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An Intrinsic Characterization of Properness for Linear Time-varying Systems*

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Abstract

The notion of properness, well-known for constant linear systems, is generalized to the time-varying case. In a module theoretic framework intrinsic characterizations of input-output system properness are derived. For this, filtrations and the associated Hilbert polynomials are used. Classical criteria for properness of polynomial and state representations for constant linear systems are generalized to the time-varying case. As a measure for the non-properness, an input-output index of a system is defined intrinsically which is directly related to the index of implicit systems of differential equations.

Key words: filtrations, Hilbert polynomial, index, linear time-varying systems, properness

AMS Subject Classifications: 93A25, 93B25, 93C05, 93C50, 65D20

1 Introduction

Most of constant linear control theory concerns proper systems. The following reasons are common knowledge: the modeling of the system behaviour for high frequencies requires the use of a proper model. For constant continuous time systems properness is equivalent to causality, even if it rather concerns smoothness (see also Willems' discussion of this question in [29]). Also it is well-known that proper linear systems can be "realized" without using differentiators.

Though properness is well established for *constant* linear systems, this subject has not attracted a lot of attention in the *time-varying* case [15, 16, 18, 31, 32].

Our investigation is based on the module theoretic approach to linear systems as it has been introduced by Fliess during the last years [5, 9].

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This is a more general use of module theory than the classical one since the work of Kalman [19]. Here the system itself is considered as a module. The approach comprises systems over an arbitrary differential field and easily allows us to include time-varying systems. It can be, to some extent, generalized to distributed parameter systems [7], to systems with delays [13], and to discrete time systems [8]. It can also be used in an elegant way for the study of several questions on nonlinear systems (see e.g. [11, 23, 27] or [25, 26] where an extension of the notion of properness to the nonlinear case has been proposed).

This module theoretic approach is very much related to the classical polynomial approach to constant linear systems. Thus, it is not surprising that we are able to recover the classical properness criteria: for a left (resp. right) factorization $A^{-1}B$ (resp. ND^{-1}) of the system transfer matrix, the row-degrees of the matrix A (resp. the column-degrees of D) must be greater than or equal to those of B (resp. N) when A is row-reduced (resp. D is column-reduced) [2, 18].

We first interpret the classical transfer matrix definition of properness in the module theoretic approach. For this, we use the transfer matrix of [10] which is well defined also in the time-varying case. From this, we develop a characterization of properness on the system module. This definition uses the classical algebraic concepts of filtrations and of the Hilbert polynomial from dimension theory [1]. The system is proper if and only if the "constant term" of the Hilbert polynomial of a certain input-output filtration equals the state dimension. Thus there is only *one intrinsically defined* integer which "decides" about properness. This allows us to look at properness *independently of a representation*. This is important especially for time-varying linear systems where the computation of the transfer matrix is quite tedious.

We define an input-output index as an intrinsic measure for the non-properness of a system. Using the system theoretic interpretation of the index of a set of differential equations[1] recently given by Fliess, Lévine and Rouchon [12] we show that our input-output index and the index from [12] coincide. Consequently, a slight modification of our properness criteria allows us to calculate these indexes. An electrical network example illustrates the methods.

[1] This notion is important for numerical integration. See [12] for references.

2 Mathematical Background

Let k be an *ordinary differential field* (*see* [20]), *i.e.*, a commutative field equipped with a single derivation $\frac{d}{dt} =$ "$\cdot$" such that

$$\forall a, b \in k, \frac{da}{dt} \in k, \frac{d}{dt}(a+b) = \frac{da}{dt} + \frac{db}{dt}, \frac{d}{dt}(ab) = \frac{da}{dt}b + a\frac{db}{dt}.$$

The field $\mathbb{R}(t)$ of real rational functions and the set of meromorphic functions in the variable t on an open connected domain of $\mathbb{R}$ are examples of ordinary differential fields with respect to $\frac{d}{dt}$. A *constant* is an element $c \in k$ such that $\frac{d}{dt}c = 0$. A *field of constants* is a differential field which contains only constants. The fields $\mathbb{Q}$ and $\mathbb{R}$ are examples of differential fields of constants.

We consider the ring $k[\frac{d}{dt}]$ of linear differential operators of the form $\sum_{\text{finite}} a_i \frac{d^i}{dt^i}$, $a_i \in k$. This ring is commutative if k is a field of constants. A left $k[\frac{d}{dt}]$-module generated by a finite family $z = (z_1, \ldots, z_s)$ is denoted as $[z]$. The cardinal number of a maximal $k[\frac{d}{dt}]$-linearly independent family of elements of a $k[\frac{d}{dt}]$-module M is called its *rank*. It is denoted as $\operatorname{rk} M$. A set of generators of a $k[\frac{d}{dt}]$-module M which is a $k[\frac{d}{dt}]$-linearly independent family is called a *basis* of M. Of course, the cardinal number of a basis equals $\operatorname{rk} M$. Not all modules admit bases; a module which admits a basis is called *free*.

A $k[\frac{d}{dt}]$-module $\mathcal{T}$ is said to be a *torsion module*, or to be *torsion*, if for all $\tau \in \mathcal{T}$ there exists $a \in k[\frac{d}{dt}]$, $a \neq 0$, such that $a\tau = 0$. The ring $k[\frac{d}{dt}]$ is a principal ideal domain. Therefore, a $k[\frac{d}{dt}]$-module M which does not contain any torsion submodule is free. Every finitely generated $k[\frac{d}{dt}]$-module M is the direct sum, $M = \mathcal{T} \oplus \mathcal{F}$, of its torsion submodule $\mathcal{T}$ and of a free module $\mathcal{F}$, unique up to isomorphism [4].

Let us introduce the following notation:

$$|z|^r := \begin{cases} \operatorname{span}_k \left\{ z, \dot{z}, \ldots, z^{(r)} \right\}, & \text{for } r \geqslant 0 \\ \{0\}, & \text{for } r < 0 \end{cases}.$$

With this, $|z| := (|z|^r)_{r \in \mathbb{Z}}$ is a *filtration* of the $k[\frac{d}{dt}]$-module $[z]$. This is a non-decreasing sequence of k-vector spaces in $[z]$. This filtration is obviously *exhaustive* ($\bigcup_{r \in \mathbb{Z}} |z|^r = [z]$), *discrete* ($|z|^r = \{0\}$ for r small enough), and *excellent* (each $|z|^r$ is a finitely generated k-vector space, $\frac{d}{dt}|z|^r \subseteq |z|^{r+1}$ and there exists an $r \in \mathbb{Z}$ such that $\forall s > r$, $|z|^{s+1} = \frac{d}{dt}(|z|^s)$). Hence, there exists a *Hilbert polynomial*[2] $H_{|z|}(\mathfrak{r}) \in \mathbb{Q}[\mathfrak{r}]$ such that for $r \in \mathbb{Z}$, r

[2] The Hilbert polynomial is a numerical polynomial, *i.e.*, an element χ of the polynomial algebra $\mathbb{Q}[\mathfrak{r}]$ in one indeterminate $\mathfrak{r}$ and coefficients in the field $\mathbb{Q}$ of rational numbers such that $\chi(r) \in \mathbb{N}$ for sufficiently large $r \in \mathbb{N}$.

large enough $\dim |z|^r = H_{|z|}(r) = m(r+1) + b$, where m is the rank of $[z]$ (*see* [17]). Here one even has $H_{|z|} \in \mathbb{Z}[\mathfrak{r}]$.

Denote the field of fractions of $k[\frac{d}{dt}]$ by $k(\frac{d}{dt})$. This field, which is a skew field in general, exists since $k[\frac{d}{dt}]$ has the Ore property [4]. Then, by extension of scalars, from each left $k[\frac{d}{dt}]$-module M one obtains the (left) $k(\frac{d}{dt})$-vector space $\widehat{M} = k(\frac{d}{dt}) \otimes M$. The elements of $\widehat{M}$ can be understood as formal products $a^{-1}m$, where $a \in k[\frac{d}{dt}]$, $m \in M$. The kernel of the canonical mapping $M \to \widehat{M}$ defined by $m \mapsto \widehat{m} = 1 \otimes m$ is the torsion submodule of M [4]. The tensor-product $k(\frac{d}{dt}) \otimes —$ defines a functor from the category of left $k[\frac{d}{dt}]$-modules to the category of left $k(\frac{d}{dt})$-vector spaces. According to [10] it is called the *Laplace functor*. The *formal Laplace transform* of $m \in M$ is $\widehat{m} \in \widehat{M}$.

Consider in a $k[\frac{d}{dt}]$-module M two families $w = (w_1, \ldots, w_q)$ and $z = (z_1, \ldots, z_r)$ such that

$$w_i = \sum_{j=1}^{r} c_{i,j}(\tfrac{d}{dt}) z_j\,,\; i = 1, \ldots, q, \tag{2.1}$$

where $c_{ij}(\frac{d}{dt}) \in k[\frac{d}{dt}]$. For simplicity of notation we use a matrix notation to write (2.1) as

$$w = C(\tfrac{d}{dt})\, z,$$

with the matrix $C(\frac{d}{dt}) = \big(c_{i,j}(\frac{d}{dt})\big)$ over $k[\frac{d}{dt}]$ of appropriate size.

3 Linear Time-varying Systems

We first recall the definitions as introduced by Fliess (cf. [5, 9]). Then we give some details for explanation. A *system* Σ is a finitely generated left $k[\frac{d}{dt}]$-module. The field k is called the *ground field*; it contains the coefficients of the system equations. The system is said to be *constant* if its ground field is a field of constants; otherwise it is called *time-varying*.

We now develop the relation between linear systems and modules. Let there be given a set of linear differential equations in the variables $w_1, \ldots, w_r$, with coefficients in k. These differential equations can be written

$$\sum_{j=1}^{r} c_{i,j}(\tfrac{d}{dt})\, w_j = 0,\;\; i = 1, \ldots, q, \tag{3.1}$$

with $c_{i,j}(\frac{d}{dt}) \in k[\frac{d}{dt}]$. Consider the free left $k[\frac{d}{dt}]$-module generated by $W = (W_1, \ldots, W_r)$ and the submodule $[E]$ of $[W]$ generated by the family $E = (E_1, \ldots, E_q)$ such that $E = C(\frac{d}{dt})W$, with $C(\frac{d}{dt}) = (c_{i,j}(\frac{d}{dt}))$ the matrix of the coefficients from (3.1). For $i = 1, \ldots, r$, denote by w_i the canonical image of W_i in the quotient module $[W]/[E]$. Then, the system is just this quotient, *i.e.*, $\Sigma = [W]/[E]$. The linear differential equations (3.1) are a set of defining equations of Σ. They determine the (non-trivial) relations between the elements of Σ.

An *input* is a finite set $u = (u_1, \ldots, u_m)$ of elements of Σ such that the quotient $\Sigma/[u]$ is torsion. We assume the input to be *independent*, *i.e.*, the module $[u]$ is free. An *output* is a finite set $y = (y_1, \ldots, y_p)$ of elements of Σ. In the case $\Sigma = [u, y]$ we call Σ an *input-output system*. When $\Sigma \supsetneq [u, y]$ we say that $[u, y]$ is the *input-output subsystem* of Σ.

A system $\Sigma \supseteq [u, y]$ is *observable* if and only if the modules Σ and $[u, y]$ coincide. A system Σ is *controllable* if and only if the module Σ is free. Relations with classical criteria for controllability and observability are established in [6, 9]. Some of them will become apparent in the sequel.

We use the transfer matrix approach as it was recently introduced by Fliess [10]. We denote as $\widehat{u}_i$, (resp. $\widehat{y}_i$), the formal Laplace transforms, *i.e.*, the canonical images in $\widehat{\Sigma} = k(\frac{d}{dt}) \otimes \Sigma$, of u_i (resp. y_i). The fact that the quotient $\Sigma/[u]$ is a torsion module implies that $\widehat{u} = (\widehat{u}_1, \ldots, \widehat{u}_m)$ is a basis of the $k(\frac{d}{dt})$-vector space $\widehat{\Sigma}$. Especially for $\widehat{y} = (\widehat{y}_1, \ldots, \widehat{y}_p)$, this yields

$$\widehat{y} = T(\tfrac{d}{dt})\,\widehat{u}, \tag{3.2}$$

where $T(\frac{d}{dt})$ is a $p \times m$ matrix over $k(\frac{d}{dt})$ called the *transfer matrix*[3] of $[u, y]$. For simplicity of notation we do not introduce an extra notation but use the convention that it is the matrix expressing the Laplace transforms of the output y in the basis $\widehat{u}$. Even in the non commutative case, the transfer matrix of a system can be expanded in a Laurent series [4]

$$T(\tfrac{d}{dt}) = \sum_{\nu, \nu_0 \in \mathbb{Z}, \nu \geqslant \nu_0} T_\nu . (\tfrac{d}{dt})^{-\nu} \tag{3.3}$$

where the T_ν are $p \times m$ matrices over k and $T_{\nu_0} \neq 0$.

It is well-known in the constant case that the transfer matrix carries information only about the controllable and observable parts of the system. This can be shown as follows. For a general system $\Sigma \supsetneq [u, y]$ the Laplace functor yields $k(\frac{d}{dt}) \otimes - : \Sigma \Leftrightarrow\!\!\rightarrow \widehat{\Sigma}$. The transfer matrix only describes $\widehat{[y]}$ as a $k(\frac{d}{dt})$-subspace of $\widehat{[u]}$. Moreover, the kernel of $k(\frac{d}{dt}) \otimes - : \Sigma \Leftrightarrow\!\!\rightarrow \widehat{\Sigma}$ is the torsion submodule of Σ. As Σ itself, its submodule $[u, y]$ can be written as a direct sum of its torsion submodule and a free module ϕ. It follows that the transfer matrix T in (3.2) is determined by this free module ϕ. This submodule ϕ has been called the *transfer module* of Σ in [23], where it has been shown that it is a complete invariant for transfer equivalence. Instead

[3] See also [16, 15, 31] for a definition of the transfer matrix for time-varying linear systems.

of giving the proof that the kernel of $k(\frac{d}{dt})\otimes$— is the torsion submodule, which can be found in [4, p. 47], we consider a simple example: take the system defined by $\dot{y} \Leftrightarrow \dot{u} = 0$, $\dot{z} = u$. Obviously, the torsion submodule of Σ is generated by $y \Leftrightarrow u$. One may write

$$\widehat{(y \Leftrightarrow u)} = 1 \otimes (y \Leftrightarrow u) = (\tfrac{d}{dt})^{-1} \tfrac{d}{dt} \otimes (y \Leftrightarrow u) = (\tfrac{d}{dt})^{-1} \otimes \tfrac{d}{dt}(y \Leftrightarrow u) = 0.$$

This shows that $y \Leftrightarrow u$ is mapped on 0. One also sees that $1 \otimes (y \Leftrightarrow u) = 1 \otimes y \Leftrightarrow 1 \otimes u = \widehat{y} \Leftrightarrow \widehat{u} = 0$: the transfer matrix from $\widehat{u}$ to $\widehat{y}$ equals 1. The element $z \in \Sigma$ is unobservable because $z \notin [u, y]$. One has $\widehat{z} = 1 \otimes z = (\frac{d}{dt})^{-1} \otimes (\frac{d}{dt} z) = (\frac{d}{dt})^{-1} \widehat{u}$. One can show that all (unobservable) systems with the same transfer module (up to a particular isomorphism, *cf.* [23]) lead to the same relations between the formal Laplace transforms $\widehat{u}$ and $\widehat{y}$ in $\widehat{[u]}$, *i.e.*, to the same transfer matrix.

We end this section with a few words about representations of linear systems. Let there be given a controllable system Σ. Then, the controllability being equivalent to the freeness of the module, there exists a basis $w = (w_1, \ldots, w_m)$ of Σ. Therefore, all elements are $k[\frac{d}{dt}]$-linear combinations of the components of w. This implies the existence of matrices $D(\frac{d}{dt})$ and $N(\frac{d}{dt})$ over $k[\frac{d}{dt}]$ such that

$$u = D(\tfrac{d}{dt})\, w, \;\; y = N(\tfrac{d}{dt})\, w. \tag{3.4}$$

From the freeness of $[u]$ it follows that $D(\frac{d}{dt})$ is a full rank square matrix. The representation (3.4) is called a *polynomial representation* of Σ.

Remark 3.1 One can see that, via the construction described at the beginning of the section, (3.4) with $D(\frac{d}{dt})$ full rank always defines a controllable system.

With (3.4), one gets

$$\begin{aligned}
\widehat{y} = 1 \otimes y = 1 \otimes N(\tfrac{d}{dt})w &= N(\tfrac{d}{dt}) \otimes D(\tfrac{d}{dt})^{-1} D(\tfrac{d}{dt})w \\
&= N(\tfrac{d}{dt}) D(\tfrac{d}{dt})^{-1} \otimes D(\tfrac{d}{dt})w \\
&= N(\tfrac{d}{dt}) D(\tfrac{d}{dt})^{-1} (1 \otimes D(\tfrac{d}{dt})w) \\
&= N(\tfrac{d}{dt}) D(\tfrac{d}{dt})^{-1} \widehat{u}.
\end{aligned}$$

One recognizes a *right factorization* $T(\frac{d}{dt}) = N(\frac{d}{dt}) D(\frac{d}{dt})^{-1}$ of the transfer matrix.

Let there now be given an observable system $\Sigma = [u, y]$. The quotient $[u, y]/[u]$ is a torsion $k[\frac{d}{dt}]$module, and hence there exist equations

$$A(\tfrac{d}{dt})\, y = B(\tfrac{d}{dt})\, u \tag{3.5}$$

where $A(\frac{d}{dt})$ and $B(\frac{d}{dt})$ are matrices over $k[\frac{d}{dt}]$ of appropriate size, with $A(\frac{d}{dt})$ of full rank p [23]. (3.5) is called a *polynomial representation* of the system Σ.

Remark 3.2 Via the construction described in the beginning of the section, (3.5) always defines an observable linear system.

By a simple consideration analogue to the one for the right factorization above, the representation (3.5) yields a *left factorization* $T(\frac{d}{dt}) = A(\frac{d}{dt})^{-1}B(\frac{d}{dt})$ of the transfer matrix.

We have seen that (3.5) always defines an observable system and one might ask when it is controllable. Introduce $\Gamma(\frac{d}{dt})$ as a greatest common left divisor of $A(\frac{d}{dt})$ and $B(\frac{d}{dt})$, *i.e.*,

$$A(\tfrac{d}{dt}) = \Gamma(\tfrac{d}{dt})\widetilde{A}(\tfrac{d}{dt}),\ B(\tfrac{d}{dt}) = \Gamma(\tfrac{d}{dt})\widetilde{B}(\tfrac{d}{dt})$$

and if $\widetilde{A}(\frac{d}{dt}) = \bar{\Gamma}(\frac{d}{dt})\bar{A}(\frac{d}{dt}),\ \widetilde{B}(\frac{d}{dt}) = \bar{\Gamma}(\frac{d}{dt})\bar{B}(\frac{d}{dt})$ then $\bar{\Gamma}$ is $k[\frac{d}{dt}]$-unimodular. If $\Gamma(\frac{d}{dt})$ is itself unimodular then $A(\frac{d}{dt})$ and $B(\frac{d}{dt})$ are *left coprime.*

Writing

$$\begin{aligned} A(\tfrac{d}{dt})\widehat{y} - B(\tfrac{d}{dt})\widehat{u} &= \Gamma(\tfrac{d}{dt})(\widetilde{A}(\tfrac{d}{dt})\widehat{y} - \widetilde{B}(\tfrac{d}{dt})\widehat{u}) \\ &= \Gamma(\tfrac{d}{dt})\widetilde{A}(\tfrac{d}{dt})(\widehat{y} - \widetilde{A}(\tfrac{d}{dt})^{-1}\widetilde{B}(\tfrac{d}{dt})\widehat{u}) = 0 \end{aligned}$$

shows that the transfer matrix can also be written as

$$T(\tfrac{d}{dt}) = \widetilde{A}(\tfrac{d}{dt})^{-1}\widetilde{B}(\tfrac{d}{dt}).$$

Moreover, one sees that the kernel of the mapping $k(\frac{d}{dt})\otimes - : \Sigma \longrightarrow \widehat{\Sigma}$, *i.e.*, the torsion submodule of Σ, is trivial if and only if $\Gamma(\frac{d}{dt})$ is unimodular. We have recovered the result [6, 10, 15], that the system is controllable if and only if $A(\frac{d}{dt})$ and $B(\frac{d}{dt})$ are left coprime. The dual result, viz. that the system is observable if and only if $D(\frac{d}{dt})$ and $N(\frac{d}{dt})$ are right coprime can be shown, too (*cf.* [2, 9, 28]). See also [24], where a module theoretic approach to system duality is introduced.

A *(generalized) state* of a linear system Σ with input u is a family $\xi = (\xi_1, \ldots, \xi_n)$ of elements of Σ such that its canonical image in $\Sigma/[u]$ is a basis of this quotient considered as a k-vector space. The number $n := \dim \Sigma/[u]$, which is finite because $\Sigma/[u]$ is torsion, is called the *state dimension* of the system Σ with input u. The canonical image $\bar{\xi} = (\bar{\xi}_1, \ldots, \bar{\xi}_n)$ of the state ξ is a basis of $\Sigma/[u]$. Therefore, every element of $\Sigma/[u]$ can be expressed as a k-linear combination of the elements of $\bar{\xi}$. For the derivatives of the components of $\bar{\xi}$ and for the canonical images of the output components in $\Sigma/[u]$ it follows

$$\left\{ \begin{array}{rcl} \dot{\bar{\xi}} &=& F\bar{\xi} \\ \bar{y} &=& H\bar{\xi} \end{array} \right. .$$

For ξ and y this yields

$$\left\{\begin{array}{rcl} \dot{\xi} & = & F\,\xi + \sum_{i=0}^{\alpha} G_i\, u^{(i)} \\ y & = & H\,\xi + \sum_{i=0}^{\beta} J_i\, u^{(i)} \end{array}\right.$$

where F, G_i, H and J_i are matrices over k of appropriate size. Introducing the (generalized) state transformation $\xi^* = \xi \Leftrightarrow G_\alpha u^{(\alpha-1)}$ enables one to lower by one the order of derivation of the input in the dynamics equation $\dot{\xi} = F\,\xi + \sum_{i=0}^{\alpha} G_i\, u^{(i)}$. Repeated state transformations of that kind lead to a state representation

$$\left\{\begin{array}{rcl} \dot{x} & = & A\,x + B\,u \\ y & = & C\,x + \sum_{i=0}^{\gamma} D_i\, u^{(i)} \end{array}\right. \tag{3.6}$$

where $D_\gamma \neq 0$ unless $\gamma = 0$. Input derivatives may eventually be present in the output equation ($\gamma > 0$). The state $x = (x_1, \ldots, x_n)$ which appears in (3.6) is called a *Kalman state* [9].

4 Properness for Linear Time-varying Systems

In the sequel of the paper, we mainly restrict our attention to input-output systems $[u, y]$ because properness is an input-output property. However, everything we develop can be applied to general systems ($\Sigma \supsetneq [u, y]$) by considering their input-output subsystem $[u, y]$.

The well-known classical definition of properness says that a constant linear system is proper if the entries of its transfer matrix are proper rational fractions, *i.e.*, the degrees of the numerators do not exceed the denominator degrees[4].

Let us examine the non-commutative case. A *right* (resp. *left*) *factorization* of $w \in k(\frac{d}{dt})$ is the datum of $a, b \in k[\frac{d}{dt}]$ such that $w = a(b)^{-1}$ (resp. $w = (b)^{-1}a$). Consider different left and right factorizations of an element $w \in k[\frac{d}{dt}]$, for instance $w = a_1(b_1)^{-1} = a_2(b_2)^{-1} = (b_3)^{-1}a_3 = (b_4)^{-1}a_4$. From the equality $a_1(b_1)^{-1} = (b_3)^{-1}a_3$ it follows that $b_3 a_1 = a_3 b_1$ and hence $\mathrm{d}^\circ\, a_1 \Leftrightarrow \mathrm{d}^\circ\, b_1 = \mathrm{d}^\circ\, a_3 \Leftrightarrow \mathrm{d}^\circ\, b_3$. The equality $a_2(b_2)^{-1} = (b_3)^{-1}a_3$ yields $\mathrm{d}^\circ\, a_2 \Leftrightarrow \mathrm{d}^\circ\, b_2 = \mathrm{d}^\circ\, a_3 \Leftrightarrow \mathrm{d}^\circ\, b_3$ and then $\mathrm{d}^\circ\, a_1 \Leftrightarrow \mathrm{d}^\circ\, b_1 = \mathrm{d}^\circ\, a_2 \Leftrightarrow \mathrm{d}^\circ\, b_2 (= \mathrm{d}^\circ\, a_3 \Leftrightarrow \mathrm{d}^\circ\, b_3)$. In the same manner one establishes $\mathrm{d}^\circ\, a_3 \Leftrightarrow \mathrm{d}^\circ\, b_3 = \mathrm{d}^\circ\, a_4 \Leftrightarrow \mathrm{d}^\circ\, b_4$. Thus, the difference of numerator

[4] An equivalent characterization is $\lim_{|s|\to+\infty} T_{i,j}(s) < \infty$, where the $T_{i,j}$ are the entries of the transfer matrix.

and denominator degrees does not depend on the (left or right) factorization, and the notion of proper rational fractions can be extended to non-commutative $k(\frac{d}{dt})$. It is easy to verify that the (time-varying) proper rational fractions form a ring which is a subring of the field $k(\frac{d}{dt})$. We denote this ring as P.

Definition 4.1 *A (time-varying) linear system $[u, y]$ with input u and output y is* proper *if the entries of its transfer matrix are elements of the ring P.*

In other words, the system is proper if the components of $\widehat{y}$ are P-linearly dependent over $\widehat{u}$, where $\widehat{u}$ and $\widehat{y}$ denote the formal Laplace transforms of u and y respectively. This leads to the following characterization.

Proposition 4.1 *A (time-varying) linear system $[u, y]$ with input u is proper if and only if the P-module generated by $\widehat{y}$ is a submodule of the P-module generated by $\widehat{u}$.*

We give a characterization of system properness directly on the module $[u, y]$.

Lemma 4.1 *The following statements are equivalent:*

(i) *The system $[u, y]$ with input u and output y is proper.*

(ii) $|y|^r \cap [u] \subseteq |u|^r$, *for all $r \in \mathbb{N}$.*

(iii) *The Hilbert polynomial of the filtration $|u, y|$ is $H_{|u,y|}(\mathfrak{r}) = m(\mathfrak{r}+1)+n$ where n is the state dimension of the system $[u, y]$.*

Remark 4.1 When one is interested in *strict* properness this can be examined by just replacing (ii) by $|y, \dot{y}|^r \cap [u] \subseteq |u|^r$ and the filtration $|u, y|$ in (iii) by the filtration $|u, y, \dot{y}|$.

Proof: (i) $\Leftrightarrow$ (ii): By definition, the quotient $[u, y]/[u]$ is torsion. Therefore, for all components y_i, $i = 1, \ldots, p$, of the output there exists a differential equation

$$\sum_{j=0}^{r_i} a_{i,j} y_i^{(j)} = \sum_{l=1}^{m} \sum_{j=0}^{r'_{i,l}} b_{i,j,l} u_l^{(j)},$$

where $r_i, r'_{i,l} \in \mathbb{N}$, $a_{i,j}, b_{i,j,l} \in k$, with $a_{i,r_i} \neq 0$, and $b_{i,r'_{i,l},l} \neq 0$ unless $r'_{i,l} = 0$. For the formal Laplace transforms, the previous equation implies

$$\left(\sum_{j=0}^{r_i} a_{i,j} \frac{d^j}{dt^j}\right) \widehat{y}_i = \sum_{l=1}^{m} \left(\sum_{j=0}^{r'_{i,l}} b_{i,j,l} \frac{d^j}{dt^j}\right) \widehat{u}_l.$$

One has (i) $\Leftrightarrow [\widehat{y}]_P \subseteq [\widehat{u}]_P \Leftrightarrow r_i \geqslant \max_l r'_{i,l}$, $\forall i = 1, \ldots, p \Leftrightarrow$ (ii), where $[\widehat{y}]_P$ (resp. $[\widehat{u}]_P$) denotes the P-module generated by $\widehat{y}$ (resp. $\widehat{u}$).

(ii) $\Leftrightarrow$ (iii): Denote as $\grave{y}_i$, $i = 1, \ldots, p$, the canonical image of y_i in the quotient module $[u, y]/[u]$ and set $\grave{y} = (\grave{y}_1, \ldots, \grave{y}_p)$. For r large enough, $|\grave{y}|^r \cong [u, y]/[u]$ and hence the Hilbert polynomial of the filtration $|\grave{y}|$ is $H_{|\grave{y}|}(\mathfrak{r}) = n$. For all $r \in \mathbb{N}$,

$$\begin{aligned}
\dim |u, y|^r &= \dim |u|^r + \dim |y|^r - \dim (|u|^r \cap |y|^r) \\
&= \dim |u|^r + \dim |y|^r - \dim ([u] \cap |y|^r) \\
&\quad + \dim ([u^{(r+1)}] \cap |y|^r) \\
&= \dim |u|^r + \dim |\grave{y}|^r + \dim ([u^{(r+1)}] \cap |y|^r).
\end{aligned}$$

As $\dim |u|^r = m(r + 1)$, (ii) $\Leftrightarrow [u^{(r+1)}] \cap |y|^r = \{0\} \Leftrightarrow$ (iii). □

An examination of this proof allows us to state the following remarks.

Remark 4.2 For non-proper systems $[u, y]$, one gets $H_{|u,y|}(\mathfrak{r}) = m(\mathfrak{r} + 1) + b$, with $b > n$.

Remark 4.3 For unobservable systems $\Sigma \supsetneq [u, y]$, the filtration $|u, y|$ is exhaustive for the input-output subsystem $[u, y]$. Hence, the Hilbert polynomial in (iii) of Lemma 4.1 becomes $H_{|u,y|}(\mathfrak{r}) = m(\mathfrak{r} + 1) + n_{\mathrm{obs}}$. Here n_{obs} is the dimension of $[u, y]/[u]$ as k-vector space, *i.e.*, the number of observable state components.

From the proof of (i) $\Leftrightarrow$ (ii) in Lemma 4.1 we get the following proposition.

Proposition 4.2 *A time-varying linear system $[u, y]$ with input u and output y is proper if and only if the number ν_0 in the Laurent series expansion* (3.3) *of its transfer matrix $T(\frac{d}{dt})$ is non-negative.*

An application of Lemma 4.1 generalizes a classical result stemming from the polynomial approach [2, 18, 30] in the constant case. Recall that a non-singular polynomial matrix is called column-reduced (resp. row-reduced)[5] if its leading column (resp. row) coefficient matrix is non-singular [18, 30].

Proposition 4.3 *Let there be given a polynomial representation* (3.4)*, with $D(\frac{d}{dt})$ column-reduced, of a (controllable) input-output system $[u, y]$. The system is proper if and only if the column-degrees of each column of $D(\frac{d}{dt})$ are higher than or equal to those of the corresponding columns of $N(\frac{d}{dt})$.*

[5] These notions were first introduced by Wolovich who used the terms *column-proper* and *row-proper cf.* [18, p. 384].

Proof: Denote as $\gamma_i D$ (resp. $\gamma_i N$) the column-degree of the i-th column of the matrix $D(\frac{d}{dt})$ (resp. $N(\frac{d}{dt})$). First note that a column-reduced matrix $D(\frac{d}{dt})$ can be written as $D_{\mathrm{hc}}\mathrm{diag}\,((\frac{d}{dt})^{\gamma_i D})$ + a sum of terms of lower degree, where D_{hc} is a full rank matrix over k. It is then a simple observation that column-reducedness of $D(\frac{d}{dt})$ implies that $u_j^{(r)} \in \oplus_{i=1}^m |w_i|^{\gamma_i D + r}$, but $u_j^{(r)} \notin \oplus_{i=1}^m |w_i|^{\gamma_i D + r - 1}$, for all $j = 1, \ldots, m$ and $r \in \mathbb{N}$. Moreover, $|y|^r \subseteq \oplus_{i=1}^m |w_i|^{\gamma_i N + r}$. It readily follows that $\gamma_i N \leqslant \gamma_i D$, $i = 1, \ldots, p$, is equivalent to $|y|^r \cap [u] \subseteq |u|^r$, for all $r \in \mathbb{N}$. With (ii) of Lemma 4.1, this means properness. □

An analogous classical result on left factorizations of the transfer matrix can be generalized by considering row-degrees and row-reduced matrices.

Proposition 4.4 *Let be given a polynomial representation* (3.5)*, with* $A(\frac{d}{dt})$ *row-reduced, of an (observable) system* $[u, y]$*. The system is proper if and only if the row-degrees of each row of* $A(\frac{d}{dt})$ *are higher than or equal to those of the corresponding rows of* $B(\frac{d}{dt})$*.*

Proof: Denote as $\delta_i A$ (resp. $\delta_i B$) the row-degree of the i-th row of the matrix $A(\frac{d}{dt})$ (resp. $B(\frac{d}{dt})$). Introduce $z_i = \sum_{j=1}^p a_{i,j}(\frac{d}{dt}) y_j$, $i = 1, \ldots, p$. The row-reducedness of $A(\frac{d}{dt})$ then means that, for $i = 1, \ldots, p$, one can write

$$z_i = \left(a_{\mathrm{hr}}^i . (\tfrac{d}{dt})^{\delta_i A} + \text{sum of terms of lower degree}\right) y = b^i(\tfrac{d}{dt}) u$$

with a_{hr}^i the row vectors of a non singular matrix over k, and $b^i(\frac{d}{dt})$ the row vectors of $B(\frac{d}{dt})$. It follows from the regularity of A_{hr} that $\cup_{i=1}^p |z_i|^{r - \delta_i A} = |y|^r \cap [u]$. Moreover, $\cup_{i=1}^p |z_i|^{r - \delta_i B} \subseteq |u|^r$. Therefore, $\delta_i A \leqslant \delta_i B$, $i = 1, \ldots, p$ is equivalent to $|y|^r \cap [u] \subseteq |u|^r$, $\forall r \in \mathbb{N}$. □

A different way to prove the result is by using duality as introduced in [24]. The dual of a pair $(A(\frac{d}{dt}), B(\frac{d}{dt}))$ from a right factorization is given by $\bar{D}(\frac{d}{dt}) = A^T(\Leftrightarrow\frac{d}{dt})$ and $\bar{N}(\frac{d}{dt}) = B^T(\Leftrightarrow\frac{d}{dt})$. Of course, row degrees of $A(\frac{d}{dt})$ and $B(\frac{d}{dt})$ become column degrees of $\bar{N}(\frac{d}{dt})$ and $\bar{D}(\frac{d}{dt})$.

As a second application of Lemma 4.1 we get the following proposition which is a consequence of Proposition and Definition 5.1 in the next section (see there for the proof).

Proposition 4.5 *Let be given a Kalman state representation* (3.6) *of a system* Σ*. The system* Σ *is proper if and only if the integer* γ *in the output equation is zero.*

Remark 4.4 The integer γ which appears in (3.6) is related to the index of linear implicit differential systems [12], as we will see in the next section.

5 Input-output Index and Index of an Implicit System of Linear Differential Equations

From the Propositions 4.2 and 4.1, one sees that the integers ν_0 in (3.3) and γ in (3.6) both can be considered as a measure for non-properness. More precisely, with each input-output system $[u, y]$ we can associate[6] an *input-output index* $I_{[u,y]} = \max(0, 1 \Leftrightarrow \nu_0) = \max(0, 1 + \mathrm{d}^\circ\ D(\frac{d}{dt}))$, where $D(\frac{d}{dt}) = \sum_{i=0}^{\gamma} D_i . (\frac{d}{dt})^i$, with D_i the matrices in (3.6), and with the convention that $\mathrm{d}^\circ\ D(\frac{d}{dt}) = \Leftrightarrow\infty$ if $D(\frac{d}{dt}) = 0$ and $\mathrm{d}^\circ\ D(\frac{d}{dt}) = \gamma$ if $D(\frac{d}{dt}) \neq 0$. Following [12], the definition of $I_{[u,y]}$ is intrinsic because ν_0 and γ are intrinsic. However, with the language of filtrations developed here we are able to give other intrinsic definitions of the input-output index which are not based on any representation. Hence, in general, the calculation of the input-output index will be easier than the calculation of ν_0 or γ.

Proposition and Definition 5.1 *The input-output index $I_{[u,y]}$ of $[u, y]$ equals the minimal $l \in \mathbb{N}$ such that $H_{|y,\dot{y},u,\ldots,u^{(l)}|}(\mathfrak{r}) = m(\mathfrak{r} + l + 1) + \dim [y, u]/[u]$.*

Proof: First, observe that for all $l \in \mathbb{N}$, and for r large enough,

$$\begin{aligned} \dim \left(\mathrm{span}_k \{x\} + |y, \dot{y}, u, \ldots, u^{(l)}|^r\right) &= \dim |y, \dot{y}, u, \ldots, u^{(l)}|^r + \dim \mathrm{span}_k \{x\} \\ &\quad - \dim \left(\mathrm{span}_k \{x\} \cap |y, \dot{y}, u, \ldots, u^{(l)}|^r\right) \\ &= \dim |y, \dot{y}, u, \ldots, u^{(l)}|^r + n - n_{\mathrm{obs}}. \end{aligned}$$

Recall that $n_{\mathrm{obs}} = \dim [u, y]/[u]$ is the dimension of a state of the observable subsystem $[u, y]$. Equations (3.6) yield $|y, \dot{y}|^r \subseteq \mathrm{span}_k \{x\} + |u|^{r+\gamma+1}$ and, unless $I_{[u,y]} = 0$, $|y, \dot{y}|^r \not\subseteq \mathrm{span}_k \{x\} + |u|^{r+\gamma}$ for all $r \in \mathbb{N}$. Suppose first that $l = \gamma + 1$. One has $\mathrm{span}_k \{x\} + |u, \ldots, u^{(l)}|^r \subseteq \mathrm{span}_k \{x\} + |y, \dot{y}, u, \ldots, u^{(l)}|^r \subseteq \mathrm{span}_k \{x\} + |u, \ldots, u^{(l)}|^r$. Hence, $\dim (\mathrm{span}_k \{x\} + |y, \dot{y}, u, \ldots, u^{(l)}|^r) = n + m(r + l + 1)$. Suppose now that $l < \gamma + 1$. Unless $I_{[u,y]} = 0$, one has $\mathrm{span}_k \{x\} + |y, \dot{y}, u, \ldots, u^{(l)}|^r \supsetneq \mathrm{span}_k \{x\} + |u, \ldots, u^{(l)}|^r$, and hence $\dim (\mathrm{span}_k \{x\} + |y, \dot{y}, u, \ldots, u^{(l)}|^r) > n + m(r + l + 1)$. Therefore, for the case $I_{[u,y]} \neq 0$,

$$\gamma = l + 1 \Leftrightarrow H_{|y,\dot{y},u,\ldots,u^{(l)}|}(\mathfrak{r}) = m(\mathfrak{r} + l + 1) + n_{\mathrm{obs}},$$

$$\gamma > l \Leftrightarrow 1 \Leftrightarrow H_{|y,\dot{y},u,\ldots,u^{(l)}|}(\mathfrak{r}) > m(\mathfrak{r} + l + 1) + n_{\mathrm{obs}}.$$

If $I_{[u,y]} = 0$, one has $\dim (\mathrm{span}_k \{x\} + |y, \dot{y}, u|^r) = n + m(r + 1)$, whence $H_{|y,\dot{y},u|} = m(\mathfrak{r} + 1) + \dim [y, u]/[u]$. □

[6] The equality $\max(0, 1 - \nu_0) = \max(0, 1 + \mathrm{d}^\circ\ D(\frac{d}{dt}))$ has been shown in [12] and this will not be repeated here.

We can give yet another interesting intrinsic definition, which is equivalent to the previous ones. Consider the canonical injection $\imath : [u] \hookrightarrow [u, y]$.

Proposition and Definition 5.2 *The index $I_{[u,y]}$ of $[u, y]$ satisfies*

$$I_{[u,y]} = \min \left\{ l \in \mathbb{N} \mid \imath^{-1}(|u, y, \dot{y}|^r) \subseteq |u|^{l+r}, \forall r \in \mathbb{Z} \right\}.$$

Proof: One has

$$H_{|u,\ldots,u^{(l)},y,\dot{y}|}(\mathfrak{r}) = H_{|u,\ldots,u^{(l)}|}(\mathfrak{r}) + H_{|\dot{y}|}(\mathfrak{r}) = m(\mathfrak{r} + l + 1) + n,$$

with $\dot{y}_i$ the canonical image of y_i in $[y, u]/[u]$ if and only if $\imath^{-1}(|u, y, \dot{y}|^r) \subseteq |u|^{l+r}, \forall r \in \mathbb{Z}$. This is because $|u, \ldots, u^{(l)}, y, \dot{y}|^r/|u|^{r+l}$ and $|\dot{y}, \dot{y}|^r$ are isomorphic for all large r if and only if our inclusion condition holds. Moreover, $H_{|\dot{y}|}(\mathfrak{r}) = H_{|\dot{y},\dot{y}|}(\mathfrak{r}) = n$. This proves equivalence with the Proposition and Definition 5.1. (This proof can be formalized by considering short exact sequences for the filtered modules (*cf.* [17]).) □

Remark 5.1 The input-output index is a measure for non-strict properness: the input-output index of a strictly proper system is zero, the input-output index of a proper but not strictly proper system is one, and the input-output index of non-proper systems is greater than one. For that reason we use the filtration which tests for strict properness in the previous definition (*cf.* Remark 4.1).

Remark 5.2 An analogous result can be considered for the canonical injection of $[y]$ into $[u, y]$. The integer defined by this way measures the maximum order of zeros at infinity. This will be studied elsewhere.

We can use the input-output index in order to give a new definition of the index of an implicit system of linear differential equations (see [3][7]). The index of an implicit system of differential equations measures the number of times the equations have to be differentiated in order to get an associated explicit system. This notion, introduced for constant systems of differential equations, has been extended recently to time-varying systems [12, 21, 22]. We show how the two notions of index are related.

Consider the implicit system of linear time-varying differential equations

$$A(\tfrac{d}{dt})Y = 0, \tag{5.1}$$

where $Y = (Y_1, \ldots, Y_m)$ are the unknowns and $A(\frac{d}{dt})$ is an $m \times m$ matrix with full rank over $k[\frac{d}{dt}]$. In the $k[\frac{d}{dt}]$-module $[E, Y]$, with $E = (E_1, \ldots, E_m)$, define $W = (W_1, \ldots, W_m)$ such that $W_i = A^i(\frac{d}{dt})Y \Leftrightarrow$

[7] See [12] for further references.

E_i, $i = 1, \ldots, m$, where $A^i(\frac{d}{dt})$ denotes the ith row of $A(\frac{d}{dt})$. Set $[e, y] = [E, Y]/[W]$, where e and y are respectively the canonical images of E and Y in $[E, Y]/[W]$. As $A(\frac{d}{dt})$ has full rank, $[e, y]/[e]$ is torsion and $[e, y]$ can be considered as an input-output system with input e and output y associated to (5.1). With the definition of the index from [12] one gets the following.

Proposition and Definition 5.3 *The index of the implicit system of time-varying linear differential equations* (5.1) *is equal to the input-output index of the associated input-output system* $[e, y]$.

Remark 5.3 The index of (5.1) is invariant with respect to equivalent choice of equations and change of coordinates for the unknown variables. This follows from the invariance of the input-output index with respect to invertible changes of input and output, *i.e.*, $\tilde{y} = F_1\, y$ and $\tilde{u} = F_2\, u$, where F_1 and F_2 are invertible square matrices of appropriate size with entries in k.

6 Example

6.1 Input-output index of a linear network

Consider the time-varying linear network of Figure 1 (resistors R_1, R_2, R_3 and R_5 are adjustable). We utilize the linear idealized model of the operational amplifiers [14]. Writing down Kirchhoff laws yields:

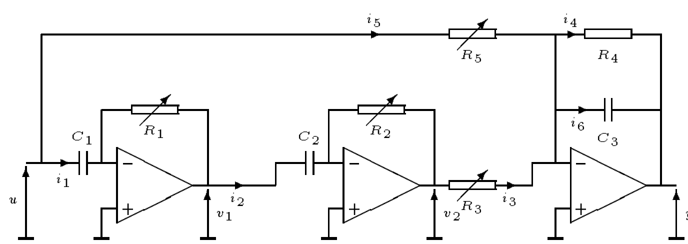

Figure 1: Linear Time-varying Network

$$
\begin{array}{rclcrclcl}
i_1 & = & C_1 \dot{u} & \quad & v_1 & = & \Leftrightarrow R_1 i_1 & & \\
i_2 & = & C_2 \dot{v}_1 & & v_2 & = & \Leftrightarrow R_2 i_2 & = & R_3 i_3 \\
i_6 & = & \Leftrightarrow C_3 \dot{y} & & y & = & \Leftrightarrow R_4 i_4 & & \\
i_3 + i_5 & = & i_4 + i_6 & & u & = & R_5 i_5 & &
\end{array}
$$

These equations are not in any standard form (*i.e.*, state-variables equations, input-output polynomial equations, transfer equations,...). However, they allow us to check properness with the filtration criterion. Variables $i_1, \ldots, i_6, v_1$ and v_2 are latent variables, u is the input, y the output. Denote as $\bar{z}$ the canonical image of $z \in \Sigma$ in $\Sigma/[u]$. This yields:

$$\begin{array}{rclrcll} \bar{i}_1 & = & 0 & \bar{v}_1 & = & \Leftrightarrow R_1 \bar{i}_1 & \\ \bar{i}_2 & = & C_2 \dot{\bar{v}}_1 & \bar{v}_2 & = & \Leftrightarrow R_2 \bar{i}_2 & = \; R_3 \bar{i}_3 \\ \bar{i}_6 & = & \Leftrightarrow C_3 \dot{\bar{y}} & \bar{y} & = & \Leftrightarrow R_4 \bar{i}_4 & \\ \bar{i}_3 + \bar{i}_5 & = & \bar{i}_4 + \bar{i}_6 & 0 & = & R_5 \bar{i}_5 . & \end{array}$$

It follows that $\bar{i}_1 = \bar{i}_2 = \bar{i}_3 = \bar{i}_5 = \bar{v}_1 = \bar{v}_2 = 0$, $\bar{i}_4 = \Leftrightarrow(1/R_4)\bar{y}$, $\bar{i}_6 = \Leftrightarrow C_3 \dot{\bar{y}}$ and $\bar{y} + R_4 C_4 \dot{\bar{y}} = 0$, and hence $\dim \Sigma/[u] = 1$. The observability is obvious.

We can check properness with the filtration $|u, y|$. We see that $|y|^1 \cap [u] \subset |u|^2$, and $|y|^1 \cap [u] \not\subset |u|^1$, and, by repeated derivation of the equations, we can check that $|y|^r \cap [u] \subset |u|^{r+1}$ and $|y|^r \cap [u] \not\subset |u|^r$, for all $r \in \mathbb{Z}$. The system is not proper. We can also calculate the Hilbert polynomial of $|u, y|$: $\dim |u, y|^0 = 2$, $\dim |u, y|^1 = 4 = (1 + 1) + 2$, $\dim |u, y|^2 = 5 = (2+1)+2, \ldots, \dim |u, y|^r = r+3 = (r+1)+2$. Thus $H_{|u,y|}(\mathfrak{r}) = \mathfrak{r}+1+2 \neq \mathfrak{r} + 1 + \dim \Sigma/[u]$.

We saw that $\forall r \in \mathbb{Z}$, $|y|^r \cap [u] \subset |u|^{r+1}$ and $|y|^r \cap [u] \not\subset |u|^r$. It follows that $\forall r \in \mathbb{Z}$, $|y, \dot{y}|^r \cap [u] \subset |u|^{r+2}$ and $|y, \dot{y}|^r \cap [u] \not\subset |u|^{r+1}$. Therefore, $\forall r \in \mathbb{Z}$, $\imath^{-1}(|u, y, \dot{y}|^r) \subseteq |u|^{r+2}$ and $\imath^{-1}(|u, y, \dot{y}|^r) \not\subset |u|^{r+1}$. The input-output index of this network is 2.

6.2 Index of a time-varying implicit system of linear differential equations

We will calculate the index of the implicit system of differential equations considered in [12], given by

$$\left\{ \begin{array}{rcl} \dot{Y}_1 + t\dot{Y}_2 & = & 0 \\ Y_1 + tY_2 & = & 0 \end{array} \right. . \tag{6.1}$$

The associated input-output system reads

$$\left\{ \begin{array}{rcl} \dot{y}_1 + t\dot{y}_2 & = & e_1 \\ y_1 + ty_2 & = & e_2 \end{array} \right. .$$

One sees that $y_1 = e_2 + te_1 \Leftrightarrow t\dot{e}_2$ and $y_2 = \Leftrightarrow e_1 + \dot{e}_2$. Hence, from Proposition and Definition 5.2, $I_{[e,y]} = \min \left\{ l \in \mathbb{N} \mid \imath^{-1}(|e, y, \dot{y}|^r) \subseteq |e|^{l+r}, \forall r \in \mathbb{Z} \right\} = 2$. The index of (6.1) is 2.

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