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H_∞ Control of Nonlinear Systems with Sampled Measurement*

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Abstract

This paper is a continuation of the paper on the H_∞ control of nonlinear systems with sampled measurement output [9]. In this paper, in addition to the results of *nonlinear* H_∞ control problem with sampled measurement output which has been reported in [9], we present results on the H_∞ control problem with sampled measurement output for *linear* systems. The results for linear systems can be obtained by specializing the methodology developed for nonlinear systems. Comparison with former work is also made for linear systems.

1 Introduction

The H_∞ control problem has been actively studied in the last years. In the state-space formulation, if the perfect continuous or sampled state is accessible, the H_∞ control problem is easily solved in the framework of differential game theory [2]. In the case where we can use only measurement output, then the H_∞ control problem is more complicated. The H_∞ control problem with continuous-time measurement output for linear systems has been solved by many researchers, and for nonlinear systems by van der Schaft [6], by Isidori and Astolfi [4], and by Ball, Helton and Walker [1]. The H_∞ control problem of linear systems with sampled measurement output was solved by Başar and Bernhard [2] over finite time horizon and by Sun, Nagpal and Khargonekar [8] for finite and infinite time horizon.

Recently, we proposed a solution of the H_∞ control problem with sampled measurement output for nonlinear systems [9]. We gave sufficient conditions for the existence of a solution. To our best knowledge, [9] is

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the first paper on the nonlinear H_∞ control with sampled measurement output.

In this paper, we review the results on the nonlinear H_∞ control problem with sampled measurement output given in [9], and present results on linear H_∞ control problem with sampled measurement output. These results can be obtained by applying the technique developed for nonlinear systems.

2 Preliminaries

In the following discussions, we treat differential equations with jumps in the form

$$\dot{x} = F(x), \quad \text{for} \quad t \neq kT, \quad \forall k \in \boldsymbol{N} \tag{2.1}$$

$$x(kT_+) = x(kT) + x_c \tag{2.2}$$

for some x_c with the same dimension of x. Here, we give a precise definition of a solution of the differential equation (2.1) with jumps (2.2). Suppose that $x(kT)$ is given, and that a solution of (2.1) for the initial condition $x(kT) + x_c$ exists on $[kT, (k+1)T]$. Let $\tilde{x}(t)$ be this solution. Then the solution of (2.1) with jumps (2.2) is the piecewise continuous function $x(\cdot)$ defined by

$$x(t) = \tilde{x}(t) \quad \forall t \in (kT, (k+1)T].$$

Once $x((k+1)T)$ is given as the final value of the solution of (2.1) on the interval $(kT, (k+1)T]$, the solution on the interval $((k+1)T, (k+2)T]$ is given in the same way, as long as the equation (2.1) has a solution on that interval with the initial condition $x((k+1)T) + x_c$.

As a specific case of a differential equation with jumps, we consider the Riccati differential equation with jumps which is quite important in the following discussion.

Consider the Riccati differential equation of the form

$$\dot{X} + XA + A^{\mathrm{T}}X + XRX + Q = 0, \quad t \neq kT \tag{2.3}$$

with jumps given by

$$X(kT_+) = X(kT) + X_{jump}, \tag{2.4}$$

where $X_{jumps} \geq 0$. Then we have the following lemma.

Lemma: *The Riccati differential equation (2.3) with jumps (2.4) has a periodic solution with period T if and only if there exists a matrix X_0 which satisfies*

$$X_0(\Theta_{11}(T) + \Theta_{12}(T)X_{0_+}) = \Theta_{21}(T) + \Theta_{22}(T)X_{0_+}, \tag{2.5}$$

$$det(\Theta_{11}(t) + \Theta_{12}(t)X_{0_+}) \neq 0, \quad t \in (0,T] \tag{2.6}$$

where

$$X_{0_+} = X_0 + X_{jump}$$

and

$$\begin{bmatrix} \Theta_{11}(t) & \Theta_{12}(t) \\ \Theta_{21}(t) & \Theta_{22}(t) \end{bmatrix} = exp \begin{bmatrix} A & R \\ -Q & -A^{\mathrm{T}} \end{bmatrix} t.$$

Furthermore, if X_0 is positive definite then $X(t)$ is also positive definite for all t.

Proof: It is easy to see that a solution $X(t)$ of (2.3) exists on $[0,T]$ if and only if (2.6) holds. If this is the case, then we have

$$\begin{bmatrix} \Theta_{11}(t) & \Theta_{12}(t) \\ \Theta_{21}(t) & \Theta_{22}(t) \end{bmatrix} \begin{bmatrix} I \\ X_{0_+} \end{bmatrix} = \begin{bmatrix} I \\ X(t) \end{bmatrix} (\Theta_{11}(t) + \Theta_{12}(t)X_{0_+}).$$

From the second row block,

$$X(t) = (\Theta_{21}(t) + \Theta_{22}(t)X_{0_+})(\Theta_{11}(t) + \Theta_{12}(t)X_{0_+})^{-1}. \tag{2.7}$$

Equating $X(T) = X(0)$, we can see that there is a periodic solution with period T if and only if (2.5) holds. If the periodic solution $X(t)$ has a property that $X_0 = X(kT)$ is positive definite, then it is easily seen from (2.4) that $X_{0_+} = X(kT_+)$ is also positive definite. Then positive definiteness of $X(t)$ for $t \in [0,T]$ follows from the fact that $X(t)$ can be expressed as

$$X(t) = \Phi^{\mathrm{T}}(t,T)X_{0_+}\Phi(t,T) + \int_t^T \Phi^{\mathrm{T}}(t,\tau)Q\Phi(t,\tau)d\tau, \tag{2.8}$$

where $\Phi(\cdot,\cdot)$ is the transition matrix associated with $-(A + \frac{1}{2}RX(t))$, i.e.

$$\frac{\partial \Phi}{\partial t}(t,\cdot) = -\Phi(t,\cdot)\left(A + \frac{1}{2}RX(t)\right).$$

In fact, differentiating both sides of (2.8) with respect to t shows that $X(t)$ defined by (2.8) satisfies (2.3). ■

3 Nonlinear Results

We consider the nonlinear systems, described by equations of the form

$$\dot{x} = f(x) + g_1(x)w + g_2(x)u \tag{3.1}$$

$$z = h_1(x) + k_{12}(x)u \tag{3.2}$$

$$y(kT) = h_2(x(kT)) + d(kT), \quad k \in \boldsymbol{N} \tag{3.3}$$

Here, $x \in \mathcal{R}^n$ is the state of the plant, $z \in \mathcal{R}^{l_1}$ is the regulated output, $y \in \mathcal{R}^{l_2}$ is the measurement output, $w \in \mathcal{R}^{m_1}$ is the disturbance input, $u \in \mathcal{R}^{m_2}$ is the control input, and $d \in \mathcal{R}^{l_2}$ is the measurement noise. T is the sampling period of measurement. We assume that the mappings f, g_1, g_2, h_1, and h_2 are all smooth (i.e. C^∞) mappings, and that $f(0) = 0$, $h_1(0) = 0$, and $h_2(0) = 0$. As seen in (3.3), the measurement is available only at $t = kT$. Our goal is to design a dynamic feedback controller, with internal state $\xi \in \mathcal{R}^n$, which locally asymptotically stabilizes the equilibrium $(x, \xi) = (0, 0)$ of the closed-loop system and is such that, for every $w(\cdot) \in \mathcal{L}_2 \cap \mathcal{L}^c_\infty$ and $d(\cdot) \in l_2 \cap l^c_\infty$, the response $z(\cdot)$ from the initial state $(x, \xi) = (0, 0)$ is in $\mathcal{L}_2$ and satisfies

$$\frac{\int_0^\infty \|z(t)\|^2 dt}{\int_0^\infty \|w(t)\|^2 dt + \sum_{i=0}^\infty \|d(iT)\|^2} \leq \gamma^2 \tag{3.4}$$

where γ is a prespecified positive number. Here, c is chosen so that the trajectory of the closed-loop system stays close enough to an equilibrium point. This problem is referred to as the H_∞ control problem with sampled measurement output. We assume the following:

(AN1) $h_1^{\mathrm{T}}(x) k_{12}(x) = 0, \;\; k_{12}^{\mathrm{T}}(x) k_{12}(x) = I$, for all x

(AN2) $\{f, h_1\}$ is locally detectable at $x = 0$ [3], i.e. any bounded trajectory of $\dot{x}(t) = f(x(t))$ satisfying $h_1(x(t)) = 0$ for all $t \geq 0$ is such that $\lim_{t\to\infty} x(t) = 0$.

First, introduce the Hamilton-Jacobi-Isaacs inequality

$$V_x f(x) + h_1^{\mathrm{T}}(x) h_1(x) + \frac{1}{4} V_x \left(\frac{1}{\gamma^2} g_1(x) g_1^{\mathrm{T}}(x) - g_2(x) g_2^{\mathrm{T}}(x)\right) V_x^{\mathrm{T}} \leq 0. \tag{3.5}$$

Assuming the existence of a (local) solution of this Hamilton-Jacobi-Isaacs inequality is a standard starting point of the nonlinear H_∞ control. We also introduce the Riccati differential equation with jumps

$$\begin{aligned} &\dot{Q} + Q\left(A + \frac{1}{\gamma^2} B_1 B_1^{\mathrm{T}} P\right) + \left(A + \frac{1}{\gamma^2} B_1 B_1^{\mathrm{T}} P\right)^{\mathrm{T}} Q \\ &\quad + \frac{1}{\gamma^2} Q B_1 B_1^{\mathrm{T}} Q + P B_2 B_2^{\mathrm{T}} P = -\epsilon_1 I, \;\; t \neq kT \end{aligned} \tag{3.6}$$

$$Q(kT_+) = Q(kT) + \gamma^2 C_2^{\mathrm{T}} C_2 - \epsilon_2 I \tag{3.7}$$

in which $\epsilon_1 > 0, \;\epsilon_2 > 0$ are suitable parameters, and $A = \frac{\partial f}{\partial x}\big|_{x=0}$, $B_1 = g_1(0)$, $B_2 = g_2(0)$, $C_2 = \frac{\partial h_2}{\partial x}\big|_{x=0}$, $P = \frac{1}{2} \cdot \frac{\partial^2 V}{\partial x^2}\big|_{x=0}$. By Lemma, this Riccati equation with jumps (3.6), (3.7) has a periodic positive definite solution if and only if there is a positive definite matrix Q_0 satisfying

$$Q_0(\Pi_{11}(T) + \Pi_{12}(T) Q_{0_+}) = \Pi_{21}(T) + \Pi_{22}(T) Q_{0_+}, \tag{3.8}$$

$$det(\Pi_{11}(t)+\Pi_{12}(t)Q_{0_+}) \neq 0, ^\forall t \in (0,T] \tag{3.9}$$

where $Q_{0_+} = Q_0 + \gamma^2 C_2^{\mathrm{T}} C_2 - \epsilon_2 I$, and

$$\begin{bmatrix} \Pi_{11}(t) & \Pi_{12}(t) \\ \Pi_{21}(t) & \Pi_{22}(t) \end{bmatrix} = \exp \begin{bmatrix} A + \frac{1}{\gamma^2} B_1 B_1^{\mathrm{T}} P & \frac{1}{\gamma^2} B_1 B_1^{\mathrm{T}} \\ -(P B_2 B_2^{\mathrm{T}} P + \epsilon_1 I) & -(A + \frac{1}{\gamma^2} B_1 B_1^{\mathrm{T}} P)^{\mathrm{T}} \end{bmatrix} t.$$

Theorem 1: *Suppose that for some $\rho_1 > 0$ a positive definite solution $V(x)$ of the Hamilton-Jacobi-Isaacs inequality (3.5) exists in $\mathcal{B}_{\rho_1}$. Suppose also that there exists a $Q_0 > 0$ satisfying the two conditions (3.8), (3.9) and that ϵ_2 is small. Under these conditions, the controller*

$$\dot{\xi} = f(\xi) + g_1(\xi)\alpha_1(\xi) + g_2(\xi)\alpha_2(\xi), \quad t \neq kT \tag{3.10}$$

$$\xi(kT_+) = \xi(kT) + \gamma^2 Q_{0_+}^{-1} C_2^{\mathrm{T}} (y(kT) - h_2(\xi(kT))) \tag{3.11}$$

$$u = \alpha_2(\xi), \tag{3.12}$$

in which $\alpha_1(x) = \frac{1}{2\gamma^2} g_1^{\mathrm{T}}(x) V_x^{\mathrm{T}}$, $\alpha_2(x) = -\frac{1}{2} g_2^{\mathrm{T}}(x) V_x^{\mathrm{T}}$ solves the H_∞ control problem with sampled observation of measurement output.

Proof: It follows from (3.5) that for any z and w,

$$\frac{dV}{dt} + \|z\|^2 - \gamma^2 \|w\|^2 \leq \|u - \alpha_2(x)\|^2 - \gamma^2 \|w - \alpha_1(x)\|^2 \tag{3.13}$$

Define $x^e = [x^{\mathrm{T}}, \xi^{\mathrm{T}}]^{\mathrm{T}}$, and for $t \neq kT$ define the extended system

$$\dot{x}^e = f^e(x^e) + g^e(x^e) r$$

$$v = h^e(x^e) = \alpha_2(\xi) - \alpha_2(x)$$

where $r = w - \alpha_1(x)$ and

$$f^e(x^e) = \begin{bmatrix} f(x) + g_1(x)\alpha_1(x) + g_2(x)\alpha_2(\xi) \\ f(\xi) + g_1(\xi)\alpha_1(\xi) + g_2(\xi)\alpha_2(\xi) \end{bmatrix}, \quad g^e(x^e) = \begin{bmatrix} g_1(x) \\ 0 \end{bmatrix}.$$

Let $e = x - \xi$ and $W(t, x^e) = e^{\mathrm{T}} Q(t) e$. Obviously, $W(t, x^e)$ is a positive semidefinite function, since $Q(t) > 0$. Define, for $t \neq kT$, $H(t, e, x)$ as follows.

$$\begin{aligned} H(t,e,x) &= W_t + W_{x^e} f^e(x^e) + h^{e\mathrm{T}}(x^e) h^e(x^e) \\ &\quad + \frac{1}{4\gamma^2} W_{x^e} g^e(x^e) g^{e\mathrm{T}}(x^e) W_{x^e}^{\mathrm{T}}. \end{aligned} \tag{3.14}$$

Then, using (3.6) we can show that

$$H(t,e,x) = e^{\mathrm{T}} M(t,e,x) e$$

for some matrix $M(t,e,x)$ satisfying $M(t,0,x) < 0$. So, there exists a $\rho_2 > 0$ such that $H(t,e,x) \le 0$ if $(e,x) \in \mathcal{B}_{\rho_2}$. Thus, since $W(t,x^e)$ is a solution to the Hamilton-Jacobi-Isaacs inequality $H(t,e,x) \le 0$, for $t \ne kT$ we have

$$\frac{dW}{dt} + \|v\|^2 - \gamma^2\|r\|^2 \le 0. \tag{3.15}$$

From (3.13) and (3.15) we have

$$\frac{dS}{dt} + \|z\|^2 - \gamma^2\|w\|^2 \le 0 \tag{3.16}$$

for $t \ne kT$, where $S(t) = V(x(t)) + W(t, x^e(t))$. Integrate both sides of (3.16) from kT_+ to $(k+1)T$.

$$S((k+1)T) - S(kT_+) + \int_{kT_+}^{(k+1)T} \|z\|^2 dt - \gamma^2 \int_{kT_+}^{(k+1)T} \|w\|^2 dt \le 0. \tag{3.17}$$

Next consider the behavior of $W(t,x^e)$ at $t = kT$. Here, for simplicity of notations, we write $W(t)$ for $W(t,x^e(t))$. It is easy to see that from (3.7) and (3.11) we have

$$\begin{aligned} W(kT_+) - W(kT) = & \\ & -\gamma^2(C_2e(kT) + d(kT))^{\mathrm{T}}(I - \gamma^2 C_2 Q_{0_+}^{-1} C_2^{\mathrm{T}})(C_2 e(kT) + d(kT)) \\ & +\gamma^2\|d(kT)\|^2 - \epsilon_2\|e(kT)\|^2 o_3(x(kT), \xi(kT)) \end{aligned} \tag{3.18}$$

where $o_3(x(kT), \xi(kT))$ consists of terms of order higher than or equal to 3 in $x(kT)$ and $\xi(kT)$ and satisfies

$$o_3(x(kT), \xi(kT))\big|_{e(kT)=0} = 0.$$

When $\epsilon_2 = 0$,

$$I - \gamma^2 C_2 Q_{0_+}^{-1} C_2^{\mathrm{T}} = (I + \gamma^2 C_2 Q_0^{-1} C_2^{\mathrm{T}})^{-1} > 0.$$

So, for $\epsilon_2 > 0$ small enough, we still have

$$I - \gamma^2 C_2 Q_{0_+}^{-1} C_2^{\mathrm{T}} > 0.$$

Hence, from (3.18),

$$W(kT_+) - W(kT) - \gamma^2\|d(kT)\|^2 \le -\epsilon_2\|e(kT)\|^2 + o_3(x(kT), \xi(kT))$$

and we can say that there is a $\rho_3 > 0$ such that

$$W(kT_+) - W(kT) - \gamma^2\|d(kT)\|^2 \le 0 \tag{3.19}$$

if $(e(kT), x(kT)) \in \mathcal{B}_{\rho_3}$.

From (3.17) and (3.19) we have

$$S((k+1)T) - S(kT) + \int_{kT}^{(k+1)T} \|z\|^2 dt \le \gamma^2 \left(\int_{kT}^{(k+1)T} \|w\|^2 dt + \|d(kT)\|^2 \right) \tag{3.20}$$

since $V(x(kT_+)) = V(x(kT))$. Summing both sides of (3.20) from 0 to K, we have

$$S(KT) - S(0) + \int_0^{KT} \|z\|^2 dt \le \gamma^2 \left(\int_0^{KT} \|w\|^2 dt + \sum_{i=0}^{K-1} \|d(iT)\|^2 \right) \tag{3.21}$$

for any integer K. Now recall that $S(0) = 0$, because $x(0) = 0$ and $\xi(0) = 0$, and that $S(KT) \ge 0$. If the disturbances $w(\cdot)$ and $d(\cdot)$ are such that $x^e(\cdot)$ is always in $\mathcal{B}_\rho$, where $\rho = min\{\rho_1, \rho_2, \rho_3\}$, we can conclude from the previous inequality that $w \in \mathcal{L}_2$ and $d \in l_2$ implies $z \in \mathcal{L}_2$ and, moreover, (3.4) holds.

To analyze (asymptotic) stability, set $w = 0$ and $d = 0$. Then the closed-loop system becomes

$$\begin{aligned} \dot{x} &= f(x) + g_2(x)\alpha_2(\xi) \\ \dot{\xi} &= f(\xi) + g_1(\xi)\alpha_1(\xi) + g_2(\xi)\alpha_2(\xi), t \ne kT \\ \xi(kT_+) &= \xi(kT) + \gamma^2 Q_{0_+}^{-1} C_2^{\mathrm{T}} (h_2(x(kT)) - h_2(\xi(kT))). \end{aligned} \tag{3.22}$$

By definition of f, α_1, α_2, h_1 and h_2, the origin $(x, \xi) = (0, 0)$ is an equilibrium point of the closed-loop system.

The function $S(t, x) = V(x) + W(t, x^e)$ is positive definite function and $S(t) = V(x(t)) + W(t, x^e(t))$ satisfies $\dot{S}(t) \le -\|z\|^2 \le 0, \quad t \ne kT$ and $S(kT_+) \le S(kT)$ along the trajectories of (3.22). Standard arguments - with minor modifications due to the presence of jumps in $x^e(t)$ - show that $S(t, x)$ is a Lyapunov function of the closed-loop system (3.22), i.e. the closed-loop system (3.22) is stable at the origin.

To prove asymptotic stability of the closed-loop system, we only have to show that the closed-loop system has the origin as an attractor. Note that, by the definition of ρ_3, $S(kT_+) = S(kT)$ if and only if $e(kT) = 0$. $\dot{S}(t) \le 0$ for $t \ne kT$ implies $S((k+1)T) \le S(kT_+)$, and therefore $S((k+1)T) \le S(kT)$ for any $k \in \boldsymbol{N}$. Thus, the sequence $\{S(kT)\}$ is monotonically nonincreasing and bounded from below by 0 ($S(t) \ge 0$). Hence, $\lim_{k\to\infty} S(kT)$ exists. Let $\lim_{k\to\infty} S(kT) = c_0$.

Choose any $x^e(0)$ in $\mathcal{B}_\rho$ and let $\Omega_{x^e(0)}$ denote the ω-limit set of the trajectory starting at $x^e(0)$. $\Omega_{x^e(0)}$ is nonempty and invariant [7] and, by definition, $S(t, x^e) = c_0$ on $\Omega_{x^e(0)}$. Thus, for any initial state in $\Omega_{x^e(0)}$, we have $\dot{S}(t) = 0$ and $S(kT_+) = S(kT)$ there. $\dot{S}(t) = 0$ implies $h_1(x(t)) = 0$ and $\alpha_2(\xi(t)) = 0$. Furthermore, $S(kT_+) = S(kT)$ implies $e(kT) = 0$, i.e. $x(kT) = \xi(kT)$ for any $k \in \boldsymbol{N}$ and this, in turn, implies $x(kT_+) = \xi(kT+)$ (see jump condition in (3.22) Thus, the closed-loop system restricted to $\Omega_{x^e(0)}$ is given by

$$\dot{x} = f(x)$$
$$\dot{\xi} = f(\xi) + g_1(\xi)\alpha_1(\xi), \quad \forall t \geq 0.$$

Note that the closed-loop system restricted on Ω has no jumps in the state. $x(t)$ being a trajectory of $\dot{x} = f(x)$ which satisfies $h_1(x(t)) = 0$, it follows from local detectability (AN2) that $\lim_{t\to\infty} x(t) = 0$, in particular $\lim_{k\to\infty} x(kT) = 0$. This, together with $x(kT) = \xi(kT)$ implies $\lim_{k\to\infty} S(kT) = 0$. Therefore, c_0 is necessarily zero and therefore, from positive definiteness of $S(t, x^e)$ we conclude that $x^e(t)$ tends to 0. ∎

Remark: The controller state ξ is regarded as an estimation of the plant state x, and $\alpha_1(\xi)$ and $\alpha_2(\xi)$ are interpreted as estimations of the worst disturbance and the optimal control input, respectively. (3.10) is a copy of the plant with these estimations as its input. This is quite understandable, because if ξ coincides x then it is well-known that the control input (3.12) is a solution of the H_∞ control with full state information.

At sampling instants, information of the plant is available through the sampled measurement output, and the estimation ξ is corrected using this information. Corrections only at the sampling instants causes jumps in ξ, and $Q(t)$ also must have jumps to compensate these jumps in ξ.

In the stability analysis, the scalar function $S(t)$ plays the roll of a Lyapunov function, although it has jumps at the sampling instants.

4 Linear Results

In this section, we consider the linear plant given by

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u \\ z &= C_1 x + D_{12} u \\ y(kT) &= C_2 x(kT) + d(kT). \end{aligned} \tag{4.1}$$

x, w, u, z, y, and d are the same as those of nonlinear plant and the matrices A, B_1, B_2, C_1, C_2, and D_{12} are supposed to have compatible dimensions. As in the nonlinear case, we assume that

(AL1) $C_1^{\mathrm{T}} D_{12} = 0, \quad D_{12}^{\mathrm{T}} D_{12} = I$

(AL2) (C_1, A) is detectable.

The goal is to design a controller which internally stabilizes the closed-loop system and renders

$$\frac{\int_0^\infty \|z(t)\|^2 dt}{\int_0^\infty \|w(t)\|^2 dt + \sum_{i=0}^\infty \|d(iT)\|^2} < \gamma^2 \tag{4.2}$$

for any $w \in \mathcal{L}_2$ and any $d \in l_2$. Note that we have strict inequality in (4.2).

In the case of linear systems, we have sufficient conditions as follows.

Theorem 2: *The H_∞ control problem of linear systems with sampled measurement output can be solved if the algebraic Riccati equation*

$$P_L A + A^{\mathrm{T}} P_L + P_L \left(\frac{1}{\gamma^2} B_1 B_1^{\mathrm{T}} - B_2 B_2^{\mathrm{T}} \right) P_L + C_1^{\mathrm{T}} C_1 = 0 \tag{4.3}$$

has a positive semidefinite solution P such that

$$\sigma \left(A + \frac{1}{\gamma^2} B_1 B_1^{\mathrm{T}} P_L - B_2 B_2^{\mathrm{T}} P_L \right) \subset \mathcal{C}^-,$$

and there is a positive definite matrix U_0 satisfying

$$U_0(\Lambda_{11}(T) + \Lambda_{12}(T) U_{0_+}) = \Lambda_{21}(T) + \Lambda_{22}(T) U_{0_+} \tag{4.4}$$

$$det(\Lambda_{11}(t) + \Lambda_{12}(t) U_{0_+}) \neq 0, \forall t \in [0, T] \tag{4.5}$$

where $U_{0_+} = U_0 + \gamma^2 C_2^{\mathrm{T}} C_2$, and

$$\begin{bmatrix} \Lambda_{11}(t) & \Lambda_{12}(t) \\ \Lambda_{21}(t) & \Lambda_{22}(t) \end{bmatrix} = \exp \begin{bmatrix} A + \frac{1}{\gamma^2} B_1 B_1^{\mathrm{T}} P_L & \frac{1}{\gamma^2} B_1 B_1^{\mathrm{T}} \\ -P_L B_2 B_2^{\mathrm{T}} P_L & -(A + \frac{1}{\gamma^2} B_1 B_1^{\mathrm{T}} P_L)^{\mathrm{T}} \end{bmatrix} t.$$

If these conditions are satisfied, then the controller

$$\dot{\xi} = \left(A + \frac{1}{\gamma^2} B_1 B_1^{\mathrm{T}} P_L - B_2 B_2^{\mathrm{T}} P_L \right) \xi, \; t \neq kT \tag{4.6}$$

$$\xi(kT_+) = \xi(kT) + \gamma^2 U_{0_+}^{-1} C_2^{\mathrm{T}} (y(kT) - C_2 \xi(kT)) \tag{4.7}$$

$$u = -B_2^{\mathrm{T}} P_L \xi \tag{4.8}$$

solves the problem of H_∞ control with sampled measurement output.

Proof: Let $V_L(x) = x^{\mathrm{T}} P_l x$, $W_L(t, x^e) = e^{\mathrm{T}} U(t) e$, and $S_L(t, x^e) = V_L(x) + W_L(t, x^e)$. Then a similar discussion as in the proof of Theorem 1 shows that the inequality (4.2) holds.

For stability analysis, set $w = 0$ and $d = 0$. Then we have

$$\dot{S_L}(t) = -\|C_1 x\|^2 - \|u\|^2 \leq 0, \quad t \neq kT \tag{4.9}$$

$$\begin{aligned} S_L(kT_+) - S_L(kT) &= -\gamma^2 e^{\mathrm{T}}(kT)C_2^{\mathrm{T}}(I + C_2U^{-1}(kT)C_2^{\mathrm{T}})^{-1}C_2e(kT) \\ &\leq 0. \end{aligned} \tag{4.10}$$

From (4.9) and (4.10) it can be seen that along the trajectory of the closed-loop system S_L is monotonically nonincreasing, and by its definition S_L is bounded from below by 0. Therefore S_L has a limit, which might depend on the initial state, if we take $t \to \infty$. Let this limit be c_0 for some initial state (x_0, ξ_0). Boundedness of S_L and positive definiteness of U imply that $x - \xi$ is also bounded. From (4.1) and (4.8), we have

$$\dot{x} = (A - B_2B_2^{\mathrm{T}}P_L)x + B_2B_2^{\mathrm{T}}P_L(x - \xi).$$

The basic results of H_∞ control problem shows that all of the eigenvalues of $A - B_2B_2^{\mathrm{T}}P_L$ is in the left half plane [5]. If $x - \xi$ is regarded as the input of this system, then we can conclude that x is bounded, and so is ξ.

Since the trajectory of the closed-loop system is bounded, its ω-limit set exists, is nonempty and invariant. Suppose the initial state is on the ω-limit set, then whole the trajectory is on the ω-limit set and $S_L(t) = c_0$. Therefore we have

$$\dot{S_L}(t) = 0, \quad S_L(kT_+) - S_L(kT) = 0.$$

From (4.9) and (4.10), on the ω-limit set,

$$C_1x = 0, \quad u = 0,$$
$$C_2(x(kT) - \xi(kT)) = 0.$$

So, the trajectory restricted on the ω-limit set is given as the solution of

$$\dot{x} = Ax$$
$$\dot{\xi} = \left(A + \frac{1}{\gamma^2}B_1B_1^{\mathrm{T}}P_L\right)\xi, \quad \forall t \geq 0.$$

Detectability of (C_1, A) implies that

$$\lim_{t\to\infty} x(t) = 0.$$

Then

$$\lim_{t\to\infty} S_L(t) = \lim_{t\to\infty} \xi^{\mathrm{T}}(t)Q(t)\xi(t) = c_0.$$

$Q(t)$ has jumps and $\xi(t)$ does not. So, the only possibility for $\xi^{\mathrm{T}}Q\xi$ to have a limit is that

$$\lim_{t\to\infty} \xi(t) = 0, \quad \text{and} \quad c_0 = 0.$$

Thus, on the ω-limit set the trajectory converges to the origin. Therefore the closed-loop system is attractive at the origin. In the case of linear systems, attractivity implies asymptotic stability. ∎

Remark: It can be shown that change of variable $\tilde{U} = \gamma^2U^{-1}$ yields the essentially equivalent equations of the controller given in [8].

5 Conclusions

In this paper, we reviewed a new approach to solve the H_∞ control problem of nonlinear systems with sampled measurement output [8]. The (sufficient) conditions for existence of such a solution are expressed in terms of the existence of a positive definite solution of the standard Hamilton-Jacobi-Isaacs inequality and by the existence of a periodic positive definite solution of a Riccati differential equation with jumps. We also showed that if we apply the method developed for the nonlinear H_∞ control problem with sampled measurement output to linear systems, then we have sufficient conditions for the existence of a solution. The controller thus obtained for linear systems is equivalent to the controller given in [8].

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