Curves of Fixed Degree

Using equivalence classes of coprime factorizations one can construct easily transformations which map an arbitrary rational matrix into a proper one: take a maximal m-submatrix of $\langle \begin{bmatrix} P \\ Q \end{bmatrix} \rangle$ as denominator and the remaining rows as numerator of the new rational matrix. This construction makes sense also for equivalence classes $\langle \begin{bmatrix} P \\ Q \end{bmatrix} \rangle$, which do not satisfy the regularity condition $\det Q \not\equiv 0$. From a system theoretic point of view the dual version of $\langle \begin{bmatrix} P \\ Q \end{bmatrix} \rangle$, namely $\langle [\tilde{Q}, \tilde{P}] \rangle \subseteq \mathbb{K}[s]^{p\times(p+m)}$ can be understood as an equivalence class of ARMA-systems. In this sense the above described transformation can be interpreted as Willems' partition of the external variables into inputs and outputs (see [20, Theorem 4.3] and also [12]).

Thus, dropping the regularity condition in (2.1), we obtain the larger space

$$\mathcal{I}_{m,r,p}(\mathbb{K}) = \{X \in \mathbb{K}[s]^{(p+m)\times m} \mid \mathrm{rk} X(s) = m \text{ for } s \in \mathbb{C}, \quad \delta(X) = r\}/_{\mathcal{U}_m(\mathbb{K})} \tag{3.1}$$

where again $\mathcal{U}_m(\mathbb{K})$ acts by multiplication from the right, i. e. if $\langle X\rangle$, $\langle Y\rangle$ denote the equivalence classes of X, $Y \in \mathbb{K}[s]^{(p+m)\times m}$ in $\mathcal{I}_{m,r,p}(\mathbb{K})$, then $\langle X\rangle = \langle Y\rangle$ iff $XU = Y$ for some $U \in \mathcal{U}_m(\mathbb{K})$.

Note that the map

$$\begin{array}{ccc} \mathcal{I}_{m,r,p}(\mathbb{R}) & \longrightarrow & \mathcal{I}_{m,r,p}(\mathbb{C}) \\ \langle X\rangle_{\mathbb{R}} & \longmapsto & \langle X\rangle_{\mathbb{C}} \end{array}$$

is injective. This can be seen as follows. From $\langle X\rangle_{\mathbb{C}} = \langle Y\rangle_{\mathbb{C}}$ with real matrices X, Y it follows $XU = Y$ for some $U \in \mathcal{U}_m(\mathbb{C})$. Without loss of generality we can assume $X = [X_1^{\mathrm{t}}, X_2^{\mathrm{t}}]^{\mathrm{t}}$, $Y = [Y_1^{\mathrm{t}}, Y_2^{\mathrm{t}}]^{\mathrm{t}}$ where X_2, $Y_2 \in \mathcal{R}_m(\mathbb{R})$. But then $X_1X_2^{-1} = Y_1Y_2^{-1} \in \mathbb{R}(s)^{p\times m}$ and by the uniqueness of coprime factorizations it follows $XV = Y$ with some $V \in \mathcal{U}_m(\mathbb{R})$. In this way we can identify the spaces $\mathcal{I}_{m,r,p}(\mathbb{R})$ and $\mathrm{Rat}_{m,r,p}(\mathbb{R})$ as subsets of $\mathcal{I}_{m,r,p}(\mathbb{C})$, which allows us to write $\langle X\rangle$ instead of $\langle X\rangle_{\mathbb{R}}$ or $\langle X\rangle_{\mathbb{C}}$.

Along the lines of [13], we will topologize the space $\mathcal{I}_{m,r,p}(\mathbb{C})$ by interpreting its elements as curves. For each $\langle X\rangle \in \mathcal{I}_{m,r,p}(\mathbb{K})$ denote by Φ_X the following map from the Riemannian sphere S^2 to the Grassmannian $\mathcal{G}_m(\mathbb{C}^{p+m})$ of m-dimensional subspaces of $\mathbb{C}^{p+m}$

$$\begin{array}{rcl} \Phi_X : S^2 & \longrightarrow & \mathcal{G}_m(\mathbb{C}^{p+m}) \\ s & \longmapsto & \begin{cases} \mathrm{span}\, X(s) & s \in \mathbb{C} \\ \mathrm{span}\, \lim_{|t|\to\infty} X(t)\Delta(t)^{-1} & s = \infty \\ \text{with a maximal } m\text{-submatrix } \Delta \text{ of } X & \end{cases} \end{array} \tag{3.2}$$

It is easy to show that this map is in fact well-defined, i. e. it depends only on the equivalence class of X in $\mathcal{I}_{m,r,p}(\mathbb{C})$ and is independent of the choice of the (non-unique) maximal m-submatrix Δ. Moreover, it is easy to see that $\Phi_X(\infty)$ is the span of the *high order coefficient matrix* of X, if we choose X as a minimal basis (in the sense of Forney [7, p. 495]).

In the special case $\langle X\rangle = \langle \left[\begin{smallmatrix} P \\ Q \end{smallmatrix}\right]\rangle$ where $Q \in \mathcal{R}_m(\mathbb{K})$ and PQ^{-1} is strictly proper, the map Φ_X fulfills the "base point condition" $\Phi_X(\infty) = \mathrm{span}\,[0, I_m]^{\mathrm{t}}$, which gives the well-known Hermann/Martin-mappings in the case of strictly proper rational matrices.

Via the Plücker-embedding of $\mathcal{G}_m(\mathbb{C}^{p+m})$ in the projective space $\mathbb{CP}^{\binom{p+m}{m}-1}$ the map Φ_X can be interpreted as a rational map from $\mathbb{CP}^1$ to $\mathbb{CP}^{\binom{p+m}{m}-1}$. Then, by definition, the McMillan degree is just the degree of this rational mapping and thus also the topological degree (in second homology). Hence $\mathcal{I}_{m,r,p}(\mathbb{C})$ is just the space of all rational maps between the algebraic varieties S^2 and $\mathcal{G}_m(\mathbb{C}^{p+m})$ of topological degree r, whereas $\mathcal{I}_{m,r,p}(\mathbb{R})$ consists of those curves which commute with the complex conjugation.

Endow $\mathcal{I}_{m,r,p}(\mathbb{C})$ with the compact-open topology, that is, a subbasis for the topology in $\mathcal{I}_{m,r,p}(\mathbb{C})$ is given by the sets $\langle K, U\rangle = \{f \in \mathcal{I}_{m,r,p}(\mathbb{C}) \mid f(K) \subseteq U\}$, where K runs over all compacta in S^2 and U over all open sets in $\mathcal{G}_m(\mathbb{C}^{p+m})$. The subsets $\mathrm{Rat}_{m,r,p}(\mathbb{C})$, $\mathcal{I}_{m,r,p}(\mathbb{R})$ and $\mathrm{Rat}_{m,r,p}(\mathbb{R})$ will be equipped with the induced topologies. Remember that the compact-open topology on this spaces is just the topology of uniform convergence since S^2 is compact and $\mathcal{G}_m(\mathbb{C}^{p+m})$ is a metric space.

Remark 3.1 In a recent paper Meyer [14] showed that the quotient space of minimal state space systems (A, B, C, D) of order r modulo similarity with the quotient Euclidian topology is homeomorphic to $\mathrm{Rat}^0_{m,r,p}(\mathbb{R}) \times \mathbb{R}^{p\times m}$ endowed with the graph topology (see Vidyasagar [18]). Since on the other side $\mathrm{Rat}^0_{m,r,p}(\mathbb{R}) \times \mathbb{R}^{p\times m}$ endowed with the topology of uniform convergence on S^2 is homeomorphic to the same quotient space, we can conclude that the graph topology on $\mathrm{Rat}^0_{m,r,p}(\mathbb{R}) \times \mathbb{R}^{p\times m}$ coincides with the topology introduced in this paper. In view of Theorem 3.5, which uses precisely this topology on $\mathrm{Rat}^0_{m,r,p}(\mathbb{R})$, we can say that we are dealing with a sort of graph topology defined on $\mathcal{I}_{m,r,p}(\mathbb{R})$ instead of the topology of uniform convergence on S^2. It is worth noting that the coincidence of these topologies holds only in the case when the McMillan degree remains fixed, since otherwise pole-zero cancellation may occur.

Remark 3.2 With the embeddings $S^1 \subset S^2$ and $\mathcal{G}_m(\mathbb{R}^{p+m}) \subset \mathcal{G}_m(\mathbb{C}^{p+m})$ it holds

$$\mathcal{I}_{m,r,p}(\mathbb{R}) = \{\langle X\rangle \in \mathcal{I}_{m,r,p}(\mathbb{C}) \mid \Phi_X(S^1) \subset \mathcal{G}_m(\mathbb{R}^{p+m})\},$$

which can be seen as follows: assume that $\langle X\rangle \notin \mathcal{I}_{m,r,p}(\mathbb{R})$ and let $X = [X_1^{\mathrm{t}}, X_2^{\mathrm{t}}]^{\mathrm{t}}$ where X_2 is a maximal m-submatrix of X. Then $X_1X_2^{-1} \in \mathbb{C}(s)^{p\times m}\backslash\mathbb{R}(s)^{p\times m}$ and hence there exists $s_0 \in \mathbb{R}$ so that $\det X_2(s_0) \neq 0$ and $\Phi_X(s_0) = \mathrm{span}\,[(X_1(s_0)X_2(s_0)^{-1})^{\mathrm{t}}, I_m]^{\mathrm{t}} \notin \mathcal{G}_m(\mathbb{R}^{p+m})$. Thus we can consider $\mathcal{I}_{m,r,p}(\mathbb{R})$ also as a space of maps from S^1 to $\mathcal{G}_m(\mathbb{R}^{p+m})$.

The point $\Phi_X(\infty)$ gives some information about $\langle X\rangle \in \mathcal{I}_{m,r,p}(\mathbb{K})$: in fact

$$\Phi_X(\infty) = \mathrm{span}\,[M^{\mathrm{t}}, I_m]^{\mathrm{t}} \iff \langle X\rangle \in \mathrm{Rat}_{m,r,p}(\mathbb{K}) \text{ is proper} \tag{3.3}$$

(and moreover, $M = 0 \iff \langle X\rangle$ strictly proper).
"$\Leftarrow$" of the above equivalence is obvious,

"$\Rightarrow$" let $X = \begin{bmatrix} P \\ Q \end{bmatrix}$, $\Phi_X(\infty) = \operatorname{span} \lim_{|t|\to\infty} [(P\Delta^{-1}(t)^{\mathrm{t}}, (Q\Delta^{-1}(t))^{\mathrm{t}}]^{\mathrm{t}} = \operatorname{span} [M^{\mathrm{t}}, I_m]^{\mathrm{t}}$, with an arbitrary maximal m-submatrix Δ of X. We have to show: $\deg \det Q = \deg \det \Delta$. From $\lim_{|t|\to\infty} Q(t)\Delta(t)^{-1} \in Gl_m$ it follows $\det Q \not\equiv 0$ and $\lim_{|t|\to\infty} \Delta(t)Q^{-1}(t) \in Gl_m$. Hence, Lemma 2.3 yields $\deg \det Q = \deg \det \Delta$. Observe that in (3.3) M is the constant part of the proper rational matrix (corresponding to) $\langle X \rangle$.

On the other side, in the non-proper case $\Phi_X(\infty)$ gives no information about the non-singularity of X_2, where $X = [X_1^{\mathrm{t}}, X_2^{\mathrm{t}}]^{\mathrm{t}}$. Of course, for this property one has to find any point $s \in S^2$ where $\det X_2(s) \neq 0$. However, we have the following

Proposition 3.3 $\mathrm{Rat}_{m,r,p}(\mathbb{K})$ *is an open subset of* $\mathcal{I}_{m,r,p}(\mathbb{K})$.

Proof:

1) $\mathbb{K} = \mathbb{C}$.
Let $l = \binom{p+m}{m} - 1$ and

$$\begin{aligned} P: \mathcal{G}_m(\mathbb{C}^{p+m}) &\longrightarrow \mathbb{CP}^l \\ \operatorname{span} M &\longmapsto [M_0 : \ldots : M_l] \end{aligned}$$

be the Plücker-embedding of the Grassmannian in projective space, where M_i are the $m \times m$-minors of M in a prescribed order and M_l is the minor computed from the last m rows of M.
Let $N = \{[x_0 : \ldots : x_{l-1} : 0] \mid (x_0, \ldots, x_{l-1}) \in \mathbb{C}^l \backslash \{0\}\}$. Then N is closed in $\mathbb{CP}^l$ and for $\langle X \rangle \in \mathcal{I}_{m,r,p}(\mathbb{C})$ it holds: $\langle X \rangle \in \mathrm{Rat}_{m,r,p}(\mathbb{C}) \Longleftrightarrow P \circ \Phi_X \notin \langle S^2, N \cap \mathcal{G}_m(\mathbb{C}^{p+m}) \rangle$. Since $\mathcal{G}_m(\mathbb{C}^{p+m}) \backslash N$ is open in $\mathcal{G}_m(\mathbb{C}^{p+m})$, it follows that

$$\mathrm{Rat}_{m,r,p}(\mathbb{C}) = \mathcal{I}_{m,r,p}(\mathbb{C}) \backslash \langle S^2, N \cap \mathcal{G}_m(\mathbb{C}^{p+m}) \rangle = \bigcup_{\substack{K \subseteq S^2 \\ \text{compact}}} \langle K, \mathcal{G}_m(\mathbb{C}^{p+m}) \backslash N \rangle$$

is open in $\mathcal{I}_{m,r,p}(\mathbb{C})$.

2) The case $\mathbb{K} = \mathbb{R}$ follows easily from $\mathrm{Rat}_{m,r,p}(\mathbb{R}) = \mathrm{Rat}_{m,r,p}(\mathbb{C}) \cap \mathcal{I}_{m,r,p}(\mathbb{R})$. $\square$

The structure of the space $\mathcal{I}_{m,r,p}(\mathbb{K})$ can be described in more detail via a precise study of the points at infinity of the curves Φ_X. In order to do so it is helpful to use the manifold structure of the Grassmannian, which will be introduced next:

Definition 3.4 *Let* p, $m \in \mathbb{N}$ *be given.*
a) Define $\mathcal{J} = \{(j_1, \ldots, j_m) \mid j_i \in \mathbb{N}, 1 \leq j_1 < \ldots < j_m \leq p+m\}$.

b) For $J = (j_1, \ldots, j_m) \in \mathcal{J}$ *let* $j_1^*, \ldots, j_p^* \in \underline{p+m}$ *so that* $j_1^* < \ldots < j_p^*$ *and* $\{j_1, \ldots, j_m\} \cup \{j_1^*, \ldots, j_p^*\} = \underline{p+m}$ *holds. Further put*

$$\sigma_J = [a_1, \ldots, a_{p+m}] \in \mathcal{P}(p+m) \text{ with } a_i = \begin{cases} e_{p+l} & \text{for } i = j_l,\ l \in \underline{m}, \\ e_l & \text{for } i = j_l^*,\ l \in \underline{p} \end{cases}$$

where e_i *denotes the* i*-th standard basis vector.*

c) For $J = (j_1, \ldots, j_m) \in \mathcal{J}$ *let* $k(J) = \#\{i \in \underline{m} \mid j_i \leq p\}$.

d) For $k \in \underline{\min\{m,p\}}$ *put*

$$V_k = \begin{bmatrix} I_{p-k} & 0 & 0 & 0 \\ 0 & 0 & I_k & 0 \\ 0 & I_k & 0 & 0 \\ 0 & 0 & 0 & I_{m-k} \end{bmatrix}.$$

Note that σ_J is the permutation which in the matrix $\sigma_J \begin{bmatrix} X \\ I \end{bmatrix} \in \mathbb{K}^{(p+m)\times m}$ places the identity I_m on the rows with indices $j_1, \ldots, j_m$, whereas the rows of $X \in \mathbb{K}^{p\times m}$ are, according to their ordering, placed on the rows with indices $j_1^*, \ldots, j_p^*$. Moreover, for $\sigma_J \in \mathcal{P}(p+m)$ there exist a unique representation of the form

$$\sigma_J = \begin{bmatrix} \tau & 0 \\ 0 & \varrho \end{bmatrix} \begin{bmatrix} I_{p-k(J)} & 0 & 0 & 0 \\ 0 & 0 & I_{k(J)} & 0 \\ 0 & I_{k(J)} & 0 & 0 \\ 0 & 0 & 0 & I_{m-k(J)} \end{bmatrix} \tag{3.4}$$

with $\tau \in \mathcal{P}(p)$ and $\varrho \in \mathcal{P}(m)$.

The domains of the charts of the manifold $\mathcal{G}_m(\mathbb{K}^{p+m})$ are given by

$$\mathrm{ch}_{\mathbb{K}}(J) := \{\mathrm{span}\, \sigma_J \begin{bmatrix} X \\ I_m \end{bmatrix} \mid X \in \mathbb{K}^{p\times m}\} \tag{3.5}$$

where J runs through the set $\mathcal{J}$ of all so-called Schubert-coordinates. With the obvious coordinate mappings $\mathrm{ch}_{\mathbb{K}}(J) \to \mathbb{K}^{p\times m}$ the Grassmannian $\mathcal{G}_m(\mathbb{K}^{p+m})$ becomes an analytic manifold of dimension pm over $\mathbb{K}$ (see [19, S. 176]).

Using this notation, (3.3) may be written in the more general form

$$\Phi_X(\infty) \in \mathrm{ch}_{\mathbb{K}}(J) \iff \langle X \rangle = \langle \sigma_J \begin{bmatrix} P \\ Q \end{bmatrix} \rangle$$
$$\text{with a maximal } m\text{-submatrix } Q \in \mathcal{R}_m(\mathbb{K}).$$

In this case PQ^{-1} is proper and it holds $\Phi_X(\infty) = \mathrm{span}\, \sigma_J \begin{bmatrix} PQ^{-1}(\infty) \\ I_m \end{bmatrix}$.

This leads to the observation that — up to a permutation of rows — the elements of $\mathcal{I}_{m,r,p}(\mathbb{K})$ can be considered as proper rational matrices. More precisely, they are given uniquely by a strictly proper rational matrix and a point on the Grassmannian, which contains just an additional constant part of the rational matrix and the permutation of rows (see (3.5)). This pointwise description can be made global for $\mathcal{I}_{m,r,p}(\mathbb{K})$ in a continuous way, as it is stated by the following theorem.

Theorem 3.5 *The map*

$$\begin{aligned} \Pi_\infty^{\mathbb{K}} : \mathcal{I}_{m,r,p}(\mathbb{K}) &\longrightarrow \mathcal{G}_m(\mathbb{K}^{p+m}) \\ \langle X \rangle &\longmapsto \Phi_X(\infty) \end{aligned}$$

is continuous and gives $\mathcal{I}_{m,r,p}(\mathbb{K})$ the structure of a locally trivial fibration with fibers homeomorphic to $\mathrm{Rat}^0_{m,r,p}(\mathbb{K})$:
for $J \in \mathcal{J}$ it is $(\Pi_\infty^{\mathbb{K}})^{-1}(\mathrm{ch}_{\mathbb{K}}(J)) \overset{\text{homeo}}{\sim} \mathrm{ch}_{\mathbb{K}}(J) \times \mathrm{Rat}^0_{m,r,p}(\mathbb{K})$.

Proof: We consider first the complex case $\mathbb{K} = \mathbb{C}$; let $\Pi_\infty^{\mathbb{C}} = \Pi_\infty$ and $\mathrm{ch}_{\mathbb{C}}(J) = \mathrm{ch}(J)$, for short.

a) For $U \subseteq \mathcal{G}_m(\mathbb{C}^{p+m})$ it is $\Pi_\infty^{-1}(U) = \langle \infty, U \rangle \subseteq \mathcal{I}_{m,r,p}(\mathbb{C})$, thus Π_∞ is continuous.

b) For given $J \in \mathcal{J}$ define

$$\begin{aligned} \Psi_J : \quad \mathrm{ch}(J) \times \mathrm{Rat}^0_{m,r,p}(\mathbb{C}) &\longrightarrow \Pi_\infty^{-1}(\mathrm{ch}(J)) \\ (\mathrm{span}\, \sigma_J \begin{bmatrix} D \\ I_m \end{bmatrix}, \langle \begin{bmatrix} P \\ Q \end{bmatrix} \rangle) &\longmapsto \langle \sigma_J \begin{bmatrix} P + DQ \\ Q \end{bmatrix} \rangle \end{aligned} \tag{3.6}$$

It is not hard to show the bijectivity of Ψ_J and the commutativity of the diagram

$$\begin{tikzcd} \mathrm{ch}(J) \times \mathrm{Rat}^0_{m,r,p}(\mathbb{C}) \arrow[rr, "\Psi_J"] \arrow[dr, "\mathrm{pr}_1"'] & & \Pi_\infty^{-1}(\mathrm{ch}(J)) \arrow[dl, "\Pi_\infty"] \\ & \mathrm{ch}(J) & \end{tikzcd} \tag{3.7}$$

with pr_1 being the projection onto the first component. Hence it remains to be proven the bicontinuity of Ψ_J.

Since S^2 is compact and $\mathcal{G}_m(\mathbb{C}^{p+m})$ a metric space, the compact-open topology on $\mathcal{I}_{m,r,p}(\mathbb{C})$ and its subspaces coincides with the topology of uniform convergence. Hence one can prove the bicontinuity of Ψ_J by considering sequences. Note that for $\langle X \rangle \in \mathrm{Rat}^0_{m,r,p}(\mathbb{C})$ it is

$$\Psi_J(\mathrm{span}\, \sigma_J \begin{bmatrix} D \\ I \end{bmatrix}, \langle X \rangle) =: \langle Y \rangle \text{ with } \Phi_Y = \overline{\sigma_J \begin{bmatrix} I & D \\ 0 & I \end{bmatrix}} \circ \Phi_X,$$

where $\overline{G}$ denotes the homeomorphism on $\mathcal{G}_m(\mathbb{C}^{p+m})$ induced by $G \in Gl_{p+m}(\mathbb{C})$. Now the continuity of Ψ_J follows from the action of Gl_{p+m} on the Grassmannian manifold, namely, if $G_l,\, G \in Gl_{p+m}$ with $\lim_{l\to\infty} G_l = G$ in Gl_{p+m}, then it holds $\lim_{l\to\infty} \overline{G_l} = \overline{G}$ in the compact-open topology of $\mathcal{C}(\mathcal{G}_m(\mathbb{C}^{p+m}), \mathcal{G}_m(\mathbb{C}^{p+m}))$. The continuity of Ψ_J^{-1} follows by symmetry, if one realizes, that the fact $\lim_{l\to\infty} \langle \sigma_J \begin{bmatrix} P_l + D_l Q_l \\ Q_l \end{bmatrix} \rangle = \langle \sigma_J \begin{bmatrix} P + DQ \\ Q \end{bmatrix} \rangle$ implies (by considering the point at ∞): $\lim_{l\to\infty} D_l = D$.

In the case $\mathbb{K} = \mathbb{R}$, the continuity of $\Pi_\infty^{\mathbb{R}}$ follows from the complex case, since $\mathcal{I}_{m,r,p}(\mathbb{R})$ and $\mathcal{G}_m(\mathbb{R}^{p+m})$ are endowed with the subspace topologies. Analogously to (3.6) we can define the map $\Psi_J^{\mathbb{R}}$ with domain $\mathrm{ch}_{\mathbb{R}}(J) \times \mathrm{Rat}^0_{m,r,p}(\mathbb{R})$ and range $(\Pi_\infty^{\mathbb{R}})^{-1}(\mathrm{ch}_{\mathbb{R}}(J))$. Then it follows directly, that $\Psi_J^{\mathbb{R}}$ is bijective and a homeomorphism, since Ψ_J is. $\square$

In the following we will always write $\mathcal{I}_{m,r,p}$, $\mathrm{Rat}_{m,r,p}$ etc. and $\mathrm{ch}(J)$ and Π_∞ without any specification of the field $\mathbb{K}$, since we do not have to distinguish between the real and complex case anymore.

For the subspace $\mathrm{Rat}_{m,r,p} \subseteq \mathcal{I}_{m,r,p}$ the structure is much more difficult to describe. This is not surprising, since the representation

$$\Pi_\infty \langle \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \rangle = \mathrm{span}\, \sigma_J \begin{bmatrix} D \\ I \end{bmatrix} \in \mathrm{ch}(J)$$

gives no information about the determinant of X_2. In this case the structure of the fibers $\Pi_\infty^{-1}(d)$ depends on the point d on the Grassmannian. We will give a brief sketch of this: let

$$d = \mathrm{span} \begin{bmatrix} \tau & 0 \\ 0 & \varrho \end{bmatrix} \begin{bmatrix} D_1 & D_2 \\ I_k & 0 \\ 0 & D_4 \\ 0 & I_{m-k} \end{bmatrix} \in \mathcal{G}_m(\mathbb{K}^{p+m})$$

with suitable $k \le \min\{m, p\}$ and $\tau \in \mathcal{P}(p)$, $\varrho \in \mathcal{P}(m)$. Such a representation exists for each $d \in \mathcal{G}_m(\mathbb{K}^{p+m})$, taking for instance $d = \mathrm{span}\, M$ with M in column echelon form (or, in other words, remembering the cell decomposition of the Grassmannian). Then one can prove

$$\Pi_\infty^{-1}(d) \cap \mathrm{Rat}_{m,r,p} \overset{\text{homeo}}{\sim} \{ \langle \begin{bmatrix} P_1 \\ P_2 \\ Q_1 \\ Q_2 \end{bmatrix} \rangle \in \mathrm{Rat}^0_{m,r,p} \mid \begin{bmatrix} P_2 \\ Q_2 \end{bmatrix} \in \mathcal{R}_m \},$$

where $P_2 \in \mathbb{K}[s]^{k\times m}$, $Q_2 \in \mathbb{K}[s]^{(m-k)\times m}$. Applying Lemma 2.5 and the usual description for rational matrices the above can be written as

$$\Pi_\infty^{-1}(d) \cap \mathrm{Rat}_{m,r,p} \overset{\text{homeo}}{\sim}$$
$$\{G = \begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix} \in \mathrm{Rat}^0_{m,r,p} \mid G_3 \in \mathbb{K}(s)^{k\times k},\ \det G_3 \not\equiv 0\}.$$

4 A Manifold Structure for $\mathcal{I}_{m,r,p}(\mathbb{K})$

Since the work of Hazewinkel [9] and Byrnes/Duncan [3] it is known that the space of strictly proper transfer matrices of fixed McMillan degree can be given the structure of an analytic manifold. This can be done by forming the quotient space of minimal state space realizations modulo similarity.

Let $\tilde\Sigma_{m,r,p}$ be as in (1.1), $\Sigma_{m,r,p} = \tilde\Sigma_{m,r,p}/\overset{s}{\sim}$ the quotient modulo similarity action $\overset{s}{\sim}$, and $\Pi : \tilde\Sigma_{m,r,p} \to \Sigma_{m,r,p}$ the canonical projection. Endow $\Sigma_{m,r,p}$ with the quotient topology.

The manifold structure of $\Sigma_{m,r,p}$ can be described via local canonical forms:
Let $[(A,B,C)] = \Pi(A,B,C)$ be the similarity orbit of (A,B,C) and put $K(r;m) = \{(\alpha_1,\ldots,\alpha_m) \mid \alpha_i \in \mathbb{N}_0, \sum_{i=1}^m \alpha_i = r\}$ (the set $K(r;m)$ will parametrize the charts of the manifold $\Sigma_{m,r,p}$). Define for $\alpha = (\alpha_1,\ldots,\alpha_m) \in K(r;m)$ and $(A,B,C) \in \tilde\Sigma_{m,r,p}$ with $B = [b_1,\ldots,b_m]$

$$R(A,B,C)_\alpha = [b_1, Ab_1, \ldots, A^{\alpha_1-1}b_1, \ldots, b_m, \ldots, A^{\alpha_m-1}b_m] \in \mathbb{K}^{r\times r}$$

and finally $V_\alpha = \{(A,B,C) \in \tilde\Sigma_{m,r,p} \mid \det R(A,B,C)_\alpha \neq 0\}$.

We summarize the known results in the following, the proof of them can be found e.g. in Hazewinkel [9, Theorem 2.5.17] and Byrnes/Duncan [3, p. 43, p. 46]

Theorem 4.1 *$\Sigma_{m,r,p}$ is an analytic non-compact manifold of dimension $r(m+p)$ over $\mathbb{K}$ with charts $(\Pi(V_\alpha), \eta_\alpha)$, $\alpha \in K(r;m)$, where*

$$\begin{array}{rrcl} \eta_\alpha : & \Pi(V_\alpha) & \longrightarrow & \mathbb{K}^{r(m+p)} \\ & [(A,B,C)] & \longmapsto & (R(A,B,C)_\alpha^{-1}[A^{\alpha_1}b_1,\ldots,A^{\alpha_m}b_m], CR(A,B,C)_\alpha) \end{array}$$

Moreover, the map $\Sigma_{m,r,p} \to \mathrm{Rat}^0_{m,r,p}$, $[(A,B,C)] \mapsto C(sI-A)^{-1}B$ is a homeomorphism, if $\mathrm{Rat}^0_{m,r,p}(\mathbb{R}) \subset \mathrm{Rat}^0_{m,r,p}(\mathbb{C}) \subsetneq \mathcal{I}_{m,r,p}(\mathbb{C})$ is endowed with the compact-open topology, as introduced in Section 3.

Since $\mathcal{I}_{m,r,p}$ is locally homeomorphic to $\mathrm{ch}(J) \times \mathrm{Rat}^0_{m,r,p}$, the above theorem implies at once, that the space $\mathcal{I}_{m,r,p}$ is locally an analytic manifold of dimension $r(m+p)+mp$. This analytic structure will also be global if the coordinate changes between two charts $\mathrm{ch}(J) \times \Pi(V_\alpha)$ and $\mathrm{ch}(\tilde J) \times \Pi(V_\beta)$

are shown to be analytic. In order to do so we can restrict ourselves to the case $\tilde{J} = (p+1,\ldots,p+m) \in \mathcal{J}$. So let

$$d = \operatorname{span}\begin{bmatrix}\tilde{D}\\ I\end{bmatrix} = \operatorname{span}\sigma_J\begin{bmatrix}D\\ I\end{bmatrix} \in \operatorname{ch}(\tilde{J}) \cap \operatorname{ch}(J)$$

for some given $J \in \mathcal{J}$. Thanks to the manifold structure of $\mathcal{G}_m(\mathbb{K}^{p+m})$, the coordinate change $\tilde{D} \mapsto D$ is analytic. Thus it remains to be proven the analyticity of the map

$$\begin{array}{rccc} \beta_J: & \mathrm{Rat}^0_{m,r,p} & \longrightarrow & \mathrm{Rat}^0_{m,r,p} \\ & \langle\begin{bmatrix}\tilde{P}\\ \tilde{Q}\end{bmatrix}\rangle & \longmapsto & \langle\begin{bmatrix}P\\ Q\end{bmatrix}\rangle \end{array} \tag{4.1}$$

where

$$\langle\sigma_J\begin{bmatrix}P+DQ\\ Q\end{bmatrix}\rangle = \langle\begin{bmatrix}\tilde{P}+\tilde{D}\tilde{Q}\\ \tilde{Q}\end{bmatrix}\rangle. \tag{4.2}$$

The condition (4.2) is just

$$\Psi_{\tilde{J}}^{-1} \circ \Psi_J(d, \langle\begin{bmatrix}P\\ Q\end{bmatrix}\rangle) = (d, \langle\begin{bmatrix}\tilde{P}\\ \tilde{Q}\end{bmatrix}\rangle)$$

with Ψ_J as in (3.6).

The study of the map (4.1) is divided into two parts: in a first step the application of β_J on $\langle\begin{bmatrix}\tilde{P}\\ \tilde{Q}\end{bmatrix}\rangle$ is translated into a transformation of $\tilde{P}\tilde{Q}^{-1}$ into PQ^{-1}, in a second lemma this transformation is expressed in terms of minimal state space representation so that the analytic structure of $\mathrm{Rat}^0_{m,r,p}$ comes in.

Lemma 4.2 *Fix $J \in \mathcal{J}$ with $k(J) = k$, $\sigma_J = \begin{bmatrix}\tau & 0\\ 0 & \varrho\end{bmatrix}V_k$, $\tau \in \mathcal{P}(p)$, $\varrho \in \mathcal{P}(m)$ and*

$$d = \operatorname{span}\begin{bmatrix}\tilde{D}\\ I\end{bmatrix} = \operatorname{span}\begin{bmatrix}\tau & 0\\ 0 & \varrho\end{bmatrix}\begin{bmatrix}D_1 & D_2\\ I_k & 0\\ D_3 & D_4\\ 0 & I_{m-k}\end{bmatrix} \in \mathcal{G}_m(\mathbb{K}^{p+m}).$$

Further let (4.2) hold and put $\tilde{D} = \tau\begin{bmatrix}\tilde{D}_1 & \tilde{D}_2\\ \tilde{D}_3 & \tilde{D}_4\end{bmatrix}\varrho$ and $\begin{bmatrix}Z_1 & Z_2\\ Z_3 & Z_4\end{bmatrix} = \tau^{-1}[\tilde{P}\tilde{Q}^{-1} + \tilde{D}]\varrho$ where $\tilde{D}_3$ and Z_3 are $k \times k$-matrices. Then:

a) $D_3,\ \tilde{D}_3 \in Gl_k$ and $\begin{bmatrix}D_1 & D_2\\ D_3 & D_4\end{bmatrix} = \begin{bmatrix}\tilde{D}_1\tilde{D}_3^{-1} & \tilde{D}_2 - \tilde{D}_1\tilde{D}_3^{-1}\tilde{D}_4\\ \tilde{D}_3^{-1} & -\tilde{D}_3^{-1}\tilde{D}_4\end{bmatrix}$

b) $PQ^{-1} = \begin{bmatrix}Z_1Z_3^{-1} & Z_2 - Z_1Z_3^{-1}Z_4\\ Z_3^{-1} & -Z_3^{-1}Z_4\end{bmatrix} - \begin{bmatrix}D_1 & D_2\\ D_3 & D_4\end{bmatrix}$.

Proof: a) From

$$\begin{aligned}\operatorname{span}\begin{bmatrix} D_1 & D_2 \\ I_k & 0 \\ D_3 & D_4 \\ 0 & I_{m-k}\end{bmatrix} &= \operatorname{span}\begin{bmatrix}\tau^{-1} & 0 \\ 0 & \varrho^{-1}\end{bmatrix}\begin{bmatrix}\tilde{D} \\ I\end{bmatrix} = \operatorname{span}\begin{bmatrix}\tau^{-1} & 0 \\ 0 & \varrho^{-1}\end{bmatrix}\begin{bmatrix}\tilde{D} \\ I\end{bmatrix}\varrho \\ &= \operatorname{span}\begin{bmatrix}\tilde{D}_1 & \tilde{D}_2 \\ \tilde{D}_3 & \tilde{D}_4 \\ I_k & 0 \\ 0 & I_{m-k}\end{bmatrix}\end{aligned}$$

it follows that $D_3,\ \tilde{D}_3 \in Gl_k$ and so $D =$

$$\begin{bmatrix} D_1 & D_2 \\ D_3 & D_4\end{bmatrix} = \begin{bmatrix}\tilde{D}_1 & \tilde{D}_2 \\ I_k & 0\end{bmatrix}\begin{bmatrix}\tilde{D}_3 & \tilde{D}_4 \\ 0 & I_{m-k}\end{bmatrix}^{-1} = \begin{bmatrix}\tilde{D}_1\tilde{D}_3^{-1} & \tilde{D}_2 - \tilde{D}_1\tilde{D}_3^{-1}\tilde{D}_4 \\ \tilde{D}_3^{-1} & -\tilde{D}_3^{-1}\tilde{D}_4\end{bmatrix}.$$

b) The invertibility of $\tilde{D}_3$ implies $\det Z_3 \not\equiv 0$. Moreover, by (4.2)

$$\langle V_k \begin{bmatrix}\tau^{-1} & 0 \\ 0 & \varrho^{-1}\end{bmatrix}\begin{bmatrix}\tilde{P} + \tilde{D}\tilde{Q} \\ \tilde{Q}\end{bmatrix}\rangle = \langle \sigma_J^{-1}\begin{bmatrix}\tilde{P} + \tilde{D}\tilde{Q} \\ \tilde{Q}\end{bmatrix}\rangle = \langle \begin{bmatrix} P + DQ \\ Q\end{bmatrix}\rangle,$$

and the claim follows thanks to Lemma 2.5. □

The following lemma can be easily verified.

Lemma 4.3 *Let* $T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4\end{bmatrix} \in \mathbb{K}^{p\times m}$ *be proper with* $T_3 \in \mathbb{K}(s)^{k\times k}$, $\det T_3 \not\equiv 0$ *and put* $\hat{T} = \begin{bmatrix} T_1T_3^{-1} & T_2 - T_1T_3^{-1}T_4 \\ T_3^{-1} & -T_3^{-1}T_4\end{bmatrix}$. *If* $T(s) = \begin{bmatrix} C_1 \\ C_2\end{bmatrix}(sI - A)^{-1}[B_1, B_2] + \begin{bmatrix} D_1 & D_2 \\ D_3 & D_4\end{bmatrix}$ *is a minimal state space realization and* $\det D_3 \neq 0$, *then* $\hat{T}(s) =$

$$\begin{bmatrix} C_1 - D_1D_3^{-1}C_2 \\ -D_3^{-1}C_2\end{bmatrix}(sI - A + B_1D_3^{-1}C_2)^{-1}[B_1D_3^{-1}, B_2 - B_1D_3^{-1}D_4] + \begin{bmatrix} D_1D_3^{-1} & D_2 - D_1D_3^{-1}D_4 \\ D_3^{-1} & -D_3^{-1}D_4\end{bmatrix}$$

is a minimal state space realization as well.

Combining these two lemmata, the map β_J in (4.1), (4.2) can be written as

$$\begin{array}{rccc} f_J: & \Sigma_{m,r,p} & \longrightarrow & \Sigma_{m,r,p} \\ & [(A, [B_1, B_2], \begin{bmatrix} C_1 \\ C_2\end{bmatrix})] & \longmapsto & [(\tilde{A}, \tilde{B}, \tilde{C})]\end{array}$$

where $(\tilde{A}, \tilde{B}, \tilde{C}) =$

$$(A - B_1 D_3^{-1} C_2, [B_1 D_3^{-1}, B_2 - B_1 D_3^{-1} D_4], \begin{bmatrix} C_1 - D_1 D_3^{-1} C_2 \\ -D_3^{-1} C_2 \end{bmatrix}).$$

This is in fact analytic, so we get:

Corollary 4.4 *The space $\mathcal{I}_{m,r,p}(\mathbb{K})$ is an analytic manifold of dimension $r(m+p)+mp$ over $\mathbb{K}$. The underlying topology is the compact-open topology.* $\mathrm{Rat}_{m,r,p}(\mathbb{K})$ *is an open submanifold.*

Using the results on $\mathrm{Rat}^0_{m,r,p}$ we can derive at once some consequences.

Corollary 4.5

a) $\mathcal{I}_{m,r,p}$ *and* $\mathrm{Rat}_{m,r,p}$ *are non-compact for* $r \geq 1$ *(*$\mathcal{I}_{m,0,p} \overset{\text{homeo}}{\sim} \mathcal{G}_m(\mathbb{K}^{p+m})$, $\mathrm{Rat}_{m,0,p} \overset{\text{homeo}}{\sim} \mathbb{K}^{p\times m}$*).*

b) $\mathrm{Rat}^0_{m,r,p} \times \mathbb{K}^{p\times m}$ *and* $\mathrm{Rat}_{m,r,p}$ *are dense subsets of* $\mathcal{I}_{m,r,p}$*.*

c) $\mathcal{I}_{m,r,p}(\mathbb{C})$ *is connected for all* $m, p \geq 1$, $\mathcal{I}_{m,r,p}(\mathbb{R})$ *is connected for* $\max\{m,p\} > 1$. $\mathcal{I}_{1,r,1}(\mathbb{R}) = \mathrm{Rat}_{1,r,1}(\mathbb{R})$ *falls into* $r+1$ *connected components, parametrized by the winding number of the maps from* S^1 *to* S^1 *induced by the elements of* $\mathrm{Rat}_{1,r,1}(\mathbb{R})$*.*

Proof:

a) Follows from the non-compactness of $\mathrm{Rat}^0_{m,r,p}$, which is homeomorphic to the fibers of the continuous map Π_∞.

   $\mathcal{I}_{m,0,p}$ (resp. $\mathrm{Rat}_{m,0,p}$) is the space of constant maps from $\mathbb{KP}^1$ to $\mathcal{G}_m(\mathbb{K}^{p+m})$ (resp. to $\mathrm{ch}(J) \subset \mathcal{G}_m(\mathbb{K}^{p+m})$, where $J = (p+1, \ldots, p+m)$).

b) Follows from the openess of the map Π_∞ and the fact that $\mathrm{ch}(J)$ is dense in $\mathcal{G}_m(\mathbb{K}^{p+m})$ for all $J \in \mathcal{J}$.

c) The first statements hold since $\mathrm{Rat}^0_{m,r,p}(\mathbb{C})$ is connected and $\mathrm{Rat}^0_{m,r,p}(\mathbb{R})$ is connected whenever $\max\{m,p\} > 1$.

For the second part, remember that Brockett [1] has shown that $\mathrm{Rat}^0_{1,r,1}(\mathbb{R})$ consists of $r+1$ connected components indexed by the winding number (Cauchy index) of the elements of $\mathrm{Rat}^0_{1,r,1}(\mathbb{R})$, viewed as maps from S^1 to $S^1 \overset{\text{homeo}}{\sim} \mathcal{G}_1(\mathbb{R}^2)$. Indeed, it can be shown that for each $f \in \mathrm{Rat}^0_{1,r,1}(\mathbb{R})$ the Cauchy index is an element of $\{-r, -r+2, \ldots, r-2, r\}$ and that each of these numbers occurs as Cauchy index of a suitable element in $\mathrm{Rat}^0_{1,r,1}(\mathbb{R})$. Brockett showed the continuity of the Cauchy index and the arcwise connectivity of those subsets of $\mathrm{Rat}^0_{1,r,1}(\mathbb{R})$ in which the Cauchy index is fixed. From this the claimed number of connected components of $\mathrm{Rat}^0_{1,r,1}(\mathbb{R})$

follows. The case $\mathrm{Rat}_{1,r,1}(\mathbb{R})$ can easily be reduced to the proper case $\mathrm{Rat}^0_{1,r,1}(\mathbb{R}) \times \mathbb{R}$. □

We want to close the paper with a brief comparison of the above results with those of Cobb [4]. Cobb studied the space of rational matrices with fixed degree, but where the degree is defined as follows:
let $G = G_- + G_+$ with G_- strictly proper and G_+ polynomial and define

$$\hat{\delta}(G) = d(G_-) + \max\{k + \deg T \mid T \; k \times k\text{-minor} \not\equiv 0 \,\mathrm{of}\, G_+\}.$$

It is easy to see that this can be rewritten as

$$\hat{\delta}(G) = d(G_-) + d(s^{-1}G_+(s^{-1})).$$

The reason for this definition is that $\hat{\delta}(G)$ is precisely the minimal dimension of a singular system realization $G(s) = C(sE - A)^{-1}B$, with the size of the matrix E as dimension, see [4, Prop. 4.3]. Remember that the McMillan degree defined in Definition 2.2 is just $\mathrm{rk}E$ in strongly irreducible realizations $G(s) = C(sE - A)^{-1}B + D$ (see [17, p. 822]). So the difference of the definitions comes in by treating proper (but not strictly proper) systems differently. In the single-input single-output case the definition of Cobb becomes

$$\hat{\delta}(pq^{-1}) = \max\{\deg q, \deg p + 1\}$$

if p and q are coprime. Thus $\{f \in \mathbb{R}(s) \mid \hat{\delta}(f) = r\} = \mathrm{Rat}^0_{1,r,1}(\mathbb{R}) \cup \{f \in \mathbb{R}(s) \mid f^{-1} \in \mathrm{Rat}_{1,r-1,1}(\mathbb{R})\,\mathrm{proper}\}$. Cobb proved that the space $\{G \in \mathbb{R}(s)^{p\times m} \mid \hat{\delta}(G) = r\}$ is connected for every m, r, $p \geq 1$ if it is topologized in a suitable way as a subspace of $\mathbb{RP}^{n(pm+1)}$ (see [4, Theorem 4.4]).

References

[1] R. W. Brockett. Some geometric questions in the theory of linear systems, *IEEE Transactions on Automatic Control* **AC-21** (1976), 449–455.

[2] C. I. Byrnes. On compactifications of spaces of systems and dynamic compensation, In *Proceedings of the 22th Conference on Decision and Control*, 1983.

[3] C. I. Byrnes and T. E. Duncan. On certain topological invariants arising in system theory, In *New Directions in Applied Mathematics*, pages 29–71. New York: Springer Verlag, 1982.

[4] J. D. Cobb. Topological aspects of controllability and observability on the manifold of singular and regular systems, *Journal of Mathematical Analysis and Applications* **138** (1989), 21–42.

[5] D. F. Delchamps. *State Space and Input-Output Linear Systems.* New York: Springer-Verlag, 1988.

[6] C. A. Desoer and M. Vidyasagar. *Feedback Systems: Input-Output Properties.* New York: Academic Press, New York, 1975.

[7] G. D. Forney, Jr. Minimal bases of rational vector spaces, with applications to multivariable linear systems, *SIAM Journal on Control and Optimization* **13** (1975), 493–520.

[8] J. Grimm. Realization and canonicity for implicit systems, *SIAM Journal on Control and Optimization* **26** (1988), 1331–1347.

[9] M. Hazewinkel. Moduli and canonical forms of linear dynamical systems II: The topological case, *Mathematical System Theory* **10** (1977), 363–385.

[10] P. H. M. Janssen. General results of the McMillan degree and the Kronecker indices of ARMA and MFD models, *International Journal of Control* **48(2)** (1988), 591–608.

[11] M. Kuijper and J. M. Schumacher. Realization of autoregressive equations in pencil and descriptor form, *SIAM Journal on Control and Optimization* **28(5)** (1990), 1162–1189.

[12] M. Kuijper and J. M. Schumacher. Minimality of descriptor representations under external equivalence, *Automatica* **27(6)** (1991), 985–995.

[13] C. Martin and R. Hermann. Applications of algebraic geometry to systems theory: The McMillan degree and Kronecker indices of transfer functions as topological and holomorphic system invariants, *SIAM Journal on Control and Optimization* **16** (1978), 743–755.

[14] D. G. Meyer. The graph topology on n-th-order systems is quotient Euclidian, *IEEE Transactions on Automatic Control* **AC-36** (1991), 338–340.

[15] H. H. Rosenbrock. *State Space and Multivariable Theory.* Nelson-Wiley, 1970.

[16] J. Rosenthal. On minimal order dynamical compensators of low order systems, In *Proceedings of the 1st European Control Conference, Grenoble*, pages 374–378, 1991.

[17] G. C. Verghese, B. C. Lévy, and T. Kailath. A generalized state-space for singular systems, *IEEE Transactions on Automatic Control* **AC-26** (1981), 811–830.

[18] M. Vidyasagar. *Control System Synthesis: a Factorization Approach.* Cambridge, MA: MIT Press, 1985.

[19] H. Whitney. *Complex Analytic Varieties.* Reading, MA: Addison-Wesley Publishing Company, 1972.

[20] J. C. Willems. Models for dynamics, *Dynamics Reported* **2** (1989), 171–269.

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