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An Algorithm for Viability Kernels in Hölderian Case: Approximation by Discrete Dynamical Systems*

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Abstract

In this paper, we study two new methods for approximating the viability kernel of a given set for a Hölderian differential inclusion. We approximate this kernel by viability kernels for discrete dynamical systems. We prove a convergence result when the differential inclusion is replaced by a sequence of recursive inclusions. Furthermore, when the given set is approached by a sequence of suitable finite sets, we prove our second main convergence result. This paper is the first step to obtain numerical methods.

1 Introduction and Notations

Let X be a finite dimensional vector space and K be closed subset of X. Consider the differential inclusion:

$$x'(t) \in F(x(t)), \text{ for almost all } t \geq 0. \tag{1.1}$$

We want to study the viability kernel of K for F (denoted by $Viab_F(K)$) which is the largest closed set contained in K such that starting at any point of K there exists at least one *viable* solution (i.e. a solution such that $\forall\ t \geq 0,\ x(t) \in K$). This viability kernel plays a crucial role in various domains. In control theory, it has been introduced by Aubin in [2], studied by Byrnes-Isidori under the name of *zero dynamics* (cf [3, 14]) and used for target problems in [17] (see also [8, 4, 19]).

It is well-known (see [2, 13]) that when F is a Marchaud-map[1] a closed

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[1]A set-valued map $F : X \rightsquigarrow Y$ is a Marchaud map when F is upper-semicontinuous, with convex compact nonempty values and with linear growth.

set is viable if and only if it satisfies the following contingent[2] condition

$$\forall\, x \in K,\; F(x) \cap T_K(x) \neq \emptyset. \tag{1.2}$$

The viability kernel $Viab_F(K)$ is the largest closed viable subset contained in K. Our main aim is to determine this set in a constructive way by using discrete approximation (see also [12] for another way to approximate this set). For that purpose, for any $\rho > 0$, we associate to (1.1), the following discrete dynamical system

$$\frac{x^{n+1} - x^n}{\rho} \in F(x^n), \text{ for all } n \geq 1. \tag{1.3}$$

We denote by G_ρ the set-valued map $G_\rho = \mathbf{1} + \rho F$. Then the system (1.3) can be rewritten:

$$x^{n+1} \in G_\rho(x^n), \;\text{ for all } n \geq 0. \tag{1.4}$$

The Viability Theory allows us to study points $x_0 \in K$ such that there exists at least one viable solution to (1.3) starting at x_0 (i.e., a solution $\vec{x}$ to (1.4) such that $\forall\, n,\; x_n \in K$). Similarly, as in continuous case, we can define viable sets and viability kernels. Let us introduce some notations for discrete and continuous cases. We denote by

- $S_F(x_0)$ the set of solution $x(\cdot)$ to (1.1) starting at x_0;
- $\vec{S}_{G_\rho}(x_0)$ the set of solution $\vec{x} = (x_n)_n$ to (1.4) starting at x_0;
- $Viab_F(K)$ the viability kernel of K for (1.1);
- $\overrightarrow{Viab}_{G_\rho}(K)$ the discrete viability kernel of K for (1.4).

When F is a Marchaud map, we know that one can find a sequence of discrete viability kernels of K under $G\rho$, which converges to a closed subset contained in the viability kernel of K for F (see [20]).

In this paper, when the set valued-map F is furthermore regular enough (i.e., when F is a β-Hölderian[3]), we prove that the sequence of discrete viability kernels for the map $\Gamma_\rho(x) = x + \rho F(x) + l\rho^\beta B$ converges to the viability kernel for F.

In the last part of this paper, we consider a finite approximation X_h of the whole space X and we consider discrete inclusions on X_h. Then we prove that viability kernels of some subsets of X_h for suitable discrete inclusion converge to $Viab_F(K)$.

[2] The contingent cone (or Bouligand cone) $T_K(x)$ is the set of $v \in X$ such that $\liminf_{h \longrightarrow 0^+} d(x + hv, K)/h = 0$.

[3] The map F is an β- Hölderian map namely if there exists some $\beta > 0$ such that for any x, y, $F(x) \subset F(y) + l\|x - y\|^\beta B$.

This paper gives mathematical results for numerical methods which have been applied to particular examples (see [15]) for an economical example and [9] for the classical problem of a swimmer who tries to reach an island).

2 Approximation by Viability Kernels of Discrete Dynamical Systems

In this section, our goal consists in approximating $Viab_F(K)$ by discrete viability kernel of K for (1.3). First we recall some basic properties of discrete viability kernels (see [2, 20]).

2.1 Viability kernels for discrete dynamical systems

We call *discrete dynamical system* associated with a set-valued map G from $X \mapsto X$ the following system:

$$x^{n+1} \in G(x^n), \quad \text{for all } n \geq 0. \tag{2.1}$$

We denote by $\mathcal{K}$ the set of all sequences $\vec{x} := (x^0, ..., x^n, ...)$ from $I\!N$ to K. A solution $\vec{x}$ to (2.1) is viable in K if and only if for all $n \geq 0$, $x_n \in K$ (i.e., $\vec{x} \in \vec{S}_G(x) \cap \mathcal{K}$.)

A closed set A is *a discrete viability domain* for G if and only if starting from any initial points in A there exists at least one viable solution to (2.1). Let us recall[4] that A is a discrete viability domain if and only if

$$\forall\, x \in A,\ G(x) \cap A \neq \emptyset. \tag{2.2}$$

Then the *discrete viability kernel* $\overrightarrow{Viab}_G\,(K)$ of K for G is the largest closed discrete viability domain contained in K.

Let us notice that this set can be easily built in a constructive way (cf [2]):

Proposition 2.1 *Let G: $X \rightsquigarrow X$ be an upper semicontinuous set-valued map with closed values and K be a compact subset of $Dom(G)$. If the sequence $(K^n)_n$ (with $K^0 = K$) is defined as follows:*

$$K^{n+1} := \{x \in K^n \text{ such that: } G(x) \cap K^n \neq q\emptyset\}$$

then, $\overrightarrow{Viab}_G\,(K) = \bigcap_{n=0}^{+\infty} K^n$.

[4] cf [2, 20].

Let $G^r : X \rightsquigarrow X$ the *extension of* G defined by

$$G^r(x) := G(x) + rB. \tag{2.3}$$

The sequence of subsets $K^{r,0} = K$, $K^{r,1}, ..., K^{r,n}, ...$ defined by

$$K^{r,n+1} := \{x \in K^{r,n} \text{ , such that } G^r(x) \cap K^{r,n} \neq \emptyset\}$$

is again convergent to $\overrightarrow{Viab}_{G^r}(K)$. Furthermore, when r decreases to 0, the viability kernel of K for G^r converges to the viability kernel of K for G (cf [20]):

Proposition 2.2 *Let G and K satisfy assumptions of Proposition 2.1 and G^r be defined by (2.3), then*

$$\overrightarrow{Viab}_G(K) = \bigcap_{r>0} \overrightarrow{Viab}_{G^r}(K).$$

2.2 Approximation process

Let F a Marchaud map and F_ρ a sequence of set-valued maps satisfying

$$\forall \epsilon > 0,\ \exists \rho_\epsilon > 0,\ \forall \rho \in]0, \rho_\epsilon] \ : \ Graph(F_\rho) \subset Graph(F) + \epsilon B, \tag{2.4}$$

where B is the unit ball in $X \times X$. Thus, we define an *approximation process* of (1.1) by the dynamical discrete system $x^{n+1} \in x_n + \rho F_\rho(x^n)$.

Let us notice that (1.3) is an approximation process (case $F_\rho = F$) but there are many of them (see a detailed study concerning *Set-valued Runge-Kutta process* and *the thickening process* in [20]).

Assumption (2.4) implies that the graph of F contains the graphical upper limit[5] of F_ρ, that is to say that $Graph(F)$ contains the Painlevé-Kuratowski upper limit[6] of $Graph(F_\rho)$

$$\limsup_{\rho \to 0} Graph(F_\rho) \subset Graph(F). \tag{2.5}$$

Let K_ρ be a sequence of subsets of X such that $K = \limsup_{\rho > 0} K_\rho$. Possible K_ρ may be constant. We set $\Gamma_\rho := \mathbf{1} + \rho F_\rho$ and consider $\overrightarrow{Viab}_{\Gamma_\rho}(K_\rho)$

[5] The graphical upper limit is the upper limit of the sequence of $Graph(F_\rho)$.

[6] The upper limit of a sequence of subsets D_n of X is

$$D^\sharp = \limsup_{n \to \infty} D_n := \{y \in X \mid \liminf_{n \to \infty} d(y, D_n) = 0\};$$

the lower limit is defined by

$$\liminf_{n \to \infty} D_n := \{y \in X \mid lim_{n \to \infty} d(y, D_n) = 0\}.$$

the discrete viability kernel of K_ρ under Γ_ρ. We shall recall a result (cf [20] for the proof) which implies the viability property of the upper limit of discrete viability kernels $\overrightarrow{Viab}_{\Gamma_\rho}(K_\rho)$:

Theorem 2.3 *Let F be a Marchaud map and F_ρ be a sequence of set-valued maps such that $F = \overline{Co}\mathrm{Lim}^\sharp_{\rho\to 0}F_\rho$. Then the upper limit $\limsup_{\rho\to 0} Viab_{\Gamma_\rho}(K_\rho)$ is a viable subset under F:*

$$\limsup_{\rho\to 0^+} \overrightarrow{Viab}_{\Gamma_\rho}(K_\rho) \subset Viab_F(K). \tag{2.6}$$

Our main goal is to prove that it is possible to chose F_ρ and K_ρ in a such way that the inclusion (2.6) is an equality.

2.3 Convergence of the approximation process

We shall prove the convergence of the approximation process under the crucial following condition concerning set-valued maps F and F_ρ

$$\left\{ \begin{array}{ll} i) & M := \sup_{x\in K}\sup_{y\in F(x)}\|y\| < \infty \\ ii) & \exists\, \rho_0 > 0,\, \forall\, \rho \in]0,\rho_0],\, Graph(F(\cdot+\rho B)) \subset Graph F_\rho. \end{array} \right. \tag{2.7}$$

For any sequence F_ρ, let us define

$$\forall\, x,\ \Gamma_\rho(x) := x + \rho F_\rho(x).$$

Theorem 2.4 *Let K be a closed set and F be a Marchaud set-valued map. If maps F_ρ satisfies (2.4) and (2.7) then*

$$\limsup_{\rho\to 0} \overrightarrow{Viab}_{\Gamma_\rho}(K) = \liminf_{\rho\to 0^+} \overrightarrow{Viab}_{\Gamma_\rho}(K_\rho) = Viab_F(K). \tag{2.8}$$

Let us make some comments before proving the theorem:

Remark 1: Condition (2.7-i)) is fulfilled as soon as F is upper semicontinuous and K is compact

Remark 2: It is possible, when $Dom(K) = Dom(F_\rho)$, to write the condition (2.7-ii)) as follows:

$$\exists\rho_0 > 0,\ \forall\rho \in]0,\rho_0], \forall x,\ \forall y \in\ B(x,\rho),\ F(y) \subset F_\rho(x). \tag{2.9}$$

Remark 3: The theorem is still valid instead of (2.7-ii)) if we assume the following weaker condition:

$$Graph F((\cdot+\rho B)\cap K) \subset Graph F_\rho.$$

Remark 4: If F is a Marchaud and ℓ-Lipschitz map then maps $F_\rho := F + \frac{Ml\rho}{2} B$ satisfy condition (2.7) and (2.4).

Proof of Theorem 2.4: Thanks to Theorem 2.3, we only have to prove the inclusion

$$Viab_F(K) \subset \liminf_{\rho \to 0^+} \overrightarrow{Viab}_{\Gamma_\rho}(K).$$

Let $x_0 \in K$ and consider any solution $x(\cdot) \in S_F(x_0)$. Let ρ given in $]0, \rho_0]$. We have $x(s) - x(t) = \int_t^s x'(\sigma) d\sigma \in \int_t^s F(x(\sigma)) d\sigma$ but $x(\sigma) \in x(t) + \int_t^\sigma F(x(u)) du \subset x(t) + (\sigma - t) MB$.

Consequently,

$$x(t+\rho) \in x(t) + \int_t^{t+\rho} F(x(t) + (\sigma - t)MB) d\sigma.$$

Since $x(t) + (\sigma - t)MB \subset x(t) + \rho B$ and thanks to (2.7), we deduce that $x(t+\rho) \in x(t) + \rho F_\rho(x(t))$.

Then if $x(\cdot) \in S_F(x_0)$ then the following sequence

$$\xi_n = x(n\rho), \quad \forall n \geq 0 \tag{2.10}$$

is a solution to the discrete dynamical system associated with $\Gamma_\rho := \mathbf{1} + F_\rho$:

$$\xi_{n+1} \in \Gamma_\rho(\xi_n), \quad \forall n \geq 0. \tag{2.11}$$

So, if $x(\cdot)$ is a viable solution, $(\xi_n)_n$ is also a viable solution to (2.11). It implies

$$Viab_F(K) \subset \overrightarrow{Viab}_{\Gamma_\rho}(K), \; ; \; \forall \rho > 0$$

and then

$$Viab_F(K) \subset \liminf_{\rho \to 0} \overrightarrow{Viab}_{\Gamma_\rho}(K) \subset \limsup_{\rho \to 0} \overrightarrow{Viab}_{\Gamma_\rho}(K).$$

□

Corollary 2.5 *Let F be a Marchaud and ℓ-Lipschitz set-valued map and K a closed subset of X satisfying the boundedness condition (2.7-i)). Consider $F_\rho := F + \dfrac{Ml}{2}\rho B$. Then $\lim_{\rho \to 0} \overrightarrow{Viab}_{\Gamma_\rho}(K) = Viab_F(K)$.*

It is easy to extend this result to the Hölder case:

Corollary 2.6 *Let F be a convex compact set-valued map satisfying (2.7-i)) and the following Hölder condition:*

$$\exists \beta > 0, \; \exists \ell > 0, \; \forall (x,y), \quad F(y) \subset F(x) + \ell \|x - y\|^\beta B. \tag{2.12}$$

Consider $F_\rho := F + \ell \rho^\beta B$. Then

$$\lim_{\rho \to 0} \overrightarrow{Viab}_{\Gamma_\rho}(K) = Viab_F(K).$$

Assumptions (2.4) and (2.7-ii)) are contradictory in some sense. The first one means that the approximations F_ρ must not be too large so that their graph remains in a "small" extension of the graph of F. The second one means that approximations F_ρ must be large enough so that $F_\rho(x)$ contains all images $F(y)$ when y gets close to x.

3 Approximation by Finite Setvalued Maps

In this section, we want to replace X by a discrete set X_h and we shall state some convergence results.

With any $h \in \mathbb{R}$ we associate X_h a countable subset of X, which is an approximation of X in the following sens:

$$\left\{ \begin{array}{ll} i) & \forall x \in X,\ \exists x_h \in X_h \text{ such that } \ \|x - x_h\| \leq \alpha(h) \\ ii) & \lim\limits_{h \to 0} \alpha(h) = 0 \\ iii) & \text{all bounded subset of } X_h \text{ is finite.} \end{array} \right. \tag{3.1}$$

3.1 Approximation of discrete viability kernels

Let $G_h : X_h \rightsquigarrow X_h$ be a finite set-valued map and a subset $K_h \subset Dom(G_h)$. We call *finite dynamical system* associated to G_h the following system:

$$x_h^{n+1} \in G_h(x_h^n), \ \text{ for all } n \geq 0, \tag{3.2}$$

and we denote by

- $\mathcal{K}_h$ the set of all sequences from $\mathbb{N}$ to K_h.
- $\vec{x}_h := (x_h^0, \ldots, x_h^n, \ldots) \in \mathcal{X}_h$ a solution to system (3.2)
- $\vec{S}_{G_h}(x_h^0)$ the set of solutions $\vec{x}_h \in \mathcal{X}_h$ to the finite differential inclusion (3.2) starting from x_h^0.

A solution $\vec{x}_h$ is viable if and only if $\vec{x}_h \in \vec{S}_{G_h}(x_h) \cap \mathcal{K}_h$.

Let $K_h^0 = K_h$, $K_h^1, \ldots$, $K_h^n, \ldots$ defined recursively as in the second section:

$$K_h^{n+1} := \{x_h \in K_h^n \text{ such that: } \ G_h(x_h) \cap K_h^n \neq \emptyset\}.$$

The viability kernel algorithm holds true for finite dynamical systems whenever the set-valued map G_h has nonempty values and we have $\overrightarrow{Viab}_{G_h}$ $(K_h) = \bigcap_{n=0}^{+\infty} K_h^n$. This set can be empty. Moreover, there exists p finite, such that: $\overrightarrow{Viab}_{G_h}\ (K_h) = K_h^n = K_h^p$, $\forall n > p$.

When G_h is the reduction to K_h of a set-valued map G, we can apply no longer more previous results, since $G(x_h)$ may not contain any point of X_h and $G_h(x_h)$ may be empty.

To turnover this difficulty, we will consider greater set-valued maps G^r which still approximate G. For choosing G_h^r, we have two different difficulties: on one hand, G^r has to be large enough (such that $Dom(G^r \cap X_h) \supset K_h$) and on the other hand, it has to be small enough in view to apply Theorem 2.4.

Let us define some notations:

$$\forall D \subset X, \quad D_h := D \cap X_h$$
$$\forall x \in X, \; G^r(x) := G(x) + r\mathcal{B}$$
$$\forall x \in X, \; G_h^r(x) := G^r(x) \cap X_h.$$

According to definition (3.1) of $\alpha(h)$, we notice that extension $G_h^{\alpha(h)}$ satisfies the following nonemptyness property:

$$\forall x_h \in Dom(G) \cap X_h, \; G_h^{\alpha(h)}(x_h) := G^{\alpha(h)}(x_h) \cap X_h \neq \emptyset. \tag{3.3}$$

Then from Proposition 2.1, we can deduce the following:

Proposition 3.1 *Let G be a Marchaud map. Consider decreasing sequence of finite subsets $K_h^{\alpha(h),0} = K_h, \; K_h^{\alpha(h),1}, \ldots, \; K_h^{\alpha(h),n}, \ldots$ defined by*

$$K_h^{\alpha(h),n+1} := \{x \in K_h^{\alpha(h),n} \text{ , such that } \; G_h^{\alpha(h)}(x) \cap K_h^{\alpha(h),n} \neq \emptyset\}.$$

Then

$$\bigcap_{n=0}^{+\infty} K_h^{\alpha(h),n} = \overrightarrow{Viab}_{G_h^{\alpha(h)}}(K_h).$$

Let us describe a method to approximate the discrete viability kernel of K under G. First, we extend G such that $Dom(G_h^r) = Dom(G) \cap X_h$ (for doing this we choose $r = \alpha(h)$ then for any $x_h \in K_h$, the set $G_h^{\alpha(h)}(x_h)$ is nonempty). Secondly, we shall study convergence of $\overrightarrow{Viab}_{G_h^{\alpha(h)}}(K_h)$, when h converges to 0^+.

3.2 Discrete viability kernel of a discrete set

Since $\lim_{h\to 0} \alpha(h) = 0$, by applying Proposition 2.2, we obtain

$$\bigcap_{h>0} \overrightarrow{Viab}_{G^{\alpha(h)}}(K) = \overrightarrow{Viab}_G(K).$$

The following result gives a necessary and sufficient condition for $\overrightarrow{Viab}_{G_h^{\alpha(h)}}(K_h)$ to be the reduction of $\overrightarrow{Viab}_{G^{\alpha(h)}}(K)$ to X_h (proof in [20] Prop. 4.1):

Proposition 3.2 *Let G: $X \rightsquigarrow X$ be an upper semicontinuous set-valued map with closed values and K a closed subset of $Dom(G)$. Let $r > 0$ be such that for all $x \in Dom(G^r) \cap X_h$, $G^r(x) \cap X_h \neq \emptyset$. Then*

$$\overrightarrow{Viab}_{G_h^r}(K_h) \subset \overrightarrow{Viab}_{G^r}(K) \cap X_h. \tag{3.4}$$

It coincides if and only if:

$$\forall x_h \in \overrightarrow{Viab}_{G^r}(K) \cap X_h,\ G^r(x_h) \cap (\overrightarrow{Viab}_{G^r}(K) \cap X_h) \neq \emptyset. \tag{3.5}$$

3.3 Approximation for Hölderian maps

In general case, we cannot apply Proposition 3.2, but we can deduce the following approximation result when K is a viability domain:

Let $r(h) = \max(\ell\alpha(h)^\beta, \alpha(h))$.

Proposition 3.3 *Let $G : X \to X$ be a β-Hölderian set-valued map and K a nonempty discrete viability domain for G. Then $K_h^r := (K + r\mathcal{B}) \cap X_h$ is a finite viability domain for G_h^r*

$$\forall r \geq r(h),\ \ \overrightarrow{Viab}_{G_h^r}(K_h^r) = K_h^r.$$

Proof: We want to prove that K_h^r is a viability domain for G_h^r, namely $G_h^r(x) \cap K_h^r \neq \emptyset$ for any $x \in K_h^r$. But

$$G_h^r(x) \cap K_h^r = (G(x) + r\mathcal{B}) \cap X_h \cap (K + r\mathcal{B}) \supset X_h \cap (G(x) \cap K + r\mathcal{B})$$

which is nonempty as soon as $r \geq \alpha(h)$. □

We can now compare discrete viability kernel of K for G and finite viability kernel of $K \cap X_h$ for G_h.

Proposition 3.4 *Let $G : X \rightsquigarrow X$ be a β-Hölderian set-valued map with nonempty values satisfying the following property*

$$\forall \xi \in G(x),\ \exists \xi_h \in G(x) \cap X_h \text{ such that } \|\xi - \xi_h\| \leq \alpha(h). \tag{3.6}$$

Then, for all $r \geq \ell\alpha(h)^\beta$, we have

$$\overrightarrow{Viab}_G(K) \subset \overrightarrow{Viab}_{G_h^r}(K_h^\gamma) + \gamma\mathcal{B},$$

where $K_h^\gamma = (K + \gamma\mathcal{B}) \cap X_h$ and $\gamma := (r/\ell)^{\frac{1}{\beta}}$. □

Before proving Proposition 3.4, we state the following:

Lemma 3.5 *Let assumptions of Proposition 3.4 hold true. Consider* $r \geq \ell\alpha(h)^\beta$. *Then*

$$\forall \vec{\xi} \in \vec{S}_G(\xi^0),\ \exists \vec{\xi_h} \in \vec{S}_{G_h^r}(\xi_h^0),\ \forall n \geq 0,\ \|\xi_h^n - \xi^n\|^\beta \leq \frac{r}{\ell}. \tag{3.7}$$

Proof of the Lemma: From (3.1), there exists some ξ_h^0, which belongs to $(\xi^0 + (\frac{r}{\ell})^{1/\beta}\mathcal{B}) \cap X_h$. Assume that we found a sequence $\xi_h^p \in G_h^r(\xi_h^{p-1})$, which satisfies (3.7) until $p = n$.

Thanks to the nonemptyness property (3.3) and because G is a Hölderian map, we deduce:

$$G(\xi^n) \subset G(\xi_h^n + (\frac{r}{\ell})^{\frac{1}{\beta}}\mathcal{B}) \subset G(\xi_h^n) + r\mathcal{B} = G^r(\xi_h^n). \tag{3.8}$$

Since $\xi^{n+1} \in G(\xi^n)$, from (3.6), there exists some $\xi_h^{n+1} \in G(\xi^n) \cap X_h$ such that $\|\xi^{n+1} - \xi_h^{n+1}\|^\beta \leq \frac{r}{\ell}$. Thanks to (3.8), $\xi_h^{n+1} \in G^r(\xi_h^n) \cap X_h = G_h^r(\xi_h^n)$ and consequently $\vec{\xi_h} \in \vec{S}_{G_h^r}(\xi_h^0)$. By iterating this process, the proof is completed. □

Proof of Proposition 3.4: Consider $\xi^0 \in \overrightarrow{Viab}_G(K)$ and $\vec{\xi} \in \vec{S}_G(\xi^0)$ as an associated solution which is viable in K. Thanks to Lemma 3.5, there exist $\vec{\xi_h} \in \vec{S}_{G_h^r}(\xi_h^0)$ satisfying (3.7). Hence, for any $n \geq 0$,

$$\xi_h^n \in K + (\frac{r}{\ell})^{\frac{1}{\beta}} \cap X_h.$$

□

3.4 Convergence result

Before stating our main convergence result, we shall recall a useful lemma (see [20] for the proof).

Lemma 3.6 *Let* $D \subset X$ *be closed. Consider a decreasing sequence of closed subsets* D_ρ *such that* $D = \bigcap_{\rho>0} D_\rho$. *Assume that (3.1) holds true. Then*

$$D = \lim_{\rho,h\to 0} ((D_\rho + \alpha(h)\mathcal{B}) \cap X_h). \tag{3.9}$$

If D *satisfies the property* $\forall x \in D, \exists x_h \in D \cap X_h\ :\ \|x - x_h\| \leq \alpha(h)$, *then*

$$D = \lim_{h\to 0} (D \cap X_h). \tag{3.10}$$

Now we can state the following:

Theorem 3.7 *Let $F : X \rightsquigarrow X$ be a Hölder map with convex compact nonempty values and K be a closed subset of X satisfying the boundedness condition (2.7-ii)). Consider $\Gamma_\rho := 1 + \rho F + \ell\rho^{1+\beta}\mathcal{B}$ and assume that $\alpha(\cdot)$ and X_h satisfy (3.1).*

If $\alpha(h) \le \ell\rho^{1+\beta}$, then

$$Viab_F(K) = \lim_{\rho,h\to 0}(\overrightarrow{Viab}_{\Gamma_\rho}(K) + \alpha(h)\mathcal{B}) \cap X_h. \tag{3.11}$$

Consider ρ such that $\rho^{1+\beta} \ge \alpha(h)^\beta$ and define $\Gamma_{\rho,h}^{\ell\rho^{1+\beta}} := (\Gamma_\rho + \ell\rho^{1+\beta}\mathcal{B}) \cap X_h$. Then

$$Viab_F(K) = \lim_{\rho,h\to 0} \overrightarrow{Viab}_{\Gamma_{\rho,h}^{\ell\rho^{1+\beta}}}((K + \rho^{\frac{1+\beta}{\beta}}\mathcal{B}) \cap X_h). \tag{3.12}$$

Proof: From Corollary 2.6, $Viab_F(K) = \lim_{\rho\to 0} \overrightarrow{Viab}_{\Gamma_\rho}(K)$.

The decreasing sequence $\overrightarrow{Viab}_{\Gamma_\rho}(K)$ converges to $Viab_F(K)$ when ρ decreases to zero. Then applying Lemma 3.6 with $D_\rho = \overrightarrow{Viab}_{\Gamma_\rho}(K)$, we obtain (3.11).

To prove the second equality (3.12), we shall use Proposition 3.4 with $G = \Gamma_\rho$. We first notice that condition (3.6) is already satisfied because $\ell\rho^{1+\beta} \ge \alpha(h)$ and thanks to (3.1). Hence, thanks to Proposition 3.4

$$\overrightarrow{Viab}_{\Gamma_\rho}(K) \subset \overrightarrow{Viab}_{\Gamma_{\rho,h}^{\ell\rho^{1+\beta}}}((K + \rho^{\frac{1+\beta}{\beta}}\mathcal{B}) \cap X_h) + \rho^{\frac{1+\beta}{\beta}}B.$$

Consequently, thanks to (3.11), we proved that

$$Viab_F(K) \subset \liminf_{\rho,h\to 0} \overrightarrow{Viab}_{\Gamma_{\rho,h}^{\ell\rho^{1+\beta}}}((K + \rho^{\frac{1+\beta}{\beta}}\mathcal{B}) \cap X_h).$$

Let us prove the opposite inclusion. Since $\Gamma_\rho^{\ell\rho^{1+\beta}} = \Gamma_\rho + \ell\rho^{1+\beta}\mathcal{B} = 1 + \rho F_\rho + 2\ell\rho^{1+\beta}\mathcal{B}$. Observe that

$$Graph(\frac{\Gamma_{\rho h}^{\ell\rho^{1+\beta}} - 1}{\rho}) \subset Graph(F) + 2\ell\rho^{1+\beta}\mathcal{B}.$$

Hence (2.4) is satisfied and thanks to Theorem 2.3, we obtain

$$Viab_F(K) \supset \limsup_{\rho,h\to 0} \overrightarrow{Viab}_{\Gamma_{\rho,h}^{\ell\rho^{1+\beta}}}((K + \rho^{\frac{1+\beta}{\beta}}\mathcal{B}) \cap X_h).$$

This ends the proof. □

This result allows to approximate numerically viability kernels (see examples in [9] and [21]).

3.5 A numerical example

We apply our algorithm to a very simple example of linear control problem in $\mathbb{R}^2$ which dynamic is given by

$$\begin{cases} i) & (x'(t), y'(t)) = (x(t), y(t)) + c(u(t), v(t)) \\ ii) & (u(t), v(t)) \in B(0,1). \end{cases}$$

When $K = [-1,1]^2$, it is easy to see that $Viab_F(K) = cB(0,1)$.

Compute this viability kernel by approximating it by suitable discrete viability kernels (by taking $h_n := \frac{1}{2^n}$). When $n = 8$, we can refer to the enclosed figure.

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