Journal of Mathematical Systems, Estimation, and Control
Vol. 5, No. 1, 1995, pp. 1–21

Limit Theorems of Probability Theory in Linear Controlled Evolution Systems with Quadratic Cost*

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Abstract

In the control or adaptive control of linear stochastic evolution systems with complete observations of the state it is important to know the asymptotic distribution of the quadratic cost or the asymptotic bounds for the fluctuation of the average cost around the optimal average cost. In this paper stochastic evolution systems are considered. These systems arise from a semigroup description of various infinite dimensional systems such as linear partial differential equations. These systems are often controlled and the quadratic cost provides a measure of controlled performance. The optimal average cost as the limit of the average cost where time goes to infinity can be regarded as optimality of the control with respect to the Law of Large Numbers. It is shown that this optimal control is optimal with respect to all principal Limit Theorems of Probability Theory: the Central Limit Theorem, the Arcsine Law and the Law of the Iterated Logarithm.

Key words: Law of Large Numbers, Central Limit Theorem, Arcsine Law, Law of the Iterated Logarithm, linear evolution system, optimal average cost

AMS Subject Classifications: 60H30

1 Introduction

Infinite dimensional linear stochastic systems have been investigated for some time in the control theory (see [2]) and adaptive control theory (see [3]). The primary importance of these systems is that they can describe

*Received June 1, 1992; received in final form February 23, 1993. Summary appeared in Volume 5, Number 1, 1995.

†This research was supported by NSF Grants No. ECS-9102714 and ECS-9113029.

many linear delay time equations and various families of partial differential equations. In the control or adaptive control of infinite dimensional linear stochastic systems with complete observations of the state it is important to know the asymptotic distribution of the quadratic cost or the asymptotic bounds for the fluctuation of the average cost around the optimal average cost. The asymptotic distribution of the quadratic cost for the evolution systems is considered in [9]. In this paper we show that this optimal control is optimal with respect to all the principal Limit Theorems of Probability Theory: the Central Limit Theorem, the Arcsine Law and the Law of the Iterated Logarithm. Namely it is shown that under the optimal stationary control, the quadratic cost satisfies the Central Limit Theorem, the asymptotic lower bound for any α-quantile of the distribution of the cost is given, the asymptotic lower bound for the distribution of the proportion of the time spent by the average cost above the average optimal cost (the Arcsine Law) is established and the Law of the Iterated Logarithm provides bounds for the fluctuation of the average cost around the optimal average cost. For finite dimensional linear stochastic systems similar results were shown in [7, 8].

2 Preliminaries

The systems that are considered here are linear stochastic evolution systems where the infinitesimal generators generate strongly continuous semigroups. A semigroup $(G(t),\, t \geq 0)$ of bounded linear operators on a Hilbert space H is a strongly continuous semigroup if

$$\lim_{t \downarrow 0} G(t)x = x,$$

for each $x \in H$ in the strong topology. A strongly continuous semigroup of bounded linear operators on H is called a C_0 semigroup. If $(G(t),\, t \geq 0)$ is a C_0 semigroup then there are real numbers $\omega \geq 0$ and $M \geq 1$ such that

$$|G(t)| \leq M \mathrm{e}^{\omega t},$$

for $t \in [0, +\infty)$ where $|\cdot|$ is the operator norm. If A is the infinitesimal generator of a C_0 semigroup then the domain of A, $\mathcal{D}(A)$ is dense in H and A is a closed, linear operator (e.g., [10]).

We consider the following infinite dimensional linear stochastic controlled system

$$\mathrm{d}X(t) = AX(t)\mathrm{d}t + BU(t)\mathrm{d}t + \mathrm{d}W(t), \quad t \geq 0, \quad X(0) = x \tag{2.1}$$

together with the cost functional

$$C(t) = \int_0^t [\langle QX(s), X(s)\rangle + \langle U(s), U(s)\rangle]\mathrm{d}s, \quad t \geq 0$$

where $X(t) \in H$, H is a real, separable Hilbert space, x is an element of H, $(W(t),\, t \geq 0)$ is an H-valued Wiener process such that $W(1)$ has the nuclear covariance $\mathcal{Q}_W = E[W(1)W(1)^*]$ that is positive. A is the generator of a C_0 semigroup, $B \in L(H_1, H)$ where H_1 is a real separable Hilbert space, $U(t) \in L^2([0,\, t]; H_1)$ is an H_1-valued process depending in a nonanticipative way on the observation of $X(t)$.

The probability space $(\Omega, \mathcal{F}, \mathcal{P})$ can be chosen such that Ω is the Fréchet space of H-valued continuous functions on $\mathbb{R}_+ = [0,\, +\infty)$ with the seminorms of local uniform convergence, $\mathcal{P}$ is the Wiener measure on Ω for the process $(W(t),\, t \geq 0)$ such that $\mathcal{Q}_W = E[W(1)W(1)^*]$ and $\mathcal{F}$ is the $\mathcal{P}$-completion of the Borel δ-algebra on Ω.

The notion of the solution of the stochastic differential equation (2.1) is the mild solution, that is, the solution of the integral equation

$$X(t) = G(t)X(0) + \int_0^t G(t-s)BU(s)\mathrm{d}s + \int_0^t G(t-s)\mathrm{d}W(s), \tag{2.2}$$

where $(G(t),\, t \geq 0)$ is the semigroup $(\mathrm{e}^{ta},\, t \geq 0)$. For (square) integrable controls it is known that there is one and only one solution of (2.2).

Initially it is useful to consider the following deterministic optimal control problem in H. The system is described by the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = Ax(t) + Bu(t), \tag{2.3}$$

where $x(0) = x_0$, A is the infinitesimal generator of a C_0 semigroup, $u(t) \in H_1$, $x_0 \in \mathcal{D}(A)$ and $B \in L(H_1, H)$. The cost functional that is to be minimized is

$$C(t) = \int_0^t [\langle Qx(s), x(s)\rangle + \langle u(s), u(s)\rangle]\mathrm{d}s \tag{2.4}$$

over all $u \in L^2([0,\, t]; H_1)$, where $Q \in L(H, H)$ is bounded, symmetric and positive definite. The dependence of the cost C on the control has been suppressed for notational convenience.

Let $L^+(H, H)$ be the cone of bounded, linear, symmetric, nonnegative operators from H to H and let $\mathcal{C}_S([0,\, t], L^+(H, H))$ be the family of all maps $S : [0,\, t] \to L^+(H, H)$ such that $S(\cdot)x$ is strongly continuous for each $x \in H$. Consider the Riccati differential equation

$$\frac{\mathrm{d}R}{\mathrm{d}t} = A^*R + RA - RBB^*R + Q, \tag{2.5}$$

where $R(0) = R_0$. A mild solution of (2.5) is a function $R \in \mathcal{C}_S([0, t], L^+(H, H))$ such that

$$R(t)x = G^*(t)R_0G(t)x + \int_0^t G^*(t-s)[Q - R(s)BB^*R(s)]G(t-s)x\mathrm{d}s, \quad (2.6)$$

for all $x \in H$ where $G(t) = \mathrm{e}^{tA}$. There is one and only one solution of (2.6) (e.g., p. 64 [1]). Let φ be defined by the equation

$$\varphi(t, x) = \frac{1}{2}\langle R(t)x, x\rangle.$$

If $x \in \mathcal{D}(A)$, then φ is the solution of the Hamilton-Jacobi equation for the optimal control problem (3)–(4) and the optimal feedback control is

$$\hat{u}(s, x(s)) = -B^*R(t-s)x(s). \quad (2.7)$$

We assume that (A, B) is stabilizable and $(A, Q^{1/2})$ is detectable. Then the stationary Riccati equation

$$A^*R + RA - RBB^*R + Q = 0 \quad (2.8)$$

has one and only one nonnegative solution ([1], [11]). This solution R yields the optimal stationary control for system (2.1)

$$\hat{U}(t) = KX(t), \quad t \geq 0 \quad (2.9)$$

with

$$K = -B^*R. \quad (2.10)$$

Furthermore, this solution is the strong limit as $t \to \infty$ of the solutions of the Riccati differential equations. For

$$\theta = \mathrm{trace}(\mathcal{Q}_W R) \quad (2.11)$$

we have under (2.9)

$$\lim_{t\to\infty} \frac{EC(t)}{t} = \theta. \quad (2.12)$$

If $U(t)$ is any nonanticipative control such that

$$\lim_{t\to\infty} \frac{E\langle X(t), X(t)\rangle}{t} = 0$$

then

$$\liminf_{t\to\infty} \frac{EC(t)}{t} \geq \theta. \quad (2.13)$$

To show (12) and (13) let us consider the following equality

$$\begin{aligned} C(t) - \theta(t) + \langle RX(t), X(t)\rangle - \langle RX(0), X(0)\rangle \\ = 2\int_0^t \langle RX(s), \mathrm{d}W(s)\rangle, \quad t \geq 0 \end{aligned} \tag{2.14}$$

To verify (2.14) the extension (see [4]) of Itô's formula to $\langle RX(t), X(t)\rangle$ gives us

$$\begin{aligned} \langle RX(t), X(t)\rangle - \langle RX(0), X(0)\rangle = 2\int_0^t \langle X(s), R(A+BU)X(s)\rangle \mathrm{d}s \\ +2\int_0^t \langle RX(s), \mathrm{d}W(s)\rangle + \int_0^t \mathrm{tr}(\mathcal{Q}_W R)\mathrm{d}s. \end{aligned} \tag{2.15}$$

If we add $C(t)$ to both sides of (2.15) we get

$$C(t)+\langle RX(t), X(t)\rangle-\langle RX(0), X(0)\rangle = \int_0^t [\langle QX(s), X(s)\rangle+\langle U(s), U(s)\rangle]\mathrm{d}s$$

$$+2\int_0^t \langle X(s), R(A+BU)X(s)\rangle \mathrm{d}s + 2\int_0^t \langle RX(s), \mathrm{d}W(s)\rangle + \int_0^t \mathrm{tr}(\mathcal{Q}_W R)\mathrm{d}s.$$

According to (2.8)

$$2\langle RX(t), AX(t)\rangle + \langle QX(t), X(t)\rangle = \langle KX(t), KX(t)\rangle.$$

Hence using (2.11) and the above equalities we obtain

$$\begin{aligned} &C(t) + \langle RX(t), X(t)\rangle - \langle RX(0), X(0)\rangle \\ &\quad = 2\int_0^t (\langle KX(t), KX(t)\rangle + \langle U(s), U(s)\rangle - \langle KX(s), U(s)\rangle)\mathrm{d}s \\ &\qquad + 2\int_0^t \langle RX(s), \mathrm{d}W(s)\rangle + \theta t \\ &\quad = 2\int_0^t \langle U(s)KX(s), U(s)KX(s)\rangle \mathrm{d}s + 2\int_0^t \langle RX(s), \mathrm{d}W(s)\rangle + \theta t. \end{aligned}$$

From here (2.12) and (2.13) can be easily justified.

3 Central Limit Theorem

Under the optimal stationary control (2.9), the cost functional $C(t)$ satisfies the Central Limit Theorem that is

$$\lim_{t\to\infty} P\left(\frac{C(t)-\theta t}{\sqrt{t}} \leq y\right) = \Phi(y/\sqrt{\Delta}) \quad y \in (-\infty, +\infty) \tag{3.1}$$

where $\Phi(z)$ is the distribution function of the standard normal variable and Δ will be specified later. (3.1) follows immediately from Proposition 2 that will be stated later on. Let us now only sketch the proof.

The stochastic integral on the right hand side of (2.14) can be represented by means of a random time change of a Wiener process $W = \{W(t),\, t \geq 0\}$ (see [6]). The quadratic variation of the right hand side of (2.14) is

$$V(t) = 4\int_o^t \langle \mathcal{Q}_W RX(s), RX(s)\rangle \mathrm{d}s \tag{3.2}$$

and (2.14) has the following representation

$$2\int_0^t \langle RX(s), \mathrm{d}W(s)\rangle = \mathcal{W}(V(t)), \quad t \geq 0 \tag{3.3}$$

where $\mathcal{W} = \{\mathcal{W}(t),\, t \geq 0\}$ is a Wiener process.

Let $\mathcal{V}$ be the unique nonnegative definite solution of the equation

$$\mathcal{V}(A+BK) + (A+BK)^*\mathcal{V} + 2R\mathcal{Q}_W R = 0, \tag{3.4}$$

and let

$$\Delta = \mathrm{trace}(\mathcal{Q}_W \mathcal{V}).$$

Then in analogy to (2.14) we have for an arbitrary nonanticipative control $U(t)$ in (2.1)

$$\begin{gathered} V(t) - \Delta t + \langle \mathcal{V}X(t), X(t)\rangle - \langle \mathcal{V}X(0), X(0)\rangle \\ -2\int_0^t \langle \mathcal{V}X(s), B(U(s)-KX(s))\rangle \mathrm{d}s = 2\int_0^t \langle \mathcal{V}X(s), \mathrm{d}W(s)\rangle. \end{gathered} \tag{3.5}$$

To verify (3.5) apply Itô's formula to $(\langle \mathcal{V}X(t), X(t)\rangle,\ t \geq 0)$ and use (3.2) and (3.4) to obtain

$$\begin{aligned} V(t) &+ \langle \mathcal{V}X(t), X(t)\rangle - \langle \mathcal{V}X(0), X(0)\rangle \\ &= 4\int_0^t \langle \mathcal{Q}_W RX(s), RX(s)\rangle \mathrm{d}s \\ &\qquad + 2\int_0^t (\langle \mathcal{V}AX(s), X(s)\rangle + \langle \mathcal{V}X(s), BU(s)\rangle)\mathrm{d}s \\ &\qquad + 2\int_0^t \langle \mathcal{V}X(s), \mathrm{d}W(s)\rangle + \int_0^t \mathrm{tr}(\mathcal{Q}_W \mathcal{V})\mathrm{d}s \\ &= 2\int_0^t \langle \mathcal{V}X(s), B(U(s) - KX(s))\rangle \mathrm{d}s + 2\int_0^t \langle \mathcal{V}X(s), \mathrm{d}W(s)\rangle + \Delta t. \end{aligned}$$

Under (2.9) the strong law of large numbers can be applied to the martingale on the right hand side of (3.5) and we get

$$\lim_{t\to\infty} \frac{V(t)}{t} = \Delta \quad \text{a.s.} \tag{3.6}$$

From (2.14) and (3.3) we obtain

$$\frac{C(t) - \theta t}{\sqrt{t}} = \frac{\langle RX(0), X(0)\rangle}{\sqrt{t}} - \frac{\langle RX(t), X(t)\rangle}{\sqrt{t}} + \frac{\mathcal{W}(V(t))}{\sqrt{t}}.$$

The first two terms on the right hand side asymptotically are negligible and the third one converges in distribution to $N(0, \Delta)$ by (3.6). This gives us (3.1). (2.12) and (2.13) showed an optimality of (2.9) for the average cost criterion.

The following proposition describes an optimality property using a Central Limit Theorem.

Proposition 1 *If $U(t)$ is any nonanticipative control such that*

$$\lim_{t\to\infty} \frac{E\langle X(t), X(t)\rangle}{\sqrt{t}} = 0 \tag{3.7}$$

then

$$\limsup_{t\to\infty} P\left(\frac{C(t) - \theta t}{\sqrt{t}} \leq y\right) \leq \Phi(y/\sqrt{\Delta}), \quad y \in (-\infty, +\infty). \tag{3.8}$$

(3.1) *and* (3.8) *show an optimality of* (2.9) *for the average cost criterion.*

Recall the stochastic ordering of random variables. ξ is stochastically greater than or equal to η if $P(\xi \leq y) \leq P(\eta \leq y)$, $y \in (-\infty, +\infty)$. We shall use this stochastic ordering in the asymptotic sense to give an interpretation of (3.8).

The inequality (3.8) means that $(C(t)-\theta t)/\sqrt{t}$ as $t \to \infty$ is asymptotically stochastically greater than or equal to a random variable with normal distribution $N(0, \Delta)$.

The asymptotic lower bound for any α-quantile of the distribution of $C(t)$ is $\theta t + z_\alpha \sqrt{\Delta t}$ where $\Phi(z_\alpha) = \alpha$.

Proof: Let us recall the following elementary property of a Wiener process. For $r \geq 0$, $t > 0$, $b > a$

$$P\left(\inf_{|s-r|\geq t} \mathcal{W}(s) \leq a,\, \mathcal{W}(r) \geq b\right) \leq 3\Phi\left(\frac{a-b}{\sqrt{t}}\right).$$

Let us assume that $E\int_0^t \langle X(s), X(s)\rangle \mathrm{d}s < \infty$, $t \geq 0$. Set

$$A(t) = 2\int_0^t \langle U(s) - KX(s), U(s) - KX(s)\rangle \mathrm{d}s, \tag{3.9}$$

$$Z(t) = \int_0^t \langle X(s), X(t)\rangle \mathrm{d}s \tag{3.10}$$

and

$$M(t) = 2\int_0^t \langle RX(s), \mathrm{d}W(s)\rangle. \tag{3.11}$$

(3.7) implies that $EZ(t) = \mathrm{o}(t^{3/2})$. Using (3.7) we obtain

$$\begin{aligned} EM^2(t) &= E\left(\int_0^t \langle RX(s), \mathrm{d}W(s)\rangle\right)^2 \\ &= 4\int_0^t E\langle Q_W RX(s), RX(s)\rangle \mathrm{d}s = \mathrm{o}(t^{3/2}), \quad t \to \infty \end{aligned} \tag{3.12}$$

and

$$\left| 2\int_0^t \langle \mathcal{V}X(s), B(U(s)-KX(s))\rangle \mathrm{d}s \right| \leq \mathrm{const.}(Z(t))^{1/2}\cdot (A(t))^{1/2}, \quad t\geq 0 \tag{3.13}$$

Let ε, δ be sufficiently small, $0<\varepsilon<1$, $\delta>0$. (3.7) implies

$$P(\langle RX(t), X(t)\rangle > \delta\sqrt{t}) \leq \varepsilon, \quad P(Z(t) > \delta^2 t^{3/2}) \leq \varepsilon, \tag{3.14}$$

for t sufficiently large. Using (3.5) we can find an L sufficiently large so that

$$P(|V(t)-\Delta t| \leq L((A(t))^{1/2}(Z(t))^{1/2}+t^{3/4})) \geq 1-\varepsilon, \quad t\geq 1. \tag{3.15}$$

There is $\{\mathcal{W}(t),\, t\geq 0\}$ such that $M(t)=\mathcal{W}(V(t))$, $t\geq 0$. Hence for all t sufficiently large and using (3.15) we obtain

$$\begin{aligned} P(C(t)-\theta t\leq y\sqrt{t}) &\leq \varepsilon + P(M(t)+A(t)-\varepsilon\sqrt{t}\leq y\sqrt{t}) \leq 2\varepsilon \\ &+ P\left(\inf_{|s-\Delta t|\leq L((A(t))^{1/2}(Z(t))^{1/2}+t^{3/4})} \mathcal{W}(s) \leq -A(t)+(y+\varepsilon)\sqrt{t}\right) \\ &\leq 2\varepsilon + P(\mathcal{W}(\Delta t) < (y+2\varepsilon)\sqrt{t}) \\ &+ \sum_{j=0}^{+\infty} P\Big(\mathcal{W}(\Delta t)\geq (y+2\varepsilon)\sqrt{t}, j\sqrt{t}\leq A(t) < (j+1)\sqrt{t}, \\ &\qquad \inf_{|s-\Delta t|\leq L((A(t))^{1/2}(Z(t))^{1/2}+t^{3/4})} \mathcal{W}(s)\leq -A(t)+(y+\varepsilon)\sqrt{t}\Big) \\ &\leq 2\varepsilon + \Phi\left(\frac{y+2\varepsilon}{\sqrt{\Delta}}\right) + \sum_{j=0}^{+\infty} P\Big(\mathcal{W}(\Delta t)\geq (y+2\varepsilon)\sqrt{t}, \\ &\qquad \inf_{|s-\Delta t|\leq L(\delta\sqrt{j+1}t+t^{3/4})} \mathcal{W}(s)\leq -j\sqrt{t}+(y+\varepsilon)\sqrt{t}\Big). \end{aligned} \tag{3.16}$$

For t large we have

$$L(\delta\sqrt{j+1}t+t^{3/4}) \leq 2\delta L(j+1)t, \quad j=0,1,\dots\,. \tag{3.17}$$

Thus using the property of a Wiener process that was mentioned at the beginning of the proof we obtain

$$P(C(t)-\theta t\leq y\sqrt{t}) \leq 2\varepsilon + \Phi\left(\frac{y+2\varepsilon}{\sqrt{\Delta}}\right) + 3\sum_{j=0}^{\infty}\Phi\left(-\frac{(j+\varepsilon)\sqrt{t}}{\sqrt{2\delta L(j+1)t}}\right). \tag{3.18}$$

Further

$$\sum_{j=0}^{\infty}\Phi\left(-\frac{(j+\varepsilon)}{\sqrt{2\delta L(j+1)}}\right)\leq\frac{1}{\sqrt{2\pi}}\sum_{j=0}^{\infty}\frac{\sqrt{2\delta L(j+1)}}{j+\varepsilon}\exp\left\{-\frac{(j+\varepsilon)^2}{4\delta L(j+1)}\right\}$$
$$\leq\frac{1}{\sqrt{2\pi}}\sum_{j=0}^{\infty}\frac{\sqrt{2\delta L}}{\varepsilon}\exp\left\{-\frac{j}{8\delta l}\right\}=\frac{1}{\varepsilon}\sqrt{\frac{\delta L}{\pi}}\left(1-\exp\left\{-\frac{1}{8\delta L}\right\}\right)^{-1}.$$

We conclude that the last term in (3.18) can be made arbitrarily small by taking δ small. This completes the proof. □

Proposition 2 *If* $U(t)$ *is any nonanticipative control such that* (3.7) *holds and*

$$P\text{-}\lim_{t\to\infty}\frac{1}{\sqrt{t}}A(t)=0 \tag{3.19}$$

then

$$\lim_{t\to\infty}P\left(\frac{C(t)-\theta t}{\sqrt{t}}\leq y\right)=\Phi(y/\sqrt{\Delta}),\quad y\in(-\infty,+\infty), \tag{3.20}$$

where P-lim *means convergence in probability.*

Proof: Note that by (3.19) $P(A(t)>\delta\sqrt{t})\leq\varepsilon$ for large t. For all t sufficiently large

$$P(C(t)-\theta t>y\sqrt{t})\leq P(M(t)+A(t)+\varepsilon\sqrt{t}>y\sqrt{t})\leq\varepsilon$$
$$+P\left(\sup_{|s-\Delta t|\leq L((A(t))^{1/2}\cdot(2(t))^{1/2}+t^{3/4})}\mathcal{W}(s)\geq -A(t)+(y-\varepsilon)\sqrt{t}\right)$$
$$\leq 3\varepsilon+P(\mathcal{W}(\Delta t)>(y-2\varepsilon)+P\Bigg(\mathcal{W}(\Delta t)\leq(y-2\varepsilon)\sqrt{t},$$
$$\sup_{|s-\Delta t|\leq L(\delta^2 t+t^{3/4})}\mathcal{W}(s)\geq-\delta\sqrt{t}+(y-\varepsilon)\sqrt{t}\Bigg)$$
$$\leq 3\varepsilon+P(\mathcal{W}(\Delta t)>(y-2\varepsilon)\sqrt{t})+3\Phi\left(\frac{(\delta-\varepsilon)\sqrt{t}}{\sqrt{L(\delta^2 t+t^{3/4})}}\right).$$

The last term can be made arbitrarily small as $t\to\infty$ by taking δ sufficiently small. Hence we conclude that

$$\limsup_{t\to\infty}P\left(\frac{C(t)-\theta t}{\sqrt{t}}>y\right)\leq 3\varepsilon+1-\Phi\left(\frac{y-2\varepsilon}{\sqrt{\Delta}}\right).$$

Since ε is arbitrary, this together with (3.8) yields (3.20). □

4 The Arcsine Law

Let 1_A be the indicator function of the Borel set A. The average

$$\frac{1}{t}\int_0^t 1_{(\theta,\,\infty)}\left(\frac{C(s)}{s}\right)\mathrm{d}s \tag{4.1}$$

is the proportion of time that $\dfrac{C(s)}{s} > \theta$ or equivalently $C(s) > s\theta$. The Arcsine Law for the occupation time of $(0,\,\infty)$ for the Wiener process (see [5]) is applied to obtain an asymptotic lower bound for the distribution of the random variable in (4.1).

Proposition 3 *If $U(t)$ is any nonanticipative control such that*

$$\lim_{t\to\infty}\frac{E\langle X(t), X(t)\rangle}{\sqrt{t}} = 0 \tag{4.2}$$

then

$$\limsup_{t\to\infty} P\left(\frac{1}{t}\int_0^t 1_{(\theta,\,\infty)}\left(\frac{C(s)}{s}\right)\mathrm{d}s \le y\right) \le \frac{2}{\pi}\arcsin\sqrt{y}, \quad y\in[0,\,1]. \tag{4.3}$$

Proof: For notational convenience define 1_b by the equation

$$1_b = 1_{(b,\,\infty)}.$$

Recalling that $A(t) = 2\int_0^t \langle U(s) - KX(s), U(s) - KX(s)\rangle \mathrm{d}s$ and using (2.14) and (3.3) and an elementary change of variables we have that

$$\frac{1}{t}\int_0^t 1_0(C(s)-\theta s)\mathrm{d}s = \frac{1}{t}\int_0^t 1_0(\mathcal{W}(V(s)) - \langle RX(s), X(s)\rangle$$

$$+ \langle RX(0), X(0)\rangle + A(s))\mathrm{d}s = \int_0^t 1_0\Big(\frac{1}{\sqrt{t}}(\mathcal{W}(V(tz))$$

$$- \langle RX(tz), X(tz)\rangle + \langle RX(0), X(0)\rangle + A(tz)\Big)\mathrm{d}z$$

$$= \int_0^t 1_{\varepsilon/\sqrt{\Delta}}\Big(\frac{1}{\sqrt{\Delta}}\mathcal{W}_t(\Delta z)\Big)\mathrm{d}z$$

$$- \int_0^t (1_\varepsilon(\mathcal{W}_t(\Delta z))\mathrm{d}z + \int_0^t 1_0\Big(\mathcal{W}_t\Big(\frac{1}{t}V(tz)\Big)$$

$$- \frac{1}{\sqrt{t}}\langle RX(tz), X(tz)\rangle$$

$$+ \frac{1}{\sqrt{t}}\langle RX(0), X(0)\rangle + \frac{1}{\sqrt{t}}A(tz)\Big)\mathrm{d}z,$$

where $\varepsilon > 0$, $\mathcal{W}_t(u) = \dfrac{1}{\sqrt{t}}\mathcal{W}(tu)$, $u \geq 0$ is a Wiener process and $\Delta =$ trace$(\mathcal{Q}_W\mathcal{V})$. Let

$$f(t) = \left(\mathcal{W}_W\left(\frac{1}{t}V(tz)\right) - \frac{1}{\sqrt{t}}\langle RX(tz), X(tz)\rangle + \frac{1}{\sqrt{t}}\langle RX(0), X(0)\rangle + \frac{1}{\sqrt{t}}A(tz)\right).$$

For $y \in [0, 1]$ and $\delta > 0$ arbitrary we have

$$P\left(\frac{1}{t}\int_0^t 1_0(C(s)-\theta s) \leq y\right) \leq P\left(\int_0^t 1_{\varepsilon/\sqrt{\Delta}}\left(\frac{1}{\sqrt{\Delta}}\mathcal{W}_t(\Delta z)\right)\mathrm{d}z \leq y+\delta\right)$$
$$+P\left(\int_0^t (1_\varepsilon(\mathcal{W}_t(\Delta z)) - 1_0(f(tz)))\mathrm{d}z > \delta\right). \tag{4.4}$$

For each $t > o$ as $\delta \to 0$ and $\varepsilon \to 0$ the first probability on the right hand side of (4.4) satisfies the Arcsine Law for the occupation time of $(0, \infty)$, that is

$$\lim_{\varepsilon,\delta\to 0} P\left(\int_0^t 1_{\varepsilon/\sqrt{\Delta}}\left(\frac{1}{\sqrt{\Delta}}\mathcal{W}_t(\Delta z)\right)\mathrm{d}z \le y+\delta\right) = \frac{2}{\pi}\arcsin\sqrt{y}.$$

Thus it only remains to prove the asymptotic negligibility of the second term on the right hand side (4.4) as $t \to \infty$. Since it is necessary to compare $\mathcal{W}_t(\Delta z)$ and $\mathcal{W}_t\left(\frac{1}{t}V(tz)\right)$, (3.5) is used.

$$\begin{aligned}
1_\varepsilon(\mathcal{W}_t(\Delta z)) - 1_0(f(tz)) \le{}& 1_{\varepsilon/4}\left(\frac{1}{\sqrt{t}}\langle RX(tz), X(tz)\rangle\right) \\
&+ 1_{\varepsilon/4}\left(\frac{1}{\sqrt{t}}\langle RX(0), X(0)\rangle\right) \\
&+ 1_{\varepsilon/4}\left(\frac{1}{\sqrt{t}}\langle \mathcal{V}X(tz), X(tz)\rangle\right) \\
&+ 1_{\varepsilon/4}\left(\mathcal{W}_t(\Delta z) - \mathcal{W}_t\left(\frac{1}{t}v(tz)\right) - \frac{1}{\sqrt{t}}A(tz)\right)\cdot \\
&\quad\cdot\left(1 - 1_{\varepsilon/2\sqrt{t}}\left(\left|\frac{1}{t}V(tz) - \Delta z\right|\right.\right. \\
&\quad - \frac{2}{t}\left|\int_0^{tz}\langle \mathcal{V}X(s), B(U(s)-KX(s))\rangle\mathrm{d}s\right| \\
&\quad \left.\left. - \frac{2}{t}\left|\int_0^{tz}\langle \mathcal{V}X(s), \mathrm{d}W(s)\rangle\right|\right)\right).
\end{aligned} \tag{4.5}$$

Since

$$\begin{aligned}
&\frac{2}{t}\left|\int_0^{tz}\langle \mathcal{V}X(s), B(U(s)-KX(s))\rangle\mathrm{d}s\right| \\
&\le 2|\mathcal{V}||\,|B|\left(\frac{1}{t^{3/2}}\int_0^{tz}\langle X(s), X(s)\rangle\mathrm{d}s\right)^{1/2}\left(\frac{1}{\sqrt{t}}A(tz)\right)^{1/2},
\end{aligned}$$

we have for each $\gamma > 0$ that the last term on the right hand side of (4.5) is majorized by

$$\begin{aligned} & 1_\gamma \left(2|\mathcal{V}|\ |B| \left[\frac{1}{t^{3/2}} \int_0^{tz} \langle X(s), X(s) \rangle \mathrm{d}s \right]^{1/2} \right) \\ & \quad + 1_{\gamma/t^{1/4}} \left(\frac{1}{t} \left| \int_0^{tz} \langle \mathcal{V}X(s), \mathrm{d}W(s) \rangle \right| \right) \\ & \quad + 1_{\varepsilon/2+(1/\sqrt{t})A(tz)} \Bigg(\sup \Bigg\{ \mathcal{W}_t(\Delta z) - \mathcal{W}_t(s)) \\ & \qquad : |\Delta z - s| \le \frac{\varepsilon}{2\sqrt{t}} + \frac{\gamma}{t^{1/4}} + \gamma \left(\frac{1}{\sqrt{t}} A(tz) \right)^{1/2} \Bigg\} \Bigg). \end{aligned} \tag{4.6}$$

Let $\mathcal{R}(z)$ be the sum of the first three terms on the right hand side of (4.5) and the expression in (4.6). Integrating this new inequality we have for each $z \in [0, 1]$ and $\varepsilon > 0$ that

$$\lim_{t\to\infty} E 1_{\varepsilon/4} \left(\frac{1}{\sqrt{t}} \langle RX(tz), X(tz) \rangle \right) = 0 \tag{4.7}$$

$$\lim_{t\to\infty} E 1_{\varepsilon/4} \left(\frac{1}{\sqrt{t}} \langle RX(0), X(0) \rangle \right) = 0 \tag{4.8}$$

$$\lim_{t\to\infty} E 1_{\varepsilon/4} \left(\frac{1}{\sqrt{t}} \langle \mathcal{V}X(tz), X(tz) \rangle \right) = 0. \tag{4.9}$$

Since

$$\lim_{t\to\infty} \frac{1}{t^{3/2}} E \int_s^t \langle X(s), X(s) \rangle \mathrm{d}s = 0$$

we have that

$$\lim_{t\to\infty} E 1_\gamma \left(2|\mathcal{V}|\ |B| \left[\frac{1}{t^{3/2}} \int_0^{tz} \langle X(s), X(s) \rangle \mathrm{d}s \right]^{1/2} \right) = 0. \tag{4.10}$$

Similarly since

$$\lim_{t\to\infty} \frac{1}{t^{3/2}} E \int_0^{tz} \langle \mathcal{V}\mathcal{Q}_W X(s), X(s) \rangle \mathrm{d}s = 0$$

we have that

$$\lim_{t\to\infty} E1_{\gamma/t^{1/4}}\left(\frac{2}{t}\left|\int_0^{tz}\langle\mathcal{V}X(s),\mathrm{d}W(s)\rangle\right|\right)=0. \tag{4.11}$$

Recall the following well–known result for a Wiener process

$$P\left(\sup_{s\le u\le t}(W(u)-W(s))\ge a\right)=2P(W(t-s)>a).$$

Let M be the event given by

$$M=\left\{\sup\left\{\mathcal{W}_t(\Delta z)-\mathcal{W}_t(s):|\Delta z-s|\right.\right.$$
$$\left.\left.\le\frac{\varepsilon}{2\sqrt{t}}+\frac{\gamma}{t^{1/4}}+\gamma\left(\frac{1}{\sqrt{t}}A(tz)\right)^{1/2}\right\}\ge\frac{\varepsilon}{2}+\frac{A(tz)}{\sqrt{t}}\right\}.$$

For t sufficiently large we have that

$$\begin{aligned}P(M)&\le\sum_{j=0}^{\infty}P(M,j\sqrt{t}\le A(tz)<(j+1)\sqrt{t})\\&\le\sum_{j=0}^{\infty}P\left(\sup\{W(\Delta z)-W(s):|\Delta z-s|\le 3\gamma\sqrt{j+1}\}\ge\frac{\varepsilon}{2}+j\right)\\&\le\sum_{j=0}^{\infty}4\Phi\left(\frac{-\left(\frac{\varepsilon}{2}+j\right)}{\sqrt{3\gamma\sqrt{j+1}}}\right)=\psi(\gamma).\end{aligned}$$

It is clear that the infinite series that defines ψ converges and $\lim_{\gamma\to 0}\psi(\gamma)=0$ since

$$\lim_{t\to\infty}\sup P\left(\int_0^1(1_\varepsilon(W(\Delta z))-1_0(f(tz)))\mathrm{d}z>\delta\right)$$
$$\le\limsup P\left(\int_0^1\mathcal{R}(z)\mathrm{d}z>\delta\right)\le\frac{\psi(\gamma)}{\delta}$$

we have the desired negligibility of the second term on the right hand side of the inequality (4.4) □

Proposition 4 *If $U(t)$ is any nonanticipative control such that* (4.2) *and*

$$P\text{-}\lim_{t\to\infty}\frac{1}{\sqrt{t}}A(t)=0 \tag{4.12}$$

hold, then

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t 1_{(\theta,\infty)}\left(\frac{C(s)}{s}\right) \mathrm{d}s = \frac{2}{\pi} \arcsin \sqrt{t}, \quad y \in [0, 1]. \tag{4.13}$$

Proof: Since we established (4.3), now we have to verify that

$$\liminf_{t\to\infty} P\left(\frac{1}{t}\int_0^t 1_0(C(s)-\theta s)ds \le y\right) \ge \frac{2}{\pi}\arcsin\sqrt{y}, \quad y\in[0, 1]. \tag{4.14}$$

Analogously to (4.4) we deduce the following inequalities

$$\begin{aligned} &P\left(\frac{1}{t}\int_0^t 1_0(C(s)-\theta s)\mathrm{d}s \le y\right) \\ &\quad \ge P\left(\int_0^1 1_{-\varepsilon/\sqrt{\Delta}}\left(\frac{1}{\sqrt{t}}\mathcal{W}_t(\Delta z)\right)\mathrm{d}z \le y-\delta\right) \\ &\quad - P\left(\int_0^1 (1_0(f(tz)) - 1_{-\varepsilon}(\mathcal{W}(\Delta z)))\mathrm{d}z \ge \delta\right). \end{aligned} \tag{4.15}$$

To estimate the last probability we have

$$\int_0^1 (1_0(f(tz)) - 1_{-\varepsilon}(\mathcal{W}(\Delta z)))\mathrm{d}z = \int_0^1 \mathcal{R}^*(z)\mathrm{d}z,$$

where $\mathcal{R}^*(z)$ differs from $\mathcal{R}(z)$ only in the last term of (4.6) that we can modify as follows

$$\chi\left\{\sup_{|\Delta z - s| \le \frac{\varepsilon}{2\sqrt{t}} + \frac{\gamma}{t^{1/4}} + \gamma\left(\frac{A(tz)}{\sqrt{t}}\right)^{1/2}} (\mathcal{W}_t(s)\mathcal{W}_t(\Delta z)) \ge \frac{\varepsilon}{2} - \frac{A(tz)}{\sqrt{t}}\right\}, \tag{4.16}$$

where χ is the indicator function. Since (4.12) holds the probability of the event in (4.16) tends to 0 as $t\to\infty$. From this and from (4.9), (4.10), (4.11) we deduce the negligibility of the last term in (4.15). The first term on the right of (4.15) is independent of t and converges to $\frac{2}{\pi}\arcsin\sqrt{y}$ as $\varepsilon, \delta \to 0$. We conclude that (4.14) holds. □

5 The Law of the Iterated Logarithm

The Law of the Iterated Logarithm provides bounds for the fluctuation of $C(t)$ around θt. The bounds are obtained under the optimal stationary control (2.9) and can not be improved within the class of controls defined in the next proposition.

Proposition 5 *If $U(t)$ is any nonanticipative control such that for some $\varepsilon > 0$*

$$\lim_{t\to\infty} \frac{|X(t)|^{2+\varepsilon}}{\sqrt{t}} = 0 \quad \text{a.s.,} \tag{5.1}$$

then

$$\limsup_{t\to\infty} \frac{C(t)-\theta t}{\sqrt{2t\log\log t}} \geq \sqrt{\Delta} \quad \text{a.s.} \tag{5.2}$$

$$\liminf_{t\to\infty} \frac{C(t)-\theta t}{\sqrt{2t\log\log t}} \geq -\sqrt{\Delta} \quad \text{a.s.} \tag{5.3}$$

Furthermore, the equalities (5.2) and (5.3) hold if we assume additionally that

$$\lim_{t\to\infty} \int_0^t \frac{\langle U(s)-KX(s), U(s)-KX(s)\rangle \mathrm{d}s}{\sqrt{t\log\log t}} = 0 \quad \text{a.s. .} \tag{5.4}$$

Proof: Choose U and fix it. Consider (2.14), (3.3) and (3.5). Denote $H(t) = \int_0^t \langle \mathcal{V}X(s), B(U(s)-KX(s))\rangle \mathrm{d}s$ and $g(x) = \sqrt{2x\log\log x}$. Note that

$$g'(x) \leq \frac{g(x)}{x}, \quad \text{for } x \geq 6 \tag{5.5}$$

and that

$$|H(t)| \leq 2|\mathcal{V}|\;|B| \left(\int_0^t \langle X(s), X(s)\rangle \mathrm{d}s\right)^{1/2} \cdot (A(t))^{1/2}.$$

By (5.1) we have for t sufficiently large

$$|H(t)| \leq t^{3/4-\delta}(A(t))^{1/2}, \tag{5.6}$$

for some $\delta > 0$. (5.1) implies

$$2\int_0^t \langle \mathcal{V}X(s), \mathrm{d}W(s)\rangle = \mathrm{o}(t), \quad t\to\infty.$$

The time change for the Wiener process [6] and the Law of Iterated Logarithm can be used to prove this. Write according to (3.5)

$$\begin{aligned}\Delta t + \mathrm{o}(t) &= \Delta t - \langle \mathcal{V}X(t), X(t)\rangle + \langle \mathcal{V}X(0), X(0)\rangle + 2\int_0^t \langle \mathcal{V}X(s), \mathrm{d}W(s)\rangle \\ &= V(t) - H(t).\end{aligned} \tag{5.7}$$

From (2.14) and from (5.1) follows

$$\frac{C(t)-\theta t}{g(\Delta t + \mathrm{o}(t))} = \frac{\mathcal{W}(V(t))}{g(V(t))} + Y(t) + \mathrm{o}(1), \quad t \to \infty, \tag{5.8}$$

where

$$Y(t) = \frac{\mathcal{W}(V(t)) + A(t)}{g(\Delta t + \mathrm{o}(t))} - \frac{\mathcal{W}(V(t))}{g(V(t))}. \tag{5.9}$$

Assume first that $\lim_{t\to\infty} V(t) = \infty$. Applying the Law of the Iterated Logarithm to the first term on the right hand side of (5.8) we obtain

$$\limsup_{t\to\infty} \frac{1}{\sqrt{\Delta}}\,\frac{C(t)-\theta t}{g(t)} \geq 1 + \liminf_{t\to\infty} Y(t)$$

and

$$\liminf_{t\to\infty} \frac{1}{\sqrt{\Delta}}\,\frac{C(t)-\theta t}{g(t)} \geq -1 + \liminf_{t\to\infty} Y(t).$$

Consequently to demonstrate (5.2) and (5.3) it suffices to show that

$$\liminf_{t\to\infty} Y(t) \geq 0. \tag{5.10}$$

Note that (5.9) can be rewritten as

$$Y(t) = \frac{\mathcal{W}(V(t))}{g(V(t))} \cdot \left(\frac{g(\Delta t = \mathrm{o}(0) + H(t))}{g(\Delta t = \mathrm{o}(t))} - 1 \right) + \frac{A(t)}{g(\Delta t + \mathrm{o}(t))}.$$

Since

$$\limsup_{t\to\infty} \frac{|\mathcal{W}(V(t))|}{g(V(t))} = 1,$$

(5.10) will be proved by verifying

$$\liminf_{t\to\infty} \left(\mp \frac{g(\delta t + \mathrm{o}(t) + H(t))}{g(\Delta t + \mathrm{o}(t))} \pm 1 + \frac{A(t)}{g(\Delta t + \mathrm{o}(t))} \right) \geq 0, \tag{5.11}$$

where the upper signs are for $H(t) > 0$ and the lower ones are for $H(t) < 0$. Let t sufficiently large and let $H(t) > 0$. Then from (5.5) and (5.6) follows

$$\begin{aligned}
-\frac{g(\Delta t + \mathrm{o}(t) + H(t))}{g(\Delta t + \mathrm{o}(t))} + 1 + \frac{A(t)}{g(\Delta t + \mathrm{o}(t))} \le \\
-\frac{H(t) \cdot g'(\Delta t + \mathrm{o}(t))}{g(\Delta t + \mathrm{o}(t))} + \frac{A(t)}{g(\Delta t + \mathrm{o}(t))} \le \text{using}(5.6) \\
-\frac{H(t)}{\Delta t + \mathrm{o}(t)} + \frac{A(t)}{g(\Delta t + \mathrm{o}(t))} = \\
-\mathrm{o}(1)\sqrt{\frac{A(t)}{g(\Delta t + \mathrm{o}(t))}} + \frac{A(t)}{g(\Delta t + \mathrm{o}(t))} = \mathrm{o}(1).
\end{aligned}$$

This shows (5.11) with the upper signs. Consider

$$\frac{g(\Delta t + \mathrm{o}(t) + H(t))}{g(\Delta t + \mathrm{o}(t))} - 1 + \frac{A(t)}{g(deit + \mathrm{o}(t))}. \tag{5.12}$$

(5.12) is nonnegative if $A(t) \ge g(\Delta t + \mathrm{o}(t))$. Let

$$A(t) < g(\Delta t + \mathrm{o}(t)). \tag{5.13}$$

Then using (5.6)

$$|H(t)| \le t^{3/4-\delta}\sqrt{g(\Delta t + \mathrm{o}(t))} = \mathrm{o}(t).$$

Consequently, the first term in (5.12) approaches 1 as $t \to \infty$ provided that (5.13) holds. Hence (5.11) is established.

Assume now that

$$\lim_{t\to\infty} V(t) < \infty. \tag{5.14}$$

Then from (2.14) and (3.3) follows

$$\liminf_{t\to\infty} \frac{C(t) - \theta t}{g(t)} \ge 0$$

and hence we obtain (5.3). The nonvalidity of (5.2) would require

$$\limsup_{t\to\infty} \frac{A(t)}{g(t)} < \sqrt{\Delta}.$$

But this together with (5.6) implies $H(t) = \mathrm{o}(t)$ which with (5.7) yields $V(t) \to \infty$ in contradiction to (5.14).

To prove the second assertion of Proposition 5 note that from (5.4), (5.6) and (5.7) we obtain

$$\frac{A(t)}{g(\Delta t + \mathrm{o}(t))} = \mathrm{o}(1), \quad H(t) = \mathrm{o}(t), \quad V(t) = \Delta t + \mathrm{o}(1).$$

From here and from (5.9) we conclude that $Y(t)$ in (5.8) is negligible and equalities in (5.2) and (5.3) hold. □

6. Conclusions

Stochastic evolution systems that arise from a semigroup description of various infinite dimensional systems such as linear partial differential equations were considered. These systems are often controlled and the quadratic cost provides a measure of the controlled performance.

In the control or adaptive control of linear stochastic evolution systems with complete observations of the state it is important to know the asymptotic distribution of the quadratic cost or the asymptotic bounds for the fluctuation of the average cost around the optimal average cost.

The optimal average cost as the limit of the average cost where time goes to infinity can be regarded as optimality of the control with respect to the Law of Large Numbers. It has been shown that this optimal control can be regarded as optimality of the control with respect to all principal Limit Theorems of Probability Theory: the Central Limit Theorem, the Arcsine Law and the Law of the Iterated Logarithm.

Acknowledgement

The author thanks Professor P. Mandl for discussions about these problems and Professor T. Duncan for reading this paper and making useful corrections.

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Communicated by Clyde F. Martin