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Realization of Rational Matrices by Singular Systems*

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Abstract

We study the relationship between spaces of singular systems and rational matrices. In a recent paper it is shown that the space of all rational $p \times m$-matrices of fixed McMillan degree r is embedded in a space of rational curves of degree r from the Riemann sphere S^2 to a Grassmannian manifold (see [2]). This space of curves is locally homeomorphic to the space of all proper rational matrices of degree r. In this paper we study the space of square irreducible (not necessarily admissible) singular systems. It is shown that the space of these systems of order r and dimension $r + \min\{m, p\}$ modulo strong equivalence is homeomorphic to the above mentioned space of all rational curves of degree r. The homeomorphism is induced by the transfer matrix.

Key words: singular systems, rational matrix, realization theory, polynomial coprime factorization

AMS Subject Classifications: 93B15, 93B17, 93B20, 93B10, 93B25, 93C35

1 Introduction

From the realization theory for state space systems we know that the space

$$\mathrm{Rat}^0_{m,r,p} = \{G \in \mathbb{K}(s)^{p\times m} \mid d(G) = r,\ G \text{ strictly proper}\} \tag{1.1}$$

of (real or complex) strictly proper rational matrices G with *McMillan degree* $d(G) = r$ is homeomorphic to the quotient space of minimal state

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space systems of order r modulo similarity. Thus, let

$$\tilde{\Sigma}_{m,r,p} = \{(A,B,C) \in \mathbb{K}^{r^2+rm+pr} \mid (A,B,C) \text{ minimal}\} \tag{1.2}$$

be the space of minimal state space systems of order r, where $\mathbb{K}$ denotes the field $\mathbb{R}$ or $\mathbb{C}$. On $\tilde{\Sigma}_{m,r,p}$ acts the well-known similarity action, denoted by $\overset{\mathrm{s}}{\sim}$: $(A,B,C)\overset{\mathrm{s}}{\sim}(A',B',C')$ iff $(A',B',C') = (TAT^{-1}, TB, CT^{-1})$ for some $T \in Gl_n$. Then it is known from Byrnes/Duncan [1] that the map

$$\begin{array}{rccc} T: & \tilde{\Sigma}_{m,r,p} & \longrightarrow & \mathrm{Rat}^0_{m,r,p} \\ & (A,B,C) & \longmapsto & C(sI - A)^{-1}B \end{array}$$

induces a homeomorphism

$$\tilde{\Sigma}_{m,r,p}/_{\underset{\sim}{\mathrm{s}}} =: \Sigma_{m,r,p} \overset{\text{homeo}}{\sim} \mathrm{Rat}^0_{m,r,p}, \tag{1.3}$$

if $\Sigma_{m,r,p}$ is endowed with the quotient topology and $\mathrm{Rat}^0_{m,r,p}$ is topologized as a space of rational maps in a way which is described in [2, Section 3].

The goal of this paper is to generalize this result to arbitrary (i. e. not only proper) rational matrices of fixed degree on the one hand and singular systems as their realizations on the other one.

Thus, one object of our study is the space of rational $p \times m$-matrices of fixed degree r. This space was studied in a preceding paper, see De Mari/Glüsing-Lüerßen [2]. It can be viewed as a subspace of the space

$$\mathcal{I}_{m,r,p} = \{\langle X\rangle \mid X \in \mathbb{K}[s]^{(p+m)\times m},\ \mathrm{rk}X(s) = m \text{ for all } s \in \mathbb{C},\ \delta(X) = r\},$$

where for $X \in \mathbb{K}[s]^{(p+m)\times m}$ it is

$$\begin{aligned} \langle X\rangle &= \{XU \mid U \in \mathbb{K}[s]^{m\times m},\ \det U \equiv c \in \mathbb{K}^*\} \\ \delta(X) &= \max\{\deg \Delta \mid \Delta\ m \times m\text{-minor of } X\}. \end{aligned}$$

Via coprime factorizations, the space of all rational $p\times m$-matrices of degree r can be identified with the subspace

$$\mathrm{Rat}_{m,r,p} = \{\langle \begin{bmatrix} P \\ Q \end{bmatrix} \rangle \in \mathcal{I}_{m,r,p} \mid Q \in \mathbb{K}[s]^{m\times m},\ \det Q \not\equiv 0\}. \tag{1.4}$$

The equivalence classes $\langle X\rangle$ of $\mathcal{I}_{m,r,p}$ can be interpreted as rational maps from the Riemann sphere into a Grassmannian, the so-called Hermann-Martin-maps (cf. Martin/Hermann [7] for the strict proper case). In this way $\mathcal{I}_{m,r,p}$ can be endowed with the compact-open topology. For the details about the space $\mathcal{I}_{m,r,p}$ see [2].

While proper rational matrices can be realized as state space systems, improper matrices do have realizations as square singular systems, i. e. systems of the type

$$\left.\begin{array}{rcl} E\dot{x}(t) & = & Ax(t) + Bu(t) \\ y(t) & = & Cx(t) + Du(t) \end{array}\right\} \tag{1.5}$$

with $(E, A, B, C, D) \in \mathbb{K}^{2n^2+nm+pn+pm}$. Note that singular systems contain differential as well as algebraic equations. Systems of this form occur naturally as realizations for rational matrices: if $\Sigma = (E, A, B, C, D)$ is *admissible*, that is $\det(sE \Leftrightarrow A) \not\equiv 0$, the *transfer matrix* $T(\Sigma) := C(sE \Leftrightarrow A)^{-1}B + D$ is a (non-proper) rational matrix.

From Verghese et al. [9] it is known that each rational matrix G has a realization as an *irreducible* singular system $\Sigma = (E, A, B, C, D)$ which is unique up to *equivalence transformations*; the *order* $\mathrm{rk}E$ is just $d(G)$. The notions *irreducibility* and *equivalence transformations* generalize the concepts of minimality and similarity as they are introduced for state space systems (see Definition 2.1 and Definition 2.2).

A glance at the definition of the underlying equivalence transformations tells us that the *dimension* of the realizing system Σ, i. e. the size of the matrices E, A is not determined by the transfer matrix. Thus the results of [9] are not sufficient to formulate a bijection between the space $\mathrm{Rat}_{m,r,p}$ and a suitable quotient space of singular systems. However, it can easily be seen that each $G \in \mathrm{Rat}_{m,r,p}$ has an irreducible realization (E, A, B, C, D) with $\mathrm{rk}E = r$ and dimension $N = r + \min\{m, p\}$. Thus it follows that the quotient space $\mathcal{A}_{m,r,p}$ of all admissible irreducible systems (1.5) of order $\mathrm{rk}E = r$ and dimension N modulo strong equivalence is bijective to $\mathrm{Rat}_{m,r,p}$, where the bijection comes in an obvious way from the transfer matrix.

Note that, so far, there are some restrictive regularity conditions involved in this approach. On the one hand, we consider rational maps induced by matrices $[P^{\mathrm{t}}, Q^{\mathrm{t}}]^{\mathrm{t}}$ with Q non-singular, on the other hand their counterparts are systems (E, A, B, C, D) with $\det(sE \Leftrightarrow A) \not\equiv 0$. To establish the above mentioned bijection as a homeomorphism between $\mathcal{A}_{m,r,p}$ and $\mathrm{Rat}_{m,r,p}$, it is useful to omit this regularity conditions. Hence we change to the larger space $\mathcal{I}_{m,r,p}$. It is shown in [2] that this space has a nicer topological structure than $\mathrm{Rat}_{m,r,p}$:

If equipped with the topology of uniform convergence, the space $\mathcal{I}_{m,r,p}$ is locally homeomorphic to the space $\mathrm{Rat}^0_{m,r,p} \times \mathbb{K}^{p\times m}$ of all proper rational matrices of degree r (see [2, Theorem 3.5]). The local homeomorphism is given by

$$X \Leftrightarrow\!\rightarrow \sigma X = \begin{bmatrix} \bar{P} \\ \bar{Q} \end{bmatrix},$$

where σ is a permutation, which transports an $m \times m$-submatrix $\bar{Q}$ of X with maximal determinantal degree among all these submatrices to the last m rows. Then $\bar{P}\bar{Q}^{-1}$ is proper and the McMillan degree of the rational matrix $\bar{P}\bar{Q}^{-1}$ is equal to $\delta(X)$ (see [2, Lemma 2.3]). If $\langle X \rangle \in \mathcal{I}_{m,r,p}$ is viewed as an ARMA-system, this reorganization of data can be interpreted on a system theoretical level as an interchanging of inputs and outputs. Thus the elements in $\mathcal{I}_{m,r,p}$ can be viewed as reorganized proper systems. It is possible to associate with $\langle X \rangle \in \mathcal{I}_{m,r,p}$ an irreducible (non-admissible) system (E, A, B, C, D), namely by realizing $\bar{P}\bar{Q}^{-1}$ as a proper state space system and making the corresponding reorganization of data backwards with the system matrices. In the case $\langle X \rangle = \langle [P^{\mathrm{t}}, Q^{\mathrm{t}}]^{\mathrm{t}} \rangle \in \mathrm{Rat}_{m,r,p} \subseteq \mathcal{I}_{m,r,p}$, this leads in fact to the usual realization $PQ^{-1}(s) = C(sE \Leftrightarrow A)^{-1}B + D$.

Having established in this way a bijection between $\mathcal{I}_{m,r,p}$ and the quotient space $\mathcal{L}_{m,r,p}$ of irreducible systems of order r and dimension $N = r + \min\{m, p\}$ modulo strong equivalence as in the regular case, it is not hard, to prove this bijection to be a homeomorphism: one uses the local structure of these spaces and the well-known homeomorphism (1.3) of the state space case. In particular, this shows the map $(E, A, B, C, D) \mapsto C(sE \Leftrightarrow A)^{-1}B + D$ to be a homeomorphism between $\mathcal{A}_{m,r,p}$ and $\mathrm{Rat}_{m,r,p}$. The really non-trivial part to be done here is to prove this map being a bijection between $\mathcal{I}_{m,r,p}$ and $\mathcal{L}_{m,r,p}$.

We proceed as follows:

In the next section we introduce the main concepts for singular systems and establish some fundamental properties. One main point is Theorem 2.4, which shows the coincidence of strong equivalence and operations of strong equivalence (in the sense of [9]) for systems of the same dimension. This result is important for further questions about uniqueness of realizations.

Moreover, we construct a standard-form for singular systems. It is mostly the same as the "internal reduced form" given by Grimm [4] and will be used for the reorganization of data when constructing a map between rational curves and non-admissible systems. It is associated a state space system with the standard-form by interpreting the involved matrices in a different way. Then strong equivalent irreducible standard-forms yield similar minimal state space systems and vice versa.

In Section 3 the quotient space of irreducible systems modulo strong equivalence is studied. It can be shown that in the context of this paper, it is sufficient to consider systems (1.5), where E is of the form

$$\hat{E}_n = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{K}^{n\times n}, \tag{1.6}$$

so that strong equivalence becomes a group action.

In Section 4 a pseudotransfer function is defined. It generalizes the transfer function to non-admissible systems and associates to them a rational curve, i. e. an element of $\mathcal{I}_{m,r,p}$. It is shown in this section that this pseudotransfer function induces a homeomorphism on the level of quotient spaces. The very technical part of proving that in fact this map induces a bijection between the quotient space $\mathcal{L}_{m,r,p}$ and $\mathcal{I}_{m,r,p}$ is deferred to the last section. Some of the very tedious matrix calculations are a bit shortened; they can be found in more detail in [3].

To save space we will make use of the notations and results of [2], in particular note the preliminaries and Definition 3.4 of [2].

2 Preliminaries

In this section we introduce a few general concepts for singular systems. We show that strong equivalence and operations of strong equivalence, as they were introduced by Verghese et al. [9], coincide for systems with the same dimension. Secondly, it is given a standard-form for so-called irreducible systems under strong equivalence. This will be useful later in order to describe the local structure of the space of all singular systems.

Let $\mathcal{M}^n_{m,p} = \mathbb{K}^{2n^2+nm+pn+pm}$ be the set of all quintuples (E, A, B, C, D) describing singular systems of the form (1.5) over a field $\mathbb{K}$, where $\mathbb{K}$ is always $\mathbb{R}$ or $\mathbb{C}$.

Definition 2.1 *Let* $\Sigma = (E, A, B, C, D) \in \mathcal{M}^n_{m,p}$.

a) $\dim\Sigma = n$ *is the dimension,* $\mathrm{ord}\Sigma = \mathrm{rk}E$ *the order of the system.*

b) Σ *is called admissible, if* $\det(sE \Leftrightarrow A) \not\equiv 0$.

c) Σ *is called irreducible, if it holds* $\mathrm{rk}[sE \Leftrightarrow A, B] = n = \mathrm{rk}[sE^{\mathrm{t}} \Leftrightarrow A^{\mathrm{t}}, C^{\mathrm{t}}]^{\mathrm{t}}$ *for all* $s \in \mathbb{C}$ *and* $\mathrm{im}E + A\ker E + \mathrm{im}B = \mathbb{K}^n = \mathrm{im}E^{\mathrm{t}} + A^{\mathrm{t}}\ker E^{\mathrm{t}} + \mathrm{im}C^{\mathrm{t}}$.

d) Σ *is called canonical, if it is irreducible and fulfills* $A\ker E \subseteq \mathrm{im}E$.

Put

$$\mathcal{L}^n_{m,r,p} \quad = \quad \{\Sigma \in \mathcal{M}^n_{m,p} \mid \mathrm{ord}\Sigma = r\,,\ \Sigma \textit{ irreducible}\},$$

$$\begin{aligned} \mathcal{C}^n_{m,r,p} &= \{\Sigma \in \mathcal{M}^n_{m,p} \mid \mathrm{ord}\Sigma = r\,,\ \Sigma\ \textit{canonical}\}, \\ \mathcal{A}^n_{m,r,p} &= \{\Sigma \in \mathcal{L}^n_{m,r,p} \mid \Sigma\ \textit{admissible}\}. \end{aligned}$$

The condition $A\ker E \subseteq \mathrm{im}E$ is responsible for the fact that the system has no non-dynamical behaviour, see Verghese et al. [9, p. 816]. Observe that we call a system irreducible if it is strongly irreducible in the sense of [9]. The notion of canonicity is taken from Grimm [4] as well as the following definition of strong equivalence, which differs slightly from that used by Verghese et al. [9].

Definition 2.2

a) Let $\Sigma = (E, A, B, C, D)$, $\bar{\Sigma} = (\bar{E}, \bar{A}, \bar{B}, \bar{C}, \bar{D}) \in \mathcal{M}^n_{m,p}$. Σ and $\bar{\Sigma}$ are called strongly equivalent ($\stackrel{\mathrm{se}}{\sim}$), if there exist matrices $M, N \in Gl_n$, $Q \in \mathbb{K}^{p\times n}$, $R \in \mathbb{K}^{n\times m}$ such that

$$\begin{bmatrix} M & 0 \\ Q & I_p \end{bmatrix} \begin{bmatrix} sE \Leftrightarrow A & \Leftrightarrow B \\ C & D \end{bmatrix} \begin{bmatrix} N & R \\ 0 & I_m \end{bmatrix} = \begin{bmatrix} s\bar{E} \Leftrightarrow \bar{A} & \Leftrightarrow \bar{B} \\ \bar{C} & \bar{D} \end{bmatrix}. \tag{2.1}$$

b) A trivial l-inflation, $l \in \mathbb{N}$, of a system $\Sigma = (E, A, B, C, D) \in \mathcal{M}^n_{m,p}$ is a system of the form

$$\Sigma' = (\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A & 0 \\ 0 & I_l \end{bmatrix}, \begin{bmatrix} B \\ 0 \end{bmatrix}, [C, 0], D) \in \mathcal{M}^{n+l}_{m,p}\,. \tag{2.2}$$

The reverse process is called trivial l-deflation.

c) Two systems $\Sigma \in \mathcal{M}^n_{m,p}$ and $\Sigma' \in \mathcal{M}^{n'}_{m,p}$ are called equivalent ($\stackrel{\mathrm{e}}{\sim}$), if they can be transformed into each other by a finite sequence of transformations of the types a) and b), i. e. by strong equivalence transformations and trivial inflations/deflations.

Irreducibility, admissibility, and canonicity are preserved by $\stackrel{\mathrm{se}}{\sim}$, the first two also by $\stackrel{\mathrm{e}}{\sim}$. Note that (2.1) implies $QE = 0$ and $ER = 0$; thus strong equivalence generalizes the similarity action known for state space systems.

Remark 2.3 Since $\dim(E, A, B, C, D) \Leftrightarrow \dim\left({}^{A\ker E}\big/_{\mathrm{im}E \cap A\ker E}\right)$ is invariant under equivalence, it follows that equivalent canonical systems are of the same dimension.

The next theorem shows that for equivalent systems of the same dimension trivial inflations/deflations are not needed to transform one of the systems into the other one.

Theorem 2.4 *Let $\Sigma, \bar{\Sigma} \in \mathcal{M}^n_{m,p}$ and $\Sigma \stackrel{\mathrm{e}}{\sim} \bar{\Sigma}$. Then $\Sigma \stackrel{\mathrm{se}}{\sim} \bar{\Sigma}$.*

Proof: We assume that $\bar{\Sigma}$ is achieved from Σ by a sequence of transformations as in (2.1) or in (2.2) and its inverse process, so let

$$\bar{\Sigma} = S_1 \circ D_{l_1} \circ S_2 \circ D_{l_2} \circ \ldots \circ D_{l_t} \circ S_{t+1}(\Sigma), \tag{2.3}$$

where S_j are transformations of strong equivalence and D_{l_j} are trivial l_j-inflations (if $l_j > 0$) or l_j-deflations ($l_j < 0$). It holds: $\sum_{j=1}^{t} l_j = 0$ (Note that the l-deflations are only well-defined for systems in block-form as on the right hand side of (2.2); the notation in (2.3) requires the well-definedness of the transformations).

We prove the strong equivalence of Σ and $\bar{\Sigma}$ by induction on t.

For $t = 1$ it has to be $l_1 = 0$, hence there is nothing to prove.

Let $t = 2$; then (2.3) can be rewritten as $(D_{l_1})^{-1} \circ S_1^{-1}(\bar{\Sigma}) = S_2 \circ D_{l_2} \circ S_3(\Sigma)$, which means $D_{l_2} \circ S_1^{-1}(\bar{\Sigma}) \overset{\text{se}}{\sim} D_{l_2} \circ S_3(\Sigma)$.

Let $S_1^{-1}(\bar{\Sigma}) = (\bar{E}, \bar{A}, \bar{B}, \bar{C}, \bar{D})$, $S_3(\Sigma) = (E, A, B, C, D)$. If $l_2 < 0$, then it follows obviously $S_1^{-1}(\bar{\Sigma}) \overset{\text{se}}{\sim} S_3(\Sigma)$ and thus $\bar{\Sigma} \overset{\text{se}}{\sim} \Sigma$. If $l_2 = l > 0$, then there exist matrices $M, N \in Gl_{n+l}$, $Q \in \mathbb{K}^{p \times (n+l)}$, $R \in \mathbb{K}^{(n+l) \times m}$ such that

$$\begin{bmatrix} s\bar{E} - \bar{A} & 0 & -\bar{B} \\ 0 & -I_l & 0 \\ \bar{C} & 0 & \bar{D} \end{bmatrix} = \begin{bmatrix} M & 0 \\ Q & I_p \end{bmatrix} \begin{bmatrix} sE - A & 0 & -B \\ 0 & -I_l & 0 \\ C & 0 & D \end{bmatrix} \begin{bmatrix} N & R \\ 0 & I_m \end{bmatrix}. \tag{2.4}$$

It follows $\mathrm{rk} E = \mathrm{rk} \bar{E} =: r$ and

$$d = \dim \left({}^{A \ker E} \big/ {}_{\mathrm{im} E \,\cap\, A \ker E} \right) = \dim \left({}^{\bar{A} \ker \bar{E}} \big/ {}_{\mathrm{im} \bar{E} \,\cap\, \bar{A} \ker \bar{E}} \right),$$

since these numbers are preserved under strong equivalence.

Thus strong equivalence transformations on both systems in (2.4) leads to an equation of the type

$$\begin{bmatrix} s\hat{E}_{n-d} - \bar{F} & 0 & \bar{G} \\ 0 & I_k & 0 \\ \bar{H} & 0 & \bar{J} \end{bmatrix} = \begin{bmatrix} \tilde{M} & 0 \\ \tilde{Q} & I_p \end{bmatrix} \begin{bmatrix} s\hat{E}_{n-d} - F & 0 & G \\ 0 & I_k & 0 \\ H & 0 & J \end{bmatrix} \begin{bmatrix} \tilde{N} & \tilde{R} \\ 0 & I_m \end{bmatrix}, \tag{2.5}$$

with $k = d + l$ and $F = \begin{bmatrix} F_1 & F_2 \\ F_3 & 0 \end{bmatrix}$, $\bar{F} = \begin{bmatrix} \bar{F}_1 & \bar{F}_2 \\ \bar{F}_3 & 0 \end{bmatrix}$ and $\hat{E}_{n-d}$ as in (1.6). From this it follows by tedious but straightforward matrix manipulations

$$(\hat{E}_{n-d}, F, G, H, J) \overset{\text{se}}{\sim} (\hat{E}_{n-d}, \bar{F}, \bar{G}, \bar{H}, \bar{J})$$

(for the detailed computation see [3, pp. 107]). The case $t > 2$ can be handled by using the fact that for $k > 0$ a term of the form $D_k \circ S_j(\Sigma)$ resp.

$S_j \circ D_{-k}(\Sigma)$ can be written as $\hat{S}_j \circ D_k(\Sigma)$ resp. $D_{-k} \circ \tilde{S}_j(\Sigma)$ with suitable transformations of strong equivalence $\hat{S}_j$ resp. $\tilde{S}_j$. Hence the sequence in (2.3) can be reduced from length t to the case $t-1$. □

Remark 2.5 For admissible systems Pugh et al. [8] describe the equivalence of systems in a closed form: two admissible systems (E, A, B, C, D) and $(\bar{E}, \bar{A}, \bar{B}, \bar{C}, \bar{D})$ (of not necessarily the same dimension) are equivalent iff they fulfill equation (2.1) with suitable (non-square) matrices M, N, Q and R, which satisfy some coprimness conditions with respect to the given systems. Using this characterization Pugh et al. proved Theorem 2.4 for admissible systems, cf. [8, Theorem 6, Theorem 7].

In the following we give a standard-form for canonical systems under strong equivalence. The idea is taken from Grimm [4], who introduced a so-called "internal reduced form," which is only slightly different from the following standard-form (cf. [4, Definition 5a]). As in [2], let $\mathcal{P}(n)$ denote the set of $n \times n$-permutation matrices.

Definition 2.6 *Let $\varrho \in \mathcal{P}(m)$, $\tau \in \mathcal{P}(p)$. A system $\Sigma \in \mathcal{M}^n_{m,p}$ is called in (ϱ, τ)-standard-form, if $\Sigma \in \mathcal{M}^n_{m,p}$ is of the structure*

$$\Sigma = (\hat{E}_n, \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \begin{bmatrix} 0 & B_2 \\ I_{n-r} & B_4 \end{bmatrix}\varrho, \tau\begin{bmatrix} C_1 & C_2 \\ 0 & I_{n-r} \end{bmatrix}, \tau\begin{bmatrix} 0 & D_2 \\ 0 & 0 \end{bmatrix}\varrho) \tag{2.6}$$

with $\hat{E}_n$ as in (1.6), $A_4 \in \mathbb{K}^{(n-r)\times(n-r)}$, $D_2 \in \mathbb{K}^{(p-n+r)\times(m-n+r)}$ and the remaining matrices in fitting sizes.

The state space system

$$\hat{\Sigma} = (A_1, [A_2, B_2], \begin{bmatrix} C_1 \\ -A_3 \end{bmatrix}, \begin{bmatrix} C_2 & D_2 \\ -A_4 & -B_4 \end{bmatrix}) \in \mathbb{K}^{r^2+rm+pr+pm}$$

is called the associated state space system.

Note that $\Sigma \in \mathcal{M}^n_{m,p}$ can be in (ϱ, τ)-standard-form only if $n \leq \operatorname{ord}\Sigma + \min\{m, p\}$, which is always the case for canonical systems. It holds

Proposition 2.7 *a) Each system $\Sigma \in \mathcal{C}^n_{m,r,p}$ is strongly equivalent to a system as in (2.6) with $A_4 = 0$ and with suitable permutation matrices $\varrho \in \mathcal{P}(m)$, $\tau \in \mathcal{P}(p)$.*

b) Each system $\Sigma \in \mathcal{L}^n_{m,r,p}$ is strongly equivalent to a system

$$\tilde{\Sigma} = (\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A & 0 \\ 0 & I_k \end{bmatrix}, \begin{bmatrix} B \\ 0 \end{bmatrix}, [C, 0], D) \in \mathcal{L}^n_{m,r,p}$$

with $(E, A, B, C, D) \in \mathcal{C}^{n-k}_{m,r,p}$ *for some* $0 \leq k \leq n$.

The proofs can be done by simple matrix manipulations with the prescribed operations.

Note that for canonical systems the standard-form is more special than in Definition 2.6, namely in this case it is $A_4 = 0$. Since in the case $A_4 \neq 0$ the associated state space system is of the most general form, it will be useful to study these more general standard-forms also. Via the reorganization of data, the associated state space systems will describe the local structure of the irreducible singular systems, as we will see later. One necessary property for this ensures the next proposition: irreducibility (resp. strong equivalence) of the standard-forms is translated to minimality (resp. similarity) of the associated state space systems and vice versa.

Proposition 2.8 *Let* $\varrho \in \mathcal{P}(m)$, $\tau \in \mathcal{P}(p)$ *and* $\Sigma^i \in \mathcal{M}^n_{m,p}$ *be of the form*

$$\Sigma^i = (\hat{E}_n, \begin{bmatrix} A^i_1 & A^i_2 \\ A^i_3 & A^i_4 \end{bmatrix}, \begin{bmatrix} 0 & B^i_2 \\ I_{n-r} & B^i_4 \end{bmatrix} \varrho,\ \tau \begin{bmatrix} C^i_1 & C^i_2 \\ 0 & I_{n-r} \end{bmatrix}, \tau \begin{bmatrix} 0 & D^i_2 \\ 0 & 0 \end{bmatrix} \varrho)$$

for $i = 1, 2$. *Then for the associated state space systems* $\hat{\Sigma}^i$ *it holds:*
a) $\Sigma^1 \in \mathcal{L}^n_{m,r,p} \Longleftrightarrow \hat{\Sigma}^1 \in \tilde{\Sigma}_{m,r,p} \times \mathbb{K}^{p\times m}$.
b) $\Sigma^1 \overset{\mathrm{se}}{\sim} \Sigma^2 \Longleftrightarrow \hat{\Sigma}^1 \overset{\mathrm{s}}{\sim} \hat{\Sigma}^2$.

Proof:

a) For $\Sigma^1 = (\hat{E}_n, A^1, B^1, C^1, D^1)$ it holds $\mathrm{im}\hat{E}_n + A^1 \ker \hat{E}_n + \mathrm{im} B^1 = \mathbb{K}^n$. Further for all $s \in \mathbb{C}$ it is $\mathrm{rk}\begin{bmatrix} sI_r \Leftrightarrow A^1_1 & \Leftrightarrow A^1_2 & 0 & B^1_2 \\ \Leftrightarrow A^1_3 & \Leftrightarrow A^1_4 & I_{n-r} & B^1_4 \end{bmatrix} = n$ iff $\mathrm{rk}[sI_r \Leftrightarrow A^1_1, [\Leftrightarrow A^1_2, B^1_2]] = r$. The analogous conditions hold for the observability.

b) "$\Rightarrow$" First one observes that we can restrict ourselves to the case $\varrho = I_m$, $\tau = I_p$. For abbreviation let $\Sigma^i = (\hat{E}_n, A^i, B^i, C^i, D^i)$ for $i = 1, 2$ with the matrices as given in the proposition. Then from

$$\begin{bmatrix} M & 0 \\ Q & I_p \end{bmatrix} \begin{bmatrix} s\hat{E}_n \Leftrightarrow A^1 & \Leftrightarrow B^1 \\ C^1 & D^1 \end{bmatrix} \begin{bmatrix} N & R \\ 0 & I_m \end{bmatrix} = \begin{bmatrix} s\hat{E}_n \Leftrightarrow A^2 & \Leftrightarrow B^2 \\ C^2 & D^2 \end{bmatrix},$$

it follows $M = \begin{bmatrix} M_1 & M_2 \\ 0 & M_4 \end{bmatrix}$ with $M_1 \in Gl_r$ and $M_4 \in Gl_{n-r}$.
This yields after some computations

$$(M_1 A^1_1 M_1^{-1}, M_1[A^1_2, B^1_2], \begin{bmatrix} C^1_1 \\ \Leftrightarrow A^1_3 \end{bmatrix} M_1^{-1} \begin{bmatrix} C^1_2 & D^1_2 \\ \Leftrightarrow A^1_4 & \Leftrightarrow B^1_4 \end{bmatrix}) =$$

$$(A_1^2, [A_2^2, B_2^2], \begin{bmatrix} C_1^2 \\ \Leftrightarrow A_3^2 \end{bmatrix}, \begin{bmatrix} C_2^2 & D_2^2 \\ \Leftrightarrow A_4^2 & \Leftrightarrow B_4^2 \end{bmatrix}).$$

The implication "$\Leftarrow$" is trivial. $\square$

Remark 2.9 Consider the system Σ given by (2.6). The corresponding equations are

$$\left.\begin{array}{rcl} \dot{x}_1 & = & A_1x_1 + A_2x_2 + B_2u_2 \\ 0 & = & A_3x_1 + A_4x_2 + u_1 + B_4u_2 \\ y_1 & = & C_1x_1 + C_2x_2 + D_2u_2 \\ y_2 & = & x_2 \end{array}\right\} \tag{2.7}$$

with respect to suitable partitions of input, state and output. This formulation shows that some of the inputs (those described by u_1) are not really inputs, but satisfy $u_1 = \Leftrightarrow(A_3x_1 + A_4x_2 + B_4u_2)$, whereas the part y_2 of the output is free. Neglecting initial conditions, one can formulate (2.7) as

$$\begin{array}{rcl} \dot{x}_1 & = & A_1x_1 + [A_2, B_2]\begin{pmatrix} y_2 \\ u_2 \end{pmatrix} \\ \begin{pmatrix} y_1 \\ u_1 \end{pmatrix} & = & \begin{bmatrix} C_1 \\ \Leftrightarrow A_3 \end{bmatrix} x_1 + \begin{bmatrix} C_2 & D_2 \\ \Leftrightarrow A_4 & \Leftrightarrow B_4 \end{bmatrix}\begin{pmatrix} y_2 \\ u_2 \end{pmatrix}, \end{array}$$

which is just the associated state space system. Hence standard-form and associated state space system are obtained from each other by interchanging some of the inputs and outputs with each other.

This interpretation was studied in detail for ARMA-systems by Willems [11]. He defines systems via their external behaviour and divides (if possible) the external variables in inputs and outputs according to their properties with respect to the dynamical system. Kuijper/Schumacher [5, 6] followed this approach especially for singular systems. In the sense of [11] the free external variables of (2.7) are u_2 and y_2, whereas y_1 and u_1 process and do not anticipate y_2 and u_2. Further x_1 satisfies the "axiom of state." Thus, within this terminology, (2.7) is in fact a state space system (for the notions see [11, p. 216, p. 186]).

3 A Quotient Space of Singular Systems

This section is devoted to the quotient space modulo strong equivalence of all irreducible systems with fixed order r and dimension n. Analogously to the regular case, where $E \in Gl_n$ can be transformed to $E = I_n$, we will consider only those systems $(E, A, B, C, D) \in \mathcal{L}^n_{m,r,p}$, where E is of the form $\hat{E}_n$ as in (1.6). Strong equivalence reduced to this subspace of systems then becomes a group action. We show that the quotient space related to this action is homeomorphic to $\mathcal{L}^n_{m,r,p}/\underset{\sim}{\text{se}}$.

It is obvious that each system $\Sigma \in \mathcal{L}^n_{m,r,p}$ is strongly equivalent to a system $\hat{\Sigma} = (\hat{E}_n, \hat{A}, \hat{B}, \hat{C}, \hat{D})$. Therefore we introduce the space

$$\hat{\mathcal{L}}^n_{m,r,p} = \{(A, B, C, D) \in \mathbb{K}^M \mid (\hat{E}_n, A, B, C, D) \in \mathcal{L}^n_{m,r,p}\}$$

with $M := n^2 + nm + pn + pm$. There is a canonical embedding

$$\begin{array}{rrcl} \nu: & \hat{\mathcal{L}}^n_{m,r,p} & \longrightarrow & \mathcal{L}^n_{m,r,p} \\ & (A, B, C, D) & \longmapsto & (\hat{E}_n, A, B, C, D), \end{array} \tag{3.1}$$

if both spaces are equipped with the Euclidean topology. Having in mind the identification $\nu(A, B, C, D) = (\hat{E}_n, A, B, C, D)$, we also call (A, B, C, D) a system. Define also

$$\begin{array}{rcl} \hat{\mathcal{A}}^n_{m,r,p} & = & \{(A, B, C, D) \in \hat{\mathcal{L}}^n_{m,r,p} \mid \det(s\hat{E}_n - A) \not\equiv 0\}, \\ \hat{\mathcal{C}}^n_{m,r,p} & = & \{(A, B, C, D) \in \hat{\mathcal{L}}^n_{m,r,p} \mid (\hat{E}_n, A, B, C, D) \in \mathcal{C}^n_{m,r,p}\}, \end{array}$$

endowed with the topologies induced by $\hat{\mathcal{L}}^n_{m,r,p}$. Then $\hat{\mathcal{A}}^n_{m,r,p}$ is an open subset of $\hat{\mathcal{L}}^n_{m,r,p}$. The advantage of the space $\hat{\mathcal{L}}^n_{m,r,p}$ is that strong equivalence, if restricted to this space, becomes a group action: let $\mathcal{H}^n_{m,r,p} = \mathcal{H}$ be the subgroup of $Gl_{n+p} \times Gl_{n+m}$ consisting of elements of the form

$$\left(\begin{bmatrix} M_1 & M_2 & 0 \\ 0 & M_4 & 0 \\ 0 & Q_2 & I_p \end{bmatrix}, \begin{bmatrix} M_1^{-1} & 0 & 0 \\ N_3 & N_4 & R_2 \\ 0 & 0 & I_m \end{bmatrix}^{-1} \right),$$

where $M_1 \in Gl_r$, M_4, $N_4 \in Gl_{n-r}$ and M_2, N_3, Q_2, R_2 are arbitrary matrices of appropriate size. Denote by α be the algebraic action

$$\left.\begin{array}{rrcl} \alpha: & \mathcal{H} \times \hat{\mathcal{L}}^n_{m,r,p} & \longrightarrow & \hat{\mathcal{L}}^n_{m,r,p} \\ & (h, (A, B, C, D)) & \longmapsto & (MAN, M(B + AR), (C - QA)N, \\ & & & D + CR - QB - QAR) \end{array}\right\} \tag{3.2}$$

where $h = (\begin{bmatrix} M & 0 \\ Q & I_p \end{bmatrix}, \begin{bmatrix} N & R \\ 0 & I_m \end{bmatrix}^{-1}) \in \mathcal{H}$ with M, N, Q, R as in the above form. Then the equivalence relation

$$\begin{array}{l} (A, B, C, D) \overset{s}{\sim} (\bar{A}, \bar{B}, \bar{C}, \bar{D}) :\Longleftrightarrow \\ \qquad\qquad\qquad \exists\, h \in \mathcal{H} : \ \alpha(h, (A, B, C, D)) = (\bar{A}, \bar{B}, \bar{C}, \bar{D}) \end{array}$$

is just the restriction of the strong equivalence to systems in $\hat{\mathcal{L}}^n_{m,r,p}$: it is $(A,B,C,D)\overset{\mathrm{s}}{\sim}(\bar{A},\bar{B},\bar{C},\bar{D})$ iff $(\hat{E}_n,A,B,C,D)\overset{\mathrm{se}}{\sim}(\hat{E}_n,\bar{A},\bar{B},\bar{C},\bar{D})$. Note also that $\overset{\mathrm{s}}{\sim}$ generalizes the similarity action from $\tilde{\Sigma}_{m,r,p}\times\mathbb{K}^{p\times m}=\hat{\mathcal{L}}^r_{m,r,p}=\hat{\mathcal{C}}^r_{m,r,p}=\hat{\mathcal{A}}^r_{m,r,p}$ to $\hat{\mathcal{L}}^n_{m,r,p}$ for $n>r$. The restriction to $\hat{\mathcal{L}}^n_{m,r,p}$ fits also topologically as it holds:

Proposition 3.1

a) It is $\hat{\mathcal{L}}^n_{m,r,p}/\underset{\sim}{\mathrm{s}}\overset{\mathrm{homeo}}{\sim}\mathcal{L}^n_{m,r,p}/\underset{\sim}{\mathrm{se}}$ and $\hat{\mathcal{A}}^n_{m,r,p}/\underset{\sim}{\mathrm{s}}\overset{\mathrm{homeo}}{\sim}\mathcal{A}^n_{m,r,p}/\underset{\sim}{\mathrm{se}}$ if all spaces are endowed with the quotient topologies.

b) $\hat{\mathcal{A}}^n_{m,r,p}/\underset{\sim}{\mathrm{s}}$ is open embedded in $\hat{\mathcal{L}}^n_{m,r,p}/\underset{\sim}{\mathrm{s}}$ and $\hat{\mathcal{C}}^r_{m,r,p}/\underset{\sim}{\mathrm{s}}=(\tilde{\Sigma}_{m,r,p}\times\mathbb{K}^{p\times m})/\underset{\sim}{\mathrm{s}}$ is open embedded in $\hat{\mathcal{A}}^n_{m,r,p}/\underset{\sim}{\mathrm{s}}$ for all $n\geq r$.

Proof:

a) Consider the well-defined bijection

$$\begin{array}{rrcl} f: & \hat{\mathcal{L}}^n_{m,r,p}/\underset{\sim}{\mathrm{s}} & \longrightarrow & \mathcal{L}^n_{m,r,p}/\underset{\sim}{\mathrm{se}} \\ & [(A,B,C,D)]_1 & \longmapsto & [(\hat{E}_n,A,B,C,D)]_2 \end{array}$$

with $[\cdot]_i$ as the corresponding equivalence classes. The continuity of the canonical projections $\Pi:\mathcal{L}^n_{m,r,p}\rightarrow\mathcal{L}^n_{m,r,p}/\underset{\sim}{\mathrm{se}}$ and $\hat{\Pi}:\hat{\mathcal{L}}^n_{m,r,p}\rightarrow\hat{\mathcal{L}}^n_{m,r,p}/\underset{\sim}{\mathrm{s}}$ and the openness of the last one imply the continuity of f. Hence it remains to show that f is open.

If $U\subseteq\hat{\mathcal{L}}^n_{m,r,p}/\underset{\sim}{\mathrm{s}}$ is open, then also $\hat{V}=\hat{\Pi}^{-1}(U)\subseteq\hat{\mathcal{L}}^n_{m,r,p}$. Put $V=\{\Sigma\in\mathcal{L}^n_{m,r,p}\mid\exists(A,B,C,D)\in\hat{V}:\Sigma\overset{\mathrm{se}}{\sim}\nu(A,B,C,D)\}$ with ν as in (3.1). Then it is $V=\Pi^{-1}(f(U))$, thus we have to show that $V\subseteq\mathcal{L}^n_{m,r,p}$ is open. For this let $\Sigma_0=(E_0,A_0,B_0,C_0,D_0)\in V$. We will construct an open neighborhood of Σ_0 in V. Without loss of generality we can assume that E_0 is of the form $\begin{bmatrix}E_1' & E_2'\\ E_3' & E_4'\end{bmatrix}$ with $\mathrm{rk}E_1'=r=\mathrm{rk}E_0$.

Put

$$W=\{\Sigma=(\begin{bmatrix}E_1 & E_2\\ E_3 & E_4\end{bmatrix},A,B,C,D)\in\mathcal{L}^n_{m,r,p}\mid\mathrm{rk}E_1=r\}$$

and for $\Sigma\in W$ let

$$M(\Sigma)=\begin{bmatrix}E_1^{-1} & 0\\ -E_3E_1^{-1} & I\end{bmatrix},\ N(\Sigma)=\begin{bmatrix}I & -E_1^{-1}E_2\\ 0 & I\end{bmatrix}\in Gl_n.$$

Consider the map

$$\begin{array}{rrcl} \varphi: & W & \longrightarrow & \hat{\mathcal{L}}^n_{m,r,p} \\ & \Sigma=(E,A,B,C,D) & \longmapsto & (M(\Sigma)AN(\Sigma),M(\Sigma)B,CN(\Sigma),D). \end{array}$$

Then it is $\Sigma_0 \in \varphi^{-1}(\hat{V}) \subseteq W \cap V$. Thus the continuity of φ and the openness of W in $\mathcal{L}^n_{m,r,p}$ imply that $\varphi^{-1}(\hat{V})$ is an open neighborhood of Σ_0 in V.
The same proof also works for $\hat{\mathcal{A}}^n_{m,r,p}/\underset{\sim}{\mathrm{s}}$.

b) The open embedding of $\hat{\mathcal{A}}^n_{m,r,p}/\underset{\sim}{\mathrm{s}}$ in $\hat{\mathcal{L}}^n_{m,r,p}/\underset{\sim}{\mathrm{s}}$ is obvious.
For the last statement of the proposition let β be the embedding

$$\begin{array}{rrcl} \beta: & \hat{\mathcal{C}}^r_{m,r,p} & \longrightarrow & \hat{\mathcal{A}}^n_{m,r,p} \\ & (A,B,C,D) & \longmapsto & \left(\begin{bmatrix} A & 0 \\ 0 & I_{n-r} \end{bmatrix}, \begin{bmatrix} B \\ 0 \end{bmatrix}, [C,0], D\right) \end{array}$$

and let $\Pi : \hat{\mathcal{A}}^n_{m,r,p} \to \hat{\mathcal{A}}^n_{m,r,p}/\underset{\sim}{\mathrm{s}}$ and $\Pi' : \hat{\mathcal{C}}^r_{m,r,p} \to \hat{\mathcal{C}}^r_{m,r,p}/\underset{\sim}{\mathrm{s}}$ be the canonical projections. Then the map

$$\begin{array}{rrcl} \Psi: & \hat{\mathcal{C}}^r_{m,r,p}/\underset{\sim}{\mathrm{s}} & \longrightarrow & \hat{\mathcal{A}}^n_{m,r,p}/\underset{\sim}{\mathrm{s}} \\ & \Pi'(A,B,C,D) & \longmapsto & \Pi(\beta(A,B,C,D)) \end{array}$$

is well-defined, injective, and continuous. For an open set $U \subseteq \hat{\mathcal{C}}^r_{m,r,p}/\underset{\sim}{\mathrm{s}}$ it is $\Pi'^{-1}(U)$ open in $\hat{\mathcal{C}}^r_{m,r,p}$ and thus $\beta(\Pi'^{-1}(U)) = V$ open in $\beta(\hat{\mathcal{C}}^r_{m,r,p})$, endowed with the subset topology of $\hat{\mathcal{A}}^n_{m,r,p}$. With the open set

$$\mathcal{M} = \{(\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, B, C, D) \in \hat{\mathcal{A}}^n_{m,r,p} \mid A_4 \in Gl_{n-r}\}$$

it is $\Pi(\beta(\hat{\mathcal{C}}^r_{m,r,p})) = \Pi(\mathcal{M})$ open in $\hat{\mathcal{A}}^n_{m,r,p}/\underset{\sim}{\mathrm{s}}$ and $W = \{\Sigma \in \mathcal{M} \mid \exists \hat{\Sigma} \in V : \Sigma \overset{\mathrm{s}}{\sim} \hat{\Sigma}\}$ open in $\mathcal{M}$, which can be proven similar to the openness of the set V in a). It follows $\Psi(U) = \Pi(V) = \Pi(W)$ is open in $\Pi(\beta(\hat{\mathcal{C}}^r_{m,r,p}))$ and hence in $\hat{\mathcal{A}}^n_{m,r,p}/\underset{\sim}{\mathrm{s}}$.

□

At the end of this section we give a quotient space $\hat{\mathcal{A}}^n_{m,r,p}/\underset{\sim}{\mathrm{s}}$ bijective to $\mathrm{Rat}_{m,r,p}$, where the bijection is induced by the transfer matrix. In order to establish this bijection as a homeomorphism, we will extend it in the next section to a map between a quotient space $\hat{\mathcal{L}}^n_{m,r,p}/\underset{\sim}{\mathrm{s}}$ and $\mathcal{I}_{m,r,p}$ and use the local structure of the latter space.

From Verghese et al. [9], Theorem 2.4, and (1.4) it follows that the transfer function

$$\left.\begin{array}{rrcl} T: & \mathcal{A}^n_{m,r,p} & \longrightarrow & \mathrm{Rat}_{m,r,p} \\ & (E,A,B,C,D) & \longmapsto & C(sE - A)^{-1}B + D \end{array}\right\} \qquad (3.3)$$

is well-defined and satisfies $T(\Sigma) = T(\Sigma')$ iff $\Sigma \overset{\text{se}}{\sim} \Sigma'$. Observe that this holds for every $n \geq r$. The map T is also surjective if n is chosen to be $n = r + \min\{m,p\}$, as it can be seen from the following facts:

- to each $G \in \mathbb{K}(s)^{p\times m}$ there exists an irreducible realization as a singular system of order r, i. e. there is an system $(E, A, B, C, D) \in \mathcal{A}^n_{m,r,p}$ for some n, such that $G(s) = C(sE \Leftrightarrow A)^{-1}B + D$ (cf. [10, p. 242]).
- By Proposition 2.7 b) every system $\Sigma \in \mathcal{A}^{\tilde{n}}_{m,r,p}$ with arbitrary $\tilde{n} \geq r$ is equivalent to a system in some $\mathcal{C}^{n'}_{m,r,p}$. Then it is $n' \leq r + \min\{m,p\}$ since for a canonical system Σ it is $\dim\Sigma \leq \mathrm{ord}\Sigma + \min\{m,p\}$.
- Each system $\Sigma \in \mathcal{A}^{\tilde{n}}_{m,r,p}$ for some $\tilde{n} < r + \min\{m,p\}$ can transformed via trivial inflation to a system $\Sigma' \in \mathcal{A}^n_{m,r,p}$ with $n = r + \min\{m,p\}$.

The invariance of T under equivalence transformations (see [9, Theorem 5.1]) then implies the surjectivity of T in the case $n = r + \min\{m,p\}$. Thus, in this case, the above and Theorem 2.4 yield a bijective correspondence

$$\hat{\mathcal{A}}^n_{m,r,p}/\underset{\sim}{\text{s}} \overset{1-1}{\longleftrightarrow} \mathrm{Rat}_{m,r,p}. \tag{3.4}$$

4 A Homeomorphism between the Spaces $\hat{\mathcal{L}}_{m,r,p}/\underset{\sim}{\text{s}}$ and $\mathcal{I}_{m,r,p}$

For the rest of the paper fix $N = r + \min\{m,p\}$ as the dimension of the systems and put $\hat{\mathcal{L}}_{m,r,p} = \hat{\mathcal{L}}^N_{m,r,p}$, $\hat{\mathcal{A}}_{m,r,p} = \hat{\mathcal{A}}^N_{m,r,p}$.

In this section we will show that the spaces $\hat{\mathcal{L}}_{m,r,p}/\underset{\sim}{\text{s}}$ and $\mathcal{I}_{m,r,p}$ are homeomorphic, where the homeomorphism is induced by the transfer matrix for admissible systems. For this we extend the map T in (3.3) from the admissible case to the non-admissible one. This can be done by using the standard-forms and their associated state space systems introduced in Definition 2.6 and Proposition 2.7; the reorganization of data in the systems is then transported to a reordering of rows in the transfer matrix in $\mathcal{I}_{m,r,p}$. The idea is demonstrated for the admissible case and then taken as definition for the general case. One has to show a lot of well-definedness for the so-called pseudotransfer matrix, which is deferred to Section 5.

For the following notations, in particular the matrices V_k and σ_J, but also the topological structure of $\mathcal{I}_{m,r,p}$, see [2, Section 3].

Lemma 4.1 *Let* $\Sigma = (A, B, C, D) =$

$$\left(\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \begin{bmatrix} 0 & B_2 \\ I_{n-r} & B_4 \end{bmatrix}\varrho,\ \tau\begin{bmatrix} C_1 & C_2 \\ 0 & I_{n-r} \end{bmatrix},\ \tau\begin{bmatrix} 0 & D_2 \\ 0 & 0 \end{bmatrix}\varrho\right) \in \hat{\mathcal{L}}^n_{m,r,p} \tag{4.1}$$

and put $G(s) =$

$$\begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix}(s) = \begin{bmatrix} C_1 \\ \Leftrightarrow A_3 \end{bmatrix}(sI_r \Leftrightarrow A_1)^{-1}[A_2, B_2] + \begin{bmatrix} C_2 & D_2 \\ \Leftrightarrow A_4 & \Leftrightarrow B_4 \end{bmatrix}. \tag{4.2}$$

Then Σ *is admissible iff* $\det G_3 \not\equiv 0$*. If, in this case,* $PQ^{-1}(s) = C(s\hat{E}_n \Leftrightarrow A)^{-1}B + D$ *and* $\hat{P}\hat{Q}^{-1} = G$ *are polynomial coprime factorizations, then it holds*

$$\langle \begin{bmatrix} \hat{P} \\ \hat{Q} \end{bmatrix} \rangle = \langle V_{n-r} \begin{bmatrix} \tau^{-1} & 0 \\ 0 & \varrho \end{bmatrix} \begin{bmatrix} P \\ Q \end{bmatrix} \rangle \in \mathcal{I}_{m,r,p}.$$

Proof: The first statement follows from

$$s\hat{E}_n \Leftrightarrow A = \begin{bmatrix} sI \Leftrightarrow A_1 & 0 \\ \Leftrightarrow A_3 & \Leftrightarrow A_4 \Leftrightarrow A_3(sI \Leftrightarrow A_1)^{-1}A_2 \end{bmatrix} \begin{bmatrix} I & \Leftrightarrow (sI \Leftrightarrow A_1)^{-1}A_2 \\ 0 & I \end{bmatrix}.$$

If $\det G_3 \not\equiv 0$, then $(s\hat{E}_n \Leftrightarrow A)^{-1} =$

$$\begin{bmatrix} (sI \Leftrightarrow A_1)^{-1}[I + A_2 G_3^{-1} A_3 (sI \Leftrightarrow A_1)^{-1}] & (sI \Leftrightarrow A_1)^{-1} A_2 G_3^{-1} \\ G_3^{-1} A_3 (sI \Leftrightarrow A_1)^{-1} & G_3^{-1} \end{bmatrix},$$

thus

$$\tau^{-1} P Q^{-1}(s) \varrho^{-1} = \begin{bmatrix} C_1 & C_2 \\ 0 & I \end{bmatrix} (s\hat{E}_n \Leftrightarrow A)^{-1} \begin{bmatrix} 0 & B_2 \\ I & B_4 \end{bmatrix} + \begin{bmatrix} 0 & D_2 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} G_1 G_3^{-1} & G_2 \Leftrightarrow G_1 G_3^{-1} G_4 \\ G_3^{-1} & \Leftrightarrow G_3^{-1} G_4 \end{bmatrix}.$$

Partition $\hat{P} = [\hat{P}_1^{\mathrm{t}}, \hat{P}_2^{\mathrm{t}}]^{\mathrm{t}}$, $\hat{Q} = [\hat{Q}_1^{\mathrm{t}}, \hat{Q}_2^{\mathrm{t}}]^{\mathrm{t}}$ with $\hat{P}_2, \hat{Q}_1 \in \mathbb{K}[s]^{(n-r)\times m}$. Then it holds

$$\begin{bmatrix} \hat{P}_1 \\ \hat{Q}_1 \end{bmatrix} \begin{bmatrix} \hat{Q}_1 \\ \hat{Q}_2 \end{bmatrix}^{-1} \left(\begin{bmatrix} \hat{P}_2 \\ \hat{Q}_2 \end{bmatrix} \begin{bmatrix} \hat{Q}_1 \\ \hat{Q}_2 \end{bmatrix}^{-1} \right)^{-1}$$

$$= \begin{bmatrix} G_1 & G_2 \\ I & 0 \end{bmatrix} \begin{bmatrix} G_3 & G_4 \\ 0 & I \end{bmatrix}^{-1} = \begin{bmatrix} G_1 G_3^{-1} & G_2 \Leftrightarrow G_1 G_3^{-1} G_4 \\ G_3^{-1} & \Leftrightarrow G_3^{-1} G_4 \end{bmatrix}.$$

Hence the result follows. □

The lemma says just what one would expect if one looks at $\langle [P^{\mathrm{t}}, Q^{\mathrm{t}}]^{\mathrm{t}} \rangle$ as the dual of an ARMA-system: interchanging of inputs and outputs of the singular system associated with PQ^{-1} is related to an interchanging of corresponding rows in $\langle [P^{\mathrm{t}}, Q^{\mathrm{t}}]^{\mathrm{t}} \rangle$.

We use this fact for the definition of a pseudotransfer matrix for non-admissible systems in $\hat{\mathcal{L}}_{m,r,p}$.

Having introduced the larger space $\mathcal{I}_{m,r,p}$ and not only (non-proper) rational matrices, we can transform, via a permutation, the transfer matrix of the associated state space system to one of the original system, even in the non-admissible case. In the terminology of Willems [11] this construction is just the dual of ARMA-representations and first-order representations of behaviours.

Definition 4.2 *Let $\Sigma \in \hat{\mathcal{L}}^n_{m,r,p}$ be as in (4.1) with $\varrho \in \mathcal{P}(m)$ and $\tau \in \mathcal{P}(p)$, G as in (4.2), and $PQ^{-1} = G \in \mathrm{Rat}^0_{m,r,p} \times \mathbb{K}^{p\times m}$ a polynomial coprime factorization. Define*

$$\hat{T}(\Sigma) = \langle \begin{bmatrix} \tau & 0 \\ 0 & \varrho^{-1} \end{bmatrix} V_{n-r} \begin{bmatrix} P \\ Q \end{bmatrix} \rangle \in \mathcal{I}_{m,r,p}.$$

In [4, p. 1335] Grimm defines the pseudotransfer function for a system in "internal reduced form" as the transfer function of its associated state space system, which, in his approach, looks a bit different from ours. Thus, the information about the reorganization of the system, which is contained in the matrices ϱ, τ and V_{n-r}, is not included in Grimm's definition.

It is possible to define $\hat{T}(\Sigma)$ not only for systems in standard-form but for arbitrary irreducible systems via transforming them to a standard-form. For this remember the map ν as in (3.1). We have to use this map only for the formal reason that there is no notion of equivalence $\overset{\mathrm{e}}{\sim}$ introduced for systems in $\hat{\mathcal{L}}^k_{m,r,p}$.

Theorem 4.3 *The map*

$$\begin{array}{rcl} \mathcal{T} : \hat{\mathcal{L}}_{m,r,p} & \Leftrightarrow\!\!\rightarrow & \mathcal{I}_{m,r,p} \\ \Sigma & \Leftrightarrow\!\!\rightarrow & \hat{T}(\Sigma'),\ \textit{where } \nu(\Sigma)\overset{\mathrm{e}}{\sim}\nu(\Sigma'),\ \textit{and } \nu(\Sigma') \in \mathcal{L}^n_{m,r,p} \textit{ for some} \\ & & n \leq N \textit{ is in a } (\varrho,\tau)\textit{-standard-form with suitable } \varrho \in \\ & & \mathcal{P}(m),\ \tau \in \mathcal{P}(p) \end{array}$$

is well-defined, surjective, and fulfills $\Sigma \overset{\mathrm{s}}{\sim} \bar{\Sigma} \Longleftrightarrow \mathcal{T}(\Sigma) = \mathcal{T}(\bar{\Sigma})$ for $\Sigma, \bar{\Sigma} \in \hat{\mathcal{L}}_{m,r,p}$.

If $\Sigma = (A, B, C, D) \in \hat{\mathcal{A}}_{m,r,p}$ and $\mathcal{T}(\Sigma) = \langle [P^{\mathrm{t}}, Q^{\mathrm{t}}]^{\mathrm{t}} \rangle \in \mathcal{I}_{m,r,p}$, then it holds $\det Q \not\equiv 0$ and $PQ^{-1}(s) = C(s\hat{E}_n \Leftrightarrow A)^{-1}B + D$.

We call $\mathcal{T}(\Sigma)$ a pseudotransfer matrix of Σ. $\Sigma \in \hat{\mathcal{L}}_{m,r,p}$ is named a realization of $\langle T \rangle \in \mathcal{I}_{m,r,p}$, if Σ fulfills $\mathcal{T}(\Sigma) = \langle T \rangle$.

Proof: The well-definedness of $\mathcal{T}$ and the property $\Sigma \overset{s}{\sim} \bar{\Sigma} \Leftrightarrow \mathcal{T}(\Sigma) = \mathcal{T}(\bar{\Sigma})$ are rather technical to prove. We put the proof of this fact in Section 5. The surjectivity of $\mathcal{T}$ holds, since each $\langle T\rangle \in \mathcal{I}_{m,r,p}$ can be written as $\langle T\rangle = \langle \begin{bmatrix} \tau & 0 \\ 0 & \varrho^{-1} \end{bmatrix} V_k \begin{bmatrix} P \\ Q \end{bmatrix} \rangle$ with $\tau \in \mathcal{P}(p)$, $\varrho \in \mathcal{P}(m)$, $k \leq \min\{m,p\}$ and $PQ^{-1} \in \mathrm{Rat}^0_{m,r,p} \times \mathbb{K}^{p\times m}$ (remember the bijection Ψ_J in equation (3.6) of [2]). Then Definition 4.2 and the realization theory for state space systems yield at once a realization for $\langle T\rangle$ in $\hat{\mathcal{L}}^{r+k}_{m,r,p}$, which can be trivially inflated to one in $\hat{\mathcal{L}}_{m,r,p}$. □

Remark 4.4 The map $\mathcal{T}$ could have been formulated also by setting $\mathcal{T}(\Sigma) = \hat{T}(\Sigma')$ where Σ' is in addition to the given requirements also canonical. This would have made the proof of the well-definedness of $\mathcal{T}$ and of the property $\mathcal{T}(\Sigma) = \mathcal{T}(\bar{\Sigma}) \Leftrightarrow \Sigma \overset{s}{\sim} \bar{\Sigma}$ much easier. But this approach would make it much more complicated to prove that $\mathcal{T}$ induces a homeomorphism between $\hat{\mathcal{L}}_{m,r,p}/_{\overset{s}{\sim}}$ and $\mathcal{I}_{m,r,p}$. The reason for this is that the transfer matrix of the associated state space system of a canonical system in standard-form has a special structure in the direct feedthrough matrix (see Proposition 2.7 a)). Thus, with that formulation, one could not use the local structure of $\mathcal{I}_{m,r,p}$, which is (due to [2, Theorem 3.5]) homeomorphic to the space of <u>all</u> proper (minimal) state space systems.

Put $\mathcal{L}_{m,r,p} := \hat{\mathcal{L}}_{m,r,p}/_{\overset{s}{\sim}}$. Theorem 4.3 shows that

$$\begin{aligned} F:\ \mathcal{L}_{m,r,p} &\longrightarrow \mathcal{I}_{m,r,p} \\ [\Sigma] &\longmapsto \mathcal{T}(\Sigma) \end{aligned} \tag{4.3}$$

is a bijection, where $[\Sigma]$ denotes the equivalence class of Σ. In the following we will show that F is in fact a homeomorphism.

As it can be seen from [2, Theorem 3.5], the space $\mathcal{I}_{m,r,p}$ looks locally like $\mathrm{Rat}^0_{m,r,p} \times \mathbb{K}^{p\times m}$. We will give the corresponding local structure for the space $\mathcal{L}_{m,r,p}$ and then establish the bicontinuity of F via this local structure. For this remember Definition 3.4 in [2] and the definition of the matrix $\sigma_J \in \mathcal{P}(p+m)$ as in equation (3.4) of [2].

For $r \leq l \leq N$ let

$$\begin{aligned} \eta:\quad \hat{\mathcal{L}}^l_{m,r,p} &\longrightarrow \hat{\mathcal{L}}_{m,r,p} \\ (A,B,C,D) &\longmapsto \left(\begin{bmatrix} A & 0 \\ 0 & I_{N-l} \end{bmatrix}, \begin{bmatrix} B \\ 0 \end{bmatrix}, [C,0], D\right). \end{aligned}$$

For $J \in \mathcal{J} := \{(j_1, \ldots, j_m) \in \mathbb{N}^m \mid 1 \leq j_1 < \cdots < j_m \leq p + m\}$ with σ_J as in (3.4) of [2] and $k(J) = k$ put

$$\begin{aligned} \hat{\mathcal{L}}_J &= \{\Sigma \in \hat{\mathcal{L}}^{r+k}_{m,r,p} \mid \nu(\Sigma) \text{ is in a } (\varrho^{-1}, \tau)\text{-standard-form}\} \\ \tilde{\mathcal{L}}_J &= \{\Sigma \in \hat{\mathcal{L}}_{m,r,p} \mid \nu(\Sigma) \overset{\mathrm{e}}{\sim} \nu(\Sigma'), \text{ for some } \Sigma' \in \hat{\mathcal{L}}_J\}. \end{aligned}$$

Proposition 4.5 *a) It is $\hat{\mathcal{L}}_{m,r,p} = \bigcup_{J \in \mathcal{J}} \tilde{\mathcal{L}}_J$. The subsets $\tilde{\mathcal{L}}_J$ are open in $\hat{\mathcal{L}}_{m,r,p}$.*

b) Let $\Pi : \hat{\mathcal{L}}_{m,r,p} \longrightarrow \mathcal{L}_{m,r,p}$ be the canonical projection. Then it holds $\Pi(\tilde{\mathcal{L}}_J) \overset{\mathrm{homeo}}{\sim} \Sigma_{m,r,p} \times \mathbb{K}^{p \times m}$, where $\Pi(\tilde{\mathcal{L}}_J) \subseteq \mathcal{L}_{m,r,p}$ is endowed with the subset topology.

Proof:

a) The first part is obvious. For the second part, fix $J \in \mathcal{J}$ and let σ_J be as in (3.4) of [2]. Put $k = k(J)$, $l = N \Leftrightarrow r \Leftrightarrow k$ and partition $\Sigma = (A, B\varrho^{-1}, \tau C, \tau D\varrho^{-1}) \in \hat{\mathcal{L}}_{m,r,p}$ as follows:

$$\begin{aligned} \Sigma = \Big(&\Big[(A_{ij})_{i,j=1,2,3}\Big], \Big[(B_{ij})_{\substack{i=1,2,3 \\ j=1,2}}\Big] \varrho^{-1}, \tau \Big[(C_{ij})_{\substack{i=1,2 \\ j=1,2,3}}\Big], \\ &\tau \Big[(D_{ij})_{i,j=1,2}\Big] \varrho^{-1}\Big) \end{aligned} \tag{4.4}$$

with $A_{11} \in \mathbb{K}^{r \times r}$, $A_{22}, B_{21}, C_{22}, D_{21} \in \mathbb{K}^{k \times k}$ and $B_{31}, C_{23}^{\mathrm{t}} \in \mathbb{K}^{l \times k}$, which fixes the sizes of the remaining matrices as well. In the case $A_{33} \in Gl_l$ define $\tilde{A} = A_{22} \Leftrightarrow A_{23}A_{33}^{-1}A_{32}$, $\tilde{B} = B_{21} \Leftrightarrow A_{23}A_{33}^{-1}B_{31}$, $\tilde{C} = C_{22} \Leftrightarrow C_{23}A_{33}^{-1}A_{32}$ and $\tilde{D} = D_{21} \Leftrightarrow C_{23}A_{33}^{-1}B_{31}$. Let

$$\begin{aligned} \mathcal{L}_J = \{(A, B\varrho^{-1}, \tau C, \tau D\varrho^{-1}) \in \hat{\mathcal{L}}_{m,r,p} \mid \\ A_{33} \in Gl_l,\ \tilde{B},\ \tilde{C},\ I_k \Leftrightarrow \tilde{D}\tilde{B}^{-1}\tilde{A}\tilde{C}^{-1} \in Gl_k\}. \end{aligned} \tag{4.5}$$

Then it is $\eta(\hat{\mathcal{L}}_J) \subseteq \mathcal{L}_J \subseteq \tilde{\mathcal{L}}_J$. The first inclusion is obvious; the second one can be seen by transforming the system $(\hat{E}_n, A, B\varrho^{-1}, \tau C, \tau D\varrho^{-1}) \in \mathcal{L}_J$ via strong equivalence into (ϱ^{-1}, τ)-standard-form, which goes straightforward because of the non-singularity of the specified matrices in $\mathcal{L}_J$: a) invert A_{33}, b) delete (the new matrices) $A_{13}, A_{23}, A_{31}, A_{32}$, c) delete the last row of B and the last column of C, d) transform the matrices at the position (2,1) in B and at (2,2) in C to I_k, e) delete the entries D_{11}, D_{21}, D_{22} in the matrix D. The detailed transformations can be found in [3, pp. 153]. Note that all these transformations depend continuously on the given entries in $\Sigma = (A, B\varrho^{-1}, \tau C, \tau D\varrho^{-1}) \in \mathcal{L}_J$. Thus there exists a continuous map $f : \mathcal{L}_J \rightarrow \eta(\hat{\mathcal{L}}_J)$ such that $f(\Sigma) \overset{\mathrm{s}}{\sim} \Sigma$ for $\Sigma \in \mathcal{L}_J$.

Since $\mathcal{L}_J$ is open, the same holds for $\alpha(\mathcal{H} \times \mathcal{L}_J) \subseteq \hat{\mathcal{L}}_{m,r,p}$, but this is just the set $\tilde{\mathcal{L}}_J$ (remember the map α in (3.2)).

b) It is $\eta(\hat{\mathcal{L}}_J) \overset{\text{homeo}}{\sim} \hat{\mathcal{L}}_J \overset{\text{homeo}}{\sim} \tilde{\Sigma}_{m,r,p} \times \mathbb{K}^{p\times m}$ and with Proposition 2.8 $\Pi(\tilde{\mathcal{L}}_J) = \Pi(\alpha(\mathcal{H} \times \eta(\hat{\mathcal{L}}_J))) = \Pi(\eta(\hat{\mathcal{L}}_J)) \xleftrightarrow{1-1} \Sigma_{m,r,p} \times \mathbb{K}^{p\times m}$. Hence we have to prove the bicontinuity of the above bijection, where $\Pi(\eta(\hat{\mathcal{L}}_J))$ has to be endowed with the subset topology induced from $\mathcal{L}_{m,r,p}$. For this consider the commutative diagram

$$\begin{tikzcd}
\tilde{\Sigma}_{m,r,p} \times \mathbb{K}^{p\times m} \arrow[r, leftrightarrow, "\beta"] \arrow[d, "\Pi_1"'] & \eta(\hat{\mathcal{L}}_J) \arrow[r, leftrightarrow, "\iota"] \arrow[d, "\Pi|_{\eta(\hat{\mathcal{L}}_J)}"] & \alpha(\mathcal{H} \times \mathcal{L}_J) \arrow[d, "\Pi|_{\alpha(\mathcal{H}\times\mathcal{L}_J)}"] \\
\Sigma_{m,r,p} \times \mathbb{K}^{p\times m} \arrow[r, leftrightarrow, "\bar{\beta}"] & \Pi(\eta(\hat{\mathcal{L}}_J)) \arrow[r, leftrightarrow, "\mathrm{id}"] & \Pi(\alpha(\mathcal{H} \times \mathcal{L}_J)).
\end{tikzcd} \tag{4.6}$$

Here Π_1 is the canonical projection $\Pi_1 : \tilde{\Sigma}_{m,r,p} \times \mathbb{K}^{p\times m} \longrightarrow \Sigma_{m,r,p} \times \mathbb{K}^{p\times m}$, the map β is the homeomorphism $\beta : \tilde{\Sigma}_{m,r,p} \times K^{p\times m} \longrightarrow \eta(\hat{\mathcal{L}}_J)$, which is defined as

$$\beta(A_1, [B_1, B_2], \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, \begin{bmatrix} D_1 & D_2 \\ D_3 & D_4 \end{bmatrix}) = \eta(\begin{bmatrix} A_1 & B_1 \\ \Leftrightarrow C_2 & \Leftrightarrow D_3 \end{bmatrix},$$
$$\begin{bmatrix} 0 & B_2 \\ I_k & \Leftrightarrow D_4 \end{bmatrix} \varrho^{-1}, \tau \begin{bmatrix} C_1 & D_1 \\ 0 & I_k \end{bmatrix}, \tau \begin{bmatrix} 0 & D_2 \\ 0 & 0 \end{bmatrix} \varrho^{-1})$$

and ι is the canonical inclusion from $\eta(\hat{\mathcal{L}}_J)$ in $\alpha(\mathcal{H} \times \mathcal{L}_J)$. Then, by Proposition 2.8, $\bar{\beta}$ is a well-defined bijection. Further $\mathrm{id} : \Pi(\eta(\hat{\mathcal{L}}_J)) \longrightarrow \Pi(\alpha(\mathcal{H} \times \mathcal{L}_J))$ is a homeomorphism by the chosen topologies. The maps $\Pi|_{\eta(\hat{\mathcal{L}}_J)}$ and $\Pi|_{\alpha(\mathcal{H}\times\mathcal{L}_J)}$ are continuous, the latter one is also open. If we can show that $\Pi|_{\eta(\hat{\mathcal{L}}_J)}$ is an open mapping, too, then $\bar{\beta}$ is an homeomorphism and hence $\Pi(\tilde{\mathcal{L}}_J) = \Pi(\alpha(\mathcal{H} \times \mathcal{L}_J)) \overset{\text{homeo}}{\sim} \Sigma_{m,r,p} \times \mathbb{K}^{p\times m}$.

Thus it remains to be shown that $\Pi|_{\eta(\hat{\mathcal{L}}_J)}$ is open. As noted in a), there is a continuous map $f : \mathcal{L}_J \longrightarrow \eta(\hat{\mathcal{L}}_J)$ with $f(\Sigma) \overset{\text{s}}{\sim} \Sigma$. Thus, if $O \subseteq \eta(\hat{\mathcal{L}}_J)$ is open, then $f^{-1}(O) \subseteq \mathcal{L}_J$ also. But then $f^{-1}(O)$ is open in $\alpha(\mathcal{H} \times \mathcal{L}_J)$ and thus $\Pi|_{\eta(\hat{\mathcal{L}}_J)}(O) = \Pi|_{\alpha(\mathcal{H}\times\mathcal{L}_J)}(f^{-1}(O))$ is open. $\square$

With this preparation and the notations of equation (3.5) and Theorem 3.5, both in [2, Section 3], we finally get

Theorem 4.6 *The map F in (4.3) is a homeomorphism and satisfies $F(\Pi(\tilde{\mathcal{L}}_J)) = \Pi_\infty^{-1}(\mathrm{ch}(J))$. Further $F(\hat{\mathcal{A}}_{m,r,p}/_{\overset{\text{s}}{\sim}}) = \mathrm{Rat}_{m,r,p}$, so in particular $\hat{\mathcal{A}}_{m,r,p}/_{\overset{\text{s}}{\sim}}$ is homeomorphic to $\mathrm{Rat}_{m,r,p}$.*

Proof: The bijectivity of F as well as $F(\Pi(\tilde{\mathcal{L}}_J)) = \Pi_\infty^{-1}(\mathrm{ch}(J))$ and $F(\hat{\mathcal{A}}_{m,r,p}/\underset{\sim}{s}) = \mathrm{Rat}_{m,r,p}$ are fulfilled by construction. It remains to be proven the bicontinuity of F. For this consider the commutative diagram

$$\begin{array}{ccc} \Pi(\alpha(\mathcal{H} \times \mathcal{L}_J)) & \xleftrightarrow{F} & \Pi_\infty^{-1}(\mathrm{ch}(J)) \\ {\scriptstyle \bar{\beta}^{-1}}\big\downarrow & & \big\downarrow{\scriptstyle V} \\ \Sigma_{m,r,p} \times \mathbb{K}^{p\times m} & \xleftrightarrow{K} & \mathrm{Rat}^0_{m,r,p} \times \mathbb{K}^{p\times m} \end{array}$$

with $\bar{\beta}$ as in (4.6), the homeomorphism $V(\langle \sigma_J \begin{bmatrix} P \\ Q \end{bmatrix} \rangle) = \langle \begin{bmatrix} P \\ Q \end{bmatrix} \rangle$ and

$K[(A,B,C,D)] = \langle \begin{bmatrix} P \\ Q \end{bmatrix} \rangle$, where $PQ^{-1}(s) = C(sI \Leftrightarrow A)^{-1}B + D$ is a coprime factorization of the transfer matrix of (A,B,C,D). Then K is known to be a homeomorphism (see Byrnes/Duncan [1, p. 43, p. 46]). Thus $F|_{\Pi(\alpha(\mathcal{H}\times\mathcal{L}_J))}$ is a homeomorphism. Since $\mathcal{L}_{m,r,p}$ is covered by the open sets $\Pi(\alpha(\mathcal{H} \times \mathcal{L}_J))$ it follows that F is a homeomorphism too. $\square$

Of course, it is also possible to define the pseudotransfer matrix $\mathcal{T}$ for systems $\Sigma = (E,A,B,C,D)$ in $\mathcal{L}^N_{m,r,p}$, i. e. with arbitrary matrix E. Using Proposition 3.1, this leads to

Corollary 4.7 *The quotient space $\mathcal{L}^N_{m,r,p}/\underset{\sim}{e}$ of all irreducible systems modulo (strong) equivalence is homeomorphic to the space $\mathcal{I}_{m,r,p}$, the homeomorphism is induced by the pseudotransfer matrix.*

5 Proof of Theorem 4.3

We have to prove a lot of lemmata showing the invariance of the map $\mathcal{T}$ in Theorem 4.3 under various system transformations. Parts of the rather technical computations are a bit shortened; they can be found in more detail in [3].

The following relation between factorizations of the transfer matrices of state space systems, which are related by output feedback, will often be used. It is easy to prove:

$$\left.\begin{array}{l} C(sI \Leftrightarrow A)^{-1}B + D = PQ^{-1} \Longrightarrow C(sI \Leftrightarrow A + BFC)^{-1}B + \bar{D} \\ = \Big((I + \bar{D}F)P + (\bar{D} \Leftrightarrow D \Leftrightarrow \bar{D}FD)Q\Big)\Big(FP + (I \Leftrightarrow FD)Q\Big)^{-1}. \end{array}\right\} \quad (5.1)$$

In the following lemma the claimed properties of the pseudotransfer matrix are proven in the case of canonical systems.

Lemma 5.1 *Let* $\Sigma = (A, B\varrho, \tau C, \tau D\varrho)$, $\bar{\Sigma} = (\bar{A}, \bar{B}\bar{\varrho}, \bar{\tau}\bar{C}, \bar{\tau}\bar{D}\bar{\varrho}) \in \hat{\mathcal{C}}^n_{m,r,p}$ *with* $\varrho \in \mathcal{P}(m)$, $\tau \in \mathcal{P}(p)$ *and the matrices be partitioned as follows:*

$$
\begin{aligned}
(A, B, C, D) &= \left(\begin{bmatrix} A_1 & A_2 \\ A_3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & B_2 \\ I_{n-r} & B_4 \end{bmatrix}, \begin{bmatrix} C_1 & C_2 \\ 0 & I_{n-r} \end{bmatrix} \begin{bmatrix} 0 & D_2 \\ 0 & 0 \end{bmatrix}\right), \\
(\bar{A}, \bar{B}, \bar{C}, \bar{D}) &= \left(\begin{bmatrix} \bar{A}_1 & \bar{A}_2 \\ \bar{A}_3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \bar{B}_2 \\ I_{n-r} & \bar{B}_4 \end{bmatrix}, \begin{bmatrix} \bar{C}_1 & \bar{C}_2 \\ 0 & I_{n-r} \end{bmatrix} \begin{bmatrix} 0 & \bar{D}_2 \\ 0 & 0 \end{bmatrix}\right).
\end{aligned}
$$

Then it holds: $\Sigma \overset{s}{\sim} \bar{\Sigma} \iff \hat{T}(\Sigma) = \hat{T}(\bar{\Sigma})$.

Proof: First, note the following fact, which holds also, if in the system Σ the matrix A_4 is non-trivial. If

$$
G(s) = \begin{bmatrix} C_1 \\ -A_3 \end{bmatrix} (sI - A_1)^{-1} [A_2, B_2] + \begin{bmatrix} 0 & D_2 \\ -A_4 & 0 \end{bmatrix} = PQ^{-1}(s)
$$

and $S = \begin{bmatrix} I & B_4 \\ 0 & I \end{bmatrix} \varrho \in Gl_m$, $T = \tau \begin{bmatrix} I & C_2 \\ 0 & I \end{bmatrix} \in Gl_p$, then it holds $\hat{T}(\Sigma) = \langle \begin{bmatrix} T & 0 \\ 0 & S^{-1} \end{bmatrix} V_{n-r} \begin{bmatrix} P \\ Q \end{bmatrix} \rangle$. This can be verified at once by using partitions of the matrices P and Q. This fact will often be used in the following.

a) We show: $\Sigma \overset{s}{\sim} \bar{\Sigma} \Longrightarrow \hat{T}(\Sigma) = \hat{T}(\bar{\Sigma})$.

Assume

$$
\begin{bmatrix} M & 0 \\ L & I \end{bmatrix} \begin{bmatrix} s\hat{E}_n - A & -B\varrho \\ \tau C & \tau D\varrho \end{bmatrix} \begin{bmatrix} N & R \\ 0 & I \end{bmatrix} = \begin{bmatrix} s\hat{E}_n - \bar{A} & -\bar{B}\bar{\varrho} \\ \bar{\tau}\bar{C} & \bar{\tau}\bar{D}\bar{\varrho} \end{bmatrix}. \tag{5.2}
$$

Since $M\hat{E}_n N = \hat{E}_n$ and $L\hat{E}_n = 0$, $\hat{E}_n R = 0$, one can specify the following matrices

$$
\begin{aligned}
M &=: \begin{bmatrix} M_1 & M_2 \\ 0 & M_4 \end{bmatrix} & N &=: \begin{bmatrix} M_1^{-1} & 0 \\ N_3 & N_4 \end{bmatrix} \\
\begin{bmatrix} I & -\bar{C}_2 \\ 0 & I \end{bmatrix} \bar{\tau}^{-1} L &=: \begin{bmatrix} 0 & L_2 \\ 0 & L_4 \end{bmatrix} & R\bar{\varrho}^{-1} \begin{bmatrix} I & -\bar{B}_4 \\ 0 & I \end{bmatrix} &=: \begin{bmatrix} 0 & 0 \\ R_3 & R_4 \end{bmatrix},
\end{aligned}
$$

where M_i, N_i, L_i and R_i are suitable matrices. Moreover, put

$$
\left.\begin{aligned}
T &:= \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} := \begin{bmatrix} I & -\bar{C}_2 \\ 0 & I \end{bmatrix} \bar{\tau}^{-1} \tau \begin{bmatrix} I & C_2 \\ 0 & I \end{bmatrix}, \\
S &:= \begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix} := \begin{bmatrix} I & B_4 \\ 0 & I \end{bmatrix} \varrho\bar{\varrho}^{-1} \begin{bmatrix} I & -\bar{B}_4 \\ 0 & I \end{bmatrix}.
\end{aligned}\right\} \tag{5.3}
$$

Then after some computations from (5.2), one gets

$$
\begin{aligned}
&S_2 = 0 && T_2 = 0\\
&S_1 = M_4^{-1} && T_4 = N_4^{-1}\\
&L_2 = T_1 D_2 S_3 S_1^{-1} && R_4 = \Leftrightarrow T_4^{-1} T_3 D_2 S_4\\
&\bar{B}_2 = M_1(B_2 \Leftrightarrow A_2 T_4^{-1} T_3 D_2) S_4 && \bar{D}_2 = T_1 D_2 S_4\\
&\bar{C}_1 = T_1(C_1 \Leftrightarrow D_2 S_3 S_1^{-1} A_3) M_1^{-1} && \bar{A}_2 = M_1 A_2 N_4\\
&\bar{A}_1 = M_1 A_1 M_1^{-1} + M_2 A_3 M_1^{-1} + M_1 A_2 N_3 && \bar{A}_3 = M_4 A_3 M_1^{-1}.
\end{aligned}
$$

With these equations, (5.2) and the definitions

$$
\begin{aligned}
F &= \begin{bmatrix} T_4^{-1} T_3 & T_4^{-1} T_3 D_2 S_3 S_1^{-1} \\ 0 & \Leftrightarrow S_3 S_1^{-1} \end{bmatrix},\\
U &= \begin{bmatrix} T_1 & L_2 \\ 0 & S_1^{-1} \end{bmatrix} \in Gl_p, \quad V = \begin{bmatrix} T_4^{-1} & R_4 \\ 0 & S_4 \end{bmatrix} \in Gl_m,
\end{aligned} \tag{5.4}
$$

it follows

$$
\begin{aligned}
&\bar{G}(s) :=\\
&\qquad \begin{bmatrix} \bar{C}_1 \\ \Leftrightarrow \bar{A}_3 \end{bmatrix} (sI \Leftrightarrow \bar{A}_1)^{-1} [\bar{A}_2, \bar{B}_2] + \begin{bmatrix} 0 & \bar{D}_2 \\ 0 & 0 \end{bmatrix}\\
&= \; U \left(\begin{bmatrix} C_1 \\ \Leftrightarrow A_3 \end{bmatrix} (sI \Leftrightarrow A_1 + [A_2, B_2] F \begin{bmatrix} C_1 \\ \Leftrightarrow A_3 \end{bmatrix})^{-1} [A_2, B_2] + \begin{bmatrix} 0 & D_2 \\ 0 & 0 \end{bmatrix} \right) V.
\end{aligned}
$$

If

$$
G(s) := \begin{bmatrix} C_1 \\ \Leftrightarrow A_3 \end{bmatrix} (sI \Leftrightarrow A_1)^{-1} [A_2, B_2] + \begin{bmatrix} 0 & D_2 \\ 0 & 0 \end{bmatrix} = P Q^{-1}(s)
$$

is a coprime factorization of G, then by (5.1) it is

$$
\bar{G} = U \Big(P + DF(P \Leftrightarrow DQ) \Big) \Big(Q + F(P \Leftrightarrow DQ) \Big)^{-1} V.
$$

Using $DFD = 0$, one checks that this is in fact a coprime factorization of $\bar{G}$. The explicit structure of the involved matrices as given in (5.4) leads to

$$
\begin{bmatrix} U & 0 \\ 0 & V^{-1} \end{bmatrix} \begin{bmatrix} P + DF(P \Leftrightarrow DQ) \\ Q + F(P \Leftrightarrow DQ) \end{bmatrix} = V_{n-r} \begin{bmatrix} T & 0 \\ 0 & S^{-1} \end{bmatrix} V_{n-r} \begin{bmatrix} P \\ Q \end{bmatrix}.
$$

The definition of the map $\hat{T}$ (see Definition 4.2) and (5.3) now yield

$$
\hat{T}(\bar{\Sigma}) \;=\; \langle \begin{bmatrix} \bar{\tau} & 0 \\ 0 & \bar{\varrho}^{-1} \end{bmatrix} \begin{bmatrix} I & \bar{C}_2 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & \Leftrightarrow \bar{B}_4 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} T & 0 \\ 0 & S^{-1} \end{bmatrix} V_{n-r} \begin{bmatrix} P \\ Q \end{bmatrix} \rangle
$$

$$= \quad \langle \begin{bmatrix} \tau & 0 \\ 0 & \varrho^{-1} \end{bmatrix} \begin{bmatrix} I & C_2 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & \Leftrightarrow B_4 \\ 0 & 0 & 0 & I \end{bmatrix} V_{n-r} \begin{bmatrix} P \\ Q \end{bmatrix} \rangle = \hat{T}(\Sigma).$$

b) We show: $\hat{T}(\Sigma) = \hat{T}(\bar{\Sigma}) \Longrightarrow \Sigma \overset{s}{\sim} \bar{\Sigma}$.

Let S, $\bar{S} \in Gl_m$ and T, $\bar{T} \in Gl_p$ be

$$S = \begin{bmatrix} I & B_4 \\ 0 & I \end{bmatrix} \varrho,\ \bar{S} = \begin{bmatrix} I & \bar{B}_4 \\ 0 & I \end{bmatrix} \bar{\varrho},\ T = \tau \begin{bmatrix} I & C_2 \\ 0 & I \end{bmatrix},\ \bar{T} = \bar{\tau} \begin{bmatrix} I & \bar{C}_2 \\ 0 & I \end{bmatrix}. \quad (5.5)$$

Further put $PQ^{-1} =$

$$\begin{bmatrix} C_1 \\ \Leftrightarrow A_3 \end{bmatrix} (sI \Leftrightarrow A_1)^{-1} [A_2, B_2] + \begin{bmatrix} 0 & D_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix} \begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{bmatrix}^{-1}, \quad (5.6)$$

and $\bar{P}\bar{Q}^{-1} =$

$$\begin{bmatrix} \bar{C}_1 \\ \Leftrightarrow \bar{A}_3 \end{bmatrix} (sI \Leftrightarrow \bar{A}_1)^{-1} [\bar{A}_2, \bar{B}_2] + \begin{bmatrix} 0 & \bar{D}_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \bar{P}_1 & \bar{P}_2 \\ \bar{P}_3 & \bar{P}_4 \end{bmatrix} \begin{bmatrix} \bar{Q}_1 & \bar{Q}_2 \\ \bar{Q}_3 & \bar{Q}_4 \end{bmatrix}^{-1}. \quad (5.7)$$

Then it follows

$$\hat{T}(\Sigma) = \langle \begin{bmatrix} T & 0 \\ 0 & S^{-1} \end{bmatrix} V_{n-r} \begin{bmatrix} P \\ Q \end{bmatrix} \rangle = \langle \begin{bmatrix} \bar{T} & 0 \\ 0 & \bar{S}^{-1} \end{bmatrix} V_{n-r} \begin{bmatrix} \bar{P} \\ \bar{Q} \end{bmatrix} \rangle = \hat{T}(\bar{\Sigma}). \quad (5.8)$$

With the definition

$$\bar{T}^{-1} T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \in Gl_p,\ \bar{S} S^{-1} = \begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix} \in Gl_m$$

and the partition of the matrices P, $\bar{P}$, Q, $\bar{Q}$ as in (5.6) and (5.7) equation (5.8) can be rewritten as

$$\bar{P}\bar{Q}^{-1} = \left(\begin{bmatrix} T_1 & 0 \\ 0 & S_1 \end{bmatrix} PQ^{-1} + \begin{bmatrix} T_2 & 0 \\ 0 & S_2 \end{bmatrix} \right) \left(\begin{bmatrix} T_3 & 0 \\ 0 & S_3 \end{bmatrix} PQ^{-1} + \begin{bmatrix} T_4 & 0 \\ 0 & S_4 \end{bmatrix} \right)^{-1}.$$

This leads to the following equation of proper rational matrices

$$\begin{bmatrix} T_1 & 0 \\ 0 & S_1 \end{bmatrix} PQ^{-1} + \begin{bmatrix} T_2 & 0 \\ 0 & S_2 \end{bmatrix} = \bar{P}\bar{Q}^{-1} \left(\begin{bmatrix} T_3 & 0 \\ 0 & S_3 \end{bmatrix} PQ^{-1} + \begin{bmatrix} T_4 & 0 \\ 0 & S_4 \end{bmatrix} \right).$$

A comparison of the constant parts on both sides yields $S_2 = 0,\ T_2 = 0$.

Now it is possible to transform the system Σ via strong equivalence to $\bar{\Sigma}$. This transformation was first used by Grimm [4, p. 1343], who needed this result in his approach:

First, with $N = \begin{bmatrix} I & 0 \\ -T_4^{-1}T_3C_1 & T_4^{-1} \end{bmatrix}$ and $R = \begin{bmatrix} 0 & 0 \\ 0 & -T_4^{-1}T_3D_2 \end{bmatrix} S$ it holds

$$\begin{bmatrix} s\hat{E}_n - A & -B\varrho \\ \tau C & \tau D\varrho \end{bmatrix} \begin{bmatrix} N & R \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & \bar{T} \end{bmatrix} \begin{bmatrix} s\hat{E}_n - \tilde{A} & -\tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & S \end{bmatrix},$$

where

$$(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}) = \left(\begin{bmatrix} \tilde{A}_1 & \tilde{A}_2 \\ \tilde{A}_3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \tilde{B}_2 \\ I & 0 \end{bmatrix}, \begin{bmatrix} \tilde{C}_1 & 0 \\ 0 & I \end{bmatrix}, \begin{bmatrix} 0 & \tilde{D}_2 \\ 0 & 0 \end{bmatrix} \right),$$

suitably. This can be verified directly.

Secondly with

$$M = \begin{bmatrix} I & \tilde{B}_2 S_4^{-1} S_3 \\ 0 & S_1 \end{bmatrix}, L = \bar{T} \begin{bmatrix} 0 & -\tilde{D}_2 S_4^{-1} S_3 \\ 0 & 0 \end{bmatrix}$$

it can be seen that

$$\begin{bmatrix} M & 0 \\ L & I \end{bmatrix} \begin{bmatrix} s\hat{E}_n - \tilde{A} & -\tilde{B}S \\ \bar{T}\tilde{C} & \bar{T}\tilde{D}S \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & \bar{T} \end{bmatrix} \begin{bmatrix} s\hat{E}_n - \hat{A} & -\hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \bar{S} \end{bmatrix}$$

where

$$\hat{\Sigma} := (\hat{A}, \hat{B}, \hat{C}, \hat{D}) = \left(\begin{bmatrix} \hat{A}_1 & \hat{A}_2 \\ \hat{A}_3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \hat{B}_2 \\ I & 0 \end{bmatrix}, \begin{bmatrix} \hat{C}_1 & 0 \\ 0 & I \end{bmatrix}, \begin{bmatrix} 0 & \hat{D}_2 \\ 0 & 0 \end{bmatrix} \right)$$

with some matrices $\hat{A}_i, \hat{B}_2, \hat{C}_1$ and $\hat{D}_2$. Therefore it is

$$\Sigma \overset{s}{\sim} (\hat{A}, \hat{B}\bar{S}, \bar{T}\hat{C}, \bar{T}\hat{D}\bar{S}).$$

Let

$$\hat{P}\hat{Q}^{-1}(s) = \begin{bmatrix} \hat{C}_1 \\ -\hat{A}_3 \end{bmatrix} (sI - \hat{A}_1)^{-1} [\hat{A}_2, \hat{B}_2] + \begin{bmatrix} 0 & \hat{D}_2 \\ 0 & 0 \end{bmatrix}$$

be a coprime factorization. The invariance of $\hat{T}$ under $\overset{s}{\sim}$ yields $\hat{T}(\Sigma) = \langle \begin{bmatrix} \bar{T} & 0 \\ 0 & \bar{S}^{-1} \end{bmatrix} V_{n-r} \begin{bmatrix} \hat{P} \\ \hat{Q} \end{bmatrix} \rangle$. Thus with (5.8) it is $\hat{P}\hat{Q}^{-1} = \bar{P}\bar{Q}^{-1}$ and

$$(\hat{A}_1, [\hat{A}_2, \hat{B}_2], \begin{bmatrix} \hat{C}_1 \\ -\hat{A}_3 \end{bmatrix}, \begin{bmatrix} 0 & \hat{D}_2 \\ 0 & 0 \end{bmatrix})$$

and

$$(\bar{A}_1, [\bar{A}_2, \bar{B}_2], \begin{bmatrix} \bar{C}_1 \\ -\bar{A}_3 \end{bmatrix}, \begin{bmatrix} 0 & \bar{D}_2 \\ 0 & 0 \end{bmatrix}) \in \tilde{\Sigma}_{m,r,p} \times \mathbb{K}^{p \times m}$$

are similar state space systems. With Proposition 2.8 this implies $\Sigma \overset{s}{\sim} (\hat{A}, \hat{B}\bar{S}, \bar{T}\hat{C}, \bar{T}\hat{D}\bar{S}) \overset{s}{\sim} \bar{\Sigma}$. □

By the above lemma, the map

$$\left.\begin{array}{rcl} \tilde{T}: \hat{\mathcal{L}}^n_{m,r,p} & \Leftrightarrow\!\!\rightarrow & \mathcal{I}_{m,r,p} \\ \Sigma & |\!\!\Leftrightarrow\!\!\rightarrow & \hat{T}(\tilde{\Sigma}), \text{ if } \nu(\Sigma) \overset{e}{\sim} \nu(\tilde{\Sigma}) \text{ and } \nu(\tilde{\Sigma}) \in \mathcal{C}^k_{m,r,p} \text{ for} \\ & & \text{some } k \leq n \text{ is in a } (\varrho, \tau)\text{-standard form} \\ & & \text{with suitable } \varrho \in \mathcal{P}(m),\ \tau \in \mathcal{P}(p) \end{array}\right\} \quad (5.9)$$

is well-defined for every $n \geq r$. Here the map ν is as before defined as in (3.1). We will use the map $\tilde{T}$ to show the well-definedness of $\mathcal{T}$.

The next two lemmata establish the equality of $\hat{T}(\Sigma)$ and $\tilde{T}(\Sigma)$, i. e. the equality of the pseudotransfer matrices $\hat{T}(\Sigma)$ of a non-canonical system in standard-form and an equivalent canonical system.

Lemma 5.2 *Let $l + t = n \Leftrightarrow r$, $\Sigma = (A, B, C, D) =$*

$$\left(\begin{bmatrix} A_1 & A_{21} & A_{22} \\ A_{31} & 0_l & 0 \\ A_{32} & 0 & I_t \end{bmatrix}, \begin{bmatrix} 0 & 0 & B_1 \\ I_l & 0 & B_2 \\ 0 & I_t & B_3 \end{bmatrix}, \begin{bmatrix} C_1 & C_2 & C_3 \\ 0 & I_l & 0 \\ 0 & 0 & I_t \end{bmatrix}, \begin{bmatrix} 0 & D_2 \\ 0_{l+t} & 0 \end{bmatrix} \right) \in \hat{\mathcal{L}}^n_{m,r,p}$$

and

$$PQ^{-1}(s) = \begin{bmatrix} C_1 \\ \Leftrightarrow A_{31} \\ \Leftrightarrow A_{32} \end{bmatrix} (sI \Leftrightarrow A_1)^{-1} [A_{21}, A_{22}, B_1] + \begin{bmatrix} C_2 & C_3 & D_2 \\ 0 & 0 & \Leftrightarrow B_2 \\ 0 & \Leftrightarrow I_t & \Leftrightarrow B_3 \end{bmatrix} \quad (5.10)$$

be a coprime factorization of the transfer matrix of the associated state space system. Then $\tilde{T}(\Sigma) = \langle V_{l+t} \begin{bmatrix} P \\ Q \end{bmatrix} \rangle = \hat{T}(\Sigma) \in \mathcal{I}_{m,r,p}$ (the case $l = 0$ is included).

Proof: By direct computation one gets $(\hat{E}_n, A, B, C, D) \overset{e}{\sim}$

$$\left(\hat{E}_{r+l}, \begin{bmatrix} A_1 \Leftrightarrow A_{22}A_{32} & A_{21} \\ A_{31} & 0_l \end{bmatrix}, \begin{bmatrix} 0 & \Leftrightarrow A_{22} & B_1 \Leftrightarrow A_{22}B_3 \\ I_l & 0 & B_2 \end{bmatrix}, \right.$$
$$\left. M \begin{bmatrix} C_1 \Leftrightarrow C_3A_{32} & C_2 \\ \Leftrightarrow A_{32} & 0 \\ 0 & I_l \end{bmatrix}, M \begin{bmatrix} 0 & \Leftrightarrow C_3 & D_2 \Leftrightarrow C_3B_3 \\ 0 & \Leftrightarrow I_t & \Leftrightarrow B_3 \\ 0 & 0 & 0 \end{bmatrix} \right)$$

with $M = \begin{bmatrix} I_{p-l-t} & 0 & 0 \\ 0 & 0 & I_l \\ 0 & I_t & 0 \end{bmatrix} \in \mathcal{P}(p)$, which gives a canonical system. The transfer matrix $\hat{G}$ of the associated state space system of this canonical system is

$$\hat{G}(s) =$$
$$F_1\Big(\begin{bmatrix} C_1 \\ \Leftrightarrow A_{31} \\ \Leftrightarrow A_{32}\end{bmatrix}(sI \Leftrightarrow A_1 + \hat{B}F\begin{bmatrix} C_1 \\ \Leftrightarrow A_{31} \\ \Leftrightarrow A_{32}\end{bmatrix})^{-1}\hat{B} + \begin{bmatrix} C_2 & 0 & D_2 \\ 0 & 0 & \Leftrightarrow B_2 \\ 0 & I_t & 0\end{bmatrix}\Big)F_2$$

with the matrices $\hat{B} = [A_{21}, A_{22}, B_1]$ and

$$F_1 = \begin{bmatrix} I_{p-l+t} & 0 & C_3 \\ 0 & 0 & I_t \\ 0 & I_l & 0\end{bmatrix},\ F_2 = \begin{bmatrix} I_l & 0 & 0 \\ 0 & \Leftrightarrow I_t & \Leftrightarrow B_3 \\ 0 & 0 & I_{m-l-t}\end{bmatrix},\ F = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \Leftrightarrow I_t \\ 0 & 0 & 0\end{bmatrix}.$$

With (5.1) and

$$D' := \begin{bmatrix} C_2 & C_3 & D_2 \\ 0 & 0 & \Leftrightarrow B_2 \\ 0 & \Leftrightarrow I_t & \Leftrightarrow B_3\end{bmatrix}, \tilde{D} := \begin{bmatrix} C_2 & 0 & D_2 \\ 0 & 0 & \Leftrightarrow B_2 \\ 0 & I_t & 0\end{bmatrix}$$

it follows from (5.10)

$$F_1^{-1}\hat{G}(s)F_2^{-1} =$$
$$\Big((I + \tilde{D}F)P + (\tilde{D} \Leftrightarrow D' \Leftrightarrow \tilde{D}FD')Q\Big)\Big(FP + (I \Leftrightarrow FD')Q\Big)^{-1}.$$

Hence it is

$$\tilde{T}(\Sigma) = \langle\begin{bmatrix} M & 0 \\ 0 & I\end{bmatrix}V_l\begin{bmatrix} F_1 & 0 \\ 0 & F_2^{-1}\end{bmatrix}\begin{bmatrix} I + \tilde{D}F & \tilde{D} \Leftrightarrow D' \Leftrightarrow \tilde{D}FD' \\ F & I \Leftrightarrow FD'\end{bmatrix}\begin{bmatrix} P \\ Q\end{bmatrix}\rangle$$

which, after a few computations, yields $\tilde{T}(\Sigma) = \langle V_{l+t}\begin{bmatrix} P \\ Q\end{bmatrix}\rangle$. □

Lemma 5.3 *Let $r \leq n \leq N$, $l = n \Leftrightarrow r$, $k = l + t \leq \min\{m, p\}$, and $\Sigma = (A, BS, TC, TDS) \in \hat{\mathcal{C}}^n_{m,r,p}$ with matrices*

$$S = \begin{bmatrix} U & 0 \\ 0 & I_{m-k}\end{bmatrix} \in Gl_m,\ T = \begin{bmatrix} I_{p-k} & 0 \\ 0 & W\end{bmatrix} \in Gl_p,$$

and

$$(A, B, C, D) = (\begin{bmatrix} A_1 & A_2 \\ A_3 & 0\end{bmatrix}, \begin{bmatrix} 0 & B_1 & B_2 \\ I_l & B_3 & B_4\end{bmatrix}, \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \\ 0 & I_l\end{bmatrix}, \begin{bmatrix} 0 & D_1 & D_2 \\ 0 & D_3 & D_4 \\ 0_l & 0 & 0\end{bmatrix})$$

where B_3, $C_4^{\mathrm{t}} \in \mathbb{K}^{l\times t}$, $D_3 \in \mathbb{K}^{t\times t}$. *Then* $\tilde{T}(\Sigma) = \overline{\begin{bmatrix} T & 0 \\ 0 & S^{-1} \end{bmatrix}}\tilde{T}(A,B,C,D)$, *if for* $M \in Gl_{p+m}$ *the map* $\overline{M} : \mathcal{I}_{m,r,p} \to \mathcal{I}_{m,r,p}$ $\langle X\rangle \mapsto \langle MX\rangle$ *denotes the induced homeomorphism.*

Proof: Let

$$PQ^{-1}(s) = \begin{bmatrix} C_1 \\ C_3 \\ -A_3 \end{bmatrix}(sI - A_1)^{-1}[A_2, B_1, B_2] + \begin{bmatrix} C_2 & D_1 & D_2 \\ C_4 & D_3 & D_4 \\ 0_l & -B_3 & -B_4 \end{bmatrix} \quad (5.11)$$

be a coprime factorization of the transfer matrix of the state space system associated with (A,B,C,D). Then it is $\tilde{T}(A,B,C,D) = \langle V_l[P^{\mathrm{t}}, Q^{\mathrm{t}}]^{\mathrm{t}}\rangle$. Let $\tau \in \mathcal{P}(k)$ be such that with $\tau W = \begin{bmatrix} W_1 & W_2 \\ W_3 & W_4 \end{bmatrix}$ it is $\begin{bmatrix} W_1 & W_2 \\ W_3 & W_4 \end{bmatrix}\begin{bmatrix} C_4 \\ I_l \end{bmatrix} = \begin{bmatrix} \bar{V}_2 \\ \bar{V}_4 \end{bmatrix}$ where $\bar{V}_4 \in Gl_l$. With $\hat{\tau} = \begin{bmatrix} I_{p-k} & 0 \\ 0 & \tau^{-1} \end{bmatrix} \in \mathcal{P}(p)$ it holds

$$T = \hat{\tau}\begin{bmatrix} I_{p-k} & 0 \\ 0 & \tau W \end{bmatrix}. \quad (5.12)$$

Further let $\tilde{V}_1 := W_1 - \bar{V}_2\bar{V}_4^{-1}W_3$. Then one can see that $\tilde{V}_1 \in Gl_t$.

First we will consider the system $\bar{\Sigma} := (A, B, TC, TD)$. By some straightforward matrix transformations one gets $\bar{\Sigma} \overset{s}{\sim} \tilde{\Sigma} :=$

$$\left(\begin{bmatrix} \tilde{A}_1 & A_2\bar{V}_4^{-1} \\ A_3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \tilde{B}_1 & \tilde{B}_2 \\ I_l & B_3 & B_4 \end{bmatrix}, \hat{\tau}\begin{bmatrix} \tilde{C}_1 & C_2\bar{V}_4^{-1} \\ \tilde{V}_1C_3 & \bar{V}_2\bar{V}_4^{-1} \\ 0 & I_l \end{bmatrix}, \hat{\tau}\begin{bmatrix} 0 & \tilde{D}_1 & \tilde{D}_2 \\ 0 & \tilde{D}_3 & \tilde{D}_4 \\ 0_l & 0 & 0 \end{bmatrix}\right)$$

with matrices $\tilde{A}_1 = A_1 - A_2\bar{V}_4^{-1}W_3C_3$, $\tilde{C}_1 = C_1 - C_2\bar{V}_4^{-1}W_3C_3$,

$$\begin{aligned} \begin{bmatrix} \tilde{D}_1 & \tilde{D}_2 \\ \tilde{D}_3 & \tilde{D}_4 \end{bmatrix} &= \begin{bmatrix} D_1 - C_2\bar{V}_4^{-1}W_3D_3 & D_2 - C_2\bar{V}_4^{-1}W_3D_4 \\ W_1D_3 - \bar{V}_2\bar{V}_4^{-1}W_3D_3 & W_1D_4 - \bar{V}_2\bar{V}_4^{-1}W_3D_4 \end{bmatrix}, \\ [\tilde{B}_1,\, \tilde{B}_2] &= [B_1 - A_2\bar{V}_4^{-1}W_3D_3,\, B_2 - A_2\bar{V}_4^{-1}W_3D_4]. \end{aligned}$$

Let

$$\tilde{P}\tilde{Q}^{-1}(s) = \begin{bmatrix} \tilde{C}_1 \\ \tilde{V}_1C_3 \\ -A_3 \end{bmatrix}(sI - \tilde{A}_1)^{-1}[A_2\bar{V}_4^{-1}, \tilde{B}_1, \tilde{B}_2] + \begin{bmatrix} C_2\bar{V}_4^{-1} & \tilde{D}_1 & \tilde{D}_2 \\ \bar{V}_2\bar{V}_4^{-1} & \tilde{D}_3 & \tilde{D}_4 \\ 0 & -B_3 & -B_4 \end{bmatrix}$$

be a coprime factorization of the transfer matrix of the state space system associated with $\tilde{\Sigma}$.

In the next step we will establish the relation between $\tilde{P}\tilde{Q}^{-1}$ and PQ^{-1}. Put

$$F_1 = \begin{bmatrix} I & \Leftrightarrow C_2\bar{V}_4^{-1}W_3 & 0 \\ 0 & \tilde{V}_1 & 0 \\ 0 & 0 & I \end{bmatrix} \in Gl_p, F = \begin{bmatrix} 0 & \bar{V}_4^{-1}W_3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \mathbb{K}^{m\times p}$$

and

$$F_2 = \begin{bmatrix} \bar{V}_4^{-1} & \Leftrightarrow \bar{V}_4^{-1}W_3D_3 & \Leftrightarrow \bar{V}_4^{-1}W_3D_4 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \in Gl_m.$$

Further let

$$D' = \begin{bmatrix} D_1' & D_2' & D_3' \\ \tilde{V}_1^{-1}\bar{V}_2 & \tilde{V}_1^{-1}W_1D_3 & \tilde{V}_1^{-1}W_1D_4 \\ 0 & \Leftrightarrow B_3 & \Leftrightarrow B_4 \end{bmatrix}, \ \bar{D} := \begin{bmatrix} C_2 & D_1 & D_2 \\ C_4 & D_3 & D_4 \\ 0_l & \Leftrightarrow B_3 & \Leftrightarrow B_4 \end{bmatrix},$$

where

$$\begin{aligned} D_1' &= C_2(I + \bar{V}_4^{-1}W_3\tilde{V}_1^{-1}\bar{V}_2), \\ D_2' &= D_1 + C_2\bar{V}_4^{-1}W_3\tilde{V}_1^{-1}W_1D_3, \\ D_3' &= D_2 + C_2\bar{V}_4^{-1}W_3\tilde{V}_1^{-1}W_1D_4. \end{aligned}$$

Then with (5.11) and (5.1) it holds

$$\tilde{P}\tilde{Q}^{-1}(s) =$$

$$F_1\Big(\begin{bmatrix} C_1 \\ C_3 \\ \Leftrightarrow A_3 \end{bmatrix}(sI \Leftrightarrow A_1 + [A_2, B_1, B_2]F\begin{bmatrix} C_1 \\ C_3 \\ \Leftrightarrow A_3 \end{bmatrix})^{-1}[A_2, B_1, B_2] + D'\Big)F_2 =$$

$$F_1\Big((I + D'F)P + (D' \Leftrightarrow \bar{D} \Leftrightarrow D'F\bar{D})Q\Big)\Big(FP + (I \Leftrightarrow F\bar{D})Q\Big)^{-1}F_2.$$

Thus

$$\begin{aligned} \langle\begin{bmatrix} \tilde{P} \\ \tilde{Q} \end{bmatrix}\rangle &= \langle\begin{bmatrix} F_1 & 0 \\ 0 & F_2^{-1} \end{bmatrix}\begin{bmatrix} I + D'F & D' \Leftrightarrow \bar{D} \Leftrightarrow D'F\bar{D} \\ F & I \Leftrightarrow F\bar{D} \end{bmatrix}\begin{bmatrix} P \\ Q \end{bmatrix}\rangle \\ &= \langle V_l\begin{bmatrix} I_{p-k} & 0 & 0 \\ 0 & \tau W & 0 \\ 0 & 0 & I_m \end{bmatrix}V_l\begin{bmatrix} P \\ Q \end{bmatrix}\rangle, \end{aligned}$$

where the last equality can be verified by using the equations $(I+\bar{V}_4^{-1}W_3\tilde{V}_1^{-1}\bar{V}_2)\ \bar{V}_4^{-1}W_3 = \bar{V}_4^{-1}W_3\ \tilde{V}_1^{-1}W_1$ and $\Leftrightarrow\tilde{V}_1^{-1}\bar{V}_2\ \bar{V}_4^{-1}W_3 = I\Leftrightarrow\tilde{V}_1^{-1}W_1$ and making some matrix calculations.

With the equation (5.12) it follows finally

$$\begin{aligned}\tilde{T}(\bar{\Sigma}) &= \langle\begin{bmatrix}\hat{\tau} & 0\\ 0 & I\end{bmatrix}V_l\begin{bmatrix}\tilde{P}\\ \tilde{Q}\end{bmatrix}\rangle = \langle\begin{bmatrix}\hat{\tau} & 0\\ 0 & I\end{bmatrix}\begin{bmatrix}I_{p-k} & 0 & 0\\ 0 & \tau W & 0\\ 0 & 0 & I_m\end{bmatrix}V_l\begin{bmatrix}P\\ Q\end{bmatrix}\rangle\\ &= \langle\begin{bmatrix}T & 0\\ 0 & I\end{bmatrix}V_l\begin{bmatrix}P\\ Q\end{bmatrix}\rangle = \overline{\begin{bmatrix}T & 0\\ 0 & I\end{bmatrix}}\tilde{T}(A,B,C,D).\end{aligned}$$

In a further step we can assume $\bar{\Sigma} = (A,B,TC,TD) = (\bar{A},\bar{B},\bar{C},\bar{D})$ in standard-form and then we get analogously

$$\tilde{T}(\Sigma) = \tilde{T}(\bar{A},\bar{B}S,\bar{C},\bar{D}S) = \overline{\begin{bmatrix}I & 0\\ 0 & S^{-1}\end{bmatrix}}\tilde{T}(\bar{A},\bar{B},\bar{C},\bar{D}).$$

□

Now we can show the coincidence of the maps $\hat{T}$ and $\tilde{T}$ defined in Definition 4.2 and in (5.9).

Lemma 5.4 *Let $r \leq n \leq N$, $k = n\Leftrightarrow r$, and*

$$\Sigma = (\begin{bmatrix}A_1 & A_2\\ A_3 & A_4\end{bmatrix}, \begin{bmatrix}0 & B_2\\ I_k & B_4\end{bmatrix}\varrho, \tau\begin{bmatrix}C_1 & C_2\\ 0 & I_k\end{bmatrix}, \tau\begin{bmatrix}0 & D_2\\ 0 & 0\end{bmatrix}\varrho) \in \hat{\mathcal{L}}^n_{m,r,p}$$

with $D_2 \in \mathbb{K}^{(p-k)\times(m-k)}$. Then $\hat{T}(\Sigma) = \tilde{T}(\Sigma)$.

Proof: Let $\mathrm{rk}A_4 = t \leq k$, $k = l+t$, and $U, V \in Gl_k$ such that $UA_4V = H := \begin{bmatrix}0_l & 0\\ 0 & I_t\end{bmatrix}$. Put $U[A_3,B_4] = [\hat{A}_3,\hat{B}_4]$ and $[A_2^{\mathrm{t}},C_2^{\mathrm{t}}]^{\mathrm{t}}V = [\hat{A}_2^{\mathrm{t}},\hat{C}_2^{\mathrm{t}}]^{\mathrm{t}}$. It follows

$$\begin{aligned}&\Sigma\overset{s}{\sim}(\begin{bmatrix}A_1 & \hat{A}_2\\ \hat{A}_3 & H\end{bmatrix}, \begin{bmatrix}0 & B_2\\ I_k & \hat{B}_4\end{bmatrix}S, T\begin{bmatrix}C_1 & \hat{C}_2\\ 0 & I_k\end{bmatrix}, T\begin{bmatrix}0 & D_2\\ 0 & 0\end{bmatrix}S) =:\\ &\hat{\Sigma} =: (\hat{A},\hat{B}S,T\hat{C},T\hat{D}S)\end{aligned}$$

with

$$S = \begin{bmatrix}U & 0\\ 0 & I_{m-k}\end{bmatrix}\varrho \in Gl_m,\ T = \tau\begin{bmatrix}I_{p-k} & 0\\ 0 & V\end{bmatrix} \in Gl_p.$$

For the transfer matrices of the state space systems associated with Σ and $\tilde{\Sigma} = (\hat{A}, \hat{B}, \hat{C}, \hat{D})$ choose coprime factorizations

$$PQ^{-1}(s) = \begin{bmatrix} C_1 \\ \Leftrightarrow A_3 \end{bmatrix} (sI \Leftrightarrow A_1)^{-1}[A_2, B_2] + \begin{bmatrix} C_2 & D_2 \\ \Leftrightarrow A_4 & \Leftrightarrow B_4 \end{bmatrix},$$

$$\hat{P}\hat{Q}^{-1}(s) = \begin{bmatrix} C_1 \\ \Leftrightarrow \hat{A}_3 \end{bmatrix} (sI \Leftrightarrow A_1)^{-1}[\hat{A}_2, B_2] + \begin{bmatrix} \hat{C}_2 & D_2 \\ H & \Leftrightarrow \hat{B}_4 \end{bmatrix} = \hat{U} PQ^{-1}(s)\hat{V}$$

with $\hat{U} = \begin{bmatrix} I_{p-k} & 0 \\ 0 & U \end{bmatrix} \in Gl_p$, $\hat{V} = \begin{bmatrix} V & 0 \\ 0 & I_{m-k} \end{bmatrix} \in Gl_m$. Then

$$\hat{T}(\Sigma) = \langle \begin{bmatrix} \tau & 0 \\ 0 & \varrho^{-1} \end{bmatrix} V_k \begin{bmatrix} P \\ Q \end{bmatrix} \rangle = \langle \begin{bmatrix} T & 0 \\ 0 & S^{-1} \end{bmatrix} V_k \begin{bmatrix} \hat{U} & 0 \\ 0 & \hat{V}^{-1} \end{bmatrix} \begin{bmatrix} P \\ Q \end{bmatrix} \rangle \quad (5.13)$$

$$= \langle \begin{bmatrix} T & 0 \\ 0 & S^{-1} \end{bmatrix} V_k \begin{bmatrix} \hat{P} \\ \hat{Q} \end{bmatrix} \rangle = \hat{T}(\hat{\Sigma}). \quad (5.14)$$

With Lemma 5.2 it holds

$$\tilde{T}(\tilde{\Sigma}) = \hat{T}(\tilde{\Sigma}) = \langle V_k \begin{bmatrix} \hat{P} \\ \hat{Q} \end{bmatrix} \rangle. \quad (5.15)$$

Moreover, one can transform via equivalence

$$eqvil(\hat{E}_n, \hat{A}, \hat{B}, \hat{C}, \hat{D}) \overset{\text{se}}{\sim} (\hat{E}_n, \begin{bmatrix} \tilde{A} & 0 \\ 0 & I_t \end{bmatrix}, \begin{bmatrix} \tilde{B} \\ 0 \end{bmatrix}, M[\tilde{C}, 0], M\tilde{D})$$

$$\overset{\text{e}}{\sim} (\hat{E}_{n-t}, \tilde{A}, \tilde{B}, M\tilde{C}, M\tilde{D})$$

with $M = \begin{bmatrix} I_{p-k} & 0 & 0 \\ 0 & 0 & I_l \\ 0 & I_t & 0 \end{bmatrix}$ and suitable matrices

$$(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}) = (\begin{bmatrix} \tilde{A}_1 & \tilde{A}_2 \\ \tilde{A}_3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \tilde{B}_1 & \tilde{B}_2 \\ I_l & 0 & \tilde{B}_3 \end{bmatrix}, \begin{bmatrix} \tilde{C}_1 & \tilde{C}_2 \\ \tilde{C}_3 & 0 \\ 0 & I_l \end{bmatrix}, \begin{bmatrix} 0 & \tilde{D}_1 & \tilde{D}_2 \\ 0 & \tilde{D}_3 & \tilde{D}_4 \\ 0_l & 0 & 0 \end{bmatrix}).$$

Therefore $(\hat{E}_n, \hat{A}, \hat{B}S, T\hat{C}, T\hat{D}S) \overset{\text{e}}{\sim} (\hat{E}_{n-t}, \tilde{A}, \tilde{B}S, TM\tilde{C}, TM\tilde{D}S)$ and Lemma 5.3 together with the equations (5.13), (5.14), and (5.15) yield

$$\tilde{T}(\Sigma) = \tilde{T}(\tilde{A}, \tilde{B}S, TM\tilde{C}, TM\tilde{D}S) = \overline{\begin{bmatrix} T & 0 \\ 0 & S^{-1} \end{bmatrix}} \tilde{T}(\tilde{A}, \tilde{B}, M\tilde{C}, M\tilde{D})$$

$$
\begin{aligned}
&= \overline{\begin{bmatrix} T & 0 \\ 0 & S^{-1} \end{bmatrix}} \tilde{T}(\tilde{\Sigma}) = \overline{\begin{bmatrix} T & 0 \\ 0 & S^{-1} \end{bmatrix}} \hat{T}(\tilde{\Sigma}) = \langle \begin{bmatrix} T & 0 \\ 0 & S^{-1} \end{bmatrix} V_k \begin{bmatrix} \hat{P} \\ \hat{Q} \end{bmatrix} \rangle \\
&= \hat{T}(\Sigma).
\end{aligned}
$$

□

Now we can close with the

Proof of Theorem 4.3: It remains to prove the well-definedness of $\mathcal{T}$ and the property $\Sigma \overset{s}{\sim} \Sigma' \Leftrightarrow \mathcal{T}(\Sigma) = \mathcal{T}(\Sigma')$ for $\Sigma, \Sigma' \in \hat{\mathcal{L}}_{m,r,p}$. Let $\Sigma \in \hat{\mathcal{L}}_{m,r,p}$ and $\nu(\Sigma') \overset{e}{\sim} \nu(\Sigma) \overset{e}{\sim} \nu(\tilde{\Sigma})$ with $\nu(\Sigma') \in \mathcal{L}^k_{m,r,p}$ and $\nu(\tilde{\Sigma}) \in \mathcal{L}^n_{m,r,p}$ in (ϱ', τ')-standard-form resp. $(\tilde{\varrho}, \tilde{\tau})$-standard-form, where ν is as in (3.1). Then $\nu(\Sigma') \overset{e}{\sim} \nu(\tilde{\Sigma})$ and by Lemma 5.4 it is $\tilde{T}(\Sigma') = \hat{T}(\Sigma')$ and $\tilde{T}(\tilde{\Sigma}) = \hat{T}(\tilde{\Sigma})$. On the other side it is $\tilde{T}(\Sigma') = \tilde{T}(\tilde{\Sigma})$, since canonical standard-forms equivalent to Σ' and $\tilde{\Sigma}$ must be strong equivalent to each other (by Remark 2.3 and Theorem 2.4), so we can use Lemma 5.1. This shows the well-definedness of $\mathcal{T}$ and part "$\Rightarrow$" of the equivalence. For "$\Leftarrow$" note that by Lemma 5.4 $\mathcal{T}(\Sigma) = \tilde{T}(\Sigma)$, thus we can use Lemma 5.1 again. □

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