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Linear Quadratic Optimal Control of Time–Varying Systems with Indefinite Costs on Hilbert Spaces: The Finite Horizon Problem*

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Abstract

In this paper we consider time–varying linear systems on Hilbert spaces and study the optimal control problem with indefinite performance criteria over a finite horizon interval. Due to the indefiniteness of the cost function, the associated integral Riccati equation in general does not process a solution on the whole interval. Applying an operator theoretic approach due to Hinrichsen and Pritchard [10] equivalent conditions are arrived for the unique solvability of the linear quadratic optimization problem and for the existence of solutions to the integral Riccati equation. Contrary to the finite–dimensional situation these problems are not generally equivalent. The results are applied to a parameterized Riccati equation which plays an important role in robustness analysis.

Key words: infinite–dimensional systems, time–varying, mild evolution operator, Riccati equation, optimal control problem

AMS Subject Classifications: 49J22, 93C05, 93C50, 93C60

Notation

X, Z	Banach space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ with norm $\|\cdot\|_X$, etc.;
H, U, Y	Hilbert spaces over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ with norm $\|\cdot\|_H$, etc.;

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$\mathcal{L}(X, Z)$	Banach space of bounded linear operators from X to Z provided with the operator norm $\\|\cdot\\|_{\mathcal{L}(X,Z)}$;
$\mathcal{L}(X)$	$\mathcal{L}(X, X)$;
$\mathcal{H}(H)$	$\{T \in \mathcal{L}(H) \| T$ is Hermitian$\}$;
$[t_0, t_1]$	compact interval in $\mathbb{R}$;
$,_{t_0}^{t_1}$	$\{(t, s) \| t_0 \leq s \leq t \leq t_1\}$;
$C([t_0, t_1], X)$	$\{f : [t_0, t_1] \to X \| f$ is continuous;$\}$;
$L_2(t_0, t_1; X)$	$\{f : [t_0, t_1] \to X \| f$ is measurable and $\\|f\\|_2 := \left(\int_{t_0}^{t_1} \\|f(t)\\|^2 \, dt\right)^{1/2} < \infty\Big\}$;
$L_{s,\infty}(t_0, t_1; \mathcal{L}(X, Z))$	$\{F : [t_0, t_1] \to \mathcal{L}(X, Z) \| F$ is strongly measurable and $\\|F\\|_\infty := \operatorname{ess\,sup}_{t \in [t_0, t_1]} \\|F(t)\\| < \infty\Big\}$.

Note that in order to evaluate the integrals used in this paper, we need to be able to integrate functions giving values in a general Banach space. The definitions and properties which are needed to justify the evaluation of such integrals can be found in E. Hille and R. Phillips [9] or S. Bochner [2].

1 Introduction

Optimal control problems with quadratic cost criteria appear in many applications, e.g. network theory, stability theory, filtering and estimation. Different types of Riccati equations play an important role for the solution of these problems. A survey of results for the finite–dimensional Riccati equation, the finite–dimensional optimal control problem and its applications can be found in [1] and [3]. Results of infinite–dimensional systems are given for example, in [4], [12] and [14].

In this paper we consider time–varying linear systems described by mild evolution operators on Hilbert spaces. The state of these systems is given by the input–state map

$$x(t; t_0, x_0, u) = \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, \rho)B(\rho)u(\rho)\, d\rho, \qquad t \in [t_0, t_1],$$

where Φ is a mild evolution operator and $u \in L_2(t_0, t_1; U)$, for U a given Hilbert space. The performance index is of the form

$$\begin{aligned} J(t_0, t_1, x_0, u) &= \langle x(t_1; t_0, x_0, u), Gx(t_1; t_0, x_0, u)\rangle \\ &\quad + \int_{t_0}^{t_1} \langle x(t; t_0, x_0, u), C(t)x(t; t_0, x_0, u)\rangle + \|u(t)\|^2 \, dt. \end{aligned}$$

In this context, due to the lack of differentiability of the system trajectories, the differential Riccati equation is replaced by the integral Riccati equation

$$\begin{aligned} P(t)z &= \Phi^*(t_1,t)G\Phi(t_1,t)z \\ &\quad + \int_t^{t_1} \Phi^*(\rho,t)[C(\rho) - P(\rho)B(\rho)B^*(\rho)P(\rho)]\Phi(\rho,t)z\,d\rho. \end{aligned} \tag{1.1}$$

Gibson [7] has shown that (1.1) has a solution P on the interval $[t_0, t_1]$ if the operators G and $C(t)$ are positive semi–definite.

In this paper we only require that G and $C(t)$ are self–adjoint. Hence, we obtain an integral Riccati equation with indefinite state penalty. One motivation to investigate this generalized integral Riccati equation lies in possible applications to robustness analysis. This analysis will be the subject of another article which deals with the infinite horizon problem on the basis of results obtained in this paper. These results, however, are also of independent interest. Due to the lack of positive semi–definiteness, our finite horizon linear quadratic optimization problem does not necessarily have a solution on $[t_0, t_1]$. Hence necessary and sufficient conditions for the solvability of the optimal control problem and the associated integral Riccati equation are required. Our aim is to extend results proved in [10] for finite–dimensional systems to an infinite–dimensional context. To reach this goal, we extend an operator theoretic approach developed by Hinrichsen and Pritchard in the unpublished manuscript [10]. In spite of the fact that crucial steps in [10] do not extend to the infinite–dimensional situation, we are able to prove several results. The main results are Theorem 3.5 and Theorem 3.6 which give equivalent condition for both problems. In fact, an example is included, showing that — contrary to the finite–dimensional case — the two problems started with are not equivalent.

The author knows only of two papers dealing with the integral Riccati equation (1.1), where G and $C(t)$ are arbitrary self–adjoint operators [5] and [13]. Dragan and Halanay [5] have considered a more general integral Riccati equation under stronger assumptions. In this situation they are able to prove only the equivalence of the conditions **(R1)** and **(R3)**. Pandolfi [13] proves the existence of solutions to (1.1) on the infinite horizon (with $G = 0$) when a suitable comparison equation is solvable. In these papers the linear quadratic optimization problem have not been considered.

We proceed as follows. In the next section we summarize some results concerning mild evolution operators. In Section 3 we give a detailed problem formulation and prove the main results. Note that some of the intermediate results required for the proofs are also of independent interest. Finally, in Section 4 we apply the previous results to a parameterized Riccati equation.

2 Mild Evolution Operators

We will make no assumptions about the differentiability of our evolution operators, and the properties we assume should leave us with a class of evolution operators of sufficient generality to include practically all well-posed linear models of realistic dynamical systems (see Curtain, Pritchard [4]). Throughout this section we assume that Φ is a function mapping $,_{t_0}^{t_1}$ into $\mathcal{L}(X)$.

Definition 2.1 *$\Phi : ,_{t_0}^{t_1} \to \mathcal{L}(X)$ is said to be a* **mild evolution operator (on X)** *if*

(a.) $\Phi(t,t) = I$ *for each* $t \in [t_0, t_1]$,

(b.) $\Phi(t,\sigma)\Phi(\sigma,s) = \Phi(t,s)$ *for* $t_0 \leq s \leq \sigma \leq t \leq t_1$,

(c.) The maps $\Phi(\cdot, s) : [s, t_1] \to \mathcal{L}(X)$ *and* $\Phi(t, \cdot) : [t_0, t] \to \mathcal{L}(X)$ *are strongly continuous.*

Remark: Let Φ be a mild evolution operator.

(a.) Using the uniform boundedness principle, we obtain $\sup_{t_0 \leq s \leq t} \|\Phi(t,s)\| < \infty$ for fixed $t \in [t_0, t_1]$ and $\sup_{s \leq t \leq t_1} \|\Phi(t,s)\| < \infty$ for fixed $s \in [t_0, t_1]$.

(b.) Let $(t,s) \in ,_{t_0}^{t_1}$ with $s < t$ and fix $\rho \in (s,t)$. Applying Definition 2.1 (c.), the factorization

$$\Phi(t,s) = \Phi(t,\rho)\Phi(\rho,s)$$

shows that $\Phi(\cdot,\cdot)$ is strongly continuous in (t,s). This implies the strong continuity of the function

$$\Phi(\cdot,\cdot) : ,_{t_0}^{t_1} \backslash \{(\tau,\tau) | t_0 \leq \tau \leq t_1\} \to \mathcal{L}(X),$$

and we obtain $\sup_{(t,s)\in\Omega} \|\Phi(t,s)\| < \infty$ for each compact set $\Omega \subseteq ,_{t_0}^{t_1} \backslash \{(\tau,\tau)\ | t_0 \leq \tau \leq t_1\}$ using the uniform boundedness principle.

In general, Definition 2.1 does not imply the uniform boundedness of $\|\Phi(\cdot,\cdot)\|$ on $,_{t_0}^{t_1}$, for a counterexample see Gibson [7], Appendix B.

Definition 2.2 *We say that a mild evolution operator Φ is* **bounded** *if*

$$M_\Phi := \sup_{(t,s)\in ,_{t_0}^{t_1}} \|\Phi(t,s)\| < \infty.$$

For example, a mild evolution operator is bounded if $\Phi(\cdot,\cdot)$ is weakly continuous on $,_{t_0}^{t_1}$.

Definition 2.3 *We say that a mild evolution operator Φ on H has the* **property $(*)$** *if*

- *The map $\Phi(\cdot,s)^* : [s,t_1] \to \mathcal{L}(H)$ is strongly measurable for fixed $s \in [t_0,t_1]$.*
- *The map $\Phi(t,\cdot)^* : [t_0,t] \to \mathcal{L}(H)$ is strongly measurable for fixed $t \in [t_0,t_1]$.*

In general, the strong measurability of $W : [t_0,t_1] \to \mathcal{L}(H)$ does not imply that $W^* : [t_0,t_1] \to \mathcal{L}(H)$ is strongly measurable. But whether every mild evolution operator has property $(*)$ is not known to the author. Using the evolution property (Definition 2.1 (b.)) of mild evolution operators with property $(*)$ it is easy to see that Φ^* is strongly measurable on $,_{t_0}^{t_1}$. In the following we write $\Phi^*(t,s)$ instead of $\Phi(t,s)^*$. We use the following perturbation theorem from Gibson [7] and Curtain, Pritchard [4]:

Theorem 2.4 *Let Φ be a bounded mild evolution operator and let $W \in L_{s,\infty}(t_0,t_1;\mathcal{L}(X))$. Then the operator integral equation*

$$\Phi_W(t,s)x = \Phi(t,s)x + \int_s^t \Phi(t,\rho)W(\rho)\Phi_W(\rho,s)x\,d\rho, \tag{2.2}$$

$x \in X$, $(t,s) \in ,_{t_0}^{t_1}$, has a unique solution $\Phi_W : ,_{t_0}^{t_1} \to \mathcal{L}(X)$ in the class of bounded mild evolution operators. Φ_W is also the unique solution in the class of bounded mild evolution operators of

$$\Phi_W(t,s)x = \Phi(t,s)x + \int_s^t \Phi_W(t,\rho)W(\rho)\Phi(\rho,s)x\,d\rho, \quad x \in X, (t,s) \in ,_{t_0}^{t_1}.$$

This theorem is a generalization of a similar perturbation result for semigroups, given by Hille and Phillips [9]. In the following we will denote by Φ_W the unique bounded mild evolution operator which satisfies (2.2). The following perturbation theorem is also useful.

Theorem 2.5 *Suppose Φ is a bounded mild evolution operator on H with property $(*)$ and let $W \in L_{s,\infty}(t_0,t_1;\mathcal{L}(H))$ be given. Furthermore, let W^* be strongly measurable. Then the operator integral equation*

$$\Phi_W(t,s)x = \Phi(t,s)x + \int_s^t \Phi(t,\rho)W(\rho)\Phi_W(\rho,s)x\,d\rho, \quad x \in H, (t,s) \in ,_{t_0}^{t_1},$$

has a unique solution $\Phi_W : ,_{t_0}^{t_1} \to \mathcal{L}(H)$ in the class of bounded mild evolution operators with property $(*)$.

Proof: The previous theorem implies that the operator integral equation has a unique solution $\Phi_W : ,_{t_0}^{t_1} \to \mathcal{L}(H)$ in the class of bounded mild evolution operators. Since

$$\Phi_W^*(t,s)x = \Phi^*(t,s)x + \int_s^t \Phi_W^*(\rho,s)W^*(\rho)\Phi^*(t,\rho)x \, d\rho, \quad x \in H, (t,s) \in ,_{t_0}^{t_1}$$

it is easy to see that Φ_W has property $(*)$. □

Since strong measurability of $W : [t_0,t_1] \to \mathcal{L}(H)$ implies only weak measurability of the adjoint $W^* : [t_0,t_1] \to \mathcal{L}(H)$, we also require that W^* is strongly measurable. This is certainly the case if either H is separable or W is measurable.

3 Problem Formulation and Main Results

Throughout this section we will assume that $\Phi : ,_{t_0}^{t_1} \to \mathcal{L}(H)$ is a bounded mild evolution operator with property **(*)**, $B \in L_{s,\infty}(t_0,t_1;\mathcal{L}(U,H))$ and B^* is strongly measurable, $G \in \mathcal{H}(H)$, $C \in L_{s,\infty}(t_0,t_1;\mathcal{H}(H))$ and $x_0 \in H$. In this section we do not require any assumptions about definiteness of $C(t)$ or G. We consider an evolution process defined by the input–state map

$$x(t;t_0,x_0,u) = \Phi(t,t_0)x_0 + \int_{t_0}^t \Phi(t,\rho)B(\rho)u(\rho)\,d\rho, \qquad t \in [t_0,t_1], \quad (3.3)$$

where $u \in L_2(t_0,t_1;U)$. In the following we call u the control or input. Simple calculations show that the integral in (3.3) is well defined and $x \in C([t_0,t_1],H)$. We are particularly interested in feedback controls of the form

$$u(t) := F(t)x(t;t_0,x_0,u),$$

where $F \in L_{s,\infty}(t_0,t_1;\mathcal{L}(H,U))$. The state trajectories of the corresponding closed–loop system are solutions of the following integral equation

$$x(t;t_0,x_0,u) = \Phi(t,t_0)x_0 + \int_{t_0}^t \Phi(t,\rho)B(\rho)F(\rho)x(\rho,t_0,x_0,u)\,d\rho, \quad t \in [t_0,t_1].$$

The next theorem shows the existence and uniqueness of x in the class $C([t_0,t_1],H)$:

Theorem 3.1 *Let $F \in L_{s,\infty}(t_0,t_1;\mathcal{L}(H,U))$, $x_0 \in H$, $\tilde{u} \in L_2(t_0,t_1;U)$ be given and suppose that the control of our system is defined by*

$$u(t) = F(t)x(t;t_0,x_0,u) + \tilde{u}(t), \qquad t \in [t_0,t_1]. \quad (3.4)$$

Then the unique solution of (3.3) in $C([t_0,t_1],H)$ is given by

$$x(t;t_0,x_0,u) = \Phi_{BF}(t,t_0)x_0 + \int_{t_0}^{t} \Phi_{BF}(t,\rho)B(\rho)\tilde{u}(\rho)\,d\rho, \tag{3.5}$$

where Φ_{BF} is defined as in Theorem 2.4.

Proof: First we show that (3.4) and (3.5) satisfy equation (3.3). This follows from

$$\begin{aligned}
&x(t;t_0,x_0,u)\\
&\quad= \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,\rho)B(\rho)F(\rho)\Phi_{BF}(\rho,t_0)x_0\,d\rho\\
&\qquad+ \int_{t_0}^{t}\left[\Phi(t,\rho)B(\rho)\tilde{u}(\rho) + \int_{\rho}^{t}\Phi(t,\tau)B(\tau)F(\tau)\Phi_{BF}(\tau,\rho)B(\rho)\tilde{u}(\rho)\,d\tau\right]d\rho\\
&\quad= \Phi(t,t_0)x_0 + \int_{t_0}^{t}\Phi(t,\rho)B(\rho)\tilde{u}(\rho)\,d\rho\\
&\qquad+ \int_{t_0}^{t}\Phi(t,\rho)B(\rho)F(\rho)\left[\Phi_{BF}(\rho,t_0)x_0 + \int_{t_0}^{\rho}\Phi_{BF}(\rho,\tau)B(\tau)\tilde{u}(\tau)\,d\tau\right]d\rho\\
&\quad= \Phi(t,t_0)x_0 + \int_{t_0}^{t}\Phi(t,\rho)B(\rho)\left[F(\rho)x(\rho;t_0,x_0,u) + \tilde{u}(\rho)\right]d\rho.
\end{aligned}$$

It remains to prove the uniqueness of x in $C([t_0,t_1],H)$. Assuming $x_1, x_2 \in C([t_0,t_1],H)$ are solutions of (3.3), where u is given by (3.4), with $x_1 \neq x_2$. Then $y := x_1 - x_2$ satisfies $y(t) = \int_{t_0}^{t}\Phi(t,\rho)B(\rho)F(\rho)y(\rho)\,d\rho$ and we obtain $\|y(t)\| \leq M_\Phi\|B\|_\infty\|F\|_\infty\int_{t_0}^{t}\|y(\rho)\|\,d\rho$. Now Gronwall's lemma implies $y = 0$, which is a contradiction to our assumption and the proof is complete. □

The performance index for our input–state system is given by

$$\begin{aligned}
J(t_0,t_1,x_0,u) \;=\;& \langle x(t_1;t_0,x_0,u), Gx(t_1;t_0,x_0,u)\rangle\\
&+ \int_{t_0}^{t_1}\langle x(t;t_0,x_0,u), C(t)x(t;t_0,x_0,u)\rangle + \|u(t)\|^2\,dt,
\end{aligned}$$

where $u \in L_2(t_0,t_1;U)$. In general, due to the lack of positive semi–definiteness of G and $C(t)$, there need not exist a function $u \in L_2(t_0,t_1;U)$ which minimizes the performance index $J(t_0,t_1,x_0,u)$. Now we write the performance index as a quadratic form. For this, we introduce the following operator functions developed by Hinrichsen and Pritchard in [10] and

Jonckheere and Silverman in [11]:

$$(\mathcal{C}_{t_0,t_1}u)(t) = B^*(t)\int_t^{t_1}\Phi^*(\rho,t)C(\rho)\int_{t_0}^{\rho}\Phi(\rho,\tau)B(\tau)u(\tau)\,d\tau\,d\rho, \tag{3.6}$$

$$(\mathcal{G}_{t_0,t_1}u)(t) = B^*(t)\Phi^*(t_1,t)G\int_{t_0}^{t_1}\Phi(t_1,\rho)B(\rho)u(\rho)\,d\rho, \tag{3.7}$$

where $u \in L_2(t_0,t_1;U)$ and $t \in [t_0,t_1]$. The proof of the next lemma is straightforward, and left to the reader:

Lemma 3.2 $\mathcal{C}_{t_0,t_1}, \mathcal{G}_{t_0,t_1} \in \mathcal{H}(L_2(t_0,t_1;U))$.

The next lemma is an immediate extension of [10], Lemma 7.1.12, and the proof is the same.

Lemma 3.3 *The performance index has the form:*

$$J(t_0,t_1,x_0,u) = \langle x_0, M_{t_0,t_1}x_0\rangle + 2Re\langle N_{t_0,t_1}x_0, u\rangle_2 + \langle u, (I+\mathcal{C}_{t_0,t_1}+\mathcal{G}_{t_0,t_1})u\rangle_2$$

where $M_{t_0,t_1} \in \mathcal{H}(H)$, $N_{t_0,t_1} \in \mathcal{L}(H, L_2(t_0,t_1;U))$ *and* $\mathcal{C}_{t_0,t_1}, \mathcal{G}_{t_0,t_1}$ $\in \mathcal{H}(L_2(t_0,t_1;U))$ *are given by*

$$\begin{aligned} M_{t_0,t_1}x &= \Phi^*(t_1,t_0)G\Phi(t_1,t_0)x + \int_{t_0}^{t_1}\Phi^*(\rho,t_0)C(\rho)\Phi(\rho,t_0)x\,d\rho, \\ N_{t_0,t_1}x(t) &= B^*(t)\Phi^*(t_1,t)G\Phi(t_1,t_0)x + \int_t^{t_1}B^*(t)\Phi^*(\rho,t)C(\rho)\Phi(\rho,t_0)x\,d\rho \end{aligned}$$

and (3.6), (3.7).

Proof:

$$\begin{aligned} &J(t_0,t_1,x_0,u) \\ &\quad= \langle x(t_1;t_0,x_0,u), Gx(t_1;t_0,x_0,u)\rangle \\ &\qquad+ \int_{t_0}^{t_1}\langle x(t;t_0,x_0,u), C(t)x(t;t_0,x_0,u)\rangle + \|u(t)\|^2\,dt \\ &\quad= \Big\langle \Phi(t_1,t_0)x_0 + \int_{t_0}^{t_1}\Phi(t_1,\rho)B(\rho)u(\rho)\,d\rho, \\ &\qquad\qquad G\Big[\Phi(t_1,t_0)x_0 + \int_{t_0}^{t_1}\Phi(t_1,\rho)B(\rho)u(\rho)\,d\rho\Big]\Big\rangle \\ &\qquad+ \int_{t_0}^{t_1}\Big\{\Big\langle \Phi(t,t_0)x_0 + \int_{t_0}^{t}\Phi(t,\rho)B(\rho)u(\rho)\,d\rho, C(t)\Big[\Phi(t,t_0)x_0 \end{aligned}$$

$$
\begin{aligned}
& + \left. \left. \int_{t_0}^{t} \Phi(t,\rho)B(\rho)u(\rho)\,d\rho \right] \right\rangle + \|u(t)\|^2 \Big\} \, dt \\
&= \langle \Phi(t_1,t_0)x_0, G\Phi(t_1,t_0)x_0\rangle + 2\mathrm{Re}\int_{t_0}^{t_1} \langle \Phi(t_1,t_0)x_0, G\Phi(t_1,\rho)B(\rho)u(\rho)\rangle\, d\rho \\
&\quad + \int_{t_0}^{t_1} \left\langle u(\rho), B^*(\rho)\Phi^*(t_1,\rho)G\int_{t_0}^{t_1} \Phi(t_1,\tau)B(\tau)u(\tau)\,d\tau \right\rangle d\rho \\
&\quad + \int_{t_0}^{t_1} \langle \Phi(t,t_0)x_0, C(t)\Phi(t,t_0)x_0\rangle\, dt + \int_{t_0}^{t_1} \|u(t)\|^2\, dt \\
&\quad + 2\mathrm{Re}\int_{t_0}^{t_1} \left\langle \Phi(t,t_0)x_0, C(t)\int_{t_0}^{t} \Phi(t,\rho)B(\rho)u(\rho)\,d\rho \right\rangle dt \\
&\quad + \int_{t_0}^{t_1} \left\langle \int_{t_0}^{t} \Phi(t,\rho)B(\rho)u(\rho)\,d\rho, C(t)\int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)\,d\tau \right\rangle dt \\
&= \langle x_0, M_{t_0,t_1}x_0\rangle + 2\mathrm{Re}\int_{t_0}^{t_1} \langle B^*(\rho)\Phi^*(t_1,\rho)G\Phi(t_1,t_0)x_0, u(\rho)\rangle\, d\rho \\
&\quad + 2\mathrm{Re}\int_{t_0}^{t_1} \left\langle \int_{\rho}^{t_1} B^*(\rho)\Phi^*(t,\rho)C(t)\Phi(t,t_0)x_0\, dt, u(\rho) \right\rangle d\rho \\
&\quad + \langle u, (I+\mathcal{G}_{t_0,t_1})u\rangle_2 \\
&\quad + \int_{t_0}^{t_1}\int_{t_0}^{t} \left\langle u(\rho), B^*(\rho)\Phi^*(t,\rho)C(t)\int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)\,d\tau \right\rangle d\rho\, dt \\
&= \langle x_0, M_{t_0,t_1}x_0\rangle + 2\mathrm{Re}\langle N_{t_0,t_1}x_0, u\rangle_2 + \langle u, (I+\mathcal{C}_{t_0,t_1}+\mathcal{G}_{t_0,t_1})u\rangle_2 .
\end{aligned}
$$

□

We study the following optimization problem:

Finite Horizon Linear Quadratic Optimization Problem: Does there exist for every $x_0 \in H$ a unique input $u \in L_2(t_0,t_1;U)$ which minimizes the performance index $J(t_0,t_1,x_0,u)$?

The integral Riccati equation associated with the finite horizon linear quadratic optimization problem has the form

$$
\begin{aligned}
P(t)z &= \Phi^*(t_1,t)G\Phi(t_1,t)z \\
&\quad + \int_t^{t_1} \Phi^*(\rho,t)[C(\rho) - P(\rho)B(\rho)B^*(\rho)P(\rho)]\Phi(\rho,t)z\, d\rho, \quad z \in H.
\end{aligned}
\tag{3.8}
$$

Definition 3.4 *P is said to be a* **solution of the integral Riccati equation (IRE) on $[t_0,t_1]$(with final condition $P(t_1) = G$)** *if $P \in L_{s,\infty}(t_0,t_1;\mathcal{H}(H))$ and for each $z \in H$ and $t \in [t_0,t_1]$ the equation (3.8) is satisfied.*

If the IRE (3.8) has a solution on $[t_0,t_1]$, then it also has a solution on $[\tilde{t},t_1]$ with $t_0 \le \tilde{t} \le t_1$. Since G and $C(t)$ are only self–adjoint, such a solution does not necessarily exist, as Example 3.7 shows. The main results of this paper are presented in the following two theorems which give sufficient and necessary conditions for the solvability of the integral Riccati equation, and of the finite horizon linear quadratic optimization problem:

Theorem 3.5 *The following statements are equivalent:*

(O1) *For every $x_0 \in H$ there exists a unique input $u \in L_2(t_0,t_1;U)$ which minimizes the performance index $J(t_0,t_1,x_0,u)$.*

(O2) *$I+\mathcal{C}_{t_0,t_1}+\mathcal{G}_{t_0,t_1} > 0$ and the range of N_{t_0,t_1} is a subset of the range of $I+\mathcal{C}_{t_0,t_1}+\mathcal{G}_{t_0,t_1}$.*

Let these condition be satisfied. Then for every $x_0 \in H$ the optimal control u_{t_0,x_0} fulfill

$$(I+\mathcal{C}_{t_0,t_1}+\mathcal{G}_{t_0,t_1})u_{t_0,x_0} + N_{t_0,t_1}x_0 = 0. \tag{3.9}$$

Proof: For $x_0 \in H$ and for any pair $u, \hat{u} \in L_2(t_0,t_1;U)$ using Lemma 3.3, we have

$$\begin{aligned} &J(t_0,t_1,x_0,\hat{u}+u) \\ &\quad = J(t_0,t_1,x_0,\hat{u}) + 2\mathrm{Re}\langle N_{t_0,t_1}x_0 + (I+\mathcal{C}_{t_0,t_1}+\mathcal{G}_{t_0,t_1})\hat{u}, u\rangle_2 \qquad (3.10) \\ &\qquad + \langle u, (I+\mathcal{C}_{t_0,t_1}+\mathcal{G}_{t_0,t_1})u\rangle_2. \end{aligned}$$

Let $x_0 \in H$ be given and assume that $u_{t_0,x_0} \in L_2(t_0,t_1;U)$ is a minimizing solution, but does not satisfy equation (3.9). If we choose $u_\varepsilon := -\varepsilon((I+\mathcal{C}_{t_0,t_1}+\mathcal{G}_{t_0,t_1})u_{t_0,x_0} + N_{t_0,t_1}x_0)$, $\varepsilon > 0$ small, we obtain

$$\begin{aligned} &J(t_0,t_1,x_0,u_{t_0,x_0}+u_\varepsilon) \\ &\quad = J(t_0,t_1,x_0,u_{t_0,x_0}) - \frac{2}{\varepsilon}\|u_\varepsilon\|_2^2 + \langle u_\varepsilon, (I+\mathcal{C}_{t_0,t_1}+\mathcal{G}_{t_0,t_1})u_\varepsilon\rangle_2 \\ &\quad \leq J(t_0,t_1,x_0,u_{t_0,x_0}) - \left[\frac{2}{\varepsilon} - \|I+\mathcal{C}_{t_0,t_1}+\mathcal{G}_{t_0,t_1}\|\right]\|u_\varepsilon\|_2^2. \end{aligned}$$

Thus u_{t_0,x_0} cannot be optimal which is a contradiction to our assumption. Hence every optimal control u_{t_0,x_0} satisfy (3.9).

(O1)⇒(O2) Since for every $x_0 \in H$ the optimal control u_{t_0,x_0} satisfies (3.9) the range of N_{t_0,t_1} is a subset of the range of $I+\mathcal{C}_{t_0,t_1}+\mathcal{G}_{t_0,t_1}$. It remains to prove that $I+\mathcal{C}_{t_0,t_1}+\mathcal{G}_{t_0,t_1} > 0$. Suppose there exists a function $u \in L_2(t_0,t_1;U)$ with $\langle u, (I+\mathcal{C}_{t_0,t_1}+\mathcal{G}_{t_0,t_1})u\rangle < 0$. Then using Lemma 3.3 we see

$$\inf_{\gamma>0} J(t_0,t_1,0,\gamma u) = -\infty,$$

which contradicts our assumption. Since there exists a unique input $u \in L_2(t_0, t_1; U)$ which minimizes the performance index $J(t_0, t_1, 0, u)$, we have $I + \mathcal{C}_{t_0,t_1} + \mathcal{G}_{t_0,t_1} > 0$.

(O2)⇒(O1) **(O2)** implies that for every $x_0 \in H$, equation (3.9) has a unique solution and it is easy to see using (3.10) that this solution minimizes the performance index. Since every optimal control satisfies (3.9), statement **(O1)** is proved. □

Concerning the solvability of the integral Riccati equation we have the following equivalent conditions:

Theorem 3.6 *The following statements are equivalent:*

(R1) *The IRE (3.8) has a unique solution on* $[t_0, t_1]$.

(R2) *There exists a constant* $\beta > 0$, *such that for each* $t_0' \in [t_0, t_1]$ *and each* $x_0 \in H$

$$\inf_{u \in L_2(t_0', t_1; U)} J(t_0', t_1, x_0, u) \geq -\beta \|x_0\|^2.$$

(R3) *There exists a constant* $\varepsilon > 0$ *such that for every* $u \in L_2(t_0, t_1; U)$*:*

$$J(t_0, t_1, 0, u) \geq \varepsilon \|u\|_2^2.$$

(R4) $I + \mathcal{C}_{t_0,t_1} + \mathcal{G}_{t_0,t_1}$ *is positive definite and invertible in* $\mathcal{L}(L_2(t_0, t_1; U))$.

In the finite–dimensional situation, this theorem has been proven by Hinrichsen and Pritchard [10]. In this case, in the problem formulation differential Riccati equations may be considered, instead of the IRE. Further, in the theorem, **(R2)** and **(R4)** may be replaced by **(R2')** and **(R4')** respectively, where

(R2') There exists a constant $\beta > 0$, such that for each $x_0 \in H$

$$\inf_{u \in L_2(t_0, t_1; U)} J(t_0, t_1, x_0, u) \geq -\beta \|x_0\|^2.$$

(R4') $I + \mathcal{C}_{t_0,t_1} + \mathcal{G}_{t_0,t_1} > 0$.

It is possible to replace **(R2)** by **(R2')** because for every $(t, s) \in \, , {}_{t_0}^{t_1}$, $\Phi(t, s)$ is invertible. Since in the finite–dimensional situation the operators $\mathcal{C}_{t_0,t_1}$ and $\mathcal{G}_{t_0,t_1}$ are compact, it is easy to see that **(R4')** implies **(R4)**. Clearly if there exists a solution to the IRE on $[t_0, t_1]$, then the finite horizon linear quadratic optimization problem is uniquely solvable. Furthermore, in the finite–dimensional case the converse direction is true. The next example shows that in the infinite–dimensional situation in general **(R4')** does not imply **(R4)** and **(O1)** not **(R1)**. At the end of this section we give sufficient conditions such that from **(R4')** follows **(R4)** in the infinite–dimensional situation.

Example 3.7: We choose $U = H = L_2(0,1;\mathbb{R})$, $[t_0, t_1] = [0,1]$ and define for $k \in [0,1]$ the shift–operator $S_k : H \to H$ by

$$(S_k x)(t) = \begin{cases} 0 \quad , & t < k \\ x(t-k), & t \geq k \end{cases} .$$

It is easy to see that $S_k \in \mathcal{L}(H)$ with $\|S_k\| = \delta_{k,1}$ and the adjoint operator is given by

$$(S_k^* x)(t) = \begin{cases} x(t+k), & t \leq 1-k \\ 0 \quad , & t > 1-k \end{cases} .$$

Then $\Phi : \,, {}_0^1 \to \mathcal{L}(H)$, $\Phi(t,s) = S_{t-s}$, defines a bounded mild evolution operator with property $(*)$. Furthermore, we set $B(t) \equiv I$, $C(t) \equiv 0$ and $G = -I$. In this situation, the operators $\mathcal{C}_{0,1}, \mathcal{G}_{0,1}$, $M_{0,1}$ and $N_{0,1}$ are given by $\mathcal{C}_{0,1} = 0$, $(\mathcal{G}_{0,1}u)(t) = -S_{1-t}^* \int_0^1 S_{1-\rho}[u(\rho)]\, d\rho$, $M_{0,1} = 0$ and $N_{0,1} = 0$. The estimate

$$\begin{aligned} \langle u, (I + \mathcal{G}_{0,1})u \rangle &= \langle u, u \rangle + \langle u, \mathcal{G}_{0,1} u \rangle \\ &= \langle u, u \rangle - \left\langle \int_0^1 S_{1-t} u(t)\, dt, \int_0^1 S_{1-t} u(t)\, dt \right\rangle_U \\ &= \int_0^1 \|u(t)\|_U^2\, dt - \left\| \int_0^1 S_{1-t} u(t)\, dt \right\|_U^2 \\ &\geq \int_0^1 \underbrace{\|u(t)\|_U^2 - \|S_{1-t} u(t)\|_U^2}_{\geq 0, \text{ since } \|S_{1-t}\| \leq 1} dt \geq 0 \end{aligned}$$

proves $I + \mathcal{G}_{0,1} \geq 0$.

Let $a \in L_2(0,1;U)$ with $a(t)(s) = a \in \mathbb{R}$, $s, t \in [0,1]$, and assume that there exists $u \in L_2(0,1;U)$ such that $[I + \mathcal{G}_{0,1}]u = a$. Since $L_2(0,1;U) \cong L_2([0,1]^2, \mathbb{R})$ ([6], III.11.17) there exists a set $\,,_1 \subseteq [0,1]^2$ of measure zero in $[0,1]^2$ such that for every $(t,s) \in [0,1]^2 \backslash \,,_1$ we have

$$([I + \mathcal{G}_{0,1}]u)(t)(s) = a.$$

Furthermore, by [6], III.11.16 there exists a set $\Delta \subset [0,1]$ of measure zero in $\mathbb{R}$ such that for every $s \in [0,1] \backslash \Delta$

$$\int_0^1 [S_{1-t}^* S_{1-\rho} u(\rho)](s)\, d\rho = \left(\int_0^1 S_{1-t}^* S_{1-\rho} u(\rho)\, d\rho \right)(s).$$

Define $\Gamma_2 := [0,1]\times\Delta$. Then Γ_2 and $\Gamma_3 := \Gamma_1 \cup \Gamma_2$ have vanishing measure. Let

$$\Lambda := \{\delta \in [0,1] : \{h \in [0, 1-\delta] : (\delta + h, h) \in \Gamma_3\} \text{ have positive measure}\}.$$

Then Λ has measure zero in $[0,1]$ because Γ_3 has vanishing measure. Define $\Gamma_4 := \{(\delta+h, h) | \delta \in \Lambda, h \in [0, 1-\delta]\}$ and $\Gamma = \Gamma_3 \cup \Gamma_4$. Thus Γ has measure zero. Moreover, choose $(t,s) \in [0,1] \backslash \Gamma$. Hence we obtain

$$u(t)(s) = a + \int_0^1 [S^*_{1-t} S_{1-\rho} u(\rho)](s)\, d\rho.$$

It is easy to see that

$$[S^*_{1-t} S_{1-\rho} u(\rho)](s) = \begin{cases} 0 & , \quad s < t - \rho \\ u(\rho)(s + \rho - t), & \quad t - \rho \le s \le t \\ 0 & , \quad t < s \end{cases}.$$

Thus we have $u(t)(s) = a$ if $s > t$. Now we consider the case that $s \le t$. This implies

$$u(t)(s) = a + \int_{t-s}^1 u(\rho)(s + \rho - t)\, d\rho,$$

which shows

$$u(t)(s) = u(t+h)(s+h) \qquad \text{for } h \in [-s, 1-t] \text{ and } (t+h, s+h) \notin \Gamma_3.$$

Since $(t,s) \notin \Gamma_4$, we obtain

$$u(t)(s) = a + \int_{t-s}^1 u(t)(s)\, d\rho = a + (1 - t + s) u(t)(s)$$

and thus

$$u(t)(s) = \begin{cases} a & , s > t, (t,s) \notin \Gamma \\ a/(t-s) & , s < t, (t,s) \notin \Gamma \end{cases}.$$

Choosing $a = 0$ we get that $I + \mathcal{G}_{0,1}$ is injective which implies using $I + \mathcal{G}_{0,1} \ge 0$ that $I + \mathcal{G}_{0,1} > 0$. Furthermore, the range of $N_{0,1}$ is a subset of the range of $I + \mathcal{G}_{0,1}$ (Note: $N_{0,1} = 0$). Thus Theorem 3.5 proves that for every $x_0 \in H$, there exists a unique input $u \in L_2(0,1;U)$ which minimizes the performance index $J(0,1,x_0,u)$. Now we prove that $I + \mathcal{G}_{0,1}$ is not surjective. Assume there exists an input $u \in L_2(0,1;U)$ such that

$$[I + \mathcal{G}_{0,1}]u = 1 \qquad \text{in } L_2(0,1;U).$$

The calculations above ($a = 1$) prove that $u \notin L_2(0,1;U)$, which contradicts our assumption and $I + \mathcal{G}_{0,1}$ is not surjective. This example shows that in general the solvability of the optimal control problem does not imply the existence of solutions to the IRE and **(R4')** does not imply **(R4)**.

The next theorem proves that the solvability of the optimal control problem implies **(R2')**. But in general the solvability of the optimal control problem does not imply **(R2)** as the previous example shows. Thus contrary to the finite–dimensional situation, in general we can not derive **(R2)** from **(R2')**.

Theorem 3.7 **(O1)** *implies* **(R2')**.

Proof: First Theorem 3.5 implies that there exists an operator $T \in \mathcal{H}(L_2(t_0,t_1;U))$ such that $I + \mathcal{C}_{t_0,t_1} + \mathcal{G}_{t_0,t_1} = T^2$. Clearly the range of N_{t_0,t_1} is a subset of the range of T [use Theorem 3.5]. By [4], Theorem 3.3, there exists a constant $\gamma > 0$ such that

$$\|Tu\|_2 \geq \gamma \|N^*_{t_0,t_1} u\|_H, \qquad u \in L_2(t_0,t_1;U).$$

Hence

$$\begin{aligned} J(t_0,t_1,x_0,u) &= \langle x_0, M_{t_0,t_1} x_0\rangle + 2\mathrm{Re}\langle x_0, N^*_{t_0,t_1} u\rangle + \|Tu\|_2^2 \\ &\geq \langle x_0, M_{t_0,t_1} x_0\rangle + 2\mathrm{Re}\langle x_0, N^*_{t_0,t_1} u\rangle + \gamma^2 \|N^*_{t_0,t_1} u\|_H^2 \\ &= \langle x_0, M_{t_0,t_1} x_0\rangle + \gamma^2 \|N^*_{t_0,t_1} u + \gamma^{-2} x_0\|^2 - \gamma^{-2}\|x_0\|^2 \\ &\geq -(\|M_{t_0,t_1}\| + \gamma^{-2})\|x_0\|^2. \end{aligned}$$

□

For the proof of Theorem 3.6 we need some preliminary results, which are also of independent interest. Some of the proofs in [10] do not extend to the infinite–dimensional situation, and so the generalization is far from being trivial. The reasons are that the evolution operators are not differentiable, and **(R4)** must be used instead of **(R4')**.

Lemma 3.8 *Suppose P is a solution of the IRE (3.8) on $[t_0,t_1]$ and $u \in L_2(t_0,t_1;U)$ is a given control. Then*

$$\begin{aligned} &\langle x_0, P(t_0) x_0\rangle \\ &\quad = \langle x(t_1;t_0,x_0,u), Gx(t_1;t_0,x_0,u)\rangle \\ &\qquad + \int_{t_0}^{t_1} \{\langle x(\rho;t_0,x_0,u), [C(\rho) - P(\rho)B(\rho)B^*(\rho)P(\rho)] x(\rho;t_0,x_0,u)\rangle \\ &\qquad\qquad - 2Re\langle x(\rho;t_0,x_0,u), P(\rho)B(\rho)u(\rho)\rangle\}\, d\rho, \end{aligned}$$

where x is given by (3.3).

Proof: First we set $M(\rho) := [C(\rho) - P(\rho)B(\rho)B^*(\rho)P(\rho)]$. Then by (3.3),

$$
\begin{aligned}
&\langle x(t_1;t_0,x_0,u), Gx(t_1;t_0,x_0,u)\rangle \\
&= \underbrace{\langle \Phi(t_1,t_0)x_0, G\Phi(t_1,t_0)x_0\rangle}_{=:I_1} + 2\mathrm{Re}\left\langle \Phi(t_1,t_0)x_0, G\int_{t_0}^{t_1}\Phi(t_1,\rho)B(\rho)u(\rho)\,d\rho\right\rangle \\
&\quad + \left\langle \int_{t_0}^{t_1}\Phi(t_1,\rho)B(\rho)u(\rho)\,d\rho, G\int_{t_0}^{t_1}\Phi(t_1,\tau)B(\tau)u(\tau)\,d\tau\right\rangle \\
&= I_1 + 2\,\underbrace{\mathrm{Re}\int_{t_0}^{t_1}\langle \Phi(t_1,\rho)\underline{\Phi(\rho,t_0)x_0}, G\Phi(t_1,\rho)\underline{B(\rho)u(\rho)}\rangle\,d\rho}_{=:I_2} \\
&\quad + \int_{t_0}^{t_1}\int_{t_0}^{\rho}\langle \Phi(t_1,\rho)B(\rho)u(\rho), G\Phi(t_1,\rho)\Phi(\rho,\tau)B(\tau)u(\tau)\rangle\,d\tau\,d\rho \\
&\quad + \int_{t_0}^{t_1}\int_{\rho}^{t_1}\langle \Phi(t_1,\tau)\Phi(\tau,\rho)B(\rho)u(\rho), G\Phi(t_1,\tau)B(\tau)u(\tau)\rangle\,d\tau\,d\rho \\
&= I_1 + 2I_2 \\
&\quad + 2\,\underbrace{\mathrm{Re}\int_{t_0}^{t_1}\int_{t_0}^{\rho}\langle \Phi(t_1,\rho)\underline{\Phi(\rho,\tau)B(\tau)u(\tau)}, G\Phi(t_1,\rho)\underline{B(\rho)u(\rho)}\rangle\,d\tau\,d\rho}_{=:I_3} \\
&= I_1 + 2I_2 + 2I_3,
\end{aligned}
$$

$$
\begin{aligned}
&\int_{t_0}^{t_1}\langle x(\rho;t_0,x_0,u), M(\rho)x(\rho;t_0,x_0,u)\rangle\,d\rho \\
&= \underbrace{\int_{t_0}^{t_1}\langle \Phi(\rho,t_0)x_0, M(\rho)\Phi(\rho,t_0)x_0\rangle\,d\rho}_{=:I_4} \\
&\quad + 2\mathrm{Re}\int_{t_0}^{t_1}\left\langle \Phi(\rho,t_0)x_0, M(\rho)\int_{t_0}^{\rho}\Phi(\rho,\tau)B(\tau)u(\tau)\,d\tau\right\rangle\,d\rho \\
&\quad + \int_{t_0}^{t_1}\left\langle \int_{t_0}^{\rho}\Phi(\rho,\tau)B(\tau)u(\tau)\,d\tau, M(\rho)\int_{t_0}^{\rho}\Phi(\rho,\varepsilon)B(\varepsilon)u(\varepsilon)\,d\varepsilon\right\rangle\,d\rho \\
&= I_4 + 2\,\underbrace{\mathrm{Re}\int_{t_0}^{t_1}\int_{\rho}^{t_1}\langle \Phi(\tau,\rho)\underline{\Phi(\rho,t_0)x_0}, M(\tau)\Phi(\tau,\rho)\underline{B(\rho)u(\rho)}\rangle\,d\tau\,d\rho}_{=:I_5} \\
&\quad + \int_{t_0}^{t_1}\int_{\rho}^{t_1}\int_{t_0}^{\varepsilon}\langle \Phi(\varepsilon,\tau)B(\tau)u(\tau), M(\varepsilon)\Phi(\varepsilon,\rho)B(\rho)u(\rho)\rangle\,d\tau\,d\varepsilon\,d\rho \\
&= I_4 + 2I_5
\end{aligned}
$$

$$
\begin{aligned}
& +\int_{t_0}^{t_1}\int_{\rho}^{t_1}\left[\int_{t_0}^{\rho}+\int_{\rho}^{\varepsilon}\right]\langle\Phi(\varepsilon,\tau)B(\tau)u(\tau), M(\varepsilon)\Phi(\varepsilon,\rho)B(\rho)u(\rho)\rangle\, d\tau\, d\varepsilon\, d\rho \\
= \quad & I_4 + 2I_5 \\
& \underbrace{+\int_{t_0}^{t_1}\int_{t_0}^{\rho}\int_{\rho}^{t_1}\langle\Phi(\varepsilon,\rho)\underline{\Phi(\rho,\tau)B(\tau)u(\tau)}, M(\varepsilon)\Phi(\varepsilon,\rho)\underline{B(\rho)u(\rho)}\rangle\, d\varepsilon\, d\tau\, d\rho}_{=:I_{6,a}} \\
& + \int_{t_0}^{t_1}\int_{\rho}^{t_1}\int_{\tau}^{t_1}\langle\Phi(\varepsilon,\tau)B(\tau)u(\tau), M(\varepsilon)\Phi(\varepsilon,\rho)B(\rho)u(\rho)\rangle\, d\varepsilon\, d\tau\, d\rho \\
= \quad & I_4 + 2I_5 + I_{6,a} \\
& +\int_{t_0}^{t_1}\int_{t_0}^{\rho}\int_{\rho}^{t_1}\langle\Phi(\varepsilon,\rho)B(\rho)u(\rho), M(\varepsilon)\Phi(\varepsilon,\tau)B(\tau)u(\tau)\rangle\, d\varepsilon\, d\tau\, d\rho \\
= \quad & I_4 + 2I_5 + I_{6,a} \\
& +\int_{t_0}^{t_1}\int_{t_0}^{\rho}\int_{\rho}^{t_1}\overline{\langle\Phi(\varepsilon,\rho)\underline{\Phi(\rho,\tau)B(\tau)u(\tau)}, M(\varepsilon)\Phi(\varepsilon,\rho)\underline{B(\rho)u(\rho)}\rangle}\, d\varepsilon\, d\tau\, d\rho \\
= \quad & I_4 + 2I_5 + 2I_6,
\end{aligned}
$$

where $I_6 = \mathrm{Re} I_{6,a}$, and

$$
\begin{aligned}
& -2\mathrm{Re}\int_{t_0}^{t_1}\langle x(\rho;t_0,x_0,u), P(\rho)B(\rho)u(\rho)\rangle\, d\rho \\
& \quad = \quad \underbrace{-2\,\mathrm{Re}\int_{t_0}^{t_1}\langle\underline{\Phi(\rho,t_0)x_0}, P(\rho)\underline{B(\rho)u(\rho)}\rangle\, d\rho}_{=:I_7} \\
& \qquad\qquad \underbrace{-2\,\mathrm{Re}\int_{t_0}^{t_1}\int_{t_0}^{\rho}\langle\underline{\Phi(\rho,\tau)B(\tau)u(\tau)}, P(\rho)\underline{B(\rho)u(\rho)}\rangle\, d\tau\, d\rho}_{=:I_8} \\
& \quad = \quad -2I_7 - 2I_8.
\end{aligned}
$$

Since P is a solution of the IRE (3.8) on $[t_0, t_1]$, we obtain

$$
\begin{aligned}
I_1 + I_4 &= \langle x_0, P(t_0)x_0\rangle, \\
2I_2 + 2I_5 - 2I_7 &= 0, \\
2I_3 + 2I_6 - 2I_8 &= 0.
\end{aligned}
$$

This proves the lemma. □

Lemma 3.9 *Suppose P is a solution of the IRE (3.8) on $[t_0, t_1]$. Then*

$$
\langle x_0, P(t_0)x_0\rangle = J(t_0, t_1, x_0, -B^*(\cdot)P(\cdot)\Phi_{-BB^*P}(\cdot, t_0)x_0).
$$

Proof: For $u(t) = -B^*(t)P(t)\Phi_{-BB^*P}(t,t_0)x_0$ we obtain $x(t;t_0,x_0,u) = \Phi_{-BB^*P}(t,t_0)x_0$, hence $u(t) = -B^*(t)P(t)x(t;t_0,x_0,u)$. Now Lemma 3.8 implies

$$\begin{aligned}
&\langle x_0, P(t_0)x_0\rangle \\
&\quad = \langle x(t_1;t_0,x_0,u), Gx(t_1;t_0,x_0,u)\rangle \\
&\qquad + \int_{t_0}^{t_1} \langle x(\rho;t_0,x_0,u), [C(\rho) - P(\rho)B(\rho)B^*(\rho)P(\rho)]x(\rho;t_0,x_0,u)\rangle \\
&\qquad\qquad +2\mathrm{Re}\langle x(\rho;t_0,x_0,u), P(\rho)B(\rho)B^*(\rho)P(\rho)x(\rho;t_0,x_0,u)\rangle \, d\rho \\
&\quad = \langle x(t_1;t_0,x_0,u), Gx(t_1;t_0,x_0,u)\rangle \\
&\qquad\qquad + \int_{t_0}^{t_1} \langle x(\rho;t_0,x_0,u), C(\rho)x(\rho;t_0,x_0,u)\rangle \\
&\qquad\qquad + \| - B^*(\rho)P(\rho)x(\rho;t_0,x_0,u)\|^2 \, d\rho \\
&\quad = J(t_0,t_1,x_0,-B^*(\cdot)P(\cdot)\Phi_{-BB^*P}(\cdot,t_0)x_0).
\end{aligned}$$

□

Theorem 3.10 *Suppose P is a solution of the IRE (3.8) on $[t_0,t_1]$, and $u \in L_2(t_0,t_1;U)$, with the corresponding state being given by (3.3). Then*

$$\begin{aligned}
&J(t_0,t_1,x_0,-B^*(\cdot)P(\cdot)\Phi_{-BB^*P}(\cdot,t_0)x_0) \\
&\quad = \langle x_0, P(t_0)x_0\rangle \\
&\quad = J(t_0,t_1,x_0,u) - \int_{t_0}^{t_1} \|u(\rho) + B^*(\rho)P(\rho)x(\rho;t_0,x_0,u)\|^2 \, d\rho;
\end{aligned}$$

hence

$$\begin{aligned}
\min_{u\in L_2(t_0,t_1;U)} J(t_0,t_1,x_0,u) &= J(t_0,t_1,x_0,-B^*(\cdot)P(\cdot)\Phi_{-BB^*P}(\cdot,t_0)x_0) \\
&= \langle x_0, P(t_0)x_0\rangle.
\end{aligned}$$

Furthermore, the finite horizon linear quadratic optimization problem is (uniquely) solvable with optimal control u_{t_0,x_0} given by

$$u_{t_0,x_0}(t) = -B^*(t)P(t)\Phi_{-BB^*P}(t,t_0)x_0$$

and optimal cost $\langle x_0, P(t_0)x_0\rangle$.

Proof: Using Lemma 3.8 and Lemma 3.9 the result follows from

$$\begin{aligned}
&J(t_0,t_1,x_0,-B^*(\cdot)P(\cdot)\Phi_{-BB^*P}(\cdot,t_0)x_0) \\
&\quad = \langle x_0, P(t_0)x_0\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \langle x(t_1;t_0,x_0,u), Gx(t_1;t_0,x_0,u)\rangle \\
&\quad + \int_{t_0}^{t_1} \langle x(\rho;t_0,x_0,u), [C(\rho) - P(\rho)B(\rho)B^*(\rho)P(\rho)]x(\rho;t_0,x_0,u)\rangle \\
&\qquad\qquad -2\mathrm{Re}\langle x(\rho;t_0,x_0,u), P(\rho)B(\rho)u(\rho)\rangle \, d\rho \\
&= \langle x(t_1;t_0,x_0,u), Gx(t_1;t_0,x_0,u)\rangle \\
&\quad + \int_{t_0}^{t_1} \left[\langle x(\rho;t_0,x_0,u), C(\rho)x(\rho;t_0,x_0,u)\rangle + \|u(\rho)\|^2\right] \, d\rho \\
&\quad - \int_{t_0}^{t_1} \|u(\rho) + B^*(\rho)P(\rho)x(\rho;t_0,x_0,u)\|^2 \, d\rho.
\end{aligned}
$$

□

Corollary 3.11 *The IRE (3.8) has at most one solution on $[t_0, t_1]$ with final condition $P(t_1) = G$.*

Proof: Let $P_1, P_2 \in L_{s,\infty}(t_0, t_1; \mathcal{H}(H))$ be solutions of (3.8) on $[t_0, t_1]$. Then the previous theorem implies $\langle P_1(t)x_0, x_0\rangle = \langle P_2(t)x_0, x_0\rangle$ for each $t \in [t_0, t_1]$, $x_0 \in H$. Hence $P_1 = P_2$. □

Lemma 3.13 gives a lower bound for the length of the integral such that (3.8) has a solution without further assumption. We need this result for the construction of a solution to the IRE on a given interval $[t_0, t_1]$ under the assumption **(R2)**.

Lemma 3.12 *Let $F : X \to X$, $v \in X$ and $\alpha > 0$ be given. Further, suppose $F(0) = 0$, $\|v\| \le \frac{1}{2}\alpha$ and*

$$\|F(x_1) - F(x_2)\| \le \frac{1}{2}\|x_1 - x_2\| \qquad \text{for every } \|x_1\|, \|x_2\| \le \alpha.$$

Then there exists a unique solution $x \in X$ with $\|x\| \le \alpha$ of the equation $F(x) + v = x$.

Proof: Consider the function $\tilde{F} : B(0,\alpha) \to B(0,\alpha)$, $B(0,\alpha) := \{x \in X \| \|x\| \le \alpha\}$, defined by $\tilde{F}(x) := F(x) + v$. Then Banach's fixpoint theorem implies the result. □

Lemma 3.13 *Suppose there exists a constant $\alpha > 0$ such that*

$$2\alpha\|B\|_\infty^2 M_\Phi^2(t_1 - t_0) \le \frac{1}{2}, \tag{3.11}$$

$$\|G\|M_\Phi^2 + \|C\|_\infty M_\Phi^2(t_1 - t_0) \le \frac{\alpha}{2}, \tag{3.12}$$

then the IRE (3.8) has a unique solution on $[t_0, t_1]$.

If the interval $[t_0, t_1]$ is suitably small then there exists an $\alpha > 0$ such that (3.11) and (3.12) are satisfied.

Proof: Consider, for $P \in L_{s,\infty}(t_0, t_1; \mathcal{H}(H))$, $t \in [t_0, t_1]$ and $x \in H$, F and v defined by

$$\begin{aligned} F(P)(t)x &= -\int_t^{t_1} \Phi^*(\rho, t)P(\rho)B(\rho)B^*(\rho)P(\rho)\Phi(\rho, t)x \, d\rho, \\ v(t)x &= \Phi^*(t_1, t)G\Phi(t_1, t)x + \int_t^{t_1} \Phi^*(\rho, t)C(\rho)\Phi(\rho, t)x \, d\rho. \end{aligned}$$

It is easy to see that F maps $L_{s,\infty}(t_0, t_1; \mathcal{H}(H))$ to itself and that $v \in L_{s,\infty}(t_0, t_1; \mathcal{H}(H))$ is satisfied [Note: Using the properties of Φ, it is not difficult to show that for fixed $x \in H$ the functions $F(P)(\cdot)x$ and $v(\cdot)x$ are measurable on $[t_0, t_1]$]. For $p_1, p_2 \in L_{s,\infty}(t_0, t_1; \mathcal{H}(H))$ with $\|p_1\|, \|p_2\| \leq \alpha$ and $\rho \in [t_0, t_1]$ we obtain

$$\begin{aligned} &\|p_1(\rho)B(\rho)B^*(\rho)p_1(\rho) - p_2(\rho)B(\rho)B^*(\rho)p_2(\rho)\| \\ &\quad \leq \|[p_1(\rho) - p_2(\rho)]B(\rho)B^*(\rho)p_1(\rho)\| + \|p_2(\rho)B(\rho)B^*(\rho)[p_1(\rho) - p_2(\rho)]\| \\ &\quad \leq 2\alpha\|B\|_\infty^2\|p_1 - p_2\|_\infty, \end{aligned}$$

which implies

$$\|F(p_1) - F(p_2)\| \leq (t_1 - t_0)M_\Phi^2 2\alpha\|B\|_\infty^2\|p_1 - p_2\|_\infty \overset{(3.11)}{\leq} \frac{1}{2}\|p_1 - p_2\|_\infty.$$

Furthermore, by (3.12)

$$\|v\|_\infty \leq \|G\|M_\Phi^2 + \|C\|_\infty M_\Phi^2(t_1 - t_0) \leq \frac{\alpha}{2},$$

and therefore it follows from Lemma 3.12 that the equation

$$p = v + F(p) \tag{3.13}$$

has a unique solution in the class $\{P \in L_{s,\infty}(t_0, t_1; \mathcal{H}(H)) | \|P\|_\infty \leq \alpha\}$. This completes the proof since (3.8) is equivalent to (3.13) and the uniqueness follows by Corollary 3.11. □

Lemma 3.14 *Suppose P is a solution of the IRE (3.8) on $[t_0, t_1]$. Let $X^0_{t_0,t_1}, X^P_{t_0,t_1} : L_2(t_0, t_1; U) \to L_2(t_0, t_1; U)$ be the maps given by*

$$\begin{aligned} (X^0_{t_0,t_1}u)(t) &:= B^*(t)P(t)\int_{t_0}^t \Phi(t, s)B(s)u(s) \, ds, \\ (X^P_{t_0,t_1}u)(t) &:= B^*(t)P(t)\int_{t_0}^t \Phi_{-BB^*P}(t, s)B(s)u(s) \, ds. \end{aligned}$$

Then $I+\mathcal{C}_{t_0,t_1}+\mathcal{G}_{t_0,t_1} = (I+X^0_{t_0,t_1})^*(I+X^0_{t_0,t_1})$ *and* $(I-X^P_{t_0,t_1})(I+X^0_{t_0,t_1}) = I$, $(I+X^0_{t_0,t_1})(I-X^P_{t_0,t_1}) = I$.

The operator functions $X^0_{t_0,t_1}$ and $X^P_{t_0,t_1}$ are also introduced in [10]. Furthermore, the proof of this result is exactly the same as in [10], Proposition 7.1.16.

Proof: It is easy to show that

$$((X^0_{t_0,t_1})^*u)(t) = B^*(t)\int_t^{t_1} \Phi^*(\rho,t)P(\rho)B(\rho)u(\rho)\,d\rho.$$

Hence

$$\begin{aligned}
&((X^0_{t_0,t_1})^* X^0_{t_0,t_1} u)(t) \\
&\quad= B^*(t)\int_t^{t_1} \Phi^*(\rho,t)P(\rho)B(\rho)B^*(\rho)P(\rho)\int_{t_0}^{\rho}\Phi(\rho,s)B(s)u(s)\,ds\,d\rho \\
&\quad= B^*(t)\int_t^{t_1}\Phi^*(s,t)\int_s^{t_1}\Phi^*(\rho,s)P(\rho)B(\rho)B^*(\rho)P(\rho)\Phi(\rho,s)B(s)u(s)\,d\rho\,ds \\
&\qquad +B^*(t)\int_{t_0}^{t}\int_t^{t_1}\Phi^*(\rho,t)P(\rho)B(\rho)B^*(\rho)P(\rho)\Phi(\rho,t)\Phi(t,s)B(s)u(s)\,d\rho\,ds \\
&\quad\overset{\text{IRE}}{=} B^*(t)\int_t^{t_1}\Phi^*(s,t)\Big[-P(s)B(s)u(s)+\Phi^*(t_1,s)G\Phi(t_1,s)B(s)u(s) \\
&\qquad\qquad +\int_s^{t_1}\Phi^*(\rho,s)C(\rho)\Phi(\rho,s)B(s)u(s)\,d\rho\Big]\,ds \\
&\qquad +B^*(t)\int_{t_0}^{t}\Big[-P(t)\Phi(t,s)B(s)u(s)+\Phi^*(t_1,t)G\Phi(t_1,s)B(s)u(s) \\
&\qquad\qquad +\int_t^{t_1}\Phi^*(\rho,t)C(\rho)\Phi(\rho,s)B(s)u(s)\,d\rho\Big]\,ds \\
&\quad= -B^*(t)\int_t^{t_1}\Phi^*(s,t)P(s)B(s)u(s)\,ds - B^*(t)\int_{t_0}^{t}P(t)\Phi(t,s)B(s)u(s)\,ds \\
&\qquad +B^*(t)\int_{t_0}^{t_1}\Phi^*(t_1,t)G\Phi(t_1,s)B(s)u(s)\,ds \\
&\qquad +B^*(t)\int_t^{t_1}\Phi^*(\rho,t)C(\rho)\int_{t_0}^{\rho}\Phi(\rho,s)B(s)u(s)\,ds\,d\rho \\
&\quad= (\mathcal{G}_{t_0,t_1}u)(t)+(\mathcal{C}_{t_0,t_1}u)(t)-((X^0_{t_0,t_1})^*u)(t)-(X^0_{t_0,t_1}u)(t),
\end{aligned}$$

which proves the first equation. Furthermore,

$(X^P_{t_0,t_1}u)(t)$

$$
\begin{aligned}
&= (X^0_{t_0,t_1}u)(t) \\
&\quad -B^*(t)P(t)\int_{t_0}^{t}\int_{s}^{t}\Phi_{-BB^*P}(t,\rho)B(\rho)B^*(\rho)P(\rho)\Phi(\rho,s)B(s)u(s)\,d\rho\,ds \\
&= (X^0_{t_0,t_1}u)(t) \\
&\quad -B^*(t)P(t)\int_{t_0}^{t}\Phi_{-BB^*P}(t,\rho)B(\rho)B^*(\rho)P(\rho)\int_{t_0}^{\rho}\Phi(\rho,s)B(s)u(s)\,ds\,d\rho \\
&= (X^0_{t_0,t_1}u)(t)-(X^P_{t_0,t_1}X^0_{t_0,t_1}u)(t)=((I-X^P_{t_0,t_1})X^0_{t_0,t_1})(u)(t).
\end{aligned}
$$

A similar calculation shows that $(X^P_{t_0,t_1}u)(t) = X^0_{t_0,t_1}(I - X^P_{t_0,t_1})(u)(t)$, which yields the second and third equation. □

We have now presented all the results necessary for the proof of Theorem 3.6. The proof is not an immediate extension from the finite–dimensional situation, as mentioned previously.

Proof of Theorem 3.6:

(R1)⇒(R4) Follows immediately from Lemma 3.14.

(R4)⇒(R3) Since $I + \mathcal{C}_{t_0,t_1} + \mathcal{G}_{t_0,t_1} > 0$, there exists an operator $T \in \mathcal{H}(L_2(t_0,t_1;U))$ with

$$I + \mathcal{C}_{t_0,t_1} + \mathcal{G}_{t_0,t_1} = T^2.$$

The invertibility of $I + \mathcal{C}_{t_0,t_1} + \mathcal{G}_{t_0,t_1}$ implies that T is invertible in $\mathcal{L}(L_2(t_0,t_1;U))$ with $\|(I + \mathcal{C}_{t_0,t_1} + \mathcal{G}_{t_0,t_1})^{-1}\| = \|T^{-1}\|^2$. Thus there exists a constant $\varepsilon > 0$ such that

$$J(t_0,t_1,0,u) = \langle u,(I + \mathcal{C}_{t_0,t_1} + \mathcal{G}_{t_0,t_1})u\rangle_2 = \|Tu\|_2^2 \geq \varepsilon\|u\|_2^2.$$

(R3)⇒(R2) Let $t_0' \in [t_0,t_1]$, $x_0 \in H$ and $u \in L_2(t_0',t_1;U)$. Setting

$$\bar{u}(t) := \begin{cases} 0 & ,t \in [t_0,t_0') \\ u(t) & ,t \in [t_0',t_1] \end{cases} .$$

It is not difficult to see that $J(t_0',t_1,0,u) = J(t_0,t_1,0,\bar{u})$. Then by assumption

$$
\begin{aligned}
J(t_0',t_1,x_0,u) &= \langle x_0, M_{t_0',t_1}x_0\rangle + 2\mathrm{Re}\langle N_{t_0',t_1}x_0,u\rangle + J(t_0',t_1,0,u) \\
&\geq \langle x_0, M_{t_0',t_1}x_0\rangle + 2\mathrm{Re}\langle N_{t_0',t_1}x_0,u\rangle + \varepsilon\|u\|_2^2 \\
&= \langle x_0, M_{t_0',t_1}x_0\rangle + \varepsilon\|u+\varepsilon^{-1}N_{t_0',t_1}x_0\|_2^2 \\
&\quad -\varepsilon^{-1}\|N_{t_0',t_1}x_0\|_2^2 \\
&\geq -(\|M_{t_0',t_1}\| + \varepsilon^{-1}\|N_{t_0',t_1}\|^2)\|x_0\|^2,
\end{aligned}
$$

where ε is independent of t_0'. Since the functions $t_0' \mapsto \|M_{t_0',t_1}\|$ and $t_0' \mapsto \|N_{t_0',t_1}\|$ are bounded on $[t_0, t_1]$ there exists a constant $\beta > 0$ such that **(R2)** holds.

(R2)⇒(R1) First we set

$$\alpha := 2\max\left\{\|G\|M_\Phi^4 + \|C\|_\infty M_\Phi^4(t_1 - t_0), \beta M_\Phi^2 + \|C\|_\infty M_\Phi^2(t_1 - t_0)\right\} \tag{3.14}$$

and we choose a number $N \in \mathbb{N}$ such that

$$2\alpha\|B\|_\infty^2 M_\Phi^2 \frac{t_1 - t_0}{N} \le \frac{1}{2}. \tag{3.15}$$

Now we divide up the interval $[t_0, t_1]$ in the following way

$$t_0 =: s_N < t_0 + \frac{t_1 - t_0}{N} =: s_{N-1} < \cdots < t_1 - \frac{t_1 - t_0}{N} =: s_1 < s_0 := t_1.$$

We will now prove the following statement by induction over n, $n \in \{1, \cdots, N\}$:

The equation

$$P(t)z = \Phi^*(t_1,t)G\Phi(t_1,t)z + \int_t^{t_1} \Phi^*(\rho,t)\left[C(\rho) - P(\rho)B(\rho)B^*(\rho)P(\rho)\right]\Phi(\rho,t)z\,d\rho, \tag{3.16}$$

$z \in H$, has a solution $P \in L_{s,\infty}(s_n, t_1; \mathcal{H}(H))$ with

$$\|P(s_n)\| \le \max\left\{\|G\|M_\Phi^2 + \|C\|_\infty M_\Phi^2(t_1 - s_n), \beta\right\}.$$

First let $n = 1$. Then

$$\begin{aligned} 2\alpha\|B\|_\infty^2 M_\Phi^2(t_1 - s_1) = 2\alpha\|B\|_\infty^2 M_\Phi^2 \frac{t_1 - t_0}{N} &\le \frac{1}{2} \\ \text{and} \quad \|G\|M_\Phi^2 + \|C\|_\infty M_\Phi^2(t_1 - s_1) &\le \frac{\alpha}{2}, \end{aligned}$$

and therefore Lemma 3.13 implies that equation (3.16) has a solution $P \in L_{s,\infty}(s_1, t_1; \mathcal{H}(H))$. Lemma 3.9 shows

$$\langle x_0, P(s_1)x_0\rangle = J(s_1, t_1, x_0, -B^*(\cdot)P(\cdot)\Phi_{-BB^*P}(\cdot, s_1)x_0)$$

for each $x_0 \in H$. Hence, using equation (3.16) and **(R2)**, we obtain for every $x_0 \in H$

$$\begin{aligned} -\beta\|x_0\|^2 &\le \langle x_0, P(s_1)x_0\rangle \\ &\le \left[\|G\|M_\Phi^2 + \|C\|_\infty M_\Phi^2(t_1 - s_1)\right]\|x_0\|^2. \end{aligned} \tag{3.17}$$

This shows $\|P(s_1)\| \leq \max\left\{\|G\|M_\Phi^2 + \|C\|_\infty M_\Phi^2(t_1 - s_1), \beta\right\}$ and the statement is true for $n = 1$. Now we assume that the result is true for $n \in \{1, \cdots, N-1\}$. By the assumption, there exists a $P_1 \in L_{s,\infty}(s_n, t_1; \mathcal{H}(H))$, which solves (3.16) with $\|P_1(s_n)\| \leq \max\left\{\|G\|M_\Phi^2 + \|C\|_\infty M_\Phi^2(t_1 - s_n), \beta\right\}$. Since the inequalities

$$2\alpha\|B\|_\infty^2 M_\Phi^2(s_n - s_{n+1}) \leq \frac{1}{2}$$

and

$$\begin{aligned}
&\|P_1(s_n)\|M_\Phi^2 + \|C\|_\infty M_\Phi^2(s_n - s_{n+1}) \\
&\quad \leq \max\left\{\|G\|M_\Phi^4 + \|C\|_\infty M_\Phi^4(t_1 - s_{n+1}), \beta M_\Phi^2 + \|C\|_\infty M_\Phi^2(s_n - s_{n+1})\right\} \\
&\quad \leq \frac{\alpha}{2}
\end{aligned}$$

are satisfied, Lemma 3.13 implies the existence of a solution $P_2 \in L_{s,\infty}(s_{n+1}, s_n; \mathcal{H}(H))$ of the equation

$$\begin{aligned}
P_2(t)z &= \Phi^*(s_n, t)P_1(s_n)\Phi(s_n, t)z \\
&\quad + \int_t^{s_n} \Phi^*(\rho, t)\left[C(\rho) - P_2(\rho)B(\rho)B^*(\rho)P_2(\rho)\right]\Phi(\rho, t)z\, d\rho.
\end{aligned}$$

We define

$$P(t) := \begin{cases} P_1(t), & t \in [s_n, t_1] \\ P_2(t), & t \in [s_{n+1}, s_n] \end{cases}.$$

It is easy to see that $P \in L_{s,\infty}(s_{n+1}, t_1; \mathcal{H}(H))$ solves (3.16) on the interval $[s_{n+1}, t_1]$. It remains to prove that

$$\|P(s_{n+1})\| \leq \max\left\{\|G\|M_\Phi^2 + \|C\|_\infty M_\Phi^2(t_1 - s_{n+1}), \beta\right\}.$$

Lemma 3.9 shows

$$\langle x_0, P(s_{n+1})x_0\rangle = J(s_{n+1}, t_1, x_0, -B^*(\cdot)P(\cdot)\Phi_{-BB^*P}(\cdot, s_{n+1})x_0)$$

for all $x_0 \in H$. Hence for each $x_0 \in H$ using (3.16) and **(R2)**, we obtain

$$\begin{aligned}
-\beta\|x_0\|^2 &\leq \langle x_0, P(s_{n+1})x_0\rangle \qquad (3.18)\\
&\leq \left[\|G\|M_\Phi^2 + (t_1 - s_{n+1})M_\Phi^2\|C\|_\infty\right]\|x_0\|^2,
\end{aligned}$$

which shows $\|P(s_{n+1})\| \leq \max\left\{\|G\|M_\Phi^2 + \|C\|_\infty M_\Phi^2(t_1 - s_{n+1}), \beta\right\}$ This completes the proof because the uniqueness is implied by Corollary 3.11. □

Corollary 3.15 *If $C(t) \geq 0$ a. e. and $G \geq 0$, then all the statements* **(R1)**–**(R4)**, **(O1)**, **(O2)**, **(R2')** *and* **(R4')** *are true.*

The next theorem gives a sufficient condition such that **(R4')** implies **(R4)**. In this situation the statements **(R1)**–**(R4)**, **(O1)**, **(O2)** and **(R4')** are equivalent.

Theorem 3.16 *Let U be a separable Hilbert space. Furthermore, suppose that $B(t)$ is a compact operator for every $t \in [t_0, t_1]$. Then* **(R4')** *implies* **(R4)**. *Hence we can replace* **(R4)** *by* **(R4')** *in Theorem 3.6. Furthermore, the finite horizon linear quadratic optimization problem is uniquely solvable if and only if there exists a solution of the IRE on $[t_0, t_1]$.*

Proof: First we define the functions $k_{\mathcal{C}}, k_{\mathcal{G}} : [t_0, t_1]^2 \to \mathcal{L}(U)$ by

$$\begin{aligned} k_{\mathcal{C}}(t,\rho)u &= B^*(t)\int_{\max\{t,\rho\}}^{t_1} \Phi^*(\tau,t)C(\tau)\Phi(\tau,\rho)B(\rho)u\,d\tau \\ k_{\mathcal{G}}(t,\rho)u &= B^*(t)\Phi^*(t_1,t)G\Phi(t_1,\rho)B(\rho)u, \end{aligned}$$

where $(t,\rho) \in [0,1]^2$ and $u \in U$. Then it is easy to see that

- $(\mathcal{C}_{t_0,t_1}u)(t) = \int_{t_0}^{t_1} k_{\mathcal{C}}(t,\rho)u(\rho)\,d\rho$ and
  $(\mathcal{G}_{t_0,t_1}u)(t) = \int_{t_0}^{t_1} k_{\mathcal{G}}(t,\rho)u(\rho)\,d\rho,$
- $\sup_{(t,\rho)\in[t_0,t_1]^2} \|k_{\mathcal{C}}(t,\rho)\| < \infty$ and $\sup_{(t,\rho)\in[t_0,t_1]^2} \|k_{\mathcal{G}}(t,\rho)\| < \infty,$
- the maps $k_{\mathcal{C}}$ and $k_{\mathcal{G}}$ are strongly measurable.

By assumption, $B(t)$ is a compact operator, hence using Schauder's theorem we have that $B^*(t)$ is a compact operator for every $t \in [t_0, t_1]$. By the ideal property of the compact operators we obtain that $k_{\mathcal{C}}([t_0,t_1]^2)$ is a subset of the set of compact operators in $\mathcal{L}(U)$. Since the compact operators in $\mathcal{L}(U)$ form a separable subspace of $\mathcal{L}(U)$ using that U is separable, we obtain $k_{\mathcal{C}}$ is measurable. Hence $k_{\mathcal{C}} \in L_2([t_0,t_1]^2, \mathcal{L}(U))$. The compact operators in $\mathcal{L}(U)$ provided with the operator norm form a Banach space and the finite–dimensional operators are dense in the set of compact operators. Furthermore, by [6], III.11.17 we have $L_2([t_0,t_1]^2; \mathcal{L}(U)) \cong L_2(t_0,t_1; L_2(t_0,t_1; \mathcal{L}(U)))$. Let $\varepsilon > 0$. Thus there exists $\tilde{k}_{\mathcal{C}} \in L_2([t_0,t_1]^2, \mathcal{L}(U))$ such that $\|\tilde{k}_{\mathcal{C}} - k_{\mathcal{C}}\|_2 < \varepsilon$, $\tilde{k}_{\mathcal{C}}$ is a countable function in $L_2(t_0,t_1; L_2(t_0,t_1; \mathcal{L}(U)))$ and in $L_2([t_0,t_1]^2; \mathcal{L}(U))$ and $\tilde{k}_{\mathcal{C}}(t,s)$ is a finite-dimensional operator for every $(t,s) \in [t_0,t_1]^2$. Define

$$(\tilde{\mathcal{C}}_{t_0,t_1}u)(t) = \int_{t_0}^{t_1} \tilde{k}_{\mathcal{C}}(t,\rho)u(\rho)\,d\rho.$$

Then it is easy to see that $\tilde{\mathcal{C}}_{t_0,t_1}$ is a finite–dimensional operator. Now from $\|\tilde{\mathcal{C}}_{t_0,t_1} - \mathcal{C}_{t_0,t_1}\| \leq \|\tilde{k}_{\mathcal{C}} - k_{\mathcal{C}}\|_2$ derive that $\mathcal{C}_{t_0,t_1}$ is a compact operator. That $\mathcal{G}_{t_0,t_1}$ is a compact operator can be proven in a similar way. This implies that $I+\mathcal{C}_{t_0,t_1}+\mathcal{G}_{t_0,t_1}$ is a Fredholm operator and, using $I+\mathcal{C}_{t_0,t_1}+\mathcal{G}_{t_0,t_1} > 0$, Fredholm's alternative proves that $I + \mathcal{C}_{t_0,t_1} + \mathcal{G}_{t_0,t_1}$ is bijective. Now by the open mapping theorem we obtain that $I + \mathcal{C}_{t_0,t_1} + \mathcal{G}_{t_0,t_1}$ is invertible in $\mathcal{L}(L_2(t_0, t_1; U))$. □

Remark: The assumptions of Theorem 3.16 hold if the input space U is a finite dimensional Hilbert space. We need the assumptions in Theorem 3.16 in order to prove that the kernels $k_{\mathcal{C}}$ and $k_{\mathcal{G}}$ are measurable, hence in $L_2([t_0, t_1]^2, \mathcal{L}(U))$, and compact operator valued.

4 A Parameterized Riccati Equation

In this section we will examine how the previous result can be applied to a parameterized Riccati equation, which plays an important role in robustness analysis. Throughout this section we will assume that $\Phi : ,_{t_0}^{t_1} \rightarrow \mathcal{L}(H)$ is a bounded mild evolution operator with property $(*)$, $B \in L_{s,\infty}(t_0, t_1; \mathcal{L}(U, H))$, B^* strongly measurable, $G \in \mathcal{H}(H)$, $C \in L_{s,\infty}(t_0, t_1; \mathcal{L}(H, Y))$ and $x_0 \in H$. We consider the input–output operator $\mathbb{L} : L_2(t_0, t_1; U) \rightarrow L_2(t_0, t_1; Y)$ defined by

$$u \mapsto C(\cdot) \int_{t_0}^{\cdot} \Phi(\cdot, \rho) B(\rho) u(\rho)\, d\rho.$$

In the finite dimensional case it has been shown in [8] that

$$\begin{aligned} \|\mathbb{L}\|^{-1} := \ & \sup\{\lambda \in \mathbb{R} | \mathrm{DRE}_\lambda \text{has a bounded self–adjoint} \\ & \text{solution on} [t_0, \infty)\}, \end{aligned}$$

where DRE_λ denotes the parameterized differential Riccati equation given by

$$\dot{P}(t) + A^*(t)P(t) + P(t)A(t) - \lambda^2 C^*(t)C(t) - P(t)B(t)B^*(t)P(t) = 0,$$

$t \in [t_0, \infty)$.

Now we will extend some results proved in [8] for finite–dimensional systems to the infinite–dimensional situation. But we will consider finite horizon intervals instead of infinite. Because we have no differentiability assumptions on our evolution operator, we replace the differential Riccati equation by the parameterized integral Riccati equation

$$P_\lambda(t)z = -\int_t^{t_1} \Phi^*(\rho,t)[\lambda^2 C^*(\rho)C(\rho) + P_\lambda(\rho)B(\rho)B^*(\rho)P_\lambda(\rho)]\Phi(\rho,t)z\,d\rho, \tag{IRE$_\lambda$}$$

$z \in H$.

In the previous section we have shown

Theorem 4.1 *The following statements are equivalent*

(a) *(IRE$_\lambda$) has a unique solution $P_\lambda \in L_{s,\infty}(t_0,t_1;\mathcal{H}(H))$.*

(b) *There exists a constant $\varepsilon > 0$ such that for every $u \in L_2(t_0,t_1;U)$*

$$J_\lambda(t_0,t_1,0,u) := \int_{t_0}^{t_1} -\lambda^2\|C(\rho)x(\rho;t_0,0,u)\|^2 + \|u(\rho)\|^2\,d\rho \geq \varepsilon\|u\|_2^2,$$

where

$$x(t;t_0,0,u) = \int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)\,d\tau.$$

We now apply this result to the parameterized Riccati equation

Theorem 4.2 *The following statements are equivalent*

(a) *(IRE$_\lambda$) has a unique solution $P_\lambda \in L_{s,\infty}(t_0,t_1;\mathcal{H}(H))$ on $[t_0,t_1]$.*

(b) *$|\lambda| < \|\mathbb{L}\|^{-1}$.*

Proof: Using Theorem 4.1 and

$$\begin{aligned} J_\lambda(t_0,t_1,0,u) &= \int_{t_0}^{t_1} -\lambda^2\|C(\rho)x(\rho;t_0,0,u)\|^2 + \|u(\rho)\|^2\,d\rho \\ &= -\lambda^2\|\mathbb{L}u\|_2^2 + \|u\|_2^2 \end{aligned}$$

it remains to prove the equivalence of

(i) There exists a constant $\varepsilon > 0$ such that for every $u \in L_2(t_0,t_1;U)$

$$-\lambda^2\|\mathbb{L}u\|_2^2 + \|u\|_2^2 \geq \varepsilon\|u\|_2^2.$$

(ii) $|\lambda| < \|\mathbb{L}\|^{-1}$.

(iii) implies the existence of a constant $\varepsilon > 0$ such that

$$\|\mathbb{L}u\|_2^2 \leq \left[\frac{1}{|\lambda|^2}(1-\varepsilon)\right]\|u\|_2^2. \tag{18}$$

Hence (i) holds. Conversely, (i) implies (18) and thus (ii). □

Theorem 4.2 implies the following characterization of $\|\mathbb{L}\|$:

$$\|\mathbb{L}\|^{-1} := \sup\{\lambda \in \mathbb{R} \,|\, (\text{IRE}_\lambda) \text{ has a unique solution on } [t_0,t_1]\}.$$

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