Journal of Mathematical Systems, Estimation, and Control
Vol. 5, No. 1, 1995, pp. 1–27

A Unified Representation for Nonlinear Discrete-Time and Sampled Dynamics*

S. Monaco† D. Normand-Cyrot†

Abstract

This paper deals with exponential representations which provide a unified framework to study discrete-time and sampled dynamics. This is illustrated by the study of the problems of linear and linear feedback equivalences for nonlinear discrete-time as well as sampled dynamics.

Key words: Nonlinear discrete-time systems, nonlinear sampling, feedback linearization, linear equivalence.

1 Introduction

Nonlinear control theory has been initialized with the study of bilinear or polynomial state dynamics which present the interest of involving matrices algebra (see [3, 4], [23, 24], for example). In fact, at first, bilinear equations were considered, in both continuous-time and discrete-time domains, as rather general with respect to their nonlinearity but quite structurally simple "nearly linear systems." This feeling was rapidly denied by the difficulties already encountered when studying bilinear dynamics and encouraged specific study in continuous-time and discrete-time contexts respectively.

In spite of recent developments and an increasing interest (see in various mathematical frameworks the references, [2], [5] - [7], [9, 10], [13] - [17], [19], [21], related to the problem here studied), nonlinear control theory in discrete time is still less understood than in continuous time. There are many reasons for this, such as the difficulty of setting local concepts,

*Received November 20, 1992; received in final form January 19, 1993. Summary appeared in Volume 5, Number 1, 1995.
†Work supported by the Italian MURST-40 % Project on "Teoria dei Sistemi e del Controllo" and the French MEN-DRED Grant on "Théorie des Systèmes Discrets."

extensively studied in the continuous-time case, and the loss of geometric properties even assuming an affine structure of the state equations with respect to the control.

In the linear context, many of the above-mentioned difficulties obviously do not occur, since the linear structure is preserved under sampling. The result is that digital control problems, set on the basis of continuous-time models, are solved with respect to the sampled dynamics interpreted as a discrete-time model. This is quite difficult in a nonlinear context where structural properties are generally not preserved under sampling. Bilinear equations are transformed into state affine equations and linear analytic equations are transformed into general nonlinear difference equations.

A recently developed orientation of research concerns sampled nonlinear systems and the preservation of control properties under sampling ([1], [18], [20], [25, 26]). The motivation is very practical since most of the physical systems are modelled by differential equations while controlled by digital computers.

The object of this paper is to associate an exponential representation to nonlinear discrete-time and sampled dynamics. In both cases, these representations are identical except with reference to the vector fields involved, referred to as canonical vector fields. The terminology is due to the fact that these canonical vector fields and their Lie algebras have been shown to characterize controllability, invariance and control properties of the dynamics to which they are associated ([2], [15], [20, 21], [26]).

The unified representation introduced here facilitates comparative studies between discrete-time, sampled and continuous-time dynamics.

The first results making use of vector fields in the study of discrete time systems were obtained in [7], [14], [17].

In [21], it has been shown how linear and bilinear dynamics reflect in the structure of these exponential representations. This analysis is reinforced in the present paper and illustrated by the study of linear equivalence and feedback linear equivalence of discrete-time systems as well as systems under sampling.

In a discrete-time context, it is shown that necessary and sufficient conditions for linear and linear feedback equivalences can be checked in terms of the canonical vector fields associated to the considered discrete-time dynamics. A comparison between the stated results and those known in a continuous-time domain enables to stress very strong analogies with continuous linear equivalences of generally nonlinear differential equations ([27, 28]).

When sampled systems are considered, many properties of the associated canonical vector fields can be stressed. This concerns their expressions as series expansions in the sampling period δ which satisfy combinatoric recurrent relations ([20]). In particular, when linear analytic systems are sampled, these relations give peculiar conditions for solving linear, and linear feedback equivalences reinforcing a conjecture stated in [10] and solved in [1] for $n = 2$: "*linear feedback equivalence under sampling implies linear equivalence*". With respect to this last problem, the results already obtained in [1] are here expressed in terms of these canonical vector fields.

The paper is organized as follows. The unified exponential representation for drift invertible dynamics is introduced in Sections 2 and 3 for discrete-time and sampled systems. Sections 4 and 5 deal with linear and linear feedback equivalences in both discrete-time and sampled contexts. Throughout the paper single input single output systems are investigated but all the results can be extended to the multi input multi output case, just requiring extended notations.

2 Exponential Representation of Discrete-Time Dynamics

Consider a nonlinear drift invertible discrete-time system

$$x(k+1) = F(x(k), u(k)) \tag{2.1}$$

$$y = H(x), \tag{2.2}$$

where $x \in M$ is a smooth n-dimensional manifold, $F : M \times \mathbb{R} \to \mathbb{M}$ and $H : M \to \mathbb{R}$ are analytic functions. Assume the drift term $F_0(.) := F(., 0)$ invertible and denote by $(x_e, 0)$ a equilibrium point of F . Denoting by "I_d" the identity function and by I the identity operator, the following main result can be stated:

Theorem 2.1 *The discrete-time dynamics (2.1) admits the exponential representation*

$$x(k+1) = e^{u(k)G^0(.,u(k))}[I_d]\Big|_{F_0(x(k))}, \tag{2.3}$$

where $G^0(., u) := M \to T_x M$ is a smooth vector field parametrized by u.

The proof follows from Propositions 2.1, 2.2 and 2.3 below. For this, let us introduce the vector field

$$G_1^0(x,u) := \frac{\partial}{\partial \epsilon}\Big|_{\epsilon=0} F(F^{-1}(x,u), u+\epsilon) = (\frac{\partial}{\partial \epsilon}\Big|_{\epsilon=0} F(., u+\epsilon))\Big|_{F^{-1}(x,u)} \tag{2.4}$$

and define

$$G_1^0(x) := G_1^0(x,0) \tag{2.5}$$

$$G_i^0(x) := \frac{\partial^{i-1}}{\partial u^{i-1}}\Big|_{u=0} G_1^0(x,u) \quad \forall i > 1, \tag{2.6}$$

so that

$$G_1^0(x,u) = G_1^0(x) + \sum_{i\geq 1} \frac{u^i}{i!} G_{i+1}^0(x). \tag{2.7}$$

Denoting by G_i^0 the Lie derivative associated to the vector field $G_i^0(.)$, such operator is manipulated in the sequel as a formal operator. By convention, G_i^0 is said to be of degree i, so that homogeneous polynomials, Lie polynomials and series with respect to these operators can be defined.

Denoting by $P_i(G_1^0, \ldots, G_i^0)$ a generic homogeneous polynomial of degree i, one defines $(P_{i+1})^+(G_1^0, \ldots, G_{i+1}^0)$ as the homogeneous polynomial of degree $i+1$, deduced from P_i by substituting $(G_{i_1+1}^0 G_{i_2}^0 \ldots G_{i_m}^0 + G_{i_1}^0 G_{i_2+1}^0 \ldots G_{i_m}^0 + \ldots + G_{i_1}^0 G_{i_2}^0 \ldots G_{i_m+1}^0)$ to the generic mononomial $G_{i_1}^0 G_{i_2}^0 \ldots G_{i_m}^0$ appearing in $P_i(G_1^0, \ldots, G_i^0)$. According to these notations, it has been proved in [19] that

Proposition 2.1: ([19]) *$F(F_0^{-1}(x), u)$ admits the series expansion*

$$F(F_0^{-1}(x),u) = (I + \sum_{i\geq 1} \frac{u^i}{i!} P_i(G_1^0, \ldots, G_i^0))[I_d]\Big|_x, \tag{2.8}$$

where the $P_i(G_1^0, \ldots, G_i^0)$ are homogeneous polynomials of degree i which can be recursively computed from $P_1(G_1^0) = G_1^0$ according to the relation

$$P_i(G_1^0, \ldots, G_i^0) = G_1^0 \circ P_{i-1}(G_1^0, \ldots, G_{i-1}^0) + (P_{i-1})^+(G_1^0, \ldots, G_{i-1}^0). \tag{2.9}$$

Moreover,

Proposition 2.2: ([19]) *Any polynomial $P_i(G_1^0, \ldots, G_i^0)$ for $i \geq 1$ admits the decomposition*

$$P_i(G_1^0, \ldots, G_i^0) = \sum_{\substack{i_1,\ldots,i_m \\ \Sigma i_j = i}} c(i_1, \ldots, i_m) G_{i_1}^0 \circ \ldots \circ G_{i_m}^0, \tag{2.10}$$

where the real coefficients $c(i_1, \ldots, i_m)$ verify the shuffle relations

$$\begin{aligned} c(i_1)c(i_2) &= c(i_1,i_2) + c(i_2,i_1) = c(i_1 \omega\ i_2) \\ c(i_1)c(i_2,i_3) &= c(i_1,i_2,i_3) + c(i_2,i_1,i_3) + c(i_2,i_3,i_1) = c(i_1\omega\ i_2 i_3) \\ \ldots. &\quad \ldots.\ . \end{aligned}$$

The shuffle product "ω " is defined in a recursive way as follows ([4], [22])

$$\begin{aligned} 1\omega\ i_1 &= i_1 \omega\ 1 = i_1 \\ i_1 \omega\ i_2 &= i_2 \omega\ i_1 = i_1 i_2 + i_1 i_2 \\ i_1 \ldots i_m \omega\ j_1 \ldots j_p &= i_1(i_2 \ldots i_m \omega\ j_1 \ldots j_p) + j_1(i_1 \ldots i_m \omega\ j_2 \ldots j_p) \\ \ldots\ldots\ldots &\quad \ldots\ldots\ldots\ . \end{aligned}$$

Sketches of Proofs: First, from (2.4) one deduces

$$\frac{\partial}{\partial u}F(x,u) = \frac{\partial}{\partial \epsilon}\Big|_{\epsilon=0} F(x,u+\epsilon) = G_1^0(F(x,u),u); \tag{2.11}$$

that is for $u = 0$

$$\frac{\partial}{\partial u}\Big|_{u=0} F(x,u) = G_1^0(F_0(x),0) = G_1^0[\mathrm{I}_d]\Big|_{F_0(x)},$$

or, equivalently in (2.8),

$$P_1(G_1^0) = G_1^0.$$

Successively deriving $G_1^0(F(x,u),u)$ with respect to u , one computes from (2.11)

$$\begin{aligned}\frac{\partial^2}{\partial u^2}\Big|_{u=0} F(x,u) &= \frac{\partial}{\partial u}\Big|_{u=0}(G_1^0(F(x,u),u)) \\ &= \frac{\partial}{\partial x}G_1^0(.)\Big|_{F_0(x)}.\frac{\partial}{\partial u}\Big|_{u=0}F(x,u) + \frac{\partial}{\partial u}\Big|_{u=0}[G_1^0(F_0(x),u)];\end{aligned}$$

that is, because of (2.6),

$$\frac{\partial^2}{\partial u^2}\Big|_{u=0} F(x,u) = G_1^0 \circ G_1^0[\mathrm{I}_d]\Big|_{F_0(x)} + G_2^0[\mathrm{I}_d]\Big|_{F_0(x)},$$

or, equivalently in (2.8),

$$P_2(G_1^0,G_2^0) = G_1^0 \circ P_1(G_1^0) + (P_1)^+(G_1^0) = G_1^0 \circ G_1^0 + G_2^0,$$

according to $G_2^0 := (P_1)^+(G_1^0) := (G_1^0)^+$.

Iterating the reasoning, one recovers (2.8) and (2.9) in Proposition (2.1).

Q.E.D.

The proof of Proposition (2.2) is an immediate consequence of recurrence (2.9). In fact, keeping in mind that the operation "+" corresponds to the derivation with respect to u., (2.9) exactly reproduces the derivation of a composed function and thus helps to recover the shuffle relations for the coefficients appearing in the right-hand side of (2.10).

Remark 2.1: (2.10) gives a series representation of the function $F(F_0^{-1}(.), u)$ in terms of compositions of first order differential operators applied to the identity function.

Proposition 2.3: *$uG^0(.,u)$ in (2.3) is a Lie element in the vector fields G_i^0's which takes the form*

$$uG^0(.,u) = \sum_{i\geq 1}\frac{u^i}{i!}B_i(G_1^0,\ldots,G_i^0), \tag{2.12}$$

where $B_i(G_1^0, \ldots, G_i^0)$ is a homogeneous Lie polynomial of degree i for $i \geq 1$.

Proof: Introducing the formal operator P, the right-hand side of (2.8) can be written as

$$(I + P)[\mathrm{I}_d]\Big|_x = (I + \sum_{k \geq 1} \frac{u^k}{k!} P_k)[\mathrm{I}_d]\Big|_x. \tag{2.13}$$

Interpreting the powers of P as successive compositions of non-commuting differential operators, the usual formal logarithmic expansion can be applied

$$\mathrm{Log}(1 + P) = P - \frac{P^2}{2} + \frac{P^3}{3} - \frac{P^4}{4} + \ldots \quad . \tag{2.14}$$

Now, substituting (2.10) intoP_k in (2.14), one deduces from ([22]) that $uG^0(., u)$ is a Lie element in the G_i^0's because the coefficients $c(i_1, \ldots, i_m)$ in (2.10) verify the shuffle relations. This means that $\mathrm{Log}(1 + P)$ admits a series representation in powers of u, the coefficients of which are homogeneous Lie polynomials in the G_i^0's. Q.E.D.

Remark 2.2: Taking the formal logarithmic expansion (2.14) of (2.8), the coefficient of the i-th power in u in (2.12) can be computed as

$$B_i(G_1^0, \ldots, G_i^0) = i! \sum_{m=1}^{i} \frac{(-1)^{m+1}}{m} \sum_{\substack{i_1, \ldots, i_m \geq 1 \\ i_1 + \ldots + i_m = i}} \frac{P_{i_1} \circ \ldots \circ P_{i_m}}{i_1! \ldots i_m!}, \tag{2.15}$$

which completely specifies B_i as a polynomial of degree i in the G_j^0's once every polynomial P_k is replaced by its expression set in (2.10). As previously stated, from [22] it is known that an adequate regrouping of the terms in (2.15) gives rise to a homogeneous Lie polynomial of degree i.

Even if a simple Lie decomposition of the general polynomial B_i is not available, one obtains for the first terms

$$B_1 = G_1^0, \quad B_2 = G_2^0, \quad B_3 = G_3^0 + 1/2[G_1^0, G_2^0], \quad B_4 = G_4^0 + [G_1^0, G_3^0]. \tag{2.16}$$

Proof of Theorem 2.1: The construction of the exponential representation, stated in Theorem 2.1 is easily deduced from Propositions (2.1) and (2.2) noting that the forced evolution around the drift of the dynamics (2.1) can be written as

$$F(F_0^{-1}(x), u) = e^{uG^0(.,u)}[\mathrm{I}_d]\Big|_x. \tag{2.17}$$

Because of Proposition (2.3), $uG^0(., u)$ is a smooth vector field. Q.E.D.

The exponential representation (2.3) is a basic instrument for expanding, around the free evolution, the state and output evolutions of (2.1), (2.2) in powers of the controls. For, as in [19] (see also [14]), let us denote by $G^p(x,u)$ the smooth vector field obtained by transporting $G^0(.,u)$ along the free evolution $F_0^p(.)$. More precisely let, for $p > 0$

$$G^p(x,u) := J_x(F_0^p)\Big|_{F_0^{-p}(x)} G^0(F_0^{-p}(x),u)). \tag{2.18}$$

Corollary 2.1: *The state evolution of (2.1) can be expressed as*

$$x(k) = e^{u(0)G^{k-1}(.,u(0))} \circ \ldots \circ e^{u(k-1)G^0(.,u(k-1))}[I_d]\Big|_{F_0^k(x_0)}, \tag{2.19}$$

where the $G^p(x, u(k-p-1)) := M \to T_xM$*, defined in (2.18), are smooth vector fields parametrized by* $u(k-p-1)$ *for* $p = 0, \ldots, k-1$*.*

Proof: Let us define from $G_1^0(x,u)$ in (2.4) the next vector fields for $p > 0$ and $i > 1$

$$\begin{aligned} G_1^p(x,u) &:= J_x(F_0^p)\Big|_{F_0^{-p}(x)} G_1^0(F_0^{-p}(x),u)) \\ G_i^p(x) &:= J_x(F_0^p)\Big|_{F_0^{-p}(x)} G_i^0(F_0^{-p}(x)). \end{aligned} \tag{2.20}$$

Based on (2.17) and (2.20), (2.8), (2.9) and (2.10) are true with the superscript p instead of 0, so that $e^{uG^p(.,u)}[\mathrm{I}_d]\Big|_x$ is an exponential representation of the function $F_0^p \circ F.(.,u) \circ F_0^{-p-1}(x)$. Denoting such a function by $R^p(x,u)$, it is easily verified that

$$x(k) = R^0(.,u(k-1)) \circ R^1(.,u(k-2)) \circ \ldots \circ R^{k-1}(.,u(0)) \circ F_0^k(x_0).$$

Since each function $R^p(.,u)$ admits the exponential representation $e^{uG^p(.,u)}$ $[\mathrm{I}_d]$, the general theory of composition of Lie series can be applied ([12]) and the identities hereunder follow

$$\begin{aligned} R^i(.,u) \circ R^j(.,u) &= R^i(.,u)(e^{uG^j(.,u)}[\mathrm{I}_d]) \\ &= e^{uG^j(.,u)}[R^i(.,u)] \\ &= e^{uG^j(.,u)} \circ e^{uG^i(.,u)}[\mathrm{I}_d]. \end{aligned}$$

It results that

$$\begin{aligned} x(k) \;=\; & (F(.,u(k-1)) \circ F_0^{-1}) \circ (F_0 \circ F(.,u(k-2)) \circ F_0^{-2}) \circ \ldots \\ & \circ (F_0^{-k-1} \circ F(.,u(0)) \circ F_0^{-k}) \circ F_0^k(x_0); \end{aligned}$$

that is

$$x(k) = e^{u(0)G^{k-1}(.,u(0))} \circ \ldots \circ e^{u(k-1)G^0(.,u(k-1))}[\mathrm{I}_d]\Big|_{F_0^k(x_0)},$$

which ends the proof of Corollary 2.1. Q.E.D.

The same arguments of the proof of Corollary 2.1 can be used to show that

Corollary 2.2: *The output evolution of (2.1)-(2.2) can be expressed as*

$$x(k) = e^{u(0)G^{k-1}(.,u(0))} \circ \ldots \circ e^{u(k-1)G^{0}(.,u(k-1))}[H]\Big|_{F_0^k(x_0)}, \tag{2.21}$$

where the $G^p(x, u(k-p-1)) := M \to T_x M$*, defined in (2.18), are smooth vector fields parametrized by* $u(k-p-1)$ *for* $p = 0, \ldots, k-1$.

It is useful to point out how an affine structure of $F(x,u)$, that is of the form $F_0(x) + uF_1(x)$, reflects into the exponential representation. For, the following result can easily be proved setting in (2.8) for $i \geq 2 : P_i[\mathrm{I}_d]\Big|_x = 0$.

Proposition 2.4: *The exponential representation (2.3) describes a drift invertible linear analytic dynamics if and only if*

$$G^0_{2+i} = -\sum_{p=0}^{i} c_i^p J_x(G^0_{i+1-p})G^0_{p+1} \quad \forall i \geq 0,$$

with $c_i^p = \frac{i!}{p!(p-i)!}$.

So far, the exponential representation (2.1) has been associated to a drift invertible nonlinear discrete-time dynamics. The vector fields G_i^k which characterize such a representation will be said to be *the canonical vector fields* associated to the discrete-time dynamics (2.1). They have been shown to characterize controlled invariant distributions and the controllability Lie algebra in the nonlinear discrete-time case in [7], [14, 15], [17]. In [21], it has been shown how linearity and bilinearity of the dynamics reflect into the structure of these canonical vector fields. Hereafter, it will be shown how the conditions of linear and linear feedback equivalences can be expressed in terms of them providing a unified approach for studying discrete-time and sampled dynamics. Some more properties will be pointed out.

Proposition 2.5: *Under the coordinate change* $z = f(x)$ *the canonical vector fields (2.6) and (2.20) associated to (2.1) are transformed into*

$$\tilde{G}_i^j(z) = J_x\Phi\Big|_{\Phi^{-1}(z)}.G_i^j(\Phi^{-1}(z)); \quad i \geq 1, \quad j \geq 0. \tag{2.22}$$

Proof: Let us first show that (2.22) holds for $\tilde{G}^0(.,u)$. For

$$z(k+1) \;=\; e^{u\tilde{G}^0(.,u)}[\mathrm{I}_d]\Big|_{\tilde{F}_0(z(k))} = \tilde{F}_0(z(k)) + u\tilde{G}^0(\tilde{F}_0(z(k)),u) + \ldots$$

$$= \Phi(e^{uG^0(.,u)}[\mathrm{I}_d]\Big|_{F_0(x(k))})\Big|_{\Phi^{-1}(z(k))} = e^{uG^0(.,u)}[\Phi]\Big|_{F_0(\Phi^{-1}(z(k)))}$$
$$= \Phi(F_0(\Phi^{-1}(z(k)))) + u(J_x\Phi.G^0(.,u))\Big|_{F_0(\Phi^{-1}(z(k)))} + \dots$$

and

$$(J_x\Phi.G^0(.,u))\Big|_{F_0(\Phi^{-1}(z(k)))} = ((J_x\Phi.G^0(.,u))\Big|_{\Phi^{-1}(\tilde{F}_0(z(k)))}.$$

It follows from (2.6) that

$$\tilde{G}_i^0(z) = (J_x\Phi.G_i^0(.))\Big|_{\Phi^{-1}(z)}.$$

Similar arguments can be used to show that

$$\tilde{G}^1(z,u) = (J_z\tilde{F}_0.\tilde{G}^0(z,u))\Big|_{\tilde{F}_0^{-1}(z)} = (J_x\Phi.G^1(.,u))\Big|_{\Phi^{-1}(z)}$$

and finally (2.22). Q.E.D.

The relative degree associated to a discrete-time system can be set in terms of the $G_i^k(x)$. For, recall that

Definitions 2.1:

a. *The discrete-time system (2.1)-(2.2) is said to have a relative degree r at a point x_0 if*

(i) $\frac{\partial}{\partial u} H \circ F_0^k \circ F(x,u) \neq 0, \quad 0 \leq k < r-1$

(ii) $\frac{\partial}{\partial u} H \circ F_0^{r-1} \circ F(x_0,u) \neq 0.$

b. *It is said to have a strong relative degree r at a point x_0 if (ii) holds at $u = 0$.*

Proposition 2.6: *The discrete-time system (2.1)-(2.2) has a relative degree r at an equilibrium point x_e if and only if*

(i)' $L_{G_i^k}[H]\Big|_x \neq 0, \qquad i \geq 1, \qquad 0 \leq k < r-1$

(ii)' *for some* $i \geq 1, \quad L_{G_i^{r-1}[H]}\Big|_{x_e} \neq 0.$

It has a strong relative degree r if and only if condition (ii)' holds for $i = 1$.

Proof: From definitions (2.1) the output evolution initialized at x_0 do not depend on u up to time $t = r-1$ while $y(r)$ depends on $u(0)$. With this in mind, from (2.21) with $x_0 = x_e$, the necessity immediately follows.

As far as sufficiency is concerned, note that (i)' implies

$$H\circ F_0^k\circ F(x,u) = e^{uG^k(.,u)}[H]\Big|_{(F_0^{k+1}(x))} = H\circ F_0^{k+1}(x), \qquad 0\le k<r-1.$$

Similarly, note that the equality

$$H\circ F_0^{r-1}\circ F(x_e,u) = e^{uG^{r-1}(.,u)}[H]\Big|_{F_0^r(x_e)=x_e} = e^{uG^{r-1}(.,u)}[H]\Big|_{x_e},$$

implies that (ii) is satisfied, once (ii)' is assumed. Q.E.D.

Defining a regular feedback as a smooth function $\gamma : M\times\mathbb{R}\to\mathbb{R}$, such that $\gamma(x_e,0)=0$ and

$$\frac{\partial}{\partial u}\Big|_{u=0}\gamma(x_e,u)\neq 0 \tag{2.23}$$

one has

Proposition 2.7: *Given the discrete-time system (2.1)-(2.2), initialized at an equilibrium point $(x_e,0)$, its relative degree is invariant under coordinates transformations and feedback.*

Proof: The invariance under coordinates transformation follows immediately from (2.22) and Proposition 2.5.

As far as feedback is concerned, note that for $0\le k\le r-1$

$$H\circ F^k(x,\gamma(x,0)) = H\circ F_0^k(x) \tag{2.24}$$

In fact, (2.24) is obviously verified for $k=0$.

By induction, suppose it is true for some $0<k<r-1$. Then

$$\begin{aligned} H\circ F^{k+1}(x,\gamma(x,0)) &= H\circ F^k(.,\gamma(.,0))\circ F(x,\gamma(x,0)) \\ &= H\circ F_0^k(.)\circ F(x,\gamma(x,0)) = H\circ F_0^{k+1}(x) \end{aligned}$$

since

$$\frac{\partial}{\partial u}H\circ F_0^k(.)\circ F(x,u)=0.$$

This shows that the equality holds for $k+1$, which means that condition (i) in definition (2.1) holds. Moreover from (2.24), because of (ii) and (2.23), one deduces that

$$\begin{aligned} &\frac{\partial}{\partial u}H\circ F^{r-1}(.,\gamma(.,0))\circ F(x_e,\gamma(x_e,u)) \\ =\;& \frac{\partial}{\partial u}H\circ F_0^{r-1}(.)\circ F(x_e,u).\frac{\partial\gamma(x_e,u)}{\partial u}\neq 0, \end{aligned}$$

which completes the proof of Proposition (2.7). Q.E.D.

3 Exponential Representation of Sampled Dynamics

Consider a linear analytic continuous-time system of the form

$$\dot{x}(t) = f(x(t)) + u(t)g(x(t)) \tag{3.1}$$

$$y = h(x), \tag{3.2}$$

where $x(t)$ belongs to $\mathbb{R}^{\kappa}$, f and g are analytic vector fields, h is an analytic function and the control is constant over time intervals of amplitude δ : $u(t) = u(k)$ for $k\delta \leq t < (k+1)\delta,\ \ k \geq 0$. The sampling time δ is supposed to belong to $]0, \delta_0[$ where δ_0 is chosen small enough to ensure the convergence of the series expansions with respect to δ manipulated throughout the paper.

Definition 3.1: *The nonlinear discrete-time dynamics*

$$x(k+1) = F^{\delta}(x(k), u(k)) = F_0^{\delta}(x(k)) + \sum_{i\geq 1} \frac{u(k)^i}{i!} F_i^{\delta}(x(k)) \tag{3.3}$$

is said to be sampled equivalent to (3.1) if, at each sampling instant $t = k\delta$, the equality $x(k) = x(t)$ is verified whenever $x(0) = x(t=0)$.

Recurrent expressions for the functions F_i^{δ} can be obtained by means of integro-differential formulae based on the Poincaré expansion of the flow associated to (3.1) or by means of combinatoric relations as proposed in [20].

The main result of this section is represented by the following theorem which parallels Theorem 2.1:

Theorem 3.1 *The sampled dynamics (3.3) can be expressed as*

$$x(k+1) = e^{u(k)E^{\delta}(.,u(k))}[I_d]\Big|_{e^{\delta f}(x(k))}, \tag{3.4}$$

where $uE^{\delta}(.,u)$ is a smooth vector field parametrized by u.

As previously stated, the proof of Theorem 3.1 follows from Propositions 3.1, 3.2 and 3.3 below, which are parallel to Propositions 2.1, 2.2 and 2.3. In fact, to enlighten this parallelism, one has to substitute to the vector fields G_i^k manipulated in Section 2 the vector fields $(E_i^{\delta})^k$ introduced in the sequel. For, let ([20])

$$E_1^{\delta}(x,u) := \frac{\partial}{\partial \epsilon}\Big|_{\epsilon=0} e^{-(\delta f + \delta u g)} e^{\delta f + \delta u g + \delta \epsilon g}[\mathrm{I}_d]\Big|_x \tag{3.5}$$

and define

$$E_1^{\delta}[\mathrm{I}_d]\Big|_x = E_1^{\delta}(x,0)$$

$$E_{i+1}^{\delta}[\mathrm{I}_d]\Big|_x : = \frac{\partial^i}{\partial u^i}\Big|_{u=0} E_1^{\delta}(x,u) \text{ for } i \geq 1,$$

so that

$$E_1^\delta(x,u) = E_1^\delta[\mathrm{I}_d]\Big|_x + \sum_{i\geq 1} \frac{u^i}{i!} E_{i+1}^\delta[\mathrm{I}_d]\Big|_x.$$

With the conventions set in Section 2 with respect to the G_i^0's and hereafter with respect to the E_i^δ's, the following result can be proved arguing as previously.

Proposition 3.1: ([20]) $e^{-\delta f} e^{\delta f + \delta u g}$ *admits the series expansion*

$$e^{-\delta f} e^{\delta f + \delta u g} = 1 + \sum_{i\geq 1} \frac{u^i}{i!} P_i^\delta(E_1^\delta, \ldots, E_i^\delta), \tag{3.6}$$

where the P_i^δ*'s are homogeneous polynomials of degree* i *which can be recursively computed from* $P_1^\delta(E_1^\delta) = E_1^\delta$ *according to the relation*

$$P_i^\delta(E_1^\delta, \ldots, E_i^\delta) = E_1^\delta \circ P_{i-1}^\delta(E_1^\delta, \ldots, E_{i-1}^\delta) + (P_{i-1}^\delta)^+(E_1^\delta, \ldots, E_{i-1}^\delta). \tag{3.7}$$

Moreover

Proposition 3.2: ([20]) *Any polynomial* $P_i^\delta(E_1^\delta, \ldots, E_i^\delta)$ *for* $i \geq 1$ *admits the decomposition*

$$P_i^\delta(E_1^\delta, \ldots, E_i^\delta) = \sum_{\substack{i_1,\ldots,i_m \\ \Sigma i_j = i}} c(i_1, \ldots, i_m) E_{i}^\delta \circ \ldots \circ E_{i_m}^\delta,$$

where the real coefficients $c(i_1, \ldots, i_m)$ *verify the shuffle relations.*

Sketches of Proofs: First, from (3.5) one deduces

$$\frac{\partial}{\partial u} e^{\delta f + u\delta g} = \frac{\partial}{\partial \epsilon}\Big|_{\epsilon=0} e^{\delta f + (u+\epsilon)\delta g} = e^{\delta f + u\delta g} \circ E_1^\delta(.,u); \tag{3.8}$$

that is, for $u = 0$

$$P_1^\delta(E_1^\delta) = E_1^\delta. \tag{3.9}$$

Moreover in (3.6), from (3.8) one computes

$$\begin{aligned} \frac{\partial^2}{\partial u^2}\Big|_{u=0} e^{\delta f + u\delta g} &= \frac{\partial}{\partial u}\Big|_{u=0} (\frac{\partial}{\partial u} e^{\delta f + u\delta g}) = \frac{\partial}{\partial u}\Big|_{u=0} e^{\delta f + u\delta g} \circ E_1^\delta(.,u) \\ &= \frac{\partial}{\partial u}\Big|_{u=0} (e^{\delta f + u\delta g}) \circ P_1^\delta(E_1^\delta) + e^{\delta f} \circ \frac{\partial}{\partial u}\Big|_{u=0} E_1^\delta(.,u) \\ &= e^{\delta f} \circ E_1^\delta \circ P_1^\delta(E_1^\delta) + e^{\delta f} \circ E_2^\delta; \end{aligned}$$

that is,

$$P_2^\delta(E_1^\delta, E_2^\delta) = E_1^\delta \circ P_1^\delta(E_1^\delta) + (P_1^\delta)^+(E_1^\delta) = E_1^\delta \circ E_1^\delta + E_2^\delta, \tag{3.10}$$

according to $E_2^\delta = (E_1^\delta)^+$ defined in (3.6). Iterating the reasoning, one recovers (3.7).

The same reasoning as in Section 2 holds to prove Proposition 3.2.

Q.E.D.

Proposition 3.3: *$E^\delta(.,u)$ in (3.3) is a Lie element in the vector fields E_i^δ's which takes the form,*

$$uE^\delta(.,u) = \sum_{i\geq 1} \frac{u^i}{i!} B_i(E_1^\delta, \ldots, E_i^\delta), \tag{3.11}$$

where $B_i(E_1^\delta, \ldots, E_i^\delta)$ is a homogeneous Lie polynomial of degree i.

For the first terms, one obtains as previously

$$B_1^\delta = E_1^\delta, \quad B_2^\delta = E_2^\delta, \quad B_3^\delta = E_3^\delta + 1/2[E_1^\delta, E_2^\delta], \quad B_4^\delta = E_4^\delta + [E_1^\delta, E_3^\delta],$$

The proof of Proposition 3.3 and Theorem 3.1. is achieved with the same arguments used in Section 2.

Defining as in (2.20) from E_i^δ the transported vector fields $(E_i^\delta)^k$, one has

$$(E_i^\delta)^k := e^{-\delta k f} \circ E_i^\delta \circ e^{\delta k f} = e^{-k\delta ad_f} E_i^\delta, \quad i \geq 1, \quad k > 0. \tag{3.12}$$

Remark 3.1: Proceeding further the analogies between the discrete-time and sampled dynamics, the analogous of Corollary 2.1 and 2.2 can be stated for a sampled dynamics. Moreover coordinates changes on f and g reflect on transformed vector fields E_i^δ as in Proposition 2.4.

Because of the sampled context, the following insight in the particular structure of the vector fields can be done arguing as in [8]. The vector fields E_i^δ are computed from the continuous dynamics by means of ([20])

$$\begin{aligned} E_1^\delta &= \frac{1 - e^{-\delta ad_f}}{\delta ad_f}(\delta g) \\ E_2^\delta &= \frac{1 - \delta ad_f - e^{-\delta ad_f}}{\delta^2 ad_f^2} \omega \ \delta ad_g(\delta g) \\ \ldots\ldots\ldots\ldots & \quad \ldots\ldots\ldots\ldots\ldots , \end{aligned} \tag{3.13}$$

where "ω " denotes the shuffle product, "ad" the usual Lie bracket operator and where the quotient must be interpreted as a formal cancellation.

Remark 3.2: Denoting by $\mathcal{L}$ the Lie algebra generated by f and g and by $\mathcal{L}_0$ the Lie ideal of $\mathcal{L}$ generated by g, it follows from (3.13) that

$$E_i^\delta \in \mathcal{L}_0^i \quad i \geq 1,$$

where $\mathcal{L}_0^1 = \mathcal{L}_0$ and for $i \geq 2$, $\mathcal{L}_0^i = [\mathcal{L}_0^1,\ \mathcal{L}_0^{i-1}]$, (decreasing sequence of ideals).

It is interesting to note that the vector fields (3.6) satisfy the following Lie decomposition, which fully characterize their structure and play a central role when studying the inverse problem of the discretization. More precisely, one can state

Theorem 3.2 *The vector fields E_i^δ satisfy*

$$E_2^\delta = \int_0^\delta [(E_1^\tau)',\ E_1^\tau]\, d\tau \tag{3.14}$$

or equivalently

$$(E_2^\delta)' = [(E_1^\delta)', E_1^\delta]. \tag{3.15}$$

Proof: Note that integrating by parts (3.10) yields to the equality

$$P_2^\delta(E_1^\delta, E_2^\delta) = \int_0^\delta (E_1^\tau)' \circ E_1^\tau\, \mathrm{d}\tau + \int_0^\delta E_1^\tau \circ (E_1^\tau)' d\tau + E_2^\delta. \tag{3.16}$$

On the other side, from Poincaré integro-differential formulae set in [20] and based on the integration of the Volterra kernels characterizing the input-state evolution associated to the dynamics (3.1), one has

$$P_2^\delta(E_1^\delta, E_2^\delta) = 2\int_0^\delta (E_1^\tau)' \circ E_1^\tau\, d\tau. \tag{3.17}$$

Combining (3.16) and (3.17), one easily obtains (3.14) and (3.15). Q.E.D.

Remarks 3.3:

(i) Denoting by $()'$ the derivative with respect to δ and $()^+$ the derivative with respect to u it can also be proved that

$$(E_{i+1}^\delta)' := ((E_i^\delta)^+)' = ((E_i^\delta)')^+ \tag{3.18}$$

and

$$(E_i^\delta)' = -(i-1) ad_g E_{i-1}^\delta - ad_f E_i^\delta \quad i \geq 2 \tag{3.19}$$

with

$$(E_1^\delta)' = g - ad_f E_1^\delta. \tag{3.20}$$

(ii) The relative degree of a sampled system is, generically with respect to δ, equal to 1 ([18]).

(iii) A necessary and sufficient condition for maintaining the linear analytic structure ($F_j^\delta(x) = 0$ for $j \geq 2$) under sampling is given by

$$E_{2+i}^\delta(x) = -\sum_{p=0}^{i} c_i^p J_x(E_{i+1-p}^\delta).G_{p+1}^\delta(x) \quad \forall i \geq 0. \tag{3.21}$$

4 Linear Equivalence

How do intrinsic properties of the dynamics reflect on the canonical vector fields so far introduced? In this section, linear equivalences in discrete-time and under sampling will be investigated from this point of view.

Definition 4.1: *A nonlinear system is locally linear equivalent if there exists a smooth coordinate change, $z = \Phi(x)$ around x_e , under which the system is transformed into a linear controllable one.*

Linear equivalence under sampling will denote,with respect to the sampled dynamics, linear equivalence for any $\delta \in]0, \delta_0[$.

4.1 Discrete-time linear equivalence

Starting from a single input linear dynamics on $\mathbb{R}^{\ltimes}$ of the form

$$x(k+1) = F(x(k), u(k)) = Ax(k) + Bu(k), \tag{4.1}$$

where A and B are matrices of suitable dimensions, it can immediately be verified that

$$G_1^0(x, u) = \frac{\partial}{\partial \epsilon}\Big|_{\epsilon=0} (AF^{-1}(x, u)) + Bu + B\epsilon = B = G_1^0(x) = \text{Cst}. \tag{4.2}$$

and

$$G_i^0(x) = 0 \quad \text{ for } i \geq 2 \tag{4.3}$$

$$G_1^k(x) = A^k G_1^0(x) = A^k B = \text{Cst}. \quad \forall k \in Z, \tag{4.4}$$

so that

$$[G_1^0, G_1^k](x) = 0 \quad \forall k \in Z. \tag{4.5}$$

From (4.3) one deduces the exponential representation of (4.1); that is,

$$x(k+1) = e^{u(k)B}[\mathrm{I}_d]\Big|_{Ax(k)}. \tag{4.6}$$

From (4.5) the input-state evolution can be expressed as

$$x(k) = e^{u(0)A^{k-1}B} \circ \ldots \circ e^{u(k-1)B}[\mathrm{I}_d]\Big|_{A^k x_0} = e^{u(0)A^{k-1}B + \ldots + u(k-1)B}[\mathrm{I}_d]\Big|_{A^k x_0} \tag{4.7}$$

From these simple comments, one concludes that the linearity of the dynamics implies in particular conditions (4.3) and (4.5). In fact, it will be shown hereafter that they are necessary and sufficient for linear equivalence.

Theorem 4.1 *The drift invertible dynamics (2.1) is locally linear equivalent if and only if the following conditions hold*

$$\begin{array}{ll} (i) & G_k^0 = 0 \quad \forall k \geq 2 \\ (ii) & [G_1^0, G_1^i] = 0 \text{ for } i \geq 0 \\ (iii) & \rho(G_1^0(x_e), \ldots, G_1^{n-1}(x_e)) = n. \end{array} \tag{4.8}$$

Proof - Necessity: follows from the invariance of (i) ÷ (iii) under coordinate transformations, moreover they are true for a linear controllable system, where $G_1^0 = B, G_k^0 = 0$ and $G_1^i = A^i B$ (see (4.3) and (4.4)).

Sufficiency: From standard results of differential geometry, (ii) and (iii) imply the existence of a local coordinates transformation $z = \Phi(x)$ such that

$$\frac{\partial \Phi}{\partial x}.(G_1^0, \ldots, G_1^{n-1}) = I_n, \tag{4.9}$$

which implies according to (i)

$$\tilde{G}^k(z,u) = \left(\frac{\partial \Phi}{\partial x}.G^k(.,u)\right)\Big|_{\Phi^{-1}(z)} = \tilde{G}_1^k = \text{Cst}, \quad 0 \leq k \leq n-1. \tag{4.10}$$

Moreover, because of (ii) and (iii), (4.10) implies $\tilde{G}_1^n = \text{Cst}$. For, assuming

$$\tilde{G}_1^n(z) = \sum_{i=0}^{n-1} \alpha_i(z) \tilde{G}_1^i$$

from

$$[\tilde{G}_1^j, \tilde{G}_1^n] = 0 \text{ for } j = 0, \ldots n-1$$

it follows that

$$\tilde{G}_1^j(\alpha_i(z)) = 0 \text{ for } i = 0, \ldots n-1 \text{ and } j = 0, \ldots n-1, \text{ i.e. } \alpha_i(z) = \text{Cst}.$$

and

$$\tilde{G}_1^n(z) = \text{Cst}.$$

Now, from

$$\tilde{G}_1^k = \left(J_z \tilde{F}_0^k\Big|_{\tilde{F}_0^{-k}(z)}\right).\tilde{G}_1^0 = \left(J_z \tilde{F}_0^1\Big|_{\tilde{F}_0^{-1}(z)}\right).\tilde{G}_1^{k-1} = \text{Cst}. \quad 0 \leq k \leq n,$$

it follows

$$\left(J_z \tilde{F}_0^1\Big|_{\tilde{F}_0^{-1}(z)}\right).\left(\tilde{G}_1^0, \ldots, \tilde{G}_1^{n-1}\right) = (\tilde{G}_1^1, \ldots, \tilde{G}_1^n),$$

which because of (iii), implies that $\tilde{F}_0(z)$ is linear with respect to z. Q.E.D.

Remark 4.1: It can be interesting to point out some analogy between the conditions for discrete-time linear equivalence and continuous-time linear equivalence when generally nonlinear differential dynamics are assumed. For, given a nonlinear differential dynamics

$$\dot{x}(t) = f(x(t), u(t)) = f(x(t)) + \sum_{i\geq 1} \frac{u^i}{i!} g_i(x(t)) \tag{4.11}$$

generalizing well-known conditions for continuous-time linear equivalence of linear analytic dynamics ([12]), it is easy to show that (4.11) is linear equivalent if and only if (see also [27]-[28])

$$\begin{aligned} &\text{(i)'} \quad g_k = 0 \quad \forall k \geq 2 \\ &\text{(ii)'} \quad [g_1, ad_f^k g_1] = 0 \text{ for } k \geq 0 \\ &\text{(iii)'} \quad \rho(g_1(x_e), \ldots, ad_f^{n-1} g_1(x_e)) = n. \end{aligned} \tag{4.12}$$

Note that (i)' means that the given dynamics must be linear analytic, as clearly expected.

Remark 4.2: Looking at the conditions in Theorem 4.1 and in Remark 4.1, the intuition suggests that a discrete-time dynamics satisfying conditions (i) in (4.8) might be the homologue of a continuous linear analytic dynamics, i.e. of the form (3.1). More precisely, such a nonlinear discrete-time dynamics admits an exponential representation of the form

$$x(k+1) = e^{u(k)G_1^0(.)}[\mathrm{I}_d]\Big|_{F_0(x(k))}, \tag{4.13}$$

which derives from particular state representations, but generically nonlinear in u.

The main feature of representation (4.13) is to allow the extension to the discrete-time situation of several results stated for linear analytic continuous control systems by simply substituting $ad_f^k g$ by G^k defined in (2.18). This fact, previously illustrated with respect to the linear equivalence concept, will be clarified in the next section too. It results that the dynamics (4.13) can be interpreted as the discrete-time equivalent of a linear analytic continuous one.

As an example, a particular case is represented by the discrete-time dynamics: $F(x,u) = F_0(x) + Bu$, for which one easily computes $G_1^0(x) = B = \mathrm{Cst}$, or more generally, from dynamics of the form

$$F(x,u) = F_0(x) + \sum_{i\geq 1} \frac{u^i}{i!} Fi(x)$$

with

$$\begin{aligned} F_2(x) &= J(F_1 \circ F_0^{-1}) F_1(x) \\ F_i(x) &= J(F_{i-1} \circ F_0^{-1}) F_1(x), \text{ for } i \geq 2 \ldots, \end{aligned}$$

which imply

$$G_1(x) = F_1 \circ F_0^{-1}(x) \text{ and } G_i(x) = 0 \text{ for } i \geq 2.$$

4.2 Linear equivalence under sampling

Let us now consider the sampled situation. For, setting in (3.1) $f(x) = Ax$ and $g(x) = B$, it can easily be verified from (3.13), (3.6) and (3.6) that

$$\begin{aligned} E_1^\delta = \frac{e^{\delta A} - I}{A} B &= \sum_{i \geq 1} \frac{\delta^i}{i!} A^{i-1} B = \text{Cst.} \\ E_i^\delta(x) &= 0, \quad \forall i \geq 2 \\ (E_1^\delta)^k &= e^{k\delta A} E_1^\delta = \text{Cst.}, \quad \forall k \in Z, \end{aligned} \tag{4.14}$$

and thus

$$[E_1^\delta, (E_1^\delta)^k](x) = 0, \quad \forall k \in Z. \tag{4.15}$$

(4.14) and (4.15) correspond to conditions (4.3) and (4.5) and characterize the linearity of the state equations which is preserved under sampling. On the basis of the analogies shown in Section 3, with the same arguments as those used for Theorem 4.1, the result stated in [1] (Theorem 2.1) can be reformulated and proved as follows:

Theorem 4.2 *The dynamics (3.1) is locally linear equivalent under sampling if and only if for any* $\delta \in]0, \delta_0[$

$$\begin{aligned} &\text{(i)} \quad E_i^\delta = 0, \quad i \geq 2 \\ &\text{(ii)} \quad [E_1^\delta, e^{-i\delta ad_f} E_1^\delta] = 0, \quad \forall i > 0 \\ &\text{(iii)} \quad \rho(E_1^\delta, e^{-\delta ad_f} E_1^\delta, \ldots, e^{-(n-1)\delta ad_f} E_1^\delta)\Big|_{x_e} = n. \end{aligned} \tag{4.16}$$

It is interesting to note that, because of particular combinatoric properties of the vector fields E_i^δ , pointed out in Section 3, conditions (i) and (ii) can be simplified.

Theorem 4.2' *The dynamics (3.1) is locally linear equivalent under sampling if and only if for any* $\delta \in]0, \delta_0[$ *the equivalent conditions A and B below are true.*

A $\quad E_2^\delta = 0$ and (iii) in (4.16)

B (ii) and (iii) in (4.16)

Proof: As far as **A** is concerned, one has to show that $E_2^\delta = 0$ implies (i) and (ii) of Theorem 4.2.

For, from (3.13)

$$E_2^\delta = \sum_{i \geq 1} (-1)^{j+1} \frac{\delta^{j+2}}{(j+2)!} ad_f^j \omega \ ad_g g = 0$$

implies

$$ad_f^j \omega \ ad_g g = ad_f^j ad_g g + ad_f ad_g ad_f^{j-1} g + \ldots + ad_g ad_f^j g = 0 \quad \forall j > 1. \quad (4.17)$$

For $j = 1$, one has $ad_g ad_f g = 0$.

Moreover, assuming $ad_g ad_f^k g = 0$, for $1 < k \leq j-1$, it follows from (4.17) that

$$ad_g ad_f^j g = 0 \text{ for } j \geq 0. \quad (4.18)$$

Clearly, from Remark 3.2 -(i) in Section 3, $\mathcal{L}_0^i \subset \mathcal{L}_0^2 = 0$ for $i > 2$; hence, $E_i^\delta = 0$ for $i > 2$ (i.e. condition (i) in Theorem 4.2). As far as (ii) of Theorem 4.2 is concerned, it is sufficient to note that

$$[E_1^\delta, e^{-i\delta ad_f} E_1^\delta] \in \mathcal{L}_0^2,$$

which was proved to be identically zero.

As far as **B** is concerned, one has to show in (4.16) that (ii) implies (i), which means, because of **A**, it is enough to show that (ii) implies $E_2^\delta = 0$. For, one can note in (4.16) that (ii) for $i = 1$ implies (4.18) and thus $E_2^\delta = 0$. In fact, because of the definitions (3.12) and (3.13), one has

$$[E_1^\delta, e^{-\delta ad_f} E_1^\delta] := \sum_{k \geq 2} (-1)^k \delta^k \sum_{i=1}^{k-1} \frac{(2^{k-i} - 1)}{i!(k-i)!} [ad_f^{i-1} g, ad_f^{k-i-1} g] = 0;$$

that is, for any $k \geq 2$

$$\sum_{i=1}^{k-1} (-1)^{j+1} \frac{(2^{k-i} - 1)}{i!(k-i)!} [ad_f^{i-1} g, ad_f^{k-i-1} g] = 0,$$

which implies the condition $ad_g ad_f^j g = 0$, for $j \geq 1$, arguing in a recurrent way and applying iteratively the Jacobi identity; that is,

$$[ad_f^{i-1} g, ad_f^{k-i-1} g] = ad_f [ad_f^{i-2} g, ad_f^{k-i-2} g] - [ad_f^{i-2} g, ad_f^{k-i-1} g].$$

Q.E.D.

Remark 4.3: Note as in [1] that (iii) of Theorem 4.2 is equivalent to

$$\rho(g(x_e), ad_f g(x_e), \ldots, ad_f^{n-1} g(x_e)) = n.$$

For,

$$e^{-i\delta ad_f} E_1^\delta = \sum_{i\geq 1} (-1)^{j+1} \frac{\delta^j}{j!} \left((i+1)^j - i^j\right) ad_f^{j-1} g \quad \forall i \geq 0 \tag{4.19}$$

implies

$$\left(E_1^\delta, \ldots, e^{-(n-1)\delta ad_f} E_1^\delta\right) = (g, ad_f g, \ldots,)D,$$

where

$$D = \{d_{kl}\}_{k\geq 1}^{l=1,\ldots,n} = \{\frac{\delta^k}{k!}(-1)^{k-1}(l^k - (l-1)^k)\}_k^l \tag{4.20}$$

$\rho(D) = n$ since, by substituting to each column the sum of the previous ones, we recover a Vandermonde matrix.

From Theorem 4.2' and Remark 4.3, taking into account that $E_2^\delta = 0$ for any $\delta \in]0, \delta_0[$ if and only if $ad_g ad_f^k g = 0$, for $k \geq 1$, it remains to prove that

Corollary 4.1: *Linear equivalence under sampling holds, if and only if the linear analytic continuous time dynamics (3.1) satisfies*

(i)' $ad_g ad_f^k g = 0$ *for* $k \geq 1$
(ii)' $\rho(g(x_e),\ ad_f g(x_e), \ldots, ad_f^{n-1} g(x_e)) = n,$

i.e. the continuous dynamics is locally linear equivalent (Remark 4.1 in [17]).

Remark 4.4: It follows from Theorem 4.2 and Corollary 4.1 that linear analytic dynamics are described under sampling by exponential representations of the form (4.13) if and only if they are linear equivalent. It is not difficult to verify that sampling a general nonlinear dynamics of the form (4.11), $E_i^\delta(x) = 0$ for $\delta \in]0, \delta_0[$ may result only if $g_i = 0$ for $i \geq 2$. So that under sampling, exponential representations of the form (4.13) are obtained if and only if the continuous-time system is linear equivalent (see Remark 4.1). This analysis confirms the limited interest of sampled dynamics of the form (4.13).

5 Linear Feedback Equivalence

Linear feedback equivalence for nonlinear discrete-time and sampled dynamics were studied in [9] and [1]. Hereafter these results are reformulated and studied in terms of the canonical vector fields, G_i^0 's and E_i^δ 's, with an approach which is parallel to the one used in the continuous context ([12]).

Definition 5.1: *A nonlinear system is (locally) linear feedback equivalent if there exist around x_e, a smooth coordinates change $z = \Phi(x)$ and a regular feedback such that the closed loop system is linear equivalent.*

Linear equivalence under sampling will denote,with respect to the sampled dynamics, linear equivalence for any $\delta \in]0, \delta_0[$.

5.1 Discrete-time linear feedback equivalence

With reference to a discrete-time dynamics of the form (2.1), denoting by "associated relative degree" to a function defined from M to R the relative degree associated to the system composed with the dynamics (2.1) and the given function, the following result can be proved.

Lemma 5.1 *The dynamics (2.1) is locally linear feedback equivalent if and only if there exists around x_e a smooth real valued function $\lambda(x)$, with $\lambda(x_e) = 0$ for which the associated strong relative degree is equal to n.*

Proof - Sufficiency: Since the strong relative degree associated to λ is equal to n, from Proposition 2.5 one computes from $x(0) \in U_x(0)$ a neighbourhood of $x(0)$

$$\begin{aligned} \lambda(x(1)) &= \lambda \circ F_0(x(0)), \lambda(x(2)) = \lambda \circ F_0^2(x(0)), \ldots, \lambda(x(n-1)) \\ &= \lambda \circ F_0^{n-1}(x(0)) \end{aligned}$$

and

$$\lambda(x(n)) = \lambda \circ F_0^{n-1} \circ F(x(0), u(0)).$$

The existence of a feedback law $u = \gamma(x, v)$, solution of the equality $\lambda(x(n)) = v$, where v is an external input follows from the implicit function theorem and because of the definition of a strong relative degree. It is now sufficient to consider the coordinates change ([18])

$$z = \Phi(x) = (\lambda(x), \lambda \circ F_0(x), \ldots, \lambda \circ F_0^{n-1}(x))^T$$

to transform (2.1) into the linear controllable Brunovsky canonical form

$$z(k+1) = \begin{pmatrix} 0 & 1 & 0 & - & 0 \\ 0 & 0 & 1 & - & 0 \\ & & \ddots & & \\ & & & & 1 \\ 0 & - & - & - & 0 \end{pmatrix} z(k) + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} v(k). \qquad (5.1)$$

Necessity Assume, without loss of generality, that the locally equivalent linear dynamics is in a Brunovsky canonical form which has an obviously strong relative degree n associated to $\lambda(z) = z_1$. The result is proved by recalling the invariance of the relative degree under coordinated transformation and regular feedback (Proposition 2.6). Q.E.D.

On these bases, the following theorem can be proved:

Theorem 5.1 *The dynamics (2.1) is locally linear feedback equivalent around x_e if and only if*

$$\begin{array}{ll} \text{(i)} & G_k^0 // G_1^0 \quad for \quad k > 1 \\ \text{(ii)} & span\{G_1^0, \ldots, G_1^{n-2}\} \quad is \quad involutive \quad around \quad x_e \\ \text{(iii)} & \rho(G_1^0(x_e), \ldots, G_1^{n-1}(x_e)) = n. \end{array} \tag{5.2}$$

Proof: It will be shown that conditions (i) - (iii) are equivalent to the existence of a function λ as in Lemma 5.1. Arguing as in the continuous-time context ([12]), let us first assume that such a function exists. From the definition of a strong relative degree, λ satisfies

$$\frac{\partial \lambda}{\partial x}.\left(G^0(.,u), \ldots, G^{n-2}(.,u)\right) = 0$$

for any u and

$$\frac{\partial}{\partial u}\Big|_{u=0}\left(\frac{\partial \lambda}{\partial x}.G^{n-1}(.,u)\right) \neq 0.$$

In particular for $u = 0$

$$\frac{\partial \lambda}{\partial x}.\left(G_1^0, \ldots G_1^{n-2}\right) = 0 \tag{5.3}$$

and

$$\frac{\partial \lambda}{\partial x}\Big|_{x_e}.G_1^{n-1} \neq 0,$$

which imply (ii) and (iii) because of the Frobenius Theorem.

Moreover, function λ satisfies, for any u

$$\frac{\partial \lambda}{\partial x} G^0(.,u) = \ldots = \frac{\partial \lambda \circ F_0^{n-2}}{\partial x} G^0(.,u) = 0,$$

which implies (i). On the other hand, because of (ii), there exists a function λ solving (5.3). Moreover, because of (i) and (iii), it is easily verified that the associated relative degree is a strong relative degree equal to n . Q.E.D.

Arguing as in the continuous-time case, the next remarks follow.

Remarks 5.1:

(i) When $n = 2$, the conditions of Theorem 5.1 reduces to

(a) $G^0(.,u_1)//G^0(.,u_2)$

(b) $\rho\left(G_1^0(x_e), G_1^1(x_e)\right) = 2.$

(ii) The conditions of Theorem 5.1 imply, around x_e, the involutivity of the distribution

$$\text{span}\,\{G_1^0, \ldots, G_1^k\}, \qquad 1 \leq k \leq n-3.$$

It is interesting to note that with the same arguments used in the proof of Theorem 5.1, the problem of linear feedback equivalence for a general nonlinear continuous-time dynamics can be solved. Formally, the analogy is evident.

Theorem 5.2 *The nonlinear continuous-time dynamics (4.11) is locally linear feedback equivalent if and only if*

$$\begin{array}{ll} \text{(i)} & g_k//g_1 \quad \textit{for} \quad k > 1 \\ \text{(ii)} & \textit{span}\{g_1, \ldots, ad_f^{n-2} g_1\} \quad \textit{is} \quad \textit{involutive} \quad \textit{around} \quad x_e \\ \text{(iii)} & \rho\left(g_1(x_e), \ldots, ad_f^{n-1} g_1(x_e)\right) = n. \end{array}$$

Remark 5.2 From Theorem 5.1, particularly from condition (i) in (5.2), a dynamics which admits an exponential representation of the form

$$x(k+1) = e^{\alpha(x,u)G(.)}[\mathrm{I}_d]\Big|_{\Phi_0^{-1}(x(k))} \tag{5.4}$$

with

$$\frac{\partial \alpha(x,.)}{\partial u} \neq 0$$

might be considered for studying linear feedback equivalence. Note that such an exponential representation generalizes (4.13).

5.2 Linear feedback equivalence under sampling

All the results previously discussed hold when sampled systems are investigated. For, it suffices to substitute E_i^δ to G_i^0 and $e^{-k\delta ad_f} E_i^\delta$ to G_i^k. As far as linear feedback equivalence under sampling is concerned, let us reformulate Theorem 2.2 in [1] in terms of the vector fields E_i^δ's.

Theorem 5.3 *A nonlinear continuous system is locally linear feedback equivalent under sampling if and only if for any* $\delta \in]0, \delta_0[$

$$\begin{array}{ll} \text{(i)} & E_i^\delta // E_1^\delta \quad \textit{for} \quad i \geq 2 \\ \text{(ii)} & \textit{span}\{E_1^\delta, \ldots, e^{-\delta(n-2)ad_f} E_1^\delta\} \textit{is involutive around } x_e \\ \text{(iii)} & \rho\left(E_1^\delta, e^{-\delta ad_f} E_1^\delta, \ldots, e^{-(n-1)\delta ad_f} E_1^\delta\right)\Big|_{x_e} = n. \end{array} \tag{5.5}$$

Moreover,

Proposition 5.1: *$E_i^\delta // E_1^\delta$ for $i \geq 2$ imply*

$$E_i^\delta = (i-1)!\alpha(\delta)^{i-1} E_1^\delta, \tag{5.6}$$

where $\alpha(\delta)$ is an analytic function defined from $]0, \delta_0[$ to R.

Proof: It will be obtained recurrently. Assuming (5.6) true up to k with $E_2^\delta = \alpha(\delta) E_1^\delta$ and assuming the existence of an analytic function, say $\beta(\delta)$, such that

$$E_{k+1}^\delta = \beta(\delta) E_1^\delta, \tag{5.7}$$

one deduces that $\beta(\delta) = k!\alpha(\delta)^k$. For, because of (5.7) and (3.19) one has

$$\begin{aligned} (E_{k+1}^\delta)' &= (\beta(\delta))' E_1^\delta + \beta(\delta)(E_1^\delta)' \\ &= -k ad_g(E_k^\delta) - ad_f(E_{k+1}^\delta) \\ &= -k!\alpha(\delta)^{k-1} ad_g E_1^\delta - k! g(\alpha(\delta)^{k-1}) E_1^\delta \\ &\qquad -\beta(\delta) ad_f E_1^\delta - f(\beta(\delta)) E_1^\delta \end{aligned} \tag{5.8}$$

(3.19); that is,

$$ad_f E_1^\delta = g - (E_1^\delta)'$$

and because of (5.6), for $i = 2$, and (3.19)

$$ad_g E_1^\delta = -(\alpha(\delta))' E_1^\delta - f(\alpha(\delta)) E_1^\delta - \alpha(\delta) g, \tag{5.9}$$

one deduces from the equalities (5.8) the following condition

$$\begin{aligned} \{k!(\alpha(\delta)^{k-1}(\alpha(\delta))' + k!(\alpha(\delta)^{k-1} f(\alpha) - k! g(\alpha(\delta)^{k-1}) - f(\beta(\delta)) - (\beta(\delta))'\} E_1^\delta \\ + \{k!\alpha(\delta)^k + \beta(\delta)\} g = 0, \end{aligned}$$

which yields to $\beta(\delta) = k!\alpha(\delta)^k$ since the vector fields g and E_1^δ are linearly independent, otherwise $g // ad_f^i g$ for $i \geq 1$.

Proposition 5.2: *$E_i^\delta // E_1^\delta$ for $i \geq 2$ and (iii) in (5.5) imply the linear feedback equivalence of the continuous dynamics (3.1).*

Proof: It is enough to show that the parallelism $E_i^\delta // E_1^\delta$ for $i \geq 2$ implies the involutivity of the distribution span $\{g, \ldots, ad_f^k g\}$ for $k \geq n-2$. In fact because of (5.6) and the definition (3.13) of E_2^δ , one directly deduces from the parallelism $E_2^\delta // E_1^\delta$, the parallelism of g with $ad_g ad_f^i g$ for $i \geq 1$. Then according to the identity

$$ad_f ad_g ad_f^i g = [ad_f g, ad_f^i g] + ad_g ad_f^{i+1} g,$$

one concludes that $[ad_f g, ad_f^i g]$ belongs to span $\{g, ad_f g\}$ for $i \geq 1$. Iterating the reasoning, it follows that any vector field of the form $[ad_f^p g, ad_f^i g]$

for $0 \leq p \leq n-2$ belongs to span $\{g, \ldots, ad_f^p g\}$ which is sufficient to prove the involutivity of span $\{g, \ldots, ad_f^p g\}$ for $p \leq n-2$. Q.E.D.

From Proposition 5.2, it results that sampled nonlinear exponential representations of the form (5.4) characterize a subset of the ones obtained under sampling from linear feedback equivalent dynamics. We recall that in [10] it has been conjectured that feedback linearizability under sampling of a linear analytic continuous-time system of the form (3.1) implies its linear equivalence. A proof has been given in §1 for $n = 2$ but for $n > 2$ even if very restrictive requirements on f and g can be emphasized, no complete proof is available.

6 Conclusions

In this paper, a unified representation for nonlinear discrete-time and sampled dynamics has been proposed. An exponential form and "canonical vector fields" are associated to this representation. This provides a common framework for the study of discrete-time and sampled dynamics, as shown by discussing linear and linear feedback equivalences.

References

[1] A. Arapostathis, B. Jakubczyk, H.G. Lee, S.I. Marcus and E.D. Sontag. The effect of sampling on linear equivalence and feedback linearization, *Systems and Cont. Letters* **13** (1989), 373-381.

[2] J.P. Barbot. A forward accessibility algorithm for nonlinear discrete time systems, in *Lect. Notes in Cont. and Info. Sciences* (A. Bensoussan and J.L. Lions, eds.) **144**. New York: Springer-Verlag, 1990, 314-323.

[3] C. Bruni, G. di Pillo and G. Koch. On the mathematical models of bilinear systems, *Ricerche di Automatica* **2** (1971), 11-26.

[4] M. Fliess. Un codage non commutatif pour certains systèmes échantillonnés non linéaires, *Information Control* **38** (1978), 264-287.

[5] M. Fliess. Automatique en temps discret et algèbre aux différences, *Forum Math.* **2** (1990), 213-232.

[6] M. Fliess. Reversible linear and nonlinear discrete-time dynamics, *IEEE Trans. on Ac.* **37(8)** (1992), 1144-1153.

[7] M. Fliess and D. Normand-Cyrot. A group theoretic approach to discrete-time nonlinear controllability, in *Proc 20-th IEEE-CDC*, San Diego, 1981, 551-557.

[8] R. Goodman. Lifting vector fields to nilpotent Lie groups, *J. Math. Pures et Appl.* **57** (1978), 77-86.

[9] J.W. Grizzle. Controlled invariance for discrete-time nonlinear systems with an application to the disturbance decoupling problem, *IEEE Trans Aut. Cont.* **30** (1985), 86-874.

[10] J.W. Grizzle. Feedback linearization of discrete-time systems, in *Lect. Notes in Cont. and Info. Sciences* **83**. New York: Springer-Verlag, 1986, 273-281.

[11] Gröbner. Serie di Lie e Loro Applicazioni, *Poliedro.* Roma: Cremonese, 1973.

[12] A. Isidori. *Nonlinear Control Systems: An Introduction* (II ed.). New York: Springer-Verlag, 1989.

[13] B. Jakubczyk. Feedback linearization of discrete-time systems, *Systems and Cont. Letters* **9** (1987), 411-416.

[14] B. Jakubczyk and D. Normand-Cyrot. Orbites de pseudo groupes de difféomorphismes et commandabilité des systèmes non linéaires en temps discret, *C.R. Acad. Sc.* **298** (1984), 257-260.

[15] B. Jakubczyk and E.D. Sontag. Controllability of nonlinear discrete time systems; a Lie algebraic approach, *SIAM J. Cont. and Opt.* **28** (1990).

[16] H.G. Lee, A. Arapostathis and S.I. Marcus. On the linearization of discrete-time systems, *Int. J. Cont.* **45** (1987), 1783-1785.

[17] S. Monaco and D. Normand-Cyrot. Invariant distributions for discrete-time nonlinear systems, *Systems and Cont. Letters* **5** (1985), 191-196.

[18] S. Monaco and D. Normand-Cyrot. Zero dynamics of sampled nonlinear systems, *Systems and Cont. Letters* **11** (1988), 229-234.

[19] S. Monaco and D. Normand-Cyrot. Functional expansions for nonlinear discrete-time systems, *Math. Sys. Theo.* **21** (1989), 235-254.

[20] S. Monaco and D. Normand-Cyrot. A combinatorial approach to the nonlinear sampling problem, in *Lect. Notes in Cont. and Info. Sciences* (A. Bensoussan and J.L. Lions, eds.) **144**. New York: Springer-Verlag, 1990, 788-797.

[21] S. Monaco and D. Normand-Cyrot. Canonical representations for nonlinear discrete time systems, in *Systems, Models and Feedback, Theory and Applications* (A. Isidori and T.J. Tarn, eds.). Boston: Birkhaüser, 1992.

[22] R. Ree. Lie elements and an algebra associated with shuffles, *Ann. of Maths.* **68** (1958), 210-220.

[23] A. Ruberti, P. d'Alessandro and A. Isidori. Realization and structure theory of bilinear dynamical systems, *SIAM J. Cont.* **12** (1974), 517-537.

[24] E.D. Sontag. Polynomial response maps, in *Lect Notes in Cont. and Info. Sci.* **13**. Berlin: Springer-Verlag, 1979.

[25] E.D. Sontag. An eigenvalue for sampled weak controllability of bilinear systems, *Systems and Cont. Letters* **7** (1986), 313-316.

[26] E.D. Sontag. Orbit theorems and sampling, in *Algebraic and Geometric Methods in Nonlinear Control Theory* (M. Fliess and H. Hazenwinkel, eds.). Reidel: Dordrecht, 1986, 441-486.

[27] R. Su. On the linear equivalence of nonlinear systems, *Systems and Cont. Letters* **2** (1982), 48-52.

[28] H.J. Sussmann. Lie brackets, real analycity and geometric control, in *Differential Geometric Control Theory* (R.W. Brockett, et. al., eds.). Boston: Birkhäuser, 1983, 1-116.

Dipartimento di Informatica e Sistemistica, Università di Roma "La Spaienza," Via Eudossiana 18, 00184 Rome, Italy

Laboratoire des Signaux et Systèmes, CNRS-ESE, Plateau de Moulon, 91190 Gif-sur-Yvette, France

Communicated by Christopher I. Byrnes