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Legendre-Tau Approximations for LQR Feedback Control of Acouustic Pressure Fields*

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Abstract

Active noise control techniques have become increasingly important in problems where noise reduction is required without addition of significant amount of mass. In this work, a one dimensional example of an active noise control problem is presented. The problem is formulated as a periodic linear quadratic tracking problem. Then an approximation framework based on the Legendre-Tau method for approximating the control system is developed and stability and convergence results are given for approximation. Numerical examples are presented to illustrate convergence of the computational method and the dependence of the solutions on the various parameters which define the control problem.

1 Introduction

In this paper we investigate an active noise control problem within the context of optimal control theory. Motivation for our efforts arises in an advanced turbo-prop aircraft design which offers a significant reduction in fuel consumption. These turbo-prop engines produce noisy interiors, particularly when operated at low frequencies. The suppression of the internal sound field by introduction of additional "secondary sources" of sound offers a possible solution to the interior noise problem without adding significant weight to the aircraft.

There have been several studies in recent years investigating this problem employing frequency domain techniques (see [1],[16],[21]), but only

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recently in [8] has the problem been considered as an optimal control problem in a state space setting. Our presentation of these results offers a brief summary of that formulation. The focus of this study is the development of a framework for approximation of the active noise control problem based on the Legendre-Tau method. Stability and convergence of this numerical scheme will be discussed along with results concerning convergence of the approximate control variables to those of the original infinite-dimensional problem. Convergence results for the control problems are given in the context of an abstract semigroup convergence framework as given in [2], [3], [9], [10]. To demonstrate the feasibility of the resulting computational techniques, we also present sample numerical results which illustrate this convergence.

1.1 Formulation of the problem

Let p_1 be a pressure field in the bounded domain Ω due to exterior noise sources in the absence of control sources. Generally, p_1 is composed of a finite number of sinusoidal waves. For simplicity in illustrating the state space approach, we assume that Ω is the one dimensional domain $\Omega = (-1, 1)$, and p_1 consists of a single wave $p_1(x,t) = \hat{p}_1(x)e^{i\omega t}$.

The first assumption greatly simplifies arguments for the approximation results of Section 3, but the preliminary results of [8] given in Section 2 are valid for any open, bounded set in $\mathbf{R}^3$, and in particular, the ideas presented here can be extended to higher dimensional domains. The one-dimensional version of our problem is of interest in itself since many initial experimental studies related to modeling and control are carried out in wave ducts–e.g., see [14].

Let p_2 represent the pressure field (not necessarily harmonic) due to control sources in the absence of the offending noise. With $\alpha, \beta, \gamma > 0$ and the air density $\rho = 1$, we assume p_2 is governed by the following system:

$$\begin{cases} \partial_t^2 p_2 = \gamma^2 \bigtriangleup p_2 + f & \text{in } \Omega \times [0,\infty) \\ 0 = \alpha p_2 + \beta \partial_t p_2 + \partial_n p_2 & \text{on } \partial\Omega \times [0,\infty) \\ p_2(0) = 0 & \text{in } \Omega \\ \partial_t p_2(0) = 0 & \text{in } \Omega, \end{cases}$$

where f is taken to be $\chi(\hat{\Omega})F(t)$, with $\hat{\Omega}$ being the support of the single control of amplitude $F(t)$, (a single control source is chosen only for simplicity in exposition), and the boundary conditions are the partial absorbing, partial reflecting conditions discussed in [7],[8], [10].

It is anticipated that after the control mechanism is activated at $t = 0$, the pressure field p_2 should approach a steady periodic state p_3 in a stable manner. With $\tau = 2\pi/\omega$, the steady state p_3 can be expected to be governed by:

$$\begin{cases} \partial_t^2 p_3 = \gamma^2 \triangle p_3 + f & \text{in } \Omega \times [0,\tau] \\ 0 = \alpha p_3 + \beta \partial_t p_3 + \partial_n p_3 & \text{on } \partial\Omega \times [0,\tau] \\ p_3(0) = p_3(\tau) & \text{in } \Omega \\ \partial_t p_3(0) = \partial_t p_3(\tau) & \text{in } \Omega. \end{cases} \tag{1.1}$$

If the transient dynamics are ignored, the control problem can be formulated as finding a control F, so that it minimizes the total pressure field $|p_1 + p_3|$ in an "optimal way."

1.2 Preliminary results

In order to use well-established results in control theory, we first write (1.1) in the following first order form:

$$\begin{cases} \partial \begin{pmatrix} p_3 \\ \partial_t p_3 \end{pmatrix} = \begin{pmatrix} 0 & I \\ \gamma^2 \Delta & 0 \end{pmatrix} \begin{pmatrix} p_3 \\ \partial_t p_3 \end{pmatrix} + \begin{pmatrix} 0 \\ \chi(\hat{\Omega} F(t) \end{pmatrix} & \\ & \text{in } \Omega \times (0,\tau) \\ 0 = \alpha p_3 + \beta \partial_t p_3 + \partial_n p_3 & \text{on } \partial\Omega \times [0,\tau) \\ \begin{pmatrix} p_3(0) \\ \partial_t p_3(0) \end{pmatrix} = \begin{pmatrix} p_3(\tau) \\ \partial_t p_3(\tau) \end{pmatrix} & \text{in } \Omega \,. \end{cases}$$

Define the state space $X = H^1(-1,1) \times L^2(-1,1)$, which is a Hilbert space with the usual product topology inner product $(\cdot,\cdot)_X$. It is also a Hilbert space with the equivalent inner product $\langle \cdot,\cdot \rangle_{\alpha,\gamma}$ given by

$$\langle \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \rangle_{\alpha,\gamma} = \alpha(u_1,u_2)_{L^2(\partial\Omega)} + (\nabla u_1, \nabla u_2)_{L^2(\Omega)} + \gamma^{-2}(v_1,v_2)_{L^2(\Omega)} \tag{1.2}$$

and the associated norm, $|\cdot|^2_{\alpha,\gamma} = \langle \cdot,\cdot \rangle_{\alpha,\gamma}$.

Define

$$A = \begin{bmatrix} 0 & I \\ \gamma^2 \Delta & 0 \end{bmatrix}$$

with

$$\mathcal{D}(A) = \{(u,v) \in X \,|\, u \in H^2, v \in H^1, \alpha u + \beta v + \partial_n u = 0\}.$$

It can be shown (e.g., see [8]) that the operator A has the following properties:

(i) If X is equipped with $\langle \cdot, \cdot \rangle_{\alpha,\gamma}$, A is X- dissipative on $\mathcal{D}(A)$ with $\mathcal{R}(\lambda - A) = X$ for $\lambda > 0$. Therefore, by the Lumer-Phillips Theorem, A is the infinitesimal generator of a contraction semigroup $T(t)$ on X.

(ii) For $\lambda > 0$, A has compact resolvent $[\lambda - A]^{-1}$. Therefore, A has only discrete eigenvalues in its spectrum.

(iii) For some μ_0, $M_0 > 0$, we have that

$$|T(t)|_{\alpha,\gamma} \leq M_0 e^{-\mu_0 t} \text{ for } t \geq 0. \tag{1.3}$$

(The third result requires additional regularity constraints on $\partial\Omega$ if the domain Ω is a general bounded, open set in $\mathbf{R}^3$.)

2 A Periodic Linear Quadratic Tracking Problem

In this section, we formulate the acoustic problem as a periodic linear quadratic tracking (LQT) problem as follows: We wish to find among functions $F \in L^2(0, \tau : U)$ a function F^* which minimizes

$$J(F) = \int_0^\tau \{(M[P_1 + P_3], [P_1 + P_3])_X + \theta(F, F)_{L^2}\} dt$$

subject to

$$\begin{cases} \dot{P}_3 = AP_3 + BF \quad 0 \leq t \leq \tau \\ P_3(0) = P_3(\tau) \end{cases}$$

where M is a self-adjoint, nonnegative operator, θ is a control design parameter, and $U = \mathbf{R}^1$. Here A is the generator defined in the previous section and we define

$$BF(t) = \begin{bmatrix} 0 \\ \\ \chi(\hat{\Omega})F(t) \end{bmatrix},$$

$$P_1(x,t) = \hat{P}_1(x)e^{i\omega t} = \begin{bmatrix} \hat{p}_1 \\ i\omega\hat{p}_1 \end{bmatrix} e^{i\omega t},$$

and

$$P_3(x,t) = \begin{bmatrix} p_3 \\ \partial_t p_3 \end{bmatrix}.$$

Under the assumptions of detectability and stabilizibility which in our case follow from the decay estimate in (1.3), we have (e.g., see [9], [10]) that the *optimal control*, F^*, is given by

$$F^*(t) = -\theta^{-1}B^*GP_3(t) - \theta^{-1}B^*r(t),$$

where G satisfies the *Algebraic Riccati Equation*

$$GA + A^*G + M - \theta^{-1}GBB^*G = 0,$$

and r is a tracking variable satisfying $r(x,t) = \hat{r}(x)e^{i\omega t}$ where

$$\hat{r}(x) = -[i\omega + (A^* - \theta^{-1}GBB^*)]^{-1}M\hat{P}_1.$$

One can also show that $A - \theta^{-1}BB^*G$ generates a C_0-semigroup, $S(t)$, on X such that

$$\|S(t)\|_X \leq M_1 e^{-\mu_1 t} \qquad \forall t \geq 0 \tag{2.1}$$

for some $M_1, \mu_1 > 0$.

From the observations above, we can conclude that the *optimal state* satisfies:

$$\begin{cases} \dot{P}_3 = (A - \theta^{-1}BB^*G)P_3 - \theta^{-1}BB^*r & 0 \leq t \leq \tau \\ P_3(0) = P_3(\tau). \end{cases}$$

Moreover, there exists an $\hat{F}^*$ such that the optimal control is given by

$$F^* = \hat{F}^* e^{i\omega t}$$

and thus the optimal control in our case is sinusoidal.

These findings suggest the following control strategy for P_2:

$$\begin{cases} \dot{P}_2 = (A - \theta^{-1}BB^*G)P_2 - \theta^{-1}BB^*r & t > 0, \\ P_2(0) = 0. \end{cases}$$

From (2.1), one can argue that $|P_2 - P_3| \to 0$ as $t \to \infty$, and thus the control strategy for P_2 is at least asymptotically optimal (see [8], [10]).

3 The Finite Dimensional Approximation of the Linear Quadratic Tracking Problem

3.1 General formulation

In this section, we will apply the Legendre-Tau method to the periodic tracking problem outlined in Section 2. The idea is to use the Legendre-Tau method to construct finite-dimensional control systems which approximate the dynamics of the original system. Such spectral methods have been demonstrated to be superior to the usual spline Galerkin techniques in several problems governed by hyperbolic systems [7], [5].

After formulating a sequence of finite-dimensional periodic tracking problems, we will present sufficient conditions for the convergence of the corresponding Riccati operators and the optimal solutions. While the general ideas behind our approach (stability plus consistency implies convergence) for Galerkin type methods are becoming widely known, the details for the Legendre-Tau approximation schemes are quite different. Hence we present the arguments in some detail.

In spectral methods, in particular in the Legendre-Tau methods presented here, the approximating spaces X_N, (the trial spaces), in which we seek our solutions are taken to be finite-dimensional subspaces of $\mathcal{D}(A)$. Thus the elements in X_N satisfy the boundary conditions. The approximation scheme is defined by projecting the differential equation onto spaces Y_N, (the test spaces), which are appropriately defined finite dimensional spaces spanned by polynomials.

In Galerkin schemes, $X_N = Y_N$, but in Legendre-Tau methods the spaces Y_N in general are different from the spaces X_N. Our spectral approximations are of the form

$$\begin{cases} Q_N\{\dfrac{dw^N}{dt} - Aw^N - BF(t)\} = 0 \quad 0 \leq t \leq \tau \\ w^N(0) = w^N(\tau) = w_0^N, \end{cases} \tag{3.1}$$

where Q_N is the orthogonal projection of X onto Y_N.

In our discussions, the following spaces will be used:

$$\begin{aligned} \mathcal{P}_N &= \text{ space of polynomials of degree } \leq N \\ X_N &= \{(u,v) \in \mathcal{P}_N \times \mathcal{P}_N : \ \alpha u + \beta v + \partial_n u|_{\partial\Omega} = 0\} \\ Y_N &= \mathcal{P}_N \times \mathcal{P}_{N-2} \ . \end{aligned}$$

The orthogonal projection $Q_N : X \to Y_N$ is defined by

$$Q_N = \begin{bmatrix} P_{N,\alpha}^{(1)} & 0 \\ 0 & P_{N-2}^{(0)} \end{bmatrix},$$

where $P_{N,\alpha}^{(1)}$ is the orthogonal projection of the space H^1 onto $\mathcal{P}_N$ with respect to the H^1_α-inner product (e.g., see (1.2)), defined as

$$\langle u_1, u_2 \rangle_{H^1_\alpha} = \alpha(u_1, u_2)_{L^2(\partial\Omega)} + (\nabla u_1, \nabla u_2)_{L^2(\Omega)},$$

and $P_{N-2}^{(0)}$ is the orthogonal projection of L^2 onto $\mathcal{P}_{N-2}$ with respect to the L^2 norm. Now we define a projection operator Π_N from X to X_N for the Legendre-Tau method as follows:

$$\Pi_N \begin{pmatrix} \sum_{n=0}^{\infty} u_n \phi_n \\ \\ \sum_{n=0}^{\infty} v_n \phi_n \end{pmatrix} = \begin{pmatrix} \sum_{n=0}^{N} u_n \phi_n \\ \\ \sum_{n=0}^{N-2} v_n \phi_n + \sum_{m=1}^{2} b_m \phi_{N-2+m} \end{pmatrix} \tag{3.2}$$

where $\phi_n, n = 0, 1, \ldots,$ are Legendre polynomials of degree n, and $b_m, m = 1, 2,$ are chosen so that projected elements satisfy the boundary constraints

$$\mathcal{B}(\Pi_N \begin{pmatrix} u \\ v \end{pmatrix}) = 0 \quad \text{for all } \begin{pmatrix} u \\ v \end{pmatrix} \in X;$$

here $\mathcal{B}$ is the linear boundary operator associated with X_N. We see that Π_N as defined maps X to X_N. If we denote the restriction of Π_N to Y_N by Π_N^0, we can easily see that Π_N can be written as

$$\Pi_N \begin{pmatrix} u \\ v \end{pmatrix} = \Pi_N^0 Q_N \begin{pmatrix} u \\ v \end{pmatrix} \quad \text{for all } \begin{pmatrix} u \\ v \end{pmatrix} \in X.$$

In order to have a uniquely defined solution to the approximate system as defined in (3.1), it is essential to have X_N and Y_N of the same dimension. Since X_N and Y_N are both finite-dimensional, we need to show that there exists a bijective map between the two spaces. The following lemma establishes the needed result.

Lemma 3.1: Π_N^0 *is a bijective map from* Y_N *to* X_N .

Proof: We first need to show for any $(u, v)^T \in X_N$, there exists $(\hat{u}, \hat{v})^T \in Y_N$ such that $\Pi_N^0(\hat{u}, \hat{v})^T = (u, v)^T$. For any $(u, v)^T \in X_N$, we may take $(\hat{u}, \hat{v})^T = Q_N(u, v)^T \in Y_N$. From definition of Π_N, we have

$$\Pi_N^0(\hat{u}, \hat{v})^T = \Pi_N^0 Q_N(u, v)^T = \Pi_N(u, v)^T = (u, v)^T.$$

The arguments that Π_N^0 is one-to-one are equally straightforward (details can be found in [F, p.48]).

From the proof above we gather that Π_N^0 is actually the inverse of the orthogonal projection Q_N restricted to X_N, i.e., $\Pi_N^0 = (Q_N^0)^{-1}$, where $Q_N^0 = Q_N \mid_{X_N}$ and moreover, $\Pi_N = \Pi_N^0 Q_N = (Q_N^0)^{-1} Q_N$. Note that Π_N is non-orthogonal with

$$Q_N \Pi_N = Q_N (Q_N^0)^{-1} Q_N = Q_N$$

and

$$\Pi_N Q_N = \Pi_N^0 Q_N Q_N = \Pi_N^0 Q_N = \Pi_N.$$

These last two relationships are essential in setting up the finite-dimensional approximation scheme. Now we can proceed with defining our finite-dimensional operators.

Let $A_N = A\Pi_N^0$ and $B_N = Q_N B$ and put $z^N(t) = Q_N w^N(t)$ or $w^N(t) = \Pi_N^0 z^N(t)$. Then (3.1) is equivalent to

$$\dot{z}^N(t) = A_N z^N(t) + B_N F(t). \tag{3.3}$$

In the above, A_N is an operator form Y_N to Y_N and (3.3) is an ordinary differential equation on Y_N. The corresponding solution semigroup generated by A_N will be denoted by $T^N(t)$.

Since in Legendre-Tau methods, the dynamics of the system are defined in the sense of Y_N, we need to define our approximating operators, and the resulting control and tracking equations in the sense of Y_N as well. Before proceeding to formulate the finite-dimensional approximation of the tracking problem, we need to state the following two definitions, the first one of which is adopted from [15].

Definition 3.1: N **Inner-Product and Norm.** *For* $w_1, w_2 \in X_N$, *define the "N inner-product" by* $\langle w_1, w_2\rangle_N \overset{def}{=} (Q_N w_1, w_2)_X$, *and define the "N-norm" by* $|w|_N = |Q_N w|_X$. *Denote by* $\mathcal{L}(N)$ *the space of bounded linear-operators on* X_N *equipped with the operator norm, induced by* $|\cdot|_N$.

Note: By Lemma 3.1, one can see that the seminorm $|\cdot|_N$ is actually a norm.

We also need the following notion of convergence, i.e., "Q-convergence" which can be found in [12]. The definition has been modified appropriately to express convergence in the space Y_N, instead of X_N.

Definition 3.2: Q**-Convergence** *of a sequence of bounded linear operators* $\{D_N\}$ *on* Y_N *to a bounded linear operator* D *on* X *is achieved if*

$$|D_N Q_N u - Du|_N \to 0 \text{ for each } u \in X.$$

This is denoted by $D_N \overset{Q}{\to} D$.

In order to define a control problem in the finite-dimensional space Y_N, one needs to insure the existence of admissible controls (i.e., controls $F^N \in U$, the control space, that result in finite cost). Thus we require the following condition to hold:

Condition 3.1: *For each* ${z_0}^N \in Y_N$, *there exist admissible controls* $F^N \in L_2(0, \infty : U)$ *for (3.3) and the corresponding cost function, and any admissible control drives the corresponding state* $z^N(t)$ *to zero as* $t \to \infty$.

Under this assumption the existence of a unique optimal control and a unique nonnegative self-adjoint solution to the Algebraic Riccati Equation can be established for the approximate problems. In our problem, if we take $M_N = Q_N M$, and assume that the pair (A_N, B_N) is stabilizable and (A_N, M_N) is detectable, Condition 3.1 is guaranteed. Moreover, it can be shown that the optimal control F_N^* for the approximate control system in Y_N is given by

$$F_N^*(t) = -\theta^{-1} B_N^* (G_N \bar{w}^N(t) - r^N(t)) \tag{3.4}$$

where $G_N \in \mathcal{L}(N)$ is the unique non-negative self-adjoint solution of the Algebraic Riccati Equation in the sense of Y_N

$$A_N^* G_N + G_N A_N - G_N B_N \theta^{-1} B_N^* G_N + M_N = 0 \tag{3.5}$$

and $\bar{w}^N$ is the solution of

$$Q_N[\dot{w}^N - A_N w^N - B_N F_N^*] = 0.$$

The tracking variable r^N, in (3.4) is of the form

$$r^N(x,t) = \hat{r}^N(x) e^{i\omega t}$$

where $\hat{r}^N(x)$ satisfies the equation

$$\hat{r}^N(x) = -[i\omega + (A_N^* - \theta^{-1} G_N B_N B_N^*)]^{-1} M_N \, \hat{z}_d^N.$$

The finite-dimensional cost function is

$$J^N(F) = \int_0^\tau \{\langle M_N(z^N - z_d^N), (z^N - z_d^N)\rangle_N + \theta \langle F, F\rangle_N\} dt \tag{3.6}$$

where $z_d^N = Q_N z_d$ is assumed to be sinusoidal in time with period $\tau = \dfrac{2\pi}{\omega}$.

As in the infinite-dimensional case, one can argue that optimal control (3.4) is actually sinusoidal, i.e., there exists an $\hat{F}_N^*$ such that the optimal control is given by $F_N^* = \hat{F}_N^* e^{i\omega t}$.

3.2 Convergence of the scheme

In this section, we establish convergence of the approximation scheme presented in the previous section. Moreover, we prove strong convergence of the adjoint approximating semigroups $T^{N^*}(t)$ to $T^*(t)$ in the state space X. In order to proceed with convergence in X , a version of the Trotter-Kato theorem as given in [26] will be stated.

Let $(Y_N, |\cdot|)_N)$ be a sequence of Banach spaces which converges in the sense of Kato to a Banach space $(X, \|\cdot\|)$ (see [K]), i.e., for each N there exists a bounded linear operator $Q_N : X \to Y_N$ such that

(J1) $|Q_N| \leq c_1$ for all N with c_1 independent of N,

(J2) $lim_{N\to\infty}|Q_N y|_N = \|y\|$ for all $y \in X$,

(J3) there exists a constant $c_2 \geq 0$ such that for all N and all $y \in Y_N$ there exists an $x \in X$ with

$$y = Q_N x \text{ and } \|x\| \leq c_2 |y|_N \ .$$

Under these conditions the following result is valid:

Theorem 3.1 *Let A and A_N be the infinitesimal generators of strongly continuous semigroups $T(t)$ and $T^N(t)$ on X and Y_N , respectively. Suppose that*

(A) (Stability) there exists constants M, ω such that

$$|T^N(t)|_N \leq Me^{\omega t} \text{ for all } t \geq 0 \text{ and all } N$$

and

(B) (Consistency) there exists $\lambda \in \rho(A) \cap \bigcap_{N=1}^{\infty} \rho(A_N)$ such that for all $z \in X$

$$lim_{N\to\infty}|(\lambda I - A_N)^{-1} Q_N z - Q_N(\lambda I - A)^{-1} z|_N = 0$$

Then for all $z \in X$

$$lim_{N\to\infty}|T^N(t)Q_N z - Q_N T(t)z|_N = 0$$

uniformly for t in bounded intervals.

Using the properties of Legendre polynomials and appropriate error estimates, (see [17]), one can easily show that Q_N, the orthogonal projection from X to Y_N as defined in this section, satisfies conditions (J1)- (J3). Indeed, one readily establishes ([17],[20]) that $Q_N z \to z$ for all $z \in X$.

3.2.1 Uniform dissipativity and stability

Lemma 3.2: *For the Legendre-Tau method we have*

(a) Dissipativity: For all $w^N \in X_N$, $w^N = (u^N, v^N)$

$$\langle Aw^N(t), Q_N w^N(t)\rangle_{\alpha,\gamma} \leq 0$$

(b) Stability: For some $M \geq 1$ and $\omega \in R$

$$|T^N(t)|_N \leq Me^{\omega t}$$

where M and ω are independent of N.

Proof: For part (a) we see that

$$\langle Aw, Q_N w\rangle_{\alpha,\gamma} = \langle \begin{pmatrix} v \\ \gamma^2 u_{xx} \end{pmatrix}, Q_N \begin{pmatrix} u \\ v \end{pmatrix}\rangle_{\alpha,\gamma}$$

where

$$w = \begin{pmatrix} u \\ v \end{pmatrix} \in X_N.$$

Using the definition of Q_N, we may write this as

$$\langle Aw, Q_N w\rangle_{\alpha,\gamma} = \langle \begin{pmatrix} v \\ \gamma^2 u_{xx} \end{pmatrix}, \begin{pmatrix} P^{(1)}_{N,\alpha} u \\ P^{(0)}_{N-2} v \end{pmatrix}\rangle_{\alpha,\gamma} \tag{3.7}$$

$$= \alpha(v, P^{(1)}_{N,\alpha} u)_{0,\partial\Omega} + (v_x, (P^{(1)}_{N,\alpha} u)_x)_{L^2}$$

$$+ (u_{xx}, P^{(0)}_{N-2} v)_{L^2}.$$

Since $P^{(0)}_{N-2}$ is the L_2-projection onto $\mathcal{P}_{N-2}$, and $u_{xx} \in \mathcal{P}_{N-2}$ for $u \in \mathcal{P}_N$, from integration by parts we obtain the following:

$$(u_{xx}, P^{(0)}_{N-2} v)_{L^2} = -(u_x, v_x)_{L^2} + (\partial_n u, v)_{0,\partial\Omega}.$$

Also from the definition of $P^{(1)}_{N,\alpha}$, we have:

$$\langle P^{(1)}_{N,\alpha} u, v\rangle_{H^1_\alpha} = \langle u, v\rangle_{H^1_\alpha} \qquad \text{for all } v \in \mathcal{P}_N\ .$$

Since $w \in X_N \subseteq \mathcal{P}_N \times \mathcal{P}_N$, the first two terms in (3.7) may be written

$$\langle v, u\rangle_{H^1_\alpha} = \alpha(v, u)_{0,\partial\Omega} + (u_x, v_x)_{L^2}.$$

Combining the above we find

$$\langle A_N w, Q_N w\rangle_{\alpha,\gamma} = \alpha(v, u)_{0,\partial\Omega} + (v_x, u_x)_{L^2} + (\partial_n u, v)_{0,\partial\Omega} - (u_x, v_x)_{L^2}.$$

Since w satisfies the boundary condition ($w \in X_N$), we have

$$\langle A_N w, Q_N w\rangle_{\alpha,\gamma} = \alpha(v, u)_{0,\partial\Omega} - \alpha(u, v)_{0,\partial\Omega} - \beta(v, v)_{0,\partial\Omega}$$

$$= -\beta(v, v)_{0,\partial\Omega} \leq 0$$

for $\beta > 0$.

Considering part (b), we first observe that if the dissipativity of A_N in part (a) is satisfied, one can readily show that (see [17], sec 10.5)

$$\|Q_N w^N(t)\|^2_X \leq \|Q_N w^N_0\|^2 + \exp(t) \int_0^t \|f(\tau)\|^2_{L^2} d\tau \qquad \text{for all } t > 0$$

for $w^N = (u^N, v^N)^T \in X_N$ and $f = \chi(\Omega)F$.

For $f = 0$, we have

$$\|Q_N w^N(t)\|_X^2 \leq \|Q_N w_0^N\|_N^2 \leq C\|w_0^N\|_N^2 \quad \text{for } t \in [0, \tau]$$

since Q_N is an orthogonal projection. Now take $Q_N w^N(t) = \tilde{w}^N(t) = T^N(t)\psi$ for any $\psi \in Y_N$, where T^N is the semigroup generated by the operator A_N on Y_N. The above inequality now reads

$$\|T^N(t)\psi\|_X^2 \leq C\|\psi\|_X^2 \text{ for } t \in [0, \tau].$$

Therefore, we have a constant C independent of N such that for $t \in [0, \tau]$

$$|T^N(t)|_N \leq C$$

and M_1 and ω can be obtained from standard semigroup arguments.

3.2.2 Consistency

It is considerably easier to verify assumption (B) of Theorem 3.2.1 for A_N and A if $0 \in \rho(A) \cap \bigcap_{N=1}^{\infty} \rho(A_N)$. In our problem it can be readily shown that this is the case. We can also obtain condition (B) for $(A_N)^*$ and A^* in an analogous way.

Lemma 3.3 *Suppose* $0 \in \rho(A) \cap \bigcap_{N=1}^{\infty} \rho(A_N)$. *Then*

$$lim_{N\to\infty}|(A_N)^{-1}Q_N z - Q_N A^{-1} z|_N = 0$$

and

$$lim_{N\to\infty}|((A_N)^*)^{-1}Q_N z - Q_N (A^*)^{-1} z|_N = 0$$

for all $z \in X$.

Proof: From the definition of A_N and A_N^*, we have

$$(A_N)^{-1} = Q_N A^{-1} \text{ and } ((A_N)^*)^{-1} = Q_N (A^*)^{-1}.$$

Therefore for any $z \in X$

$$|(A_N)^{-1}Q_N z - Q_N A^{-1} z|_N = |Q_N A^{-1}(Q_N z - z)|_N.$$

The result follows from (J1) and $Q_N z \to z$. The proof for the adjoint operators is completely analogous.

The basic convergence result for the scheme in the state space X is presented in the following theorem.

Theorem 3.2 *Let $T^N(t)$ be the semigroup on Y_N generated by A_N, i.e., $T^N(t) = e^{A_N t}$, $t \geq 0$. Then for all $z \in X$*

$$lim_{N\to\infty} T^N(t)Q_N z = T(t)z \text{ and } lim_{N\to\infty} T^N(t)^* Q_N z = T^*(t)z$$

uniformly for t in bounded intervals.

Proof: By Lemmas 3.2 and 3.3, the conditions (A) and (B) of Theorem 3.1 are satisfied for $T(t)$ and $T^N(t)$, with $\lambda = 0$ in condition (B). Hence for all $z \in X$,

$$lim_{N\to\infty} |T^N(t)Q_N z - Q_N T(t)z|_N = 0$$

uniformly on bounded t-intervals. From the definition of N-norm and the fact that $\|Q_N T(t)z - T(t)z\|_X \to \infty$ as $N \to \infty$ uniformly on bounded t-intervals, we obtain

$$lim_{N\to\infty} T^N(t)Q_N z = T(t)z \ .$$

Remark: (*Rate of Convergence*) In addition to the convergence result above, a rate of convergence for the approximating semigroup can be obtained. In fact, in [20] by imposing certain regularity assumptions on the solutions it was shown that the error is bounded by an inverse polynomial in N. For details see [20].

3.3 Convergence of the approximating controls

In this section, we will discuss the convergence of the approximating feedback control system to the original infinite dimensional one and will present sufficient conditions to achieve convergence. In order to establish that we have good approximations of the optimal control system, we require the following conditions to hold.

Condition 3.2: *The operator valued functions B_N and M_N are uniformly bounded in the operator norm for all N. Therefore we can find a constant C such that $\|B_N\|_{\mathcal{L}(N)} \leq C$, $\|M_N\|_{\mathcal{L}(N)} \leq C$. The operator M_N is non-negative and self-adjoint.*

Condition 3.3:

a) *For each $z \in X$, we have $T^N(t)Q_N z \xrightarrow{Q} T(t)z$ with the convergence uniform in t on bounded subsets of* $[0, \infty)$.

b) *For each $z \in X$, we have $T^{N^*}(t)Q_N z \xrightarrow{Q} T(t)^* z$ with the convergence uniform in t on bounded subsets of* $[0, \infty)$.

c) *For each $v \in U, B_N v \xrightarrow{Q} Bv$ and for each $z \in X$*

$$B_N^* Q_N z \to B^* z \ .$$

d) For each $z \in X \quad M_N Q_N z \stackrel{Q}{\to} Mz$.

If Conditions 3.1, 3.2 and 3.3 hold, we have the following convergence result (see Theorem 2.3 in [2], [3], [10], and Theorem 4.1 in [9]).

Theorem 3.3 *Suppose conditions 3.1, 3.2 and 3.3 hold and let* G_N *denote the unique non-negative self-adjoint Riccati operator on* Y_N *for the problem involving (3.3), (3.6). Further assume that a unique non-negative self-adjoint Riccati operator on* X *for the infinite-dimensional problem exists. Let* $S(t)$ *and* $S^N(t)$ *be the semigroups generated by* $A - B\theta^{-1}B^*G$ *and* $A_N - B_N\theta^{-1}B_N^*G_N$ *on* X *and* Y_N *respectively. Suppose* $\|S(t)z\| \to 0$, $t \to \infty$ *for all* $z \in X$.

If there exist positive constants M_1, M_2, *and* ω *independent of* N *and* t *such that*

$$|S^N(t)|_N \leq M_1 e^{-\omega t} \text{ for } t \geq 0 \quad N = 1, 2, \ldots \tag{3.8}$$

and

$$|G_N|_N \leq M_2 \tag{3.9}$$

then

$$G_N Q_N z \stackrel{Q}{\to} Gz \text{ for every } z \in X,$$

$$r^N(t) \stackrel{Q}{\to} r(t),$$

$$F_N(t) \stackrel{Q}{\to} F(t),$$

where convergence is uniform in t *on bounded subsets of* $[0, \infty)$.

Condition (3.8) is essential in establishing the proof. This condition is called "Uniform Exponential Stabilizability", and can either be proved directly for the approximation schemes that define S^N or can be implied if one has both uniform detectability and uniform stabilizability. The following theorem (see e.g., Theorem 2.4 in [2] and Theorem 4.2 in [9]) illustrates this point.

Theorem 3.4: *Suppose Conditions 3.1 and 3.3 hold, that* (A_N, B_N) *are uniformly (in* N*) stabilizable and* (A_N, M_N) *are uniformly detectable. Then unique nonnegative self-adjoint solutions* $G_N \in \mathcal{L}(\mathcal{N})$ *of (3.5) exist and satisfy (3.8) and (3.9).*

After discussing these sufficient conditions for convergence of the approximate control system to the original problem, in the following section we will establish that these conditions are satisfied for the acoustic problem.

3.3.1 Approximation results for the acoustic problem

As we have seen in Section 1, by taking $P_3 = (p_3, \partial_t p_3)^T$ where p_3 denotes the steady periodic pressure field, we can write the state equation as

$$\begin{cases} \dot{P}_3 = AP_3 + BF \\ \\ P_3(0) = P_3(\tau). \end{cases} \tag{3.10}$$

The operators A and B are defined as follows:

$$A = \begin{bmatrix} 0 & I \\ \gamma^2\Delta & 0 \end{bmatrix}$$

on

$$\mathcal{D}(A) = \{(u,v) \in X = H^1 \times L^2 : u \in H^2(\Omega), v \in H^1(\Omega) \quad 0 = \alpha u + \beta v + \partial_n u\},$$

$$BF = \begin{bmatrix} 0 \\ \\ \chi(\hat{\Omega})F(t) \end{bmatrix}$$

where $\hat{\Omega} \subseteq (-1,1)$ represents the domain of action for the control.

The finite dimensional approximation to (3.10) is of the form

$$Q_N \frac{\partial}{\partial t} \begin{bmatrix} u^N \\ v^N \end{bmatrix} = Q_N(A\Pi_N) \begin{bmatrix} u^N \\ v^N \end{bmatrix} + Q_N \begin{bmatrix} 0 \\ BF \end{bmatrix} \tag{3.11}$$

for $(u^N, v^N)^T \in X_N$ and $(u^N)_t = v^N$. If we take $Q_N(u^N, v^N)^T = (\tilde{u}^N, \tilde{v}^N)^T \in Y_N$, and recall that $\Pi_N = \Pi_N^0 Q_N$, we can rewrite (3.11) as

$$\frac{\partial}{\partial t} \begin{pmatrix} \tilde{u}^N \\ \tilde{v}^N \end{pmatrix} = Q_N A \Pi_N^0 \begin{pmatrix} \tilde{u}^N \\ \tilde{v}^N \end{pmatrix} + Q_N \begin{pmatrix} 0 \\ BF \end{pmatrix}. \tag{3.12}$$

Now we can proceed to check the sufficient conditions given above in Theorem 3.3 for convergence for the acoustic problem.

So far we have established the stability and convergence of the scheme for our problem in Lemma 3.2 and Theorem 3.2 , respectively. By analogous arguments, the convergence of the adjoint semigroup can also be proved. The following lemmas demonstrate the validity of the rest of the conditions expressed in Condition 3.3.

Lemma 3.4: *For $M_N = Q_N M$, and each $z \in X$*

$$M_N Q_N z \xrightarrow{Q} Mz\ .$$

Proof: $|M_N Q_N z - Mz|_N = \|Q_N Q_N M Q_N z - Q_N M z\|_X$. By using $Q_N^2 = Q_N$, we have

$$\le \|Q_N M(Q_N z - z)\|.$$

Since M_N is a bounded operator,

$$|M_N Q_N z - Mz|_N \le C\|Q_N z - z\|_X,$$

and hence the desired convergence follows from that of $Q_N z \to z$.

In an analagous manner, we can prove the following:

Lemma 3.5: *For $B_N = Q_N B$, and each $v \in U, B_N v \xrightarrow{Q} Bv$ and for each $z \in X$*

$$B_N^* Q_N z \to B^* z.$$

Thus far, we have obtained conditions 3.2 and 3.3, but we still need to establish "uniform exponential stablizibility" for our system. This will be done in the following section.

3.4 Uniform exponential stability

In this section we obtain a uniform exponential decay rate for the approximate semigroup $T^N(t)$ following the ideas in [8] and [27]. In the analysis of the decay estimate for the solution $u = u(t)$ of (1.1), given in [8], a Liapunov function of the following form is used:

$$H(t) = \frac{1}{2}tE(u,t) + 2\gamma^{-2}(\ell \cdot \nabla u(t), \partial_t u(t))_{L^2} + \gamma^{-2}((\nabla \cdot \ell - 1)u(t), \partial_t u(t))_{L^2} \tag{3.13}$$

where

$$E(u,t) = \alpha\|u(t)\|_{0,\partial\Omega}^2 + \|\nabla u(t)\|_{L^2}^2 + \gamma^{-2}\|\partial_t u(t)\|_{L^2}^2 \tag{3.14}$$

and ℓ is a vector field defined on Ω.

The main ideas in establishing the decay estimate for the infinite dimensional case consist of arguing the following along solutions u of (1.1):

(i) There exists a constant $T > 0$ such that for all $t \ge T, H(t) \ge 0$;

(ii) There exist $\hat{T} > 0$ such that for $t \ge \hat{T}$

$$\dot{H}(t) \le -\frac{1}{4}E(u,t) + c_0\|u(t)\|_{0,\partial\Omega}^2$$

(iii) There exist a constant $\bar{c}$ independent of u such that

$$\int_0^\infty E(u,t)dt \le \bar{c}E(u,0).$$

We proceed to develop a similar theory for the approximate solutions obtained using the Legendre-Tau method. The ideas presented in this section were inspired by [8] as well as by [7]. However, the arguments are quite different from earlier results for Galerkin, finite difference or spectral methods. Indeed, these results are, to our knowledge, the first to demonstrate that Legendre-Tau methods can be analyzed with such ideas. While our treatment is for a one dimensional domain Ω, the arguments will be given in a more general form that readily generalize to cases for higher dimensional Ω satisfying certain geometric regularity (i.e.: rectangular regions in two dimensions).

First we define

$$E_N(u^N,t) = |T^N(t)Q_N P_{3,0}^N|_N^2 = |P_3^N(t)|_N^2 \tag{3.15}$$

where

$$P_3^N(t) = \begin{bmatrix} u^N(\cdot,t) \\ \\ v^N(\cdot,t) \end{bmatrix} \in X_N,\ P_{3,0}^N = \begin{bmatrix} u_0^N \\ \\ v_0^N \end{bmatrix},$$

and

$$|\cdot|_N^2 = \|Q_N(\cdot)\|_X^2,$$

so that

$$E_N(u^N,t) = \alpha\|P_{N,\alpha}^{(1)}u^N(t)\|_{0,\partial\Omega}^2 + \|\nabla P_{N,\alpha}^{(1)}u^N(t)\|_{L^2} + \gamma^{-2}\|P_{N-2}^{(0)}v^N(t)\|_{L^2} \tag{3.16}$$

which can be written in shorthand notation:

$$E_N(u^N,t) = \|u^N(t)\|_{N,H_\alpha^1}^2 + \|v^N(t)\|_{N,L^2}^2.$$

In the one-dimensional case the vector field ℓ can be taken to be x, and in our problem $\Omega = (-1,1)$; for ease in exposition we take $\gamma = 1$ below. Now define $H_N(t)$ analogously to $H(t)$:

$$H_N(t) = \frac{t}{2}E_N(u^N,t) + \int_\Omega 2P_{N-2}^{(0)}v^N(x\cdot u_x^N). \tag{3.17}$$

Differentiating $H_N(t)$ in t, and using (3.11) (with B=0) we obtain

$$\dot{H}_N(t) = \frac{1}{2}[\|P_{N,\alpha}^{(1)}u^N\|_{H_\alpha^1}^2 + \|P_{N-2}^{(0)}v^N\|_{L^2}^2] \tag{3.18}$$

$$+t[\langle P_{N,\alpha}^{(1)}\dot{u}^N, u^N\rangle_{H^1_\alpha} + (P_{N-2}^{(0)}\dot{v}^N, v^N)_{L^2}]$$
$$+2\int_{-1}^{1} P_{N-2}^{(0)}\dot{v}^N(x\cdot u_x^N) + 2\int_{-1}^{1} P_{N-2}^{(0)}v^N(x\cdot \dot{u}_x^N)\ .$$

For ease in exposition in the remainder of these arguments, hereafter we drop the superscript N, from u^N and v^N, reminding the the reader that $(u,v)\in X_N$ in the arguments below. In (3.18), for the first two terms in the bracket, by using the definition of $P_{N,\alpha}^{(1)}$, and $P_{N-2}^{(0)}$, one can readily obtain the following:

$$\langle P_{N,\alpha}^{(1)}v, u\rangle_{H^1_\alpha} + (P_{N-2}^{(0)}\partial_x^2 u, v)_{L^2} = -\beta(v,v)_{0,\partial\Omega}\ .$$

Rewriting (3.18) we find

$$\begin{aligned} \dot{H}_N(t) \quad &= \quad -\frac{1}{2}\left[\left\|P_{N,\alpha}^{(1)}u\right\|_{H^1_\alpha}^2 + \left\|P_{N-2}^{(0)}v\right\|_{L^2}^2\right] \\ &\quad + \left\|P_{N,\alpha}^{(1)}u\right\|_{H^1_\alpha}^2 + \left\|P_{N-2}^{(0)}v\right\|_{L^2}^2 \\ &\quad -t\beta(v,v)_{0,\partial\Omega} + 2\int_{-1}^{1} P_{N-2}^{(0)}\partial_x^2 u(x\cdot v_x) \\ &\quad +2\int_{-1}^{1} P_{N-2}^{(0)}v(x\cdot u_x). \end{aligned} \tag{3.19}$$

Next we consider the last two integrals in (3.19). We need the following two lemmas:

Lemma 3.6: *For $(u,v)\in X_N$, the following inequality holds:*

$$\begin{aligned} 2\int_{-1}^{1}(P_{N-2}^{(0)}v)(x\cdot v_x)dx \quad &\leq \quad -\int_{-1}^{1} v^2(x)dx \\ &\quad + \quad \int_{\partial\Omega} v^2(x)\ell\cdot\eta d\sigma \end{aligned} \tag{3.20}$$

where the above boundary integral is for the one-dimensional domain $(-1,1)$.

Proof: First, we rewrite $P_{N-2}^{(0)}v$ as $P_{N-2}^{(0)}v - v + v$. It follows that

$$2\int_{-1}^{1} P_{N-2}^{(0)}v(x\cdot v_x)dx = 2\int_{-1}^{1}(P_{N-2}^{(0)}v - v)(x\cdot v_x) + 2\int_{-1}^{1} v(x\cdot v_x).$$

For the first term on the right side, we use the special properties of the Legendre polynomials

$$P^{(0)}_{N-2}v - v = -\hat{v}_{N-1}\phi_{N-1} - \hat{v}_N\phi_N$$

where the $\hat{v}_n$'s are the Legendre coefficients in the expansion of v. Also

$$x \cdot v_x = x \cdot \sum_{n=0}^{N-1} (\hat{v}_n)^1 \phi_n$$

where

$$(\hat{v}_n)^1 = (2n+1) \sum_{\substack{p=n+1 \\ p+n\ odd}}^{N-1} \hat{v}_p .$$

Then we will use the following relation

$$\phi_{k+1}(x) = \frac{2k+1}{k+1} x\phi_k(x) - \frac{k}{k+1}\phi_{k-1}(x).$$

Hence, from orthogonality of ϕ_n's, it follows that

$$\int_{-1}^{1} (P^{(0)}_{N-2}v - v)(x \cdot v_x)dx == -(\hat{v}_{N-1})^2 \frac{(2N-1)}{2N-3} - \frac{2N}{2N-1}(\hat{v}_N)^2 \leq 0.$$

For the second term

$$2\int_\Omega v \cdot (x \cdot v_x)$$

we use the identity

$$2(\ell \cdot \nabla v(x))v(x) + \sum_{k=1}^{n} \ell_{kk}(x)v^2(x) = \text{ div } (v^2(x) \cdot \ell).$$

Noting that in our case $\ell(x) = x$ and the second term on the left side is just $v^2(x)$, we obtain

$$2\int_\Omega v(x \cdot v_x) = -\int_\Omega v^2(x) + \int_{\partial\Omega} v^2(x)\ell \cdot \eta d\sigma \ .$$

Putting all the above together, we have the desired inequality.

Lemma 3.7: *For* $(u,v) \in X_N$ *the following statement holds:*

$$2\int_{-1}^{1} (P^{(0)}_{N-2}\partial_x^2 u)(x \cdot u_x)dx = -\int_{-1}^{1} \partial_x u \cdot \partial_x v + \int_{\partial\Omega} (2(x \cdot \partial_x u)\frac{\partial u}{\partial \eta} - |\partial_x u|^2 x \cdot \eta)d\sigma. \tag{3.21}$$

Proof:

$$2\int_{-1}^{1}(P_{N-2}^{(0)}\partial_x^2 u)(x\cdot u_x)dx = 2(\partial_x^2 u, P_{N-2}^{(0)}(x\cdot u_x))_{L^2}.$$

Since $\partial_x^2 u \in \mathcal{P}_{N-2}$, by definition of $P_{N-2}^{(0)}$ we have the last term equals $2(\partial_x^2 u, x\cdot u_x)_{L^2}$. Next we use the following identity

$$\begin{aligned} 2\triangle u(\ell\cdot\nabla u) \quad &+ \sum_{k=1}^{N}\ell_{kk}u\triangle u = \text{ div }\{2\nabla u(\ell\cdot\nabla u) \\ &- \quad |\nabla u|^2\cdot\ell + u(\sum_{k=1}^{N}\ell_{kk}\nabla u)\} - 2\nabla u\cdot\nabla u\ . \end{aligned}$$

Integrating the above over the domain Ω with $\ell = x$, we have:

$$\begin{aligned} 2\int_\Omega \triangle u\cdot(x\cdot\nabla u) &= \int_{\partial\Omega}(2x\cdot\nabla u)\frac{\partial u}{\partial\eta} - |\nabla u|^2 x\cdot\eta \\ &+ \int_{\partial\Omega} u\frac{\partial u}{\partial\eta}d\sigma - \int_\Omega 2\nabla u\cdot\nabla u - \int_\Omega u\cdot\triangle u. \end{aligned}$$

Using

$$-\int_\Omega u\triangle u = \int_\Omega \nabla u\cdot\nabla u - \int_{\partial\Omega} u\cdot\frac{\partial u}{\partial\eta}d\sigma,$$

we find

$$2\int_\Omega \triangle u\cdot(x\cdot\nabla u) = -\int_\Omega \nabla u\cdot\nabla u + \int_{\partial\Omega}(2(x\cdot\nabla u)\frac{\partial u}{\partial\eta} - |\nabla u|^2 x\cdot\eta)d\sigma.$$

Interpreting the above for the one-dimensional domain $\Omega = (-1,1)$, we obtain the desired equality.

Using these two lemmas, we rewrite the expression for $\dot{H}_N(t)$ in (3.19) as

$$\begin{aligned} \dot{H}_N(t) \quad &\leq \quad -\frac{1}{2}[\|P_{N,\alpha}^{(1)}u\|_{H_\alpha^1}^2 + \|P_{N-2}^{(0)}v\|_{L^2}^2] \\ &+ \quad (u_x,u_x)_{L^2} + \alpha\|u\|_{0,\partial\Omega}^2 + (P_{N-2}^{(0)}v,v)_{L^2} \\ &- \quad t\beta(v,v)_{0,\partial\Omega} - (v,v)_{L^2} + \int_{\partial\Omega} v^2\ell\cdot\eta d\sigma \\ &- \quad (u_x,u_x)_{L^2} + \int_{\partial\Omega}(2(x\cdot u_x)\frac{\partial u}{\partial\eta} - |\nabla u|^2\ell\cdot\eta)d\sigma. \end{aligned} \tag{3.22}$$

But

$$(P^{(0)}_{N-2}v - v, v)_{L^2} = (-\hat{v}_{N-1}\phi_{N-1} - \hat{v}_N\phi_N, \sum_{n=0}^{N}\hat{v}_n\phi_n)$$

$$= -(\hat{v}_{N-1})^2\frac{2}{2N-1} - (\hat{v}_N)^2\frac{2}{2N+1} \leq 0.$$

Thus,

$$\begin{aligned}\dot{H}_N(t) \quad &\leq \quad -\frac{1}{2}[\|P^{(1)}_{N,\alpha}u\|^2_{H^1_\alpha} + \|P^{(0)}_{N-2}v\|^2_{L^2}] \\ &+ \quad \alpha\|u\|^2_{0,\partial\Omega} - t\beta(v,v)_{0,\partial\Omega} + \int_{\partial\Omega} v^2\ell\cdot\eta d\sigma \\ &+ \quad \int_{\partial\Omega}(2(x\cdot\nabla u)\frac{\partial u}{\partial\eta} - |\nabla u|^2\ell\cdot\eta)d\sigma\ .\end{aligned}$$

(Note all the above is in one-dimension.)

Lemma 3.8: *There exists a $t_0 > 0$ such that for $t \geq t_0$*

$$\dot{H}_N(t) \leq -\frac{1}{4}E_N(u,t) + C_0\|u(t)\|^2_{0,\partial\Omega} \tag{3.23}$$

and

$$H_N(t) \geq 0. \tag{3.24}$$

Proof: Let $\epsilon > 0$ be fixed. We have the following estimates

$$2|\int_{\partial\Omega}(x\cdot\nabla u)\frac{\partial u}{\partial\eta}| \leq A_\epsilon\int_{\partial\Omega}\left(\frac{\partial u}{\partial\eta}\right)^2 d\sigma + \epsilon\int_{\partial\Omega}|\nabla u|^2 d\sigma$$

and

$$\left|\int_{\partial\Omega}|v|^2\ell\cdot\eta\, d\sigma\right| \leq k\int_{\partial\Omega}v^2 d\sigma$$

for some positive constants A_ϵ and k. Hence, it follows that

$$\dot{H}_N(t) \leq -\frac{1}{2}[\|P^{(1)}_{N,\alpha}u\|^2_{H^1_\alpha} + \|P^{(0)}_{N-2}v\|^2_{L^2}]$$

$$+\alpha\|u\|^2_{0,\partial\Omega} + \|v\|^2_{0,\partial\Omega}[-t\beta + k]$$

$$+\int_{\partial\Omega}(\epsilon - \ell\cdot\eta)|\nabla u|^2_{0,\partial\Omega} + A_\epsilon\int_{\partial\Omega}\left(\frac{\partial u}{\partial\eta}\right)^2 d\sigma.$$

But

$$\int_{\partial\Omega}\left(\frac{\partial u}{\partial\eta}\right)^2 d\sigma = \int_{\partial\Omega}(\alpha u+\beta v)^2 d\sigma \le D_\epsilon(\alpha,\beta)\|u\|^2_{0,\partial\Omega} + F_\epsilon(\alpha,\beta)\|v\|^2_{0,\partial\Omega}.$$

Therefore,

$$\begin{aligned}\dot{H}_N(t) &\le -\frac{1}{2}\left[\|P^{(1)}_{N,\alpha}u\|^2_{H^1_\alpha} + \|P^{(0)}_{N-2}v\|^2_{L^2}\right]\\ &+ C_1(\alpha,\beta)\|u\|^2_{0,\partial\Omega} + \|v\|^2_{0,\partial\Omega}\left[-t\beta + k + C_2(\alpha,\beta)\right]\\ &+ \int_{\partial\Omega}(\epsilon - \ell\cdot\eta)|\nabla u|^2 d\sigma.\end{aligned}$$

By choosing ϵ sufficiently small, and t large enough then

$$\dot{H}_N(t) \le -\frac{1}{4}[\|P^{(1)}_{N,\alpha}u\|^2_{H^1_\alpha} + \|P^{(0)}_{N-2}v\|^2_{L^2}] + C(\alpha,\beta)\|u\|^2_{0,\partial\Omega}.$$

Next we argue that there exists a $t_0 > 0$ such that for $t \ge t_0$, we have $H_N(t) \ge 0$, where

$$H_N(t) = \frac{t}{2}[\|u\|^2_{N,H^1_\alpha} + \|v\|^2_{N,L^2}] + (P^{(0)}_{N-2}v, x\cdot(u_x))_{L^2}.$$

By the Cauchy-Schwartz inequality

$$2(P^{(0)}_{N-2}v, x\cdot(u_x))_{L^2} \le C_1\|P^{(0)}_{N-2}v\|^2_{L^2} + C_2\|u_x\|^2_{L^2} \le CE_N(u,t)$$

for some constant $C > 0$. Therefore,

$$H_N(t) \ge \frac{1}{2}tE_N - CE_N = (\frac{1}{2}t - C)E(t,u),$$

and for sufficiently large t

$$H_N(t) \ge 0.$$

The results of Lemma 3.8 yield (i) and (ii) for the approximate system. It remains to establish (iii). We use (3.23) and (3.24) to get an estimate for $\int_0^\infty E_N(u,t)dt$ in terms of $E_N(u,0)$. We need the following result:

Lemma 3.9: *For any $\mu > 0$ there exist positive constants C_1, C_2 independent of μ such that*

$$\int_0^\infty e^{-2\mu t}E_N(u,t)dt \le C_1E_N(u,0) + C_2\int_0^\infty e^{-2\mu t}\|u\|^2_{N,0,\partial\Omega}dt. \quad (3.25)$$

Proof: We may multiply (3.23) by $e^{-2\mu t}$ for $\mu > 0$ fixed and integrate from 0 to ∞ to obtain

$$\int_0^\infty e^{-2\mu t}\dot{H}_N dt + \frac{1}{4}\int_0^\infty e^{-2\mu t}E_N(u,t)dt$$

$$\leq C_1 \int_0^\infty \|u\|^2_{N,0,\partial\Omega} e^{-2\mu t} dt.$$

By integrating by parts and observing that

$$|H_N(t)| \leq CE_N(u,t) \leq CE_N(u,0)$$

we have

$$\begin{aligned} 2\mu \int_0^\infty e^{-2\mu t}H_N(t)dt \quad &+ \quad \frac{1}{4}\int_0^\infty e^{-2\mu t}E_N(u,t)dt \leq C_2E_N(u,0) \\ &+ \quad C_1 \int_0^\infty e^{-2\mu t}\|u\|^2_{N,0,\partial\Omega}dt. \end{aligned}$$

Since $H_N(t) \geq 0$ for t sufficiently large, it follows that

$$\int_0^\infty e^{-2\mu t}E_N(u,t)dt \leq C_2'E_N(u,0) + C_1' \int_0^\infty e^{-2\mu t}\|u\|^2_{N,0,\partial\Omega}dt$$

with constants C_1', C_2' independent of μ.

Now observing that

$$\|u\|^2_{N,0,\partial\Omega} \leq E_N(u,t)$$

we have

$$\int_0^\infty e^{-2\mu t}E_N(u,t)dt \leq C_2E_N(u,0) + C_1 \int_0^\infty e^{-2\mu t}E_N(u,t)dt.$$

But

$$\int_0^\infty e^{-2\mu t}E_N(u,t)dt \leq C\frac{|E_N(u,0)|}{|2\mu|}.$$

Therefore,

$$\int_0^\infty e^{-2\mu t}E_N(u,t)dt \leq C'E_N(u,0)$$

for some $C' > 0$. Letting $\mu \downarrow 0$, we obtain

$$\int_0^\infty E_N(u,t)dt \leq CE_N(u,0), \tag{3.26}$$

which is (iii). To complete the uniform stablizibility for the approximate systems, we argue as follows.

From (iii) it follows that for any $\epsilon > 0$, there is a time $T_\epsilon \in (0, C/\epsilon)$ such that

$$E_N(u, T_\epsilon) \leq \epsilon E_N(u, 0) \tag{3.27}$$

Let t be fixed and set $\epsilon = \dfrac{C}{t}$. Since $E(u,t)$ is nonincreasing it follows from (3.27) that

$$E_N(u,t) \leq \frac{C}{t} E_N(u, 0) \tag{3.28}$$

or equivalently

$$|T^N(t) Q_N(u^N(0), v^N(0))|_N^2 \leq \frac{C}{t} |(u^N(0), v^N(0))|_N^2$$

for all

$$(u^N(0), v^N(0)) \in X_N.$$

Following the arguments presented in [8], one can say that there exists $t^* > 0$ and a $\mu_0 > 0$ such that

$$|T^N(t^*)|_N = e^{-\mu_0 t^*} < 1.$$

Now choose $M_0 = e^{-\mu_0 t^*} \max_{0 \leq t \leq t^*} |T^N(t)|_N$. Given $t > 0$, we have $t = mt^* + r$ with $r < t^*$ and $m \in N$, and therefore by the semigroup property

$$\begin{aligned} |T^N(t)|_N &= |(T^N)^m (t^*) T^N(r)|_N \leq |T^N(t^*)|_N^m M_0 e^{-\mu_0 t^*} \\ &= M_0 e^{-\mu_0 (m+1) t^*} \leq M_0 e^{-\mu_0 (mt^* + r)} = M_0 e^{-\mu_0 t}. \end{aligned}$$

4 Approximation Schemes and Numerical Experiments

4.1 Matrix formulation of the approximate problems

In the Legendre-Tau method, the expansion functions $\phi_n, n = 0, 1, 2, \ldots,$ are assumed to be Legendre polynomials which are orthogonal with respect to the L_2-norm on $\Omega = (-1, 1)$. For a second order hyperbolic system such as the wave equation with two boundary conditions, we seek an approximation to the solution of the form

$$(\tilde{u}^N(t,x), \tilde{v}^N(t,x))^T = \sum_{i=0}^{2N-1} w_i^N(t) \Phi_i^N(x) \tag{4.1}$$

where Φ_i^N are defined as follows:

$$\Phi_i^N = \begin{cases} (\phi_i, 0)^T & 0 \leq i \leq N \\ (0, \phi_{i-N-1})^T & N+1 \leq i \leq 2N-1 \end{cases} .$$

In the previous sections, it was assumed that $(\tilde{u}^N, \tilde{v}^N)^T \in Y_N$ satisfies an equation of the form

$$\frac{\partial}{\partial t}\begin{bmatrix} \tilde{u}^N \\ \tilde{v}^N \end{bmatrix} = Q_N\, A\Pi_N^0 \begin{bmatrix} \tilde{u}^N \\ \tilde{v}^N \end{bmatrix} + Q_N \begin{bmatrix} 0 \\ \chi(\hat{\Omega})F(t) \end{bmatrix}. \tag{4.2}$$

The above can be equivalently written in a variational form as

$$\left(\frac{\partial}{\partial t}\begin{bmatrix} \tilde{u}^N \\ \tilde{v}^N \end{bmatrix}, y\right)_X = \left(A\Pi_N^0 \begin{bmatrix} \tilde{u}^N \\ \tilde{v}^N \end{bmatrix}, y\right)_X + \left(\begin{bmatrix} 0 \\ \chi(\hat{\Omega})F(t) \end{bmatrix}, y\right)_X \tag{4.3}$$

for all $y \in Y_N$. Note that $(\cdot,\cdot)_X$ denotes the usual inner product on $X = H^1 \times L^2$. Choosing $y = \Phi_i^N$ and using (4.1), we can, in turn, write (4.3) in matrix form as

$$\underline{\mathcal{D}}_N \dot{w}^N = \underline{\mathcal{D}}_N \underline{\mathcal{A}}_N w^N + \underline{B}_N F(t) \tag{4.4}$$

where

$$(\underline{\mathcal{D}}_N) = \begin{bmatrix} \underline{K} & 0 \\ 0 & \underline{L} \end{bmatrix} \quad \text{with}$$

$$(\underline{K})_{i,j} = (\phi_i, \phi_j)_{H^1(-1,1)}, \text{ for all } \begin{matrix} 0 \le i \le N \\ 0 \le j \le N \end{matrix},$$

$$\text{and } (\underline{L})_{i,j} = (\phi_i, \phi_j)_{L^2(-1,1)} \text{ for all } \begin{matrix} 0 \le i \le N-2 \\ 0 \le j \le N-2 \end{matrix}$$

$$\underline{\mathcal{A}}_N = \underline{A}_N \underline{\Pi}_N$$

where

$$\underline{A}_N = \begin{bmatrix} 0_{(N+1)\times(N+1)} & I_{(N+1)\times(N+1)} \\ S^2 & 0_{(N-1)\times(N+1)} \end{bmatrix}.$$

In the above, $I_{(N+1)\times(N+1)}$ = Identity matrix of dimension $(N+1) \times (N+1)$, and S^2 = matrix representation of $\triangle$ (second-order differentiation operator) with respect to Legendre polynomials. It is an $(N-1)$ by $(N+1)$ matrix. Also, $\underline{\Pi}_N$ is the matrix representation for Π_N^0 with dimension $(2N+2) \times 2N$. In (4.4), we also have

$$\underline{B}_N = \begin{bmatrix} \left.\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}\right\} (N+1) \\ (\phi_0, \chi(\hat{\Omega}))_{L^2(-1,1)} \\ \vdots \\ (\phi_{N-2}, \chi(\hat{\Omega}))_{L^2(-1,1)} \end{bmatrix}$$

with $w^N = \text{col}\,(w_0^N, \ldots, w_{2N-1}^N)$. From equation (4.4), we see that the matrix representations of operators $Q_N A \Pi_N^0$ and $Q_N B$ are the following

$$\underline{\mathcal{A}}_N = \underline{A}_N \underline{\Pi}_N \quad \text{and} \quad \underline{\mathcal{B}}_N = (\underline{\mathcal{D}}_N)^{-1} \underline{B}_N. \tag{4.5}$$

We can define the matrix $\underline{\mathcal{M}}_N$, the matrix representation of operator $M_N = Q_N M$ by

$$[\underline{\mathcal{M}}_N]_{i,j} = (\Phi_i, M\Phi_j)_{X=H^1\times L^2} \tag{4.6}$$

for $0 \le i \le 2N-1, \quad 0 \le j \le 2N-1$. Now let z_d^N be defined as $Q_N z_d$:

$$z_d^N(x,t) = Q_N z_d(x,t) = -\begin{bmatrix} \hat{p}_1^N(x) \\ i\omega \hat{p}_1^N(x) \end{bmatrix} e^{i\omega t} = -\hat{P}_1^N(x) e^{i\omega t} = \hat{z}_d^N(x) e^{i\omega t}. \tag{4.7}$$

Expanding $\hat{z}_d^N$ in terms of the basis elements Φ_i, we obtain $\hat{z}_d^N = \sum_{i=0}^{2N-1} z_i^N \Phi_i$. From (4.7), we have

$$(\underline{\hat{z}}_d^N)_i = -(\underline{\mathcal{D}}_N)_{i,j}^{-1}\,(R[\hat{P}_1^N(x)])_j \tag{4.8}$$

where $\underline{\hat{z}}_d^N = (z_0^N, \ldots, z_{2N-1}^N)$, and $(R[\hat{P}_1^N(x)])_j = (\Phi_j, \hat{P}_1^N(x))_X$.

Combining the above considerations and notation, we obtain matrix equations for the optimal control problem in $\mathbf{R}^{2N}$. From the arguments presented in Section 3, the unique optimal control for the control system in $\mathbf{R}^{2N}$ is given in a feedback form by

$$\underline{F}_N^*(t) = -\theta^{-1} \underline{\mathcal{B}}_N^*\,(\underline{\mathcal{G}}_N \underline{P}_3^N + \underline{r}^N(t)) \tag{4.9}$$

where $\underline{F_N^*(t)} = (F_0^N, \ldots, F_{2N-1}^N) \in \mathbf{R}^{2N}, \underline{P_3^N} = (P_{3,0}^N, P_{3,1}^N \ldots, P_{3,2N-1}^N) \in \mathbf{R}^{2N}$, is the unique solution to the matrix equation

$$\begin{cases} \underline{\dot{P}_3^N} = (\underline{\mathcal{A}}_N - \theta^{-1} \underline{\mathcal{B}}_N \underline{\mathcal{B}}_N^T \underline{\mathcal{G}}_N) \underline{P}_3^N - \theta^{-1} \underline{\mathcal{B}}_N \underline{\mathcal{B}}_N^T \underline{r}^N \\ \underline{P_3}^N(0) = \underline{P_3}^N(\tau), \end{cases} \tag{4.10}$$

and $\underline{\mathcal{G}}_N$ is the solution to the matrix Algebraic Riccati Equation

$$\underline{\mathcal{A}}_N^T \underline{\mathcal{G}}_N + \underline{\mathcal{G}}_N \underline{\mathcal{A}}_N - \theta^{-1} \underline{\mathcal{G}}_N \underline{\mathcal{B}}_N \underline{\mathcal{B}}_N^T \underline{\mathcal{G}}_N + \underline{\mathcal{M}}_N = 0 \,. \tag{4.11}$$

The tracking variable $\underline{r}^N(x,t)$ is a vector in $\mathbf{R}^{2N}$ of the form $\underline{r}^N(x,t) = \hat{\underline{r}}^N(x)e^{i\omega t}$, with $\hat{\underline{r}}^N(x) \in \mathbf{R}^{2N}$, satisfying the following matrix equation:

$$[i\omega + (\underline{\mathcal{A}}_N^T - \theta^{-1} \underline{\mathcal{G}}_N \underline{\mathcal{B}}_N \underline{\mathcal{B}}_N^T)]\, \hat{\underline{r}}^N(x) = \underline{\mathcal{M}}_N \hat{\underline{z}}_d^N \,. \tag{4.12}$$

The cost function can now be represented as

$$J^N = \int_0^\tau \{(\underline{P}_3^N(t) - \underline{z}_d^N(t))^* \underline{\mathcal{M}}_N (\underline{P}_3^N(t) - \underline{z}_d^N(t)) + \theta \underline{F}_N^*(t) \underline{F}_N(t)\} dt \tag{4.13}$$

Equations (4.9) - (4.13) are the basic equations which form the core of our computations.

4.2 Numerical experiments

In this section we summarize some of our numerical computations carried out to test the Legendre-Tau ideas for the optimal control problem characterized by the matrix equations defined in the previous section. For different sets of parameters, convergence of the control variables such as the optimal state, the optimal control and the optimal value of the cost function were studied. In addition, some preliminary calculations to determine the effect of the location of the controls on the reduction of the overall pressure field were performed. All computations were carried out on an IBM 3090 at Brown University.

The theory as outlined in Section 3 predicts that the sinusoidal nature of the primary noise , z_d^N, and the tracking variable, r^N, make the optimal control and the optimal state sinusoidal. The first set of experiments investigated the validity of the above claim numerically. Given a certain set of parameters, first equation (4.10) was solved using the IMSL (version 10) routine, DBVPFD, and then the values were compared with the results from calculating $\underline{P}_3^N$, by assuming $\underline{P}_3^N(x,t) = \hat{\underline{P}}_3^N(x)e^{i\omega t}$ where $\hat{\underline{P}}_3^N(x)$ satisfies the equation

$$\hat{\underline{P}}_3^N(x) = -[i\omega - (\underline{\mathcal{A}}_N - \theta^{-1} \underline{\mathcal{B}}_N \underline{\mathcal{B}}_N^T \underline{\mathcal{G}}_N)]^{-1} \theta^{-1} \underline{\mathcal{B}}_N \underline{\mathcal{B}}_N^T \hat{\underline{r}}^N(x). \tag{4.14}$$

The results obtained validated the assumption that P_3^N is indeed sinusoidal. Therefore, in our computationl experiments we calculated $\underline{P}_3^N$ from (4.14) instead of integrating $\underline{P}_3^N$. To calculate the optimal control, we used the expression

$$\hat{\underline{F}}_N^*(x) = -\theta^{-1} \underline{\mathcal{B}}_N^T \, [\underline{\mathcal{G}}_N \hat{\underline{P}}_3^N + \hat{\underline{r}}^N(x)]. \tag{4.15}$$

Some tests were performed to observe the effect of the observation operator on the numerical computation. A suitable choice for M_N was found to be of the form

$$\underline{\mathcal{M}}_N = m \begin{bmatrix} \underline{K} & 0 \\ 0 & \underline{L} \end{bmatrix}$$

which provided overall reduction of noise field without m being very large (m large would make the problem computationally unstable).

The question of convergence of approximate states and feedback controls in our numerical experiments was a major part of our computational efforts. Two sets of tests were done with two different values of γ. The results are presented in the following two tables. In these tests, these parameters were used: $m = 100, \hat{z}_d^N = (0.01, 0.01i\omega), \theta^{-1} = 100, \alpha = 1, \beta = 100$, Frequency of input $= \omega = 100Hz$.

N	$(\underline{\mathcal{M}}_N(\hat{\underline{P}}_3^N + \hat{\underline{P}}_1^N), \hat{\underline{P}}_3^N + \hat{\underline{P}}_1^N)$	$\lvert F \text{ optimal}\rvert$	$J_{\min}^N$	CPU (Seconds)
4	60.3567	259.5862	1.5057×10^2	0.64×10^{-1}
8	112.0624	235.2025	1.5126×10^2	0.42
10	117.5520	232.2504	1.5134×10^2	0.76
12	120.0424	230.8727	1.5138×10^2	1.30
16	121.3743	230.1327	1.5140×10^2	2.94
20	122.1776	229.6782	1.5141×10^2	5.23
24	124.5840	228.2874	1.5145×10^2	9.36

Table 1: Convergence with $\gamma^2 = 10^{-4}$.

N	$(\underline{\mathcal{M}}_N(\hat{\underline{P}}_3^N + \hat{\underline{P}}_1^N), \hat{\underline{P}}_c^N + \hat{\underline{P}}_1^N)$	$\lvert F \text{ optimal}\rvert$	$J_{\min}^N$	CPU (Seconds)
4	1293.1815	526.1586	1.1729×10^2	0.74×10^{-1}
8	1277.7260	509.2901	1.19196×10^2	.047
10	1277.8102	509.2350	1.192015×10^2	0.83
12	1277.8145	509.2354	1.192015×10^2	1.39
16	1277.8149	509.2360	1.192018×10^2	3.25
20	1277.8298	509.2361	1.192014×10^2	5.89
24	1277.8266	509.2366	1.191932×10^2	9.83

Table 2: Convergence with $\gamma^2 = 10^5$

By comparing the values in these tables, it is clear that the rate of convergence of control variables is faster for $\gamma = 10^5$. The fact that the convergence rate is affected by γ is predicted from our theoretical convergence result (to see information regarding rate of convergence, see [20]). The constant C obtained in that result was dependent on α, β and γ. Although the exact nature of this dependence was not explored in our theoretical analysis, it is clear from our numerical observations that the larger the magnitude of γ, the faster the convergence rate.

Next, we explored the dependence of our solutions on the partially absorbing, partially reflecting boundary parameters α and β. In our calculations, α and β were studied as parameters whose ratio, α/β, affects the projection operator Π_N. In the following tests, the optimal control problem was solved for $N = 16$ for different ratios of α/β. In these runs we used the values $\gamma = 346m/s, \theta^{-1} = 10^5, Freq = 100Hz$, $m = 10^4$, $\hat{z}_d^N = (\sqrt{2.0}, \sqrt{2.0}i\omega), x_{loc} = 0.0$. Table 3 summarizes our results.

α/β	$(\underline{\mathcal{M}}_N(\underline{\hat{P}}_3^N + \underline{\hat{P}}_1^N), \underline{\hat{P}}_3^N + \underline{\hat{P}}_1^N)$	$\lvert F \text{ optimal}\rvert$	$J^N_{\min}$
0.01	5.408096×10^9	1.2838×10^5	5.4073×10^7
0.1	5.408093×10^9	1.2835×10^5	5.4089×10^7
1	5.408062×10^9	1.2829×10^5	5.4324×10^7
100	5.400589×10^9	1.2860×10^5	5.4012×10^7
1000	5.405078×10^9	1.2900×10^5	5.4080×10^7

Table 3: Different α/β's.

In the calculations above, β is kept at the constant value 1 and α is varied. It is seen that varying α does not have a noticeable effect on the results. On the other hand, if α is kept constant and β is varied, it is observed that for small values of β (i.e., small damping), more control is necessary to obtain optimal reduction of noise. For example, for $\alpha = 1$ and $\beta = 10^4$, the value for $|F$ optimal$|$ is 4.9320 $\times 10^5$ and the reduced noise field is 2.055 $\times 10^9$, while the corresponding values for $\alpha = 1,\ \beta = 1$ are $|F$ optimal$|$ = 1.282 $\times 10^5$ and $|\underline{\hat{P}}_3^N + \underline{\hat{P}}_1^N| = 5.408 \times 10^9$.

In a series of tests we also investigated the dependence of level noise reduction on the location of the controls. In order to see the effect of the location on our results clearly, we magnified the value of m, and θ^{-1} in the problem. We also used parameters with physical values for these tests. The following test was performed for $N = 16, \gamma = 346m/s, \theta^{-1} = 10^5, Freq = 100Hz, \alpha = 2178.41/s, \beta = 0.76185, m = 10^4, \hat{z}_d^N = (\sqrt{2.0}, \sqrt{2.0}i\omega)$ and a, the radius of the control, equal to 0.1. Note that the purpose of the test was to observe the effect of the location on the results, not necessarily to determine the optimal location for the best reduction of noise level. The following graph and table illustrate the results.

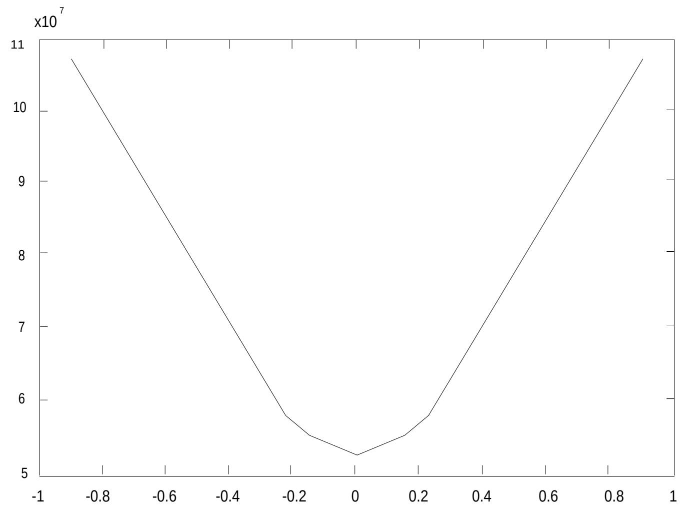

Figure 1: Effect of location of the control

$$\text{Norm of } \underline{\hat{P}}_1^N = (\underline{\mathcal{M}}_N \underline{\hat{P}}_1^N, \underline{\hat{P}}_1^N) = 3.15828 \times 10^{10}$$

X Location of control	Norm of $\underline{\hat{P}}_1^N + \underline{\hat{P}}_3^N$	$\lvert F \text{ optimal}\rvert$	$J_{\min}^N$
-0.9	1.089252×10^{10}	6.982×10^5	1.089696×10^8
-0.8	9.995285×10^9	3.645×10^5	1.000815×10^8
-0.7	9.045025×10^9	2.553×10^5	9.049933×10^7
-0.6	8.152226×10^9	2.025×10^5	8.162877×10^7
-0.5	7.353778×10^9	1.723×10^5	7.362966×10^7
-0.4	6.671984×10^9	1.534×10^5	6.672453×10^7
-0.3	6.123578×10^9	1.417×10^5	6.119115×10^7
-0.2	5.722022×10^9	1.340×10^5	5.716024×10^7
-0.1	5.476993×10^9	1.300×10^5	5.481133×10^7
0.0	5.394719×10^9	1.288×10^5	5.394764×10^7
0.1	5.477003×10^9	1.301×10^5	5.486024×10^7
0.2	5.722056×10^9	1.342×10^5	5.713618×10^7
0.4	6.672021×10^9	1.537×10^5	6.675691×10^7
0.6	8.152262×10^9	2.026×10^5	8.816427×10^7
0.8	9.995277×10^9	3.643×10^5	1.001601×10^8
0.9	1.089224×10^{10}	6.983×10^5	1.089813×10^8

Table 4: Effect of location of the control.

In Table 4, the second column represents the magnitude of the pressure field measured after it is minimized using the secondary noise source, P_3^N. A comparison between the measured value of the pressure field due to the primary noise P_1^N, and the amount of reduction of the noise pressure field, presented in the second column gives us an estimate of effectiveness of changing the location of the controls for achieving a desirable amount of reduction. We see that as the x-location changes from -0.9 to 0, the minimum cost function decreases noticeably, and our overall reduction performance improves. As we move away from $x = 0.0$, we see an increase of the minimum value of the cost function. In order to carefully study the effect of location of the controls, a more detailed and focused study is needed. The results above are only preliminary efforts in that direction.

Finally, we studied the effect of the width size of the controls (i.e., the size of $\hat{\Omega}$) on the overall performance with the following parameters: $N = 16, \gamma = 346m/s, \theta^{-1} = 10^5, Freq = 100Hz, \alpha = 2178.41/s, \beta = 0.76185, m = 10^4, \hat{z}_d^N = (\sqrt{2.0}, \sqrt{2.0}i\omega), x_{loc} = 0.0$ and a represents the radius of the control.

Radius a	Norm of $\underline{\hat{P}}_1^N + \underline{\hat{P}}_3^N$	$\lvert F$ optimal$\rvert$	$J_{\min}^N$
0.00001	5.383276×10^9	1.2824×10^5	5.384255×10^7
0.001	5.383288×10^9	1.2822×10^5	5.384746×10^7
0.01	5.383395×10^9	1.2825×10^5	5.384613×10^7
0.1	5.394714×10^9	1.2881×10^5	5.394764×10^7
0.2	5.431089×10^9	1.3031×10^5	5.447549×10^7
0.3	5.495423×10^9	1.3290×10^5	5.497428×10^7

Table 5: Effect of width of the control support.

5 Conclusions

In this paper we have formulated an active noise suppression problem (based on control through a secondary noise source) as a periodic linear quadratic tracking problem in state space. In this context we are able to give a complete state feedback theory involving a Riccati feedback gain and a tracking variable.

Approximation techniques based on Tau-Legendre ideas are presented for a one-dimensional version of this problem. It is shown that these approximations for this problem satisfy conditions sufficient to give a rather complete convergence theory. Of special interest is the use of the multiplier methods of [27] and [10] to establish the difficult uniform exponential stabilizability condition.

We also have presented sample numerical results to illustrate how such techniques can be used to investigate both qualitative and quantitative properties and characteristics of these problems.

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