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Time Minimal Synthesis for Planar Systems in the Neighborhood of a Terminal Manifold of Codimension One*

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Abstract

The goal of this article is to compute under generic assumptions the local optimal synthesis, for time optimal control in the plane, with terminal constraints belonging to a manifold of codimension one, for systems with a scalar control that enters linearly. This analysis is applied to the control of a class of chemical systems.

Key words: time optimal control, optimal synthesis, chemical systems

AMS Subject Classifications: 49B10, 93C10

1 Introduction

Consider a system in $\mathbf{R}^2$, of the form

$$\dot{v} = X(v) + uY(v) \quad , \quad |u| \leq 1, \tag{1.1}$$

where X and Y are analytic vector fields and let N be a regular analytic submanifold of $\mathbf{R}^2$ of codimension one. The set of admissible controls $\mathcal{U}$ is the set of measurable mappings with values in $[-1,+1]$. We shall study the following *local* problem. Let $v_0 \in N$. Compute in a sufficiently small open neighborhood U of v_0, the optimal synthesis for the time minimal control problem with terminal manifold N and system (1.1) restricted to U. This problem and its generalization to higher dimensional state space is motivated by the problem of controlling chemical batch reactors [2] and an example will be given at the end of this article. It is similar to the problem studied by Schättler and Sussmann in a series of articles, see for instance [8], [9], when the terminal manifold is reduced to a point (This problem

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is called the *point to point problem).* This article outlines an algorithm to compute all the syntheses in terms of the coefficients of the analytic expansions of X, Y and f, where N denotes locally the set of zeros of f. We shall mainly restrict our analysis to the generic situations and describe the *topological features* of the synthesis. In order to simplify the computations we shall use convenient normalizations and give at most *linear estimates* of the *switching curves* and of the *cut locus.* A more complete classification and more accurate estimates, without any normalizations, will be published later. Before presenting our analysis we must point out the following fact. Although any information concerning the accessibility set and the parametrization of its boundary coming from the works of Schättler and Sussmann can be used in our problem, all the classification can be done using evaluations of the solutions of the equations from Pontryagin Maximum Principle (PMP).

2 Definitions and Notations

Consider system (1.1) written as (X,Y) and let (x,y) be the coordinates of $v \in \mathbf{R}^2$. A coordinate system (U,v) such that the restriction of Y to U is $\frac{\partial}{\partial y}$ will be called *adapted.* The optimal control problem is *flat* if Y is everywhere tangent to N. Let $\hat{N} = \left\{(w,p) \in N \times \mathbf{R}^2 \ ; \ <p,v>=0 \ \forall v \in T_w N\right\}$ where $< \, , \, >$ denotes the standard inner product. An extremal (v,u) is a trajectory of (1.1), defined on $[T,0]$, $T<0$ and solution of PMP for the point to point time optimal control problem, see [6], an associated adjoint vector being denoted p, and (v,p,u) is called an *extremal lift.* It will be called a *BC-extremal* if it satisfies the boundary conditions $(v(0),p(0)) \in \hat{N}$. Let (v,u) be an extremal on $[T,0]$. A time $s \in [T,0]$ is called a *switching time* is s belongs to the closure of the set of $t's \in [T,0]$ where v is not C^1 and $v(s)$ will be called a *switching point.* We shall denote by K the set of switching points for BC-extremals. Let $v_0 \in N$, W shall denote the set of optimal switching points for the time minimal control problem for (1.1) restricted to a sufficiently small neighborhood of v_0, the terminal manifold being N. By convention, any piecewise analytic control is taken right continuous. The optimal closed loop function, if it exists, is denoted by $v \to u^*(v)$. For the concepts of synthesis we follow [10]. A stratum of the switching curve W is of *first kind* if the optimal trajectories are tangent to the stratum and of *second kind* if they are transverse. The *splitting line* L is the set of points where the optimal feedback is not unique (it will form the *cut locus*).

Let $v_1 \in \mathbf{R}^2$ and let us denote by $v(t,v_1,u)$ the trajectory of (1.1), when defined, associated to $u \in \mathcal{U}$ and starting from v_1 at time $t=0$. Let $t>0$ and $A^+(v_1,t)$ be the set of points $\{v(t,v_1,u) \ ; \ u \in \mathcal{U}\}$ *accessible* to v_1 in time t and let $A^-(v_1,t)$ the set of points v_2 such that v_1 is accessible

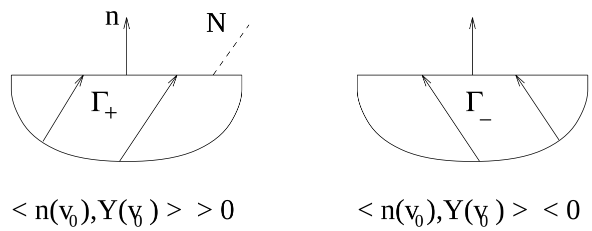

Figure 1:

to v_2, in time t. The *accessibility set* is $A^+(v_1) = U_{t>0}\ A^+(v_1, t)$ and let $A^-(v_1) = U_{t>0)}\ A^-(v_1, t)$. For $v \in \mathbf{R}^2$, we shall denote by $C(v)$ the convex set $\{X(v) + uY(v)\ ;\ |u| \leq 1\}$. Let $v_0 \in N$, in our analysis we shall assume that $Y(v_0)$ *is not zero.* Moreover it is not restrictive to suppose that $C(v_0)$ *lies entirely in one half-space limited by* $\mathrm{T_{v_0}N}$. If $X(v_0)+Y(v_0)$ or $X(v_0)-Y(v_0)$ is tangent to N, we are in the *exceptional case.* Let $n(v)$ be the normal to N at v. *In the nonexceptional case,* n(v) *will be oriented, near* v_0 *towards the half-space containing* $C(v_0)$.

3 Classification

As usual, the classification is realized by increasing the codimension of the singularities of the problem measured in the jet-space of (X, Y, f) at v_0. Here the generic situations are of codimension zero or one. The flat case is of infinite codimension, but it is analyzed because it is the generic situation in the control of batch reactors [2]. Also, it is an interesting situation because the so-called singular trajectories play generically a role in the synthesis.

3.1. Generic case Let us assume that both $X(v_0) \pm Y(v_0)$ are not tangent to N. Then with our convention, we have $< n(v), X(v) \pm Y(v) >\ > 0$, for $|v - v_0|$ small. Let (v, p, u) be a BC-extremal defined on $[T, 0]$. Since we are in the non exceptional case, one can set $p(0) = n(v(0))$. Let us assume $< n(v_0),\ Y(v_0) > \neq 0$. Then from the *transversality condition,* the optimal synthesis in a sufficiently small neighborhood of v_0 is given by Fig. 1 where $,_+$ (resp. $,_-$) are arcs corresponding to $u = 1$ (resp. $u = -1$).

3.2. Generic flat case As before, we assume that both $X(v_0) \pm Y(v_0)$ are not tangent to N. If (v, p, u) is a BC-extremal defined on $[T, 0]$, one may set $p(0) = n(v(0))$. Since Y is tangent everywhere to N, we have $< n(v),\ Y(v) >= 0$, $\forall v \in N$. Hence, the transversality condition tells us nothing about the optimal synthesis. We have to use the following result.

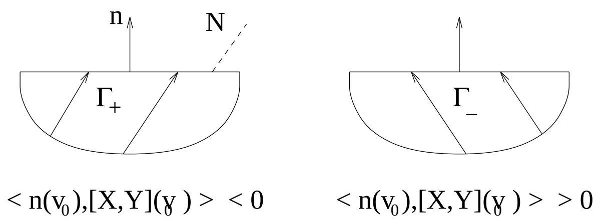

Figure 2:

3.2.1. Lemma *Let $v_0 \in N$ be such that $< n(v_0),\ Y(v_0) >= 0$ and let us assume that $< n(v_0),\ [X, Y]\,(v_0) >\neq 0$, the Lie bracket of two vector fields $Z_1,\ Z_2$ being computed with the convention $[Z_1,\ Z_2]\,(v) = \frac{\partial Z_2}{\partial x}\ (x)\ Z_1\ (v) - \frac{\partial Z_1}{\partial v}(v)\ Z_2(v)$. Then $(v, p, u \equiv 1)$* (resp.$u \equiv -1$) *is a BC-extremal with $v(0) = v_0$ if and only if $< n(v_0), [X, Y]\,(v_0) >< 0$ (resp. > 0).*

Proof: We prove for instance the assertion for $,_+$. Assume $(v, p, u \equiv 1)$ is a BC-extremal defined on $[T, 0]$ and with $v(0) = v_0$. The associated adjoint vector p can be chosen such that $p(0) = n(v_0)$. Let $\sum$ be the *switching surface* $\left\{(v, p) \in \mathbf{R}^4\ ;\ < p, Y(v) >= 0\right\}$ and let Φ^+ be the *switching function* $< p, Y(v) >$ evaluated along (p, v). If we differentiate Φ^+ with respect to t, we get $\dot{\Phi}^+ =< p, [X, Y]\,(v) >$ and hence $\Phi^+(t) = \Phi^+(0) + t < p(0),\ [X, Y]\,(v_0) > +o(t)$. Since $\Phi^+(0) = 0,\ \ p(0) = n(v_0)$ and $u(t) =$ sign $\Phi^+(t) = +1$ for $t < 0$, we must have $< p(0),\ [X, Y]\,(v_0) > < 0$.

If we apply the previous lemma, the optimal synthesis in the generic flat case is then given by Fig. 2.

3.3. Generic switching point Let $v_0 \in N$ such that $< n(v_0),\ Y(v_0) >= 0$, then $\hat{N}$ intersects the switching set $\sum$. In order to analyze this singularity, one need some preliminary lemmas.

3.3.1. Lemma *Let us assume $< n(v_0),\ [X, Y]\,(v_0) >\neq 0$. Then, the arcs $,_+$ and $,_-$ arriving at v_0 cannot be sets of input switching points.*

Proof: For instance, let us assume that $(v, p, u \equiv 1)$ is a BC-extremal on $[T, 0]$ with $v(0) = v_0$ and each point of v is a set of input switching points, then $p(0) \in \mathbf{R}n(v_0)$. From [5], $(v(0),\ p(0))$ is a normal switching point and every BC-extremal $(,\ , \lambda, u)$ which hits N near v_0 is such that $,\ = ,_+ ,_-$ or $,_- ,_+$, where $,_+ ,_-$ (resp. $,_- ,_+$) designs an arc $,_+$ (resp. $,_-$) follows by an arc $,_-$ (resp. $,_+$). Now, by assumption, there exists BC-extremals such that $,\ = ,_- ,_+$, where $,_+$ is any subarc of $v(.)$. At v_0, the associated adjoint vector belongs to $\mathbf{R}(n(v_0)) = \mathbf{R}(p(0))$. Then, we have $< p(t),\ Y(v(t)) >= 0,\ \forall t \in\ [T, 0]$. Differentiating with respect to t

and evaluating at $t = 0$, we get $< p(0),\ [X, Y](v_0) >= 0$, which is absurd since $p(0) \in \mathbf{R}n(v_0)$.

3.3.2. Lemma *Let , be an admissible trajectory arriving at v_0 and associated to a constant control u_0. Let us set $Z = X + u_0 Z$ and $\lambda(\delta, P) = \sum_{k \geq 0} \frac{(-1)^k \delta^k}{k!}\ ad^k Z(P)(v_0) - Z(v_0)$. Then if , is optimal, for each $\delta \geq 0$, small, and each vector field P of $\{X + uY\ ;\ |u| \leq 1\}$ we must have $< n, \lambda(\delta, P) > \leq 0$, where n is the unit normal at N at v_0, outwardly oriented with respect to , .*

Proof: We use a technique coming from the proof of PMP and its refinements [3] , [4] , constructing along a reference trajectory an approximation of the accessibility set. Since the terminal manifold is of codimension one, this approximation *need no to be convex* to decide about optimality.

Let , be a reference trajectory defined on $[0, T]$ and with terminal point v_0. It V is a vector field, it is convenient to denote $\{expt\ V\}$ the local one parameter group generated by V. The arc , starting from v_1 at $t = 0$ is given by $\exp\ t\ Z(v_1)$ and $\exp\ TZ\ (v_1) = v_0$.

Now, take $\delta,\ \varepsilon > 0$, sufficiently small, and any vector field P of $\{X + uY\ ;\ |u| \leq 1\}$ and consider for δ fixed, the curve

$$\alpha(\varepsilon) = (\exp\ \delta Z)\ (\exp\ \varepsilon P)\ (\exp\ (T - \delta - \varepsilon)Z)\ (v_1).$$

By construction $\alpha(0) = v_0$ and $\alpha(\varepsilon)$ lies in the accessibility set $A^+(v_1, T)$. Now, since $v_0 = \exp\ TZ\ (v_1)$, we have

$$\alpha(\varepsilon) = (\exp\ \delta Z)\ (\exp\ \varepsilon P)\ (\exp\ (-\delta - \varepsilon)Z)\ (v_0)$$

and from Baker-Campbell-Hausdorff formula we get

$$\alpha(\varepsilon) = \left(\exp\ \left(\varepsilon \left[\sum_{k \geq 0} \frac{(-1)^k \delta^k}{k!}\ ad^k Z(P) - Z\right] + o(\varepsilon)\right)\right)\ (v_0).$$

Hence $\frac{d\alpha}{d\varepsilon}_{|\varepsilon=0} = \lambda(\delta, P)$ and clearly, if $< n,\ \lambda(\delta, P) > \ > 0$, the reference trajectory , is not optimal.

3.3.3. Assumptions From now on, we shall assume that $< n(v_0),\ Y(v_0) > = 0$ and both $< n(v_0),\ X(v_0) >$ and $< n(v_0),\ [X, Y](v_0) >$ non zero.

3.3.4. Method of analysis In order to evaluate the switching curve near v_0 and the splitting line, it is convenient to make the following normalizations.

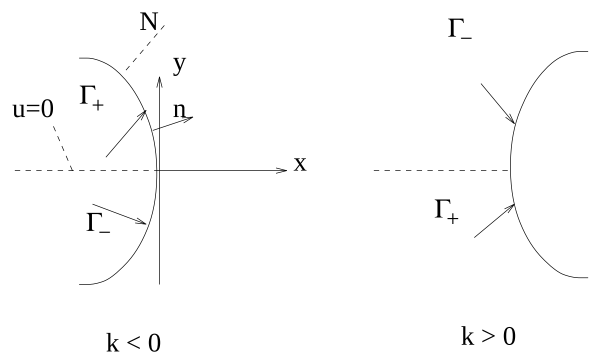

Figure 3:

First, one may set $v_0 = (0,0)$ and as in [1] , since X and Y are transverse at v, one may assume locally $Y = \frac{\partial}{\partial y}$ and that the trajectory corresponding to $u \equiv 0$ is $t \to (t,0)$. Hence, system (1.1) is written

$$\dot{x} = 1 + \sum_{i=1}^{+\infty} a_i(x)\ y^i,$$

$$\dot{y} = \sum_{i=1}^{+\infty} b_i(x) y^i + u. \tag{3.1}$$

Moreover changing if necessary y into $-y$ and u into $-u$,*one can assume* $a = a_1(0) > 0$, where $a = - < n(0,\ [X,Y](0) >,\ n(0) = (1,0)$ being the unit normal to N at 0. The terminal manifold is given locally by $s \to (c(s),\ s)$, where $c(s) = ks^2 + o(s^2)$ and k parametrized the *curvature* of N in the adapted coordinate system, since Y is tangent to N at 0. We have $n(0) = (1,0)$ and for v small, using $< n(v),\ X(v) > \ > 0$, one can set $n(v) = (n_1,\ n_2),\ n_1 = 1$ and $n_2 = -\frac{dc}{ds} = -2ks + o(s)$. Hence for s small we have : if $k < 0$, then $n_2 > 0$ if $s > 0$, and $n_2 < 0$ if $s < 0$, and conversely if $k > 0$. The hamiltonian is $H(v,p,u) =< p,\ X(v) + uY(v) >$ and if $v \in N$ is small, $p = n(v)$, its maximum over $|u| \leq 1$ is obtained as follows : if $k < 0,\ s > 0$, then $n_2 > 0$, an u maximizing H is $+1$, if $s < 0$, it is $u = -1$ and conversely if $k > 0$. Hence, we get the following important geometric behaviors : *if* k < 0, *the arcs* $,_+$ *and* $,_-$ *satisfying the transversality conditions, the normal to* N *being oriented as* n(v), *can cut themselves, contrarily to the case* k > 0, see Fig. 3.

The adjoint system associated to (3.1), with $p = (p_1,\ p_2)$ is

$$\dot{p}_1 = -p_1 \sum_{i=1}^{+\infty} a_i'(x) y^i - p_2 \sum_{i=1}^{+\infty} b_i'(x)\ y^i$$

$$\dot{p}_2 = -p_1 \sum_{i=1}^{+\infty} a_i(x)\ y^{i-1} - p_2 \sum_{i=1}^{+\infty} b_i(x)\ y^{i-1}, \tag{3.2}$$

where a_i' and b_i' are the derivatives of a_i and b_i with respect to x. *If u is a piecewise analytic control, every solution* (v, p) of (2), (3) *such that* $(v(0), p(0)) \in \hat{N}$, v *small, can be evaluated for* t *sufficiently small, by analyticity.*

3.3.5. Lemma *Near* 0, *every optimal solution is of the form* , $_+$, $_-$.

Proof: From [5] , we know that every BC-extremal is of the form , $_+$, $_-$ or , $_-$, $_+$. In fact, it follows from [9] , that every optimal solution for the point to point problem is of this form.

Since near 0, X and Y are linearly independent, in order to compare the times along the solutions of the system, we can introduce the one-form ω defined by $\omega(Y) = 0$ and $\omega(X) = 1$. Let (X_1, X_2) be the components of X, then $\omega = (1/X_1)dx$ and $d\omega = \frac{1}{X_1^2} \frac{\partial X_1}{\partial x}\ dx\ \Lambda\ dy$. Computing using (3.1), we remark that the sign of $d\omega$ near 0 is the sign of $a > 0$. Take two points v_1, v_2 near 0 and let , $_1 =$, $_+$, $_-$ and , $_2 =$, $_-$, $_+$ be two arcs joining v_1 to v_2, with respective time duration being t_1 and t_2. Using Stoke's theorem we have

$$\int_{,\,_1} \omega\ - \int_{,\,_2} \omega = t_1 - t_2 = \int_D d\omega$$

where D is the closed domain limited by , $_1 \vee$, $_2$. Since $d\omega > 0$ and the orientation of the boundary of D is clockwise, we have $t_1 < t_2$. Hence an optimal solution for the point to point problem is of the form , $_+$, $_-$ and clearly every solution for the optimal problem with N as terminal manifold has to be solution for the point to point problem.

3.3.6. Lemma *The arc* , $^0_-$ *arriving at* 0 *is not optimal.*

Proof: Let $n(0) = (1, 0)$, computing we have $< n(0),\ [X, Y](0) >= -a \neq 0$. Hence, from Lemma 3.2.1, the are , $^0_-$ is not the projection of a BC-extremal (we can also apply Lemma 3.2.2., with $Z = X - Y$ and $P = X + Y$, proving that it is not optimal).

3.3.7. Lemma *Assume* $k \neq 0$, *then the switching points of BC-extremals*

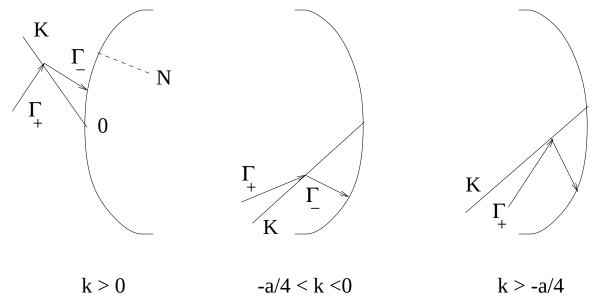

Figure 4:

$,_{+}, _{-}$ *formed an analytic curve* K *whose tangent space at* 0 *is*

$$\mathbf{R}(-2k/a,\ 1+2k/a).$$

Proof: We integrate (3.1) and (3.2), backwards in time, with initial conditions given by the boundary conditions $v(0) \in N$ and $p(0) = n(v(0)) = (1,\ -2ks) + \circ(s)$. We get $p_1(t) = 1 + o(1)$, $p_2(t) = -2ks - at + o(s,t)$. The switching times w are the solutions of $p_2(t) = 0$, $t \leq 0$. We get $w = -2ks/a + o(s)$. If $k < 0$, we must have $s < 0$ and if $k > 0$, $s > 0$. The BC-extremal $,_{-}$ is switching at $(x(w),\ y(w)) = s(-2k/a,\ 1+2k/a) + o(s)$. The lemma is then proved.

Cleary the optimal syntheses differ if a BC-extremal $,_{+}, _{-}$ is *crossing* K or is *reflecting* on K.

3.3.8. Lemma *A BC-extremal* $,_{+}, _{-}$ *is crossing* K *if* $k > 0$ *or* $\frac{-a}{4} < k < 0$ *and is reflecting on* K *if* $k < \frac{-a}{4}$.

Proof: At 0, the slope of the tangent to K is, from the previous lemma, $-1\ - a/2k$ and the respective slopes of $,^{0}_{+}$ and $,^{0}_{-}$ are 1 and -1. If $k > 0$, the slope of the tangent to K is less than -1. If $k < 0$, $-1\ - a/2k > 1$ if and only if $-a/4 < k$. Hence the geometric situations are described by Fig. 4.

3.3.9. Proposition

The optimal synthesis is given by Fig. 5.

In the first two cases, the switching curve W *is an analytic curve which coincides with* K*, the slope of its tangent at* 0 *being* $-1\ - a/2k$*. In the third case, there exists a splitting line* L *which is an analytic curve on which the optimal feedback can be* ± 1*, the slope of its tangent at* 0 *being* $-a/4k$*.*

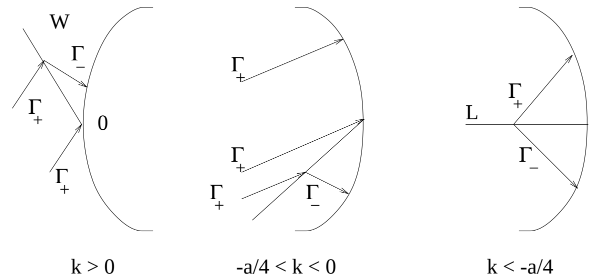

Figure 5:

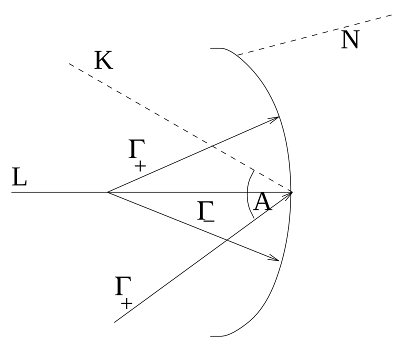

Figure 6:

Proof: In the first two cases, the situation is clear because from each point near 0, at the left of the target N, there exists only one BC-extremal , $_+$, $_-$. In the third case the situation is more complicated because more than one BC-extremal , $_+$, $_-$ is possible to reach the target. More precisely, let , $^0_+$ be the are , $_+$ arriving at 0 and let A be the acute sector delimited by K and , $^0_+$. Clearly above K, the optimal feedback is $+1$ and below , $^0_+$, it has to be -1. Let us define the splitting line L near 0 as follows : L is $\{v = (x, y),\ v$ small, $x < 0,$ such that both expt $(X \pm Y)(v)$ intersects N, see Fig. 6.

This line exists because near 0, the *value function* is continuous. By construction, L is an analytic curve located in the sector A. Clearly above L, the optimal feedback is $+1$ and below, it is -1.

In fact, everything can be evaluated using (3.1) and (3.2). If $k + a/4 < 0$, it can be shown that the arc , $^0_+$ isn't optimal. Moreoverer the slope of the tangent to L at 0 is $-a/4k \in \,]0, 1]$.

3.3.10. Remarks The generic flat case described in Section 3.2 is different from the synthesis described above. This means that a complete classification in terms of the coefficients of $c(s)$ isn't straightforward. From our analysis, we deduce that for the situations studied before, the linear estimates of K and L are given by the system $\dot{x} = 1 + ay,\ \dot{y} = u$, which serves as a *model*.

3.4. Generic fold case In this section we analyze the situation encountered when a singular extremal meets the terminal manifold at a point where Y is tangent to N. The analysis is technically relevant because we use evaluations of the accessibility set, combined with direct computations, as in the previous section.

3.4.1. Assumptions Let $v_0 \in N$ and assume Y and $[X, Y]$ are collinear at v, and Y tangent to N at v_0. Moreover suppose X transverse to N, at v_0 (these assumptions are generic, for planar systems, only in the flat case). Let S be the set of points where Y and $[X, Y]$ are colinear. We shall assume that S is a simple curve at v_0. We shall denote by U any open neighborhood of v_0 such that $S \cap U$ is a simple curve and X, Y are linearly independent on U.

3.4.2. Definition Consider a system in $\mathbf{R}^n$ of the form $\dot{v} = X + uY,\ u \in \mathbf{R}$. A singular trajectory (v, u) defined on $[0, T]$ is a solution of the system such that there exists a non trivial solution $p \in \mathbf{R}^n$, of the adjoint equation $\dot{p} = -p\left(\dfrac{\partial X}{\partial v} + u\ \dfrac{\partial Y}{\partial v}\right)$ with $< p(t),\ Y(v(t)) >= 0 \ \ \forall t \in [0, T]$.

3.4.3. Lemma *Consider a planar system $\dot{v} = X + uY,\ u \in \mathbf{R}$, restricted to an open neighborhood U of v_0 such that Assumptions 3.4.1 are satisfied. Then there exists a unique singular arc passing through v_0, located in S, and the singular control is given by the feedback law*

$$\hat{u}(v) = \frac{< p,\ ad^2X(Y)(v) >}{< p,\ ad^2Y(X)(v) >} \tag{3.3}$$

where p is such that $< p,\ Y(v) >= 0$.

Proof: By definition $< p(t),\ Y(v(t)) >= 0$ on $[0, T]$ and differentiating twice this equation with respect to t we have $< p,\ [X, Y](v) >= 0 < p,\ ad^2X(Y)(v) - u\ ad^2Y(X)(v) >= 0$.

Hence v is located on S. Moreover on U the denominator of $\hat{u}$ can't vanish. Hence the assertion is proved.

3.4.4. Definition Consider a planar system $\dot{v} = X + uY,\ u \in \mathbf{R}$, restricted to U. Let (v, u) be a singular trajectory defined on $[0, T]$ and let p an associated adjoint vector oriented using the convention $< p,\ X(v) > >$

0. Assume that $t \to v(t)$ is a simple curve. It is called *hyperbolic* if $< p(t),\ ad^2Y(X)(v(t)) > < 0\ \forall t$ and *elliptic* if $< p(t),\ ad^2Y(X)(v(t)) > > 0$.

3.4.5. Lemma *An hyperbolic (resp. elliptic) trajectory* (v,u), *defined on* $[0,T]$ *is time minimizing (resp. maximizing) with respect to all solutions of* $\dot{v} = X + uY,\ u \in \mathbf{R}$, *joining* $v(0)$ *to* $v(T)$ *and contained in a sufficiently small neighborhood of* v.

Proof: For the proof of this result see [1].

3.4.6. Proposition Let (v,u) be an hyperbolic trajectory defined on $[0,T]$, the singular control $\hat{u}$ given by (3.3) belonging to $]-1,\ +1[$. Then there exists a neighborhood V of v such that v is the unique optimal trajectory joining $v(0)$ to $v(T)$, among all solutions of (1.1) contained in V. Moreover if V is sufficiently small the accessibility set $A^+_V(v(0),\ T)$ is a closed convex set near $v(T)$ with nonempty interior whose boundary is a curve $s \to d(s)$, with $d(0) = v(T),\ d'(0) \in \mathbf{R}Y(v(T)),\ C^2$ but not in general C^3. Moreover in every adapted coordinate system ($i.e.\ Y = \dfrac{\partial}{\partial y_{|V}}$) its *curvature is zero.*

Proof: From Lemma 3.4.5, there exists a neighborhood V of v such that v is time optimal if no bound is imposed to u, hence it has to be optimal if $|u| \leq 1$. Moreover from [8] and [9], we can choose V such that every optimal trajectory, for the point to point problem, starting from $v(0)$, is a singular arc Γ_s followed by an arc Γ_+ or Γ_-. Hence near $v(T)$ the boundary of $A^+_V(v(0),\ T)$ is parametrized by $s \to d(s)$, where $s \geq 0$ and

$$d(s) = (exps\ (X \pm Y)\ (exp\ (T-s)X_s)\ (v(0)),$$

where $X_s = X + \hat{u}Y$ and $\hat{u}$ is given by (3.3).

Since, $v(T) = exp\ TX_s(v(0))$ we get

$$d(s) = (exps\ (X \pm Y)\ (exp\ -s\ X_s)\ (v(T))$$

and using Baker-Campbell-Hausdorff formula we have

$$d(s) = exp\left[s(\pm Y) + \frac{1}{2}\ s^2\,[X \pm Y,\ X_s] + o(s^2)\right]\ (v(T)).$$

The curve $d(s)$ can be evaluated using the formula

$$exps\ Z(v) = \sum_{n \geq 0}\ \frac{s^n}{n!}\ \ Z^n(id)(v),$$

for s sufficiently small, Z be any vector field acting by Lie derivative on the mappings and id is the identity mapping. In particular if $Y = \frac{\partial}{\partial y}$, we have $Y^n(id) = 0$, for $n > 1$. Since along a singular trajectory Y and $[X, Y]$ are colinear we get

$$d(s) = v(T) \pm (s + f(s))\, Y\,(v(T)) + o(s^2)$$

where $f(s) = o(s)$.

Hence setting $s' = |s + f(s)|$, the boundary of the accessibility set is given near $v(T)$ by $d : s' \to (\circ(s'^2),\ s' + \circ(s'^2))$.

Higher order dimensional expansions would tell us the nature of its singularity. For instance if the system is given by $\dot{x} = 1 - y^2,\ \dot{y} = u$, and $v(0) = (0, 0)$ we get $d(s) = \left(T - \frac{s^3}{3},\ \varepsilon s\right)$ with $\varepsilon = \pm 1$. Hence the boundary is given by the graph $x = T - \frac{|y|y^2}{3}$.

3.4.7. Normalizations Let $v_0 \in N$, one may assume $v_0 = (0, 0)$ and locally $Y = \frac{\partial}{\partial y}$. The important geometric object is the singular arc and hence we normalized system (1.1) locally as follows : S is taken as $y = 0$ and the singular arc as $t \to (t, 0)$. Hence (1.1) is given by

$$\begin{aligned} \dot{x} &= 1 + y^2 X_1(x, y) \\ \dot{y} &= -\hat{u}(x) + y X_2(x, y) + u, \ \mid u \mid \leq 1. \end{aligned} \tag{3.4}$$

Let $a = X_1(0, 0)$ and observe that the singular arc is elliptic is $a > 0$ and hyperbolic if $a < 0$. It is admissible if $\hat{u}(0) \in]-1, 1[$. The terminal manifold is parametrized by $s \to (c(s),\ s)$, where $c(s) = ks^2 + o(s^2)$.

3.4.8. Proposition *Assume the singular arc , $_s$ passing through v_0 is hyperbolic and $\hat{u}(v_0) \in\ ]-1,\ +1[$. Let N be parametrized in any adapted coordinate system by $s \to (c(s),\ s)$, $c(s) = ks^2 + \circ(s^2)$. Then the optimal synthesis is given near v_0 by Fig. 7.*

Proof: To decide if , $_s$ is optimal we use Proposition 3.4.7. One can assume the system given by (3.4). Let $v = (T, 0)$, with $T < 0$, sufficiently small. Since the boundary of the accessibility set $A^+(v, T)$ has zero curvature, we have two situations if $k \neq 0$, see Fig. 8.

In the first situation, N meets the interior of the accessibility set, hence , $_s$ is not time optimal, contrarily to the second situation (or in the flat case).

The switching points can be evaluated using (3.4). Take $v \in N$ near v_0, $v = (*, s)$ and the normal to N being $n(v) = (1, -c'(s))$. Let

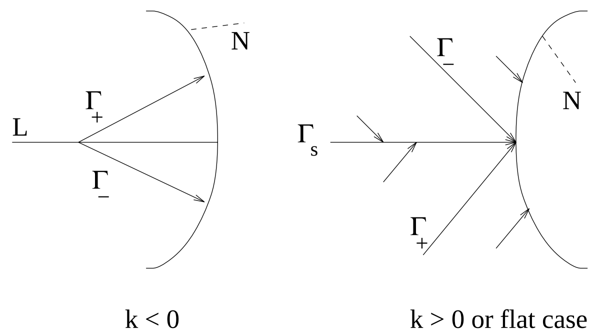

Figure 7:

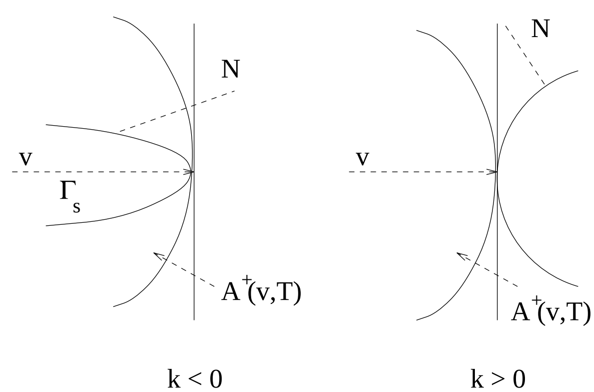

Figure 8:

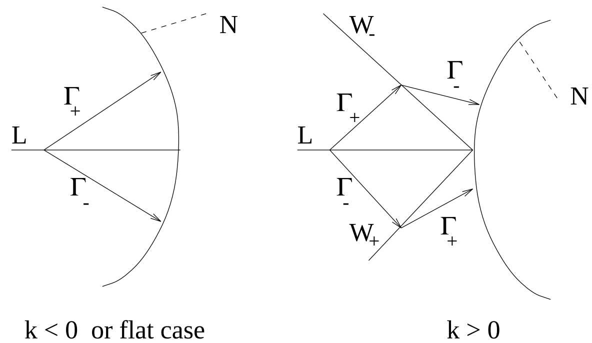

Figure 9:

$(,\ , p, u = \varepsilon)$, $\varepsilon = \pm 1$ be a BC-extremal defined on $[T, 0]$, with , $(0) = v$, $p(0) = n(v)$. If $p = (p_1,\ p_2)$, a switching time is given by solving $p_2(t) = 0$, $t < 0$. Computing we get

$$p_2(t) = -c'(s) - at^2\,(\varepsilon - \hat{u}(0)) - t\,(2as - c'(s)X_2(0)) + o(s,t)_2. \qquad (3.5)$$

From this expansion and since $a < 0$, we deduce that any BC arc $,_+$ or $,_-$ which starts from $v \neq 0$ is *without switching point* if $k \neq 0$ or in the flat case.

The synthesis follow from this analysis.

3.4.9. Proposition *Assume the singular are $,_s$ passing through v_0 elliptic and $\hat{u}(v_0) \in\ ]-1, +1[$. Let N be parametrized in any adapted coordinate system by $s \to (c(s),\ s)$, $c(s) = ks^2 + o(s^2)$. Then the optimal synthesis is given by Fig. 9.*

Proof: From Lemma 3.4.5, the arc $,_s$ is locally time maximizing, for the point to point problem, and hence it cannot be an optimal arc for our problem.

Assume now the system given locally by (3.4). Since $,_s : t \to (t, 0)$ is elliptic we have $a > 0$.

Applying Lemma 3.3.2, with $Z = X + \varepsilon Y$, $P = X - \varepsilon Y$, $\varepsilon = \pm 1$, $n = (1, 0)$, one can prove that both arcs $,_+$ and $,_-$ arriving at 0 are not optimal. Indeed we have

$$< n,\ \lambda(\delta, P) >= \varepsilon\delta^2 < n,\ \big(\varepsilon\ ad^2Y(X) - ad^2X(Y)\big)(0) > + o(\delta^2)\big)$$

and writing (3.3) at 0

$$< n,\ ad^2X(Y) - \hat{u}\ ad^2Y(X) >= 0$$

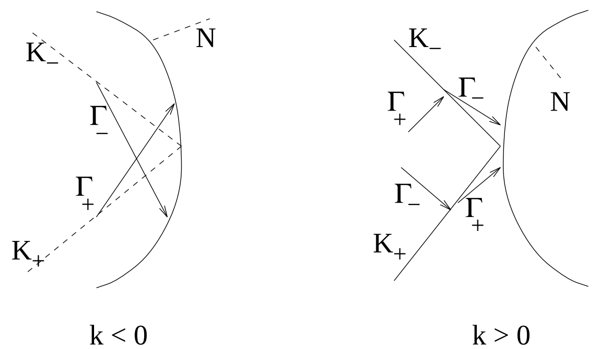

Figure 10:

we get

$$< n,\ \lambda(\delta, P) >= \delta^2 \left(1 - \frac{\hat{u}(0)}{\varepsilon}\right) < n,\ ad^2 Y(X)(0) > +o(\delta^2)$$

$$= 2a\ \delta^2 \left(1 - \frac{\hat{u}(0)}{\varepsilon}\right) + o(\delta^2)$$

which is > 0 for δ sufficiently small, since $a > 0$ and $\hat{u}(0) \in]-1,\ +1[$. This proves the assertion.

From [9] , we know that every optimal trajectory has at most one switching. To evaluate the switching points one use (3.5). We must distinguish between the case $k \neq 0$ and the flat case. Consider for instance the case $k \neq 0$. A switching time for a BC-extremal , $_+$ or , $_-$ is given by :

along , $_+$, $w_+ = -\sqrt{\dfrac{2\,|ks|}{a(1-\hat{u}(o))}} + o\,(\sqrt{s})$,

along , $_-$, $w_- = -\sqrt{\dfrac{2\,|ks|}{a(1+\hat{u}(0))}} + (o\sqrt{s})$

and the corresponding switching curves are respectively $K_+ : y = (1 - \hat{u}(0))x + o(x)$ and $K_- : y = -(1 + \hat{u}(0))x + o(x)$. Hence, if $k > 0$, the BC-extremals have the behaviors described by Fig. 10.

To construct the synthesis we proceed as in the proof of Proposition 3.3.9. If $k < 0$ the optimal switching curve is $W = W_- \cup W_+$, with $W_- = K_-$ and $W_+ = K_+$. If $k > 0$, the splitting line L can be evaluated using (3.4).

The flat case can be treated similarly, using (3.5), the switching times being of order s. More precisely we have

$$w_+ = -2s/(1 - \hat{u}(0)) + o(s),\ w_- = 2s/(1 + \hat{u}(0)) + o(s),$$

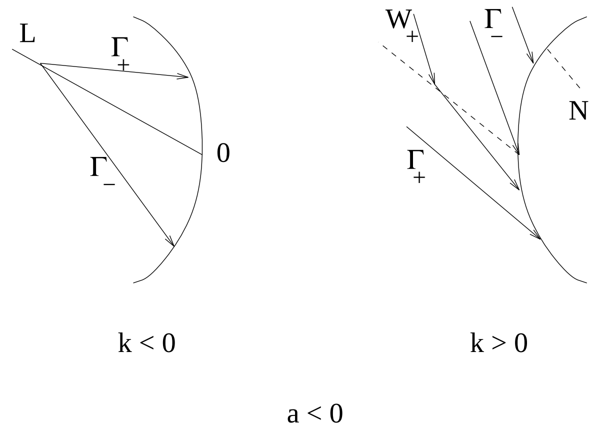

Figure 11a:

$$K_+ : y = \left(\frac{1-\hat{u}(0)}{2}\right) x + o(x),\ K_- : y = -\left(\frac{1+\hat{u}(0)}{2}\right) x + o(x).$$

All the optimal switching points are virtually concentrated on N.

3.4.10. Definitions Consider system (3.4), $v_0 \in N$ being identified to 0. We said that we are in the *parabolic case* if $\hat{u}(0) \notin]-1,\ +1[$. (This is coherent with the terminology of [5], because v_0 is lifted into a parabolic point of $\hat{N}$). Changing if necessary y into $-y$ and u into $-u$, *one can assume* $\hat{\text{u}}(0) > 1$. The geometric situation is the following : the *singular arc* , $_s$ passing through v_0, which can be fast of slow for the point to point problem, depending if $a < 0$ or $a > 0$, when $u \in \mathbf{R}$, *is not admissible.*

3.4.11. Proposition *Consider system (3.4) and assume* $\hat{\text{u}}(0) > 1$. *Let* N *be given in an adapted coordinate system near* 0 *by* $s \to (ks^2 + o(s^2),\ s)$ *and let* $a \leq n,\ ad^2Y(X)(0) >$, *where* $n = (1,0)$ *is the normal to* N *at* 0. *The optimal synthesis is given by Fig. 11.*

Proof: First $(0, n(0))$, where $n(0) = (1,0)$, is a parabolic point according to [5] and hence every BC-extremal is locally of the form , $_+$, , $_-$, , $_+$ or , $_-$, , $_+$, , $_-$. Moreover from [9] , every optimal trajectory for the point to point problem is of this form.

To decide if the arcs , $^0_+$ or , $^0_-$ are not optimal, we use Lemma 3.3.2, with $Z = X + \varepsilon Y,\ P = X - \varepsilon Y,\ \varepsilon = \pm 1,\ n = (1,0)$. We have

$$< n,\ \lambda(\delta, P) >= 2a\delta^2 \left(1 - \frac{\hat{u}(0)}{\varepsilon}\right) + o(\delta^2).$$

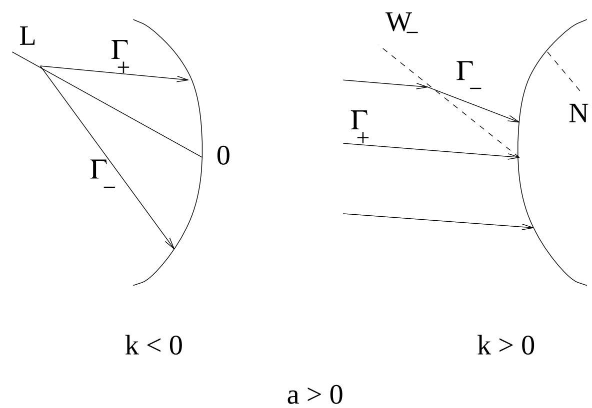

Figure 11b:

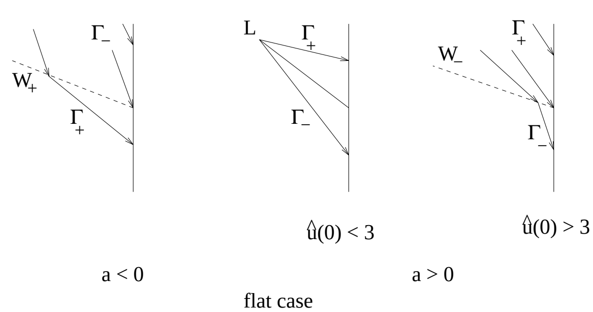

Figure 11c:

Hence if $a<0$ (resp. $a>0$) the arcs $,^{0}_{+}$ (resp. $,^{0}_{-}$) is not optimal (*The curvature of* N *is not taken into account in this criterion).*

Let us evaluate the switching points using (3.5). First, let us assume $k \neq 0$. By computing one get that a BC-extremal can have at most one switching. If $a>0$ (resp. $a<0$) it is of the form $,_{-},_{+}$ (resp. $,_{+},_{-}$), all switchings being located on a curve K_+ (resp. K_-) whose slope at 0 is $(1-\hat{u}(0))$ (resp. $-(1+\hat{u}(0))$.

Now observe that these extremals are reflecting themselves on the switching curve if $k<0$ or are crossing this curve if $k>0$. As in Section 3.3., we concude that if $k<0$, an optimal trajectory is of the form $,_{+}$ or $,_{-}$ and there exists a splitting line L. If $k>0$, an optimal trajectory is of the form $,_{-},_{+}$ (resp. $,_{+},_{-}$) if $a<0$ (resp. $a>0$), the optimal switching curves being given by $W_+ = K_+$ (resp. $W_- = K_-$).

In the flat case, the evaluations are different. If $a<0$, only a BC-extremal $,_{+}$ is switching at $w_+ = -2s/(1-\hat{u}(0)) + o(s)$ the switching curve K_+ being $y = \frac{1}{2}(1-\hat{u}(0))x + o(x)$. If $a>0$, only a BC-extremal $,_{-}$ is switching at $w_- = 2s/(1+\hat{u}(0)) + o(s)$ and the switching curve is $K_- : y = -\frac{1}{2}(1+\hat{u}(0))x + o(x)$. A BC-extremal $,_{+},_{-}$ is reflecting on K_- if $\hat{u}(0) < 3$ and crossing K_- if $\hat{u}(0) > 3$. Hence we get the corresponding synthesis.

3.5. Generic exceptional case

3.5.1. Assumptions and normalizations Let $v_0 \in N$, one may assume $v_0 = 0$. Suppose $X+Y$ tangent to N at 0. Moreover assume Y and $X-Y$ not tangent to N at 0. We can choose a coordinate system such that $Y = \frac{\partial}{\partial x}$ and N is identified to the curve $s \to (0,s)$. Hence (1.1) can be written : $\dot{x} = X_1 + u, \dot{y} = X_2$, with $X_1(0) = -1$ and $X_2(0) \neq 0$. We can suppose $X_2(0) > 0$. Moreover we assume $\frac{\partial X_1}{\partial y}(0) = a \neq 0$, which means that the contact of $,_{+}$ with N at 0 is one.

3.5.2. Proposition *Under the previous normalizations, the optimal synthesis is given by Fig. 12.*

Proof: First assume $a>0$. The arc $,^{0}_{+}$ is a BC-extremal, $n=(1,0)$ being the associated adjoint variable at 0. In order to prove it is not optimal, we apply Lemma 3.3.2, the outward normal to N with respect to $,^{0}_{+}$ being $-n, z = X+Y$ and $P = X-Y$. We get $\lambda(\delta,p) = -2Y(0) + o(1)$. Hence, for small $\delta, < -n, \lambda(\delta,p) >> 0$. This proves the assertion. Indeed, a simple computation shows the following. Assume we are at distance ε from N, in the domain $x>0$. The time to reach the target N is of order $\sqrt{\varepsilon}$ along $,^{0}_{+}$ and of order ε along $,_{-}$ because the contact of $,^{0}_{+}$ with N is one and $,_{-}$ is transverse to N. In the domain $x<0$, the optimal control is $u=+1$, the value function being not continuous.

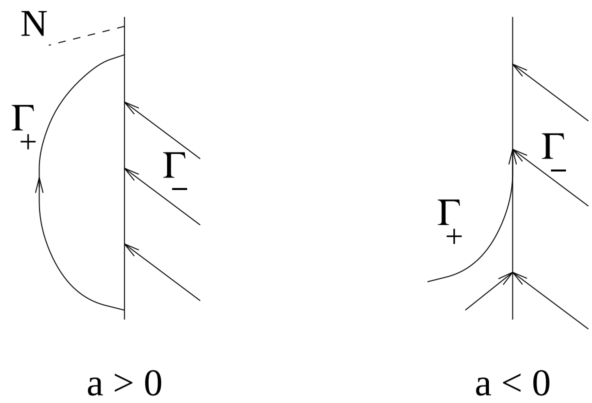

Figure 12:

When $a < 0$, the analysis is similar, but the target N is not accessible from the points in the sector $x < 0$, above Γ^0_+.

3.6. Generic flat exceptional case

3.6.1. Assumptions and normalizations The point $v_0 \in N$ is identified to $0, N$ to $s \to (0, s), Y$ is assumed tangent to N and X tangent to N at 0. Moreover we suppose $Y, X \pm Y$ not vanishing at 0 and $[X, Y]$ not tangent to N at 0. System (1.1) can be written : $\dot{x} = ax + by + \circ(x, y), \dot{y} = X_2 + u$, where $b = - < n, [X, Y](0) > \neq 0, n = (1, 0)$ normal to N. Clearly, one may assume $a = 0, b = 1$ and $1 + X_2(0) > 0$.

3.6.2. Proposition *Under the previous normalizations, the optimal synthesis is given by Fig. 13.*

Proof: According to [4], $z_0 = (0, n) \in \hat{N}$ is a normal switching point and hence every BC-extremal near z_0 is of the form $\Gamma_+ \Gamma_-$ or $\Gamma_- \Gamma_+$. Since $\hat{N}$ is contains in $< p, Y >= 0$, all the switching points are concentrated on N. Hence, near 0, every optimal trajectory is an arc Γ_+ or Γ_-. Then, the synthesis follows from Lemmas 3.2.1, 3.3.2.

4 Example

Consider the problem of obtaining in minimum time a given ratio of concentrations of species X and Y for a chemical reaction scheme of the form $X \to Y \to Z$, each reaction being of first order, the control being the derivative of the temperature of the reactions. This problem is completely solved in [2]. Let us use our classification to study the optimal synthesis

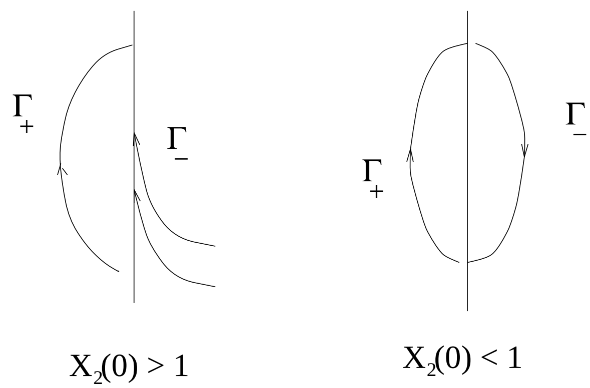

Figure 13:

near the terminal manifold. The system can be written:

$$\dot{z} = v - \beta v^{\alpha}\ z + vz,\ \dot{v} = h(v)u \tag{4.1}$$

where $z = [Y] / [X]$, $[X]$ and $[Y]$ being the respective concentrations of X, Y, $u = \dfrac{dT}{dT}$, where T is the temperature and v (and hence h) is given by *Arrhenius law* $v = A_1\ e^{-E_1/RT}$, where A_1, E_1 are physical positive parameters, R is the gaz constant and α, β parameters, with $\alpha > 1$ and $\beta > 0$. The terminal manifold is $N = \{(z,v) \in \mathbf{R}^2\ ;\ z = d\}$ where d is a given positive constant. We are in the flat case. The singular trajectories are located in $S = \left\{(z,v) \in \mathbf{R^2}\ ;\ z\left(\alpha\beta v^{\alpha-1} - 1\right) = 0\right\}$ and the singular control is given by $\hat{u} = -\dfrac{v^2}{h(v)\alpha z}$. The control u is assumed in $[u_-,\ u_+]$ with $u_- < 0 < u_+$. Moreover we shall suppose the following : $A_1 > \beta^{1/1-\alpha}$, $G = \{(z,v)\ ;\ \dot{z} = 0\}$ intersects N, and $\hat{u}$ at $S \cap N$ is in $]u_-, u_+[$ (the other cases can be analyzed similarly). Then the optimal synthesis is described near N by Fig. 14.
The two singularities of the problem have been analyzed in this article. At P_1 an hyperbolic arc $,_s$ meets N and at P_2 we are in the exceptional case.

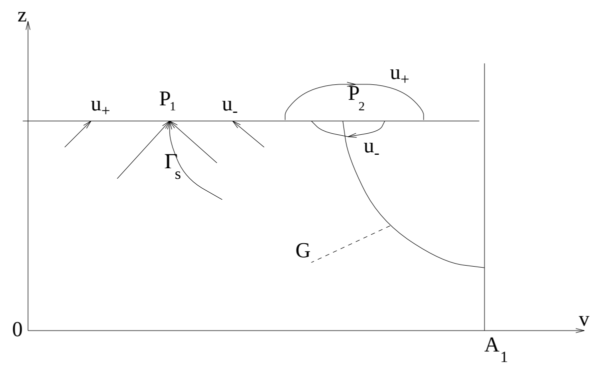

Figure 14:

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