Journal of Mathematical Systems, Estimation, and Control
Vol. 5, No. 3, 1995, pp. 1–26


# Splitting Subspaces and Acausal Spectral Factors*

György Michaletzky† Augusto Ferrante†

**Abstract**

In this paper we consider a generalization of the theory of splitting subspaces used for the construction of state space realizations of stationary time series. The usual construction provides forward or backward realizations corresponding to stable or antistable spectral factors. We study spectral factors without specification on the pole (or zero) structure and show that the usual *geometric* approach (using splitting subspaces) can be extended to this more general setting. Also we prove that the zero structure of a spectral factor is determined by the intersections of its natural state space and the past/future of the observation process, which is well-known in the stable or antistable case, and the pole structure is determined by the intersections of its state space and the past/future of the generating noise process.

**Key words**: Markovian splitting subspaces, minimal acausal spectral factors, zero structure

**AMS Subject Classifications**: 93E03, 93B27, 60G10

## 1 Introduction

Suppose that $y(k), k \in \mathbb{Z}$ is a p-dimensional, real, stationary stochastic process with zero mean and with spectral density $\Phi$. (Assume that $log(det\Phi) \in L_1$, so the spectral factorization problem can be solved.) In this paper we consider realizations of this process. It is well-known that there are several approaches and methods of analysing them. Namely, this problem is connected with spectral factorization problems; it can be solved

*Received October 8, 1993; received in final form January 25, 1994. Summary appeared in Volume 5, Number 3, 1995.

†This work was partially supported by the Hungarian Scientific Research Foundation.

using geometric, Hilbert space methods and in the rational case, the state space method can be applied, often leading to finite dimensional linear algebraic problems. The usual constructions provide forward and backward realizations corresponding to stable and antistable spectral factors. In this paper we study spectral factors without specifications on the pole (or zero) structure. This work is a continuation of the research of Picci and Pinzoni [18], Ferrante et al. [10], Ferrante [8], [9]. These papers deal with the continuous time case; in the present paper we analyse discrete time systems. Picci and Pinzoni [18] extended the positive real lemma to systems with poles outside the imaginary axis and gave a parametrization of all minimal spectral factors with a given pole and zero structure via the solutions of two ARE. Ferrante, Michaletzky and Pavon [10] gave a parametric description of all minimal square spectral factors in the case when $\Phi$ is a coercive rational function, showing that – under a mild condition – there is a one-to-one correspondence between minimal (acausal) square spectral factors and the left inner divisors of an inner function depending only on the spectrum. This also leads to a characterization via the solutions of a larger dimensional ARE. The above-mentioned condition assumes that the set of poles and the set of zeros of the stable, minimum phase spectral factor, i.e. the forward innovation realization, are disjoint. (Denote this spectral factor by $W_-$. Similarly, let $W_+, \bar{W}_-, \bar{W}_+$ denote the stable, maximum phase; antistable, minimum phase and antistable, maximum phase spectral factors, respectively.) Ferrante [9] proved that in the rational case this –unmixing type – condition is also necessary. Ferrante [8] analyses the nonrational case. He proved that if $\Phi$ is coercive and the scalar inner functions $det(W_-^{-1}W_+)$, $det(\bar{W}_+^{-1}W_+)$ are coprime then a similar characterization can be given – there is a one-to-one correspondence between minimal (acausal) square spectral factors and the left inner divisors of $\bar{W}_-^{-1}W_+$. At the MTNS'93 Conference, we also learned that P. Furhmann analysed this problem via the factorizations of the phase function $\bar{W}_+^{-1}W_-$.

In Section 2, we show that the usual geometric approach – using splitting subspaces – can be extended to this more general setting. In this way we show that the parallel structure of spectral factors and splitting subspaces remains true even in the acausal case, i.e. there is a natural way to define state spaces corresponding to acausal spectral factors, too. It is important to notice that we get the same splitting subspaces $X$ as in the earlier approaches; the pole structure is reflected in the decomposition

$$X = [X \cap H^-(w)] \oplus [X \cap H^+(w)],$$

where $w$ is the driving noise process. Since in this section we are using Hilbert space methods, the only assumption we need is the finite multiplicity property.

In Section 3 we consider the rational case and we compute the various matrices in the state space realizations, or, in other words, we analyse the so-called weak realizations. The theory of weak realizations has vast literature in the stable (antistable) case. We do not want to adapt all the methods or transform all the problems to the acausal case analysed in this paper just to give a brief insight into the similarities between the stable, antistable and instable case. So we compute the system matrices, and we analyse the Ljapunov equation. In this case the state transition matrix is not necessarily stable, so the solution of the Ljapunov equation is not necessarily positive definite. (Of course, in this case the solution is not the covariance matrix of the state vector.) On the other hand, we show that – similarly to the case considered by Faurre et al. [7] – there exists a smallest and a largest element in the set of solutions.

In Section 4 we analyse the zeros of acausal spectral factors. Since we would like to investigate the zero directions or more generally the zero structure, too, instead of working with the Smith–McMillan forms and defining the zeros of the system as the zeros of the numerator polynomials in the Smith–McMillan form, we use the concept of zero functions elaborated in the book of Ball et al. [3], and we show that this concept is also connected with the Rosenbrock matrix of the system. We also demonstrate that the zeros and zero directions describe a special connection between the state vector process $x(k)$ and the output process $y(k)$. Especially, we prove that the zeros are determined by the splitting subspace $X$. In other words, all the spectral factors sharing the same state space have the same zero structure.

In Section 5 we return to the general – nonrational – case. Using Hilbert space methods we present another proof for the theorem given by Ferrante [8] about the characterization of acausal spectral factors in terms of left inner divisors of $\bar{W}_-^{-1}W_+$. Also we show that if $W_-^{-1}W_+$ and $\bar{W}_+^{-1}W_+$ are strongly left coprime, then the condition given by Ferrante turns out to be necessary and sufficient.

## 2 Splitting Geometry – Acausal Spectral Factors

Let us introduce the following usual notations.

If $\eta \in L_2, \mathcal{A}$ is a closed subspace of $L_2$, then $E^{\mathcal{A}}\eta$ denotes the orthogonal projection of $\eta$ onto $\mathcal{A}$ . If $\mathcal{A}, \mathcal{B}$ are closed subspaces of $L_2$, then $E^{\mathcal{A}}\mathcal{B}$ denotes the *closed* subspace generated by the vectors of the form $E^{\mathcal{A}}\eta, \eta \in \mathcal{B}$.

If $z(k), k \in \mathbb{Z}$, is a q-dimensional wide sense stationary process with zero mean then denote

$$\begin{aligned} H^-(z) &=< z_i(k), i = 1, \dots, q, k \leq -1 >, \\ H^+(z) &=< z_i(k), i = 1, \dots, q, k \geq 0 >, \end{aligned} \tag{2.1}$$

where $< \cdot >$ means the generated closed subspace in $L_2$.

Shortly $H^- = H^-(y), H^+ = H^+(y)$. As an exception, in the case of the n–dimensional state process $x$ let

$$\begin{aligned} X^- &=< x_i(k), i = 1, \dots, n, k \leq 0 >, \\ X^+ &=< x_i(k), i = 1, \dots, n, k \geq 0 >, \\ X &=< x_i(0), i = 1, \dots > . \end{aligned} \tag{2.2}$$

Denote $X_- = E^{H^-}H^+, X_+ = E^{H^+}H^-, N^- = H^- \cap (H^+)^\perp, N^+ = H^+ \cap (H^-)^\perp, Z_+ = (N^+)^\perp \ominus H^-$.

The Markov property of the process $x$ can be stated as $X^-$ and $X^+$ are conditionally orthogonal with respect to $X$, i.e. $X^- \perp X^+ | X$. On the other hand $X$ is the state space of a realization of $y$; consequently,

$$H^- \vee X^- \perp H^+ \vee X^+ | X.$$

Shortly, $X$ is a Markovian splitting subspace (see [13], [14]).

Before going into the details, we motivate our construction via pointing out some important properties in the structure of state spaces, splitting subspaces, in the stable, antistable case. Assume now that $\Phi$ is a rational function and let $W(z) = D + C\left(zI - A\right)^{-1} B$ be a spectral factor of $\Phi$, i.e.

$$W(z)W(z^{-1})' = \Phi(z); \tag{2.3}$$

then (possibly extending the basic probability space) we can define a realization of the process $y$ as

$$(\Sigma) \quad \begin{cases} x(k+1) = Ax(k) + Bw(k) \\ y(k) = Cx(k) + Dw(k) \end{cases} \tag{2.4}$$

where $w(k), k \in \mathbb{Z}$, is a normalized, uncorrelated sequence.

(In the sequel of the paper we shall call this case the rational case, referring to the fact that in this case $\Phi$ is a rational function, or as the finite dimensional case, since there exists a finite dimensional realization of the process $y$.)

Let us observe that:

- if $A$ is a stable matrix (all its eigenvalues are inside the complex unit circle), then

$$w(k), w(k+1), \dots \text{ are orthogonal to } \; x(k), x(k+1), \dots \quad ; \tag{2.5}$$

- if $A$ is an antistable matrix (all its eigenvalues are outside of the closed unit circle), then

$$w(k), w(k-1), \dots \text{ are orthogonal to } \; x(k+1), x(k+2), \dots \quad ; \tag{2.6}$$

- and, in general, if $A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$, where $A_1$ is stable, $A_2$ is anti-stable, then $x$ can be separated as $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and the orthogonality properties (2.5) and (2.6) hold separately for $x_1$ – (2.5), and $x_2$ – (2.6).

Straightforward computation gives that $x(k), k \in \mathbb{Z}$, is always a Markovian process in $L_2$–sense. Moreover, the future and the past of the process $y$ are conditionally orthogonal with respect to the present of $x$.

So in the rational case for *any* spectral factor we can define a Markovian splitting subspace, and the pole structure is reflected in the orthogonality properties (2.5) and (2.6) or, in other words, in the connection between the state space $X$ and the spaces generated by the noise process $H^-(w), H^+(w)$. Namely, if we are given a stable spectral factor, then

$$X \subset H^-(w), \text{ moreover } X = E^{H^-(w)} H^+ \quad \text{(if X is minimal)} \tag{2.7}$$

in case of an antistable spectral factor

$$X \subset H^+(w), \quad \text{and} \quad X = E^{H^+(w)} H^- \quad \text{(if X is minimal).} \tag{2.8}$$

In the general case $X \cap H^-(w) \neq 0$ and $X \cap H^+(w) \neq 0$. The combination of (2.7) and (2.8) gives the idea how to define the state space in the general case.

**From geometry to spectral factors:**

Suppose that $H$ is a closed subspace of $L_2$, containing $H^-$ and $H^+$. Assume that the shift operator can be extended to $H$, denote this extension by $\mathcal{U}$. Also assume that $\mathcal{A}$, $\mathcal{B}$ are closed subspaces of $H$ and $H = \mathcal{A} \oplus \mathcal{B}$ .

**Theorem 2.1** *Define*

$$X_s = E^{\mathcal{A}} H^+, \quad X_u = E^{\mathcal{B}} H^-, \quad X = X_s \oplus X_u. \tag{2.9}$$

*Then*

*(i)* *$X$ is a splitting subspace, i.e. $H^- \perp H^+ | X$ ,*

*(ii)* *if $\mathcal{U}^{-1}\mathcal{A} \subset \mathcal{A}$, $\mathcal{U}\mathcal{B} \subset \mathcal{B}$, then $X$ is a Markovian splitting subspace,*

*(iii)* *if $\cap\, \mathcal{U}^{-j}\mathcal{A} = 0$ , and $\cap\, \mathcal{U}^{j}\mathcal{B} = 0$, and also $H$ has the finite multiplicity property, then there exists an uncorrelated sequence $w(k), k \in \mathbb{Z}$, and a spectral factor $W(z)$ such that $\mathcal{A} = H^-(w), \mathcal{B} = H^+(w)$ and $y(k) = W(z)w(k)$.*

**Proof:**

(i) If $\xi, \eta \in H$, then

$$E[(\xi - E^X\xi)(\eta - E^X\eta)] = E[(\xi - E^{X_s}\xi)(\eta - E^{X_u}\eta)] \tag{2.10}$$

because $X = X_s \oplus X_u$, and

$$E[(\xi - E^{X_s}\xi)E^{X_s}\eta)] = 0 \ ,$$

$$E[(\eta - E^{X_u}\eta)E^{X_u}\xi)] = 0 \ ,$$

$$E[E^{X_u}\xi E^{X_s}\eta)] = 0 \ .$$

Now if $\xi \in H^+, \eta \in H^-$, then $(\xi - E^{X_s}\xi) \in \mathcal{B}$ and $(\eta - E^{X_u}\eta) \in \mathcal{A}$, so they are orthogonal.

(ii) The main identity we are going to rely on is (2.10). Assume that $i, j \geq 0$.

If $\xi \in X_s, \eta \in X_u$, then $\mathcal{U}^{-i}\xi - E^{X_s}(\mathcal{U}^{-i}\xi) \in \mathcal{A}$ and $\mathcal{U}^j\eta - E^{X_u}\mathcal{U}^j\eta \in \mathcal{B}$, so they are orthogonal.

If $\xi \in X_u, \eta \in X_s$, then $\mathcal{U}^{-i}\xi \perp \mathcal{U}^{-i}(\mathcal{B} \ominus X_u) \supset (\mathcal{B} \ominus X_u)$, thus

$$E^{X_u}(\mathcal{U}^{-i}\xi) = E^{\mathcal{B}}(\mathcal{U}^{-i}\xi) \ , \text{ so } \mathcal{U}^{-i}\xi - E^{X_u}(\mathcal{U}^{-i}\xi) \in \mathcal{A} \ .$$

Similarly, $\mathcal{U}^j\eta - E^{X_s}(\mathcal{U}^j\eta) \in \mathcal{B}$; consequently, they are orthogonal.

If $\xi, \eta \in X_s$, then we can assume that $\eta = E^{\mathcal{A}}\lambda$ , where $\lambda \in H^+$, since the vectors of this type form a dense subset of $X_s$. In this case

$$\begin{aligned} &E[(\mathcal{U}^{-i}\xi - E^X(\mathcal{U}^{-i}\xi))(\mathcal{U}^j\eta - E^X(\mathcal{U}^j\eta))] \\ &\quad = E[(\mathcal{U}^{-i}\xi - E^{X_s}(\mathcal{U}^{-i}\xi))(\mathcal{U}^j\eta - E^{X_u}(\mathcal{U}^j\eta))] \\ &\quad = E[(\mathcal{U}^{-i}\xi - E^{X_s}(\mathcal{U}^{-i}\xi))\mathcal{U}^j\eta] = E[(\mathcal{U}^{-i}\xi - E^{X_s}(\mathcal{U}^{-i}\xi))\mathcal{U}^j\lambda] = 0, \end{aligned}$$

because $\mathcal{U}^{-i}\xi - E^{X_s}(U^{-i}\xi) \in (\mathcal{A} \ominus X_s) \perp H^+$, and $\mathcal{U}^j\lambda \in H^+$.

If $\xi \in X_s, \eta \in H^+$, then

$$E[(\mathcal{U}^{-i}\xi - E^{X_s}(\mathcal{U}^{-i}\xi))(\eta - E^{X_u}\eta)] = E[(\mathcal{U}^{-i}\xi - E^{X_s}(\mathcal{U}^{-i}\xi))\eta] = 0.$$

The remaining cases can be proved similarly. Thus $X$ is a splitting subspace for $H^- \vee X^-$ and $H^+ \vee X^+$, i.e., $X$ is a Markovian splitting subspace.

(iii) The usual Halmos's type wandering subspace technique proves the existence of the uncorrelated sequence $w(k)$, for which $\mathcal{A} = H^-(w), \mathcal{B} = H^+(w)$. (Let us mention that in the most general case $w$ is not necessarily finite dimensional. But, if $H$ has the finite multiplicity property, then $w$ is a finite dimensional vector process.) The equation $y(k) = W(z)w(k)$ is a direct consequence of the fact that every element in $H$ can be expressed as an infinite linear combination of the elements in the sequence $w(k), k \in \mathbb{Z}$, with coefficients in $l_2$. □

**Remark:** Observe that $X_u$ is invariant under $E^X\mathcal{U}$; also $X_s$ is invariant under $E^X\mathcal{U}^{-1}$. Consequently, they are Markovian subspaces of $X$. Also $X_u$ is invariant under $E^{\mathcal{B}}\mathcal{U}^{-1}$, $X_s$ is invariant under $E^{\mathcal{A}}\mathcal{U}$.

**From spectral factors to state spaces:**

If $W$ is a spectral factor of $\Phi$, then there exists an uncorrelated sequence $w(k), k \in \mathbb{Z}$, such that

$$y(k) = W(z)w(k) \tag{2.11}$$

(possibly extending the basic probability space). Define

$$\begin{aligned} X_s = E^{H^-(w)}H^+, \quad X_u = E^{H^+(w)}H^-, \\ X = X_s \oplus X_u. \end{aligned} \tag{2.12}$$

**Proposition 2.1** *$X$ is a Markovian splitting subspace, i.e.*

$$H^- \vee X^- \perp H^+ \vee X^+ | X. \tag{2.13}$$

**Proof:** This is a direct consequence of Theorem 2.1 (ii), because $H^-(w)$, $H^+(w)$ are invariant under the corresponding shift operators. □

**Definition 2.1** *The spectral factor $W$ is minimal, if the corresponding state space $X$ defined in (2.12) is a minimal Markovian splitting subspace.*

**Remark:** In the rational case this concept of minimality coincides with the minimality of the dimension of the state space.

Lemma (5.1) shows that in the nonrational case this definition agrees with the definition of minimality given by A. Ferrante ([8]). In case of stable/antistable spectral factors, this definition gives back the definition of minimal spectral factors used in [14], p. 274.

**Remark:** Note that $W(z)$ is stable if and only if $H^- \subset \mathcal{A}$; in other words, $X \subset \mathcal{A}$, and $W(z)$ is antistable if and only if $H^+ \subset \mathcal{B}$, i.e. $X \subset \mathcal{B}$.

In the classical theory [13], [14] a Markovian splitting subspace $X$ is always connected with two perpendicularly intersecting subspaces $S, \bar{S}$, such that

$$X = S \cap \bar{S},\ H^- \subset S,\ H^+ \subset \bar{S},\ \mathcal{U}^{-1}S \subset S,\ \mathcal{U}\bar{S} \subset \bar{S}. \tag{2.14}$$

Shortly, $X \sim (S, \bar{S})$.

In our construction a decomposition $H = \mathcal{A} \oplus \mathcal{B}$ lead us to the same Markovian splitting subspaces. What is the connection between these constructions?

**Lemma 2.1** *(i) Assume that* $H = \mathcal{A} \oplus \mathcal{B}$, *and* $\mathcal{U}^{-1}\mathcal{A} \subset \mathcal{A}, \mathcal{U}\mathcal{B} \subset \mathcal{B}$. *Define* $S = \mathcal{A} \oplus X_u, \bar{S} = \mathcal{B} \oplus X_s$. *Then* $X \sim (S, \bar{S})$.

*(ii) Assume that* $X \sim (S, \bar{S})$ *is a Markovian splitting subspace. Let* $X_u$ *be an* $E^X\mathcal{U}$ *-invariant subspace of* $X$. *Set*

$$X_s = X \ominus X_u, \mathcal{A} = (S \ominus X) \oplus X_s, \mathcal{B} = (\bar{S} \ominus X) \oplus X_u.$$

*Then*

$$\mathcal{U}^{-1}\mathcal{A} \subset \mathcal{A},\ \mathcal{U}\mathcal{B} \subset \mathcal{B}, \mathcal{A} \oplus \mathcal{B} = H \text{ and } X_u = E^{\mathcal{B}}H^-, X_s = E^{\mathcal{A}}H^+.$$

**Proof:**

(i) Because of $\mathcal{U}^{-1}\mathcal{A} \subset \mathcal{A}$, and $E^{\mathcal{B}}\mathcal{U}^{-1}X_u \subset X_u$ we obtain that $\mathcal{U}^{-1}S \subset S$. Similarly, $\mathcal{U}\bar{S} \subset \bar{S}$.

Since $H^- \perp \mathcal{B} \ominus X_u$, we get that $H^- \subset S$. In the same way $H^+ \subset \bar{S}$.

Obviously $X = S \cap \bar{S}$, $S \perp \bar{S} | X$. Since Theorem 2.1 (ii) gives that $X$ is a Markovian splitting subspace, we have that $X \sim (S, \bar{S})$.

(ii) It is immediate that $E^{\mathcal{A}}H^+ = X_s$, and $E^{\mathcal{B}}H^- = X_u$. Also $\mathcal{U}\mathcal{B} \subset \mathcal{B}$, because $\mathcal{U}(\bar{S} \ominus X) \subset \bar{S} \ominus X$, and $\mathcal{U}X_u \perp [(S \ominus X) \vee X_s]$; thus, $\mathcal{U}X_u \subset \mathcal{B}$. Similarly, $\mathcal{U}^{-1}\mathcal{A} \subset \mathcal{A}$. □

**Minimality:**

Since even in the acausal case we are considering the same Markovian splitting subspaces there is no need for changing the definition of minimality of splitting subspaces. But how can we reduce the state space in the nonminimal case *keeping* the pole structure?

**Reduction:**

We define a subspace $X_0 \subset X$, which is a Markovian splitting subspace and minimal, i.e. $X_0 \cap (H^+)^\perp = X_0 \cap (H^-)^\perp = \emptyset$. It is very natural to define it as a result of two projections

$$X_0 = E^{X_1}H^-, X_1 = E^X H^+,$$

or

$$X_0 = E^{X_2}H^+, X_2 = E^X H^-.$$

We will show that this construction gives a solution of our problem. Because of symmetry we shall analyse the result of a single projection.

**Elimination of the unobservable part:**

**Theorem 2.2** *Set* $X' = E^X H^+, X'_u = X_u \cap X', X'_s = E^{X'} X_s$. *Then*

*(i)* $X'$ *is a Markovian splitting subspace,*

*(ii)* $X_u^{'}$ *is* $E^X\mathcal{U}$*-invariant,* ${X_u}^{'} \subset X_u$*,*

*(iii)* $E^{X_s^{'}}\mathcal{U}_{|X_s^{'}}$ *is a quasiaffine transform of* $E^{X_s}\mathcal{U}_{|X_s}$*.*

**Proof:** (Observe that $X^{'} = E^{X^{'}}X_s \oplus (X^{'} \cap X_s^{\perp}) = X_u^{'} \oplus X_s^{'}$.) Define $\mathcal{B}^{'} = X_u^{'} \oplus (\mathcal{B} \ominus X_u), \mathcal{A}^{'} = H \ominus \mathcal{B}^{'} = (\mathcal{A} \ominus X_s) \oplus (X \cap (H^+)^{\perp}) \oplus X_s^{'}$. Then

$$E^{\mathcal{B}^{'}}H^- = E^{X_u^{'}}H^- = E^{X_u^{'}}E^{X_u}H^- = {X_u}' \ ,$$

$$E^{\mathcal{A}^{'}}H^+ = E^{X_s^{'}}H^+ = E^{X_s^{'}}E^{X^{'}}H^+ = {X_s}' \ .$$

Invariance properties:

$$E^X\mathcal{U}^{-1}(X \cap H^{+\perp}) =$$

$$= E^{X \vee H^+}\mathcal{U}^{-1}(X \cap H^{+\perp}) \subset E^{X \vee H^+}(\mathcal{U}^{-1}X \cap H^{+\perp}) \subset X \cap H^{+\perp}.$$

But $X^{'} = X \ominus (X \cap H^{+\perp})$, so it is invariant under the adjoint map, i.e. $E^X\mathcal{U}X^{'} \subset X^{'}$. This implies also that $E^X\mathcal{U}X_u^{'} \subset X_u^{'}$.

On the other hand, $\mathcal{B}^{'} \ominus {X_u}' = \mathcal{B} \ominus X_u$ is $\mathcal{U}$ –invariant, so we get that $\mathcal{U}\mathcal{B}^{'} \subset \mathcal{B}^{'}$. Using that $\mathcal{A}^{'} = H \ominus \mathcal{B}^{'}$, we get that $\mathcal{U}^{-1}\mathcal{A}^{'} \subset \mathcal{A}^{'}$.

Invoking Theorem 2.1 (i) and (ii), we obtain that $X_u^{'}$ is a Markovian splitting subspace.

Concerning (iii), the operator $E^{X_s}{}_{|X_s^{'}}$ is an injection with dense range and

$$E^{X_s}(E^{{X_s}'}\mathcal{U}_{|X_s^{'}}) = (E^{X_s}\mathcal{U})E^{X_s}_{|X_s^{'}} \ ,$$

so $E^{{X_s}'}\mathcal{U}_{|X_s^{'}}$ is a quasiaffine transform of $E^{X_s}\mathcal{U}_{|X_s}$. □

**Remark:** Theorem 4 in [17] implies that if $\mathcal{A} \ominus X_s$ and $\mathcal{B} \ominus X_u$ are invariant subspaces of full range of $\mathcal{A}$ and $\mathcal{B}$, respectively, then $E^{{X_s}'}\mathcal{U}_{|X_s^{'}}$ is quasisimilar to $E^{X_s}\mathcal{U}_{|X_s}$. This full range property can be expressed in terms of the spectral factor $W_a$ corresponding to $(\mathcal{A}, \mathcal{B})$. Namely, it means that $W_a(z)$ and $W_a(z^{-1})$ are strictly noncyclic (cf. Fuhrmann [11], p. 253.)

If $X$ is finite dimensional, then $E^{X_s}|X_s^{'}$ has a bounded inverse; thus the operators $E^{{X_s}'}\mathcal{U}_{|X_s^{'}}$ and $E^{X_s}\mathcal{U}_{|X_s}$ are similar.

In terms of poles, the previous proposition means that via this reduction we do not get new poles, moreover we have the same stable poles, but possibly there is a reduction in the antistable poles.

As we have mentioned earlier the unreconstructable part can be eliminated in a similar way.

**Proposition 2.2** *If* $X$ *is a minimal (Markovian) splitting subspace, then*

$$N^- \subset A, N^+ \subset B \tag{2.15}$$

*where* $N^- = H^- \cap (H^+)^{\perp}, N^+ = H^+ \cap (H^-)^{\perp}$.

**Proof:** If $X$ is minimal, then $X \perp (N^+ \oplus N^-)$. Thus, if $\eta \in N^- \subset H^-$, then $E^{\mathcal{B}}\eta \in X_u \subset X$. Consequently $\eta \perp E^{\mathcal{B}}\eta$, so $E^{\mathcal{B}}\eta = 0$, i.e. $N^- \perp \mathcal{B}$. This implies that $N^- \subset \mathcal{A}$.

The other inclusion can be proved similarly. □

Lemma 2.1 shows, that if $X \sim (S, \bar{S})$ is a proper Markovian splitting subspace, then any $E^X\mathcal{U}$–invariant subspace of $X$ generates a – in general acausal – spectral factor. What is the pole structure of this spectral factor? What is the connection between the pole structures of different spectral factors? If $W_a$ is an acausal spectral factor connected with the state space $X$ and $X'$ is another Markovian splitting subspace, then is it possible to construct another spectral factor with the same pole structure as $W$ the state space of which is $X'$? The next proposition compares the invariant subspaces of the different state spaces. It gives a geometric description of the spectral factors with the same pole structure. (Compare this with the similar result in ([18]), which describes these in an algebraic form.)

In the next section we shall answer the previous questions in the finite dimensional case. In Section 5 we return again to the infinite dimensional case.

**Proposition 2.3** *Assume that $X_1 \sim (S_1, \bar{S}_1), X_2 \sim (S_2, \bar{S}_2)$ are minimal Markovian splitting subspaces, and either $S_1 \subset S_2$ or $\bar{S}_2 \subset \bar{S}_1$. Suppose that $X_{u2} \subset X_2$ is an $E^{X_2}\mathcal{U}$-invariant subspace. Then*

$$X_{u1} = E^{X_1} X_{u2} \text{ is } E^{X_1}\mathcal{U} - \text{invariant.} \tag{2.16}$$

*Define $X_{s2} = X_2 \ominus X_{u2}, \quad X_{s1} = X_1 \ominus X_{u1}$. Then*

$$X_{s2} = E^{X_2} X_{s1}. \tag{2.17}$$

**Proof:** First consider the case when $S_1 \subset S_2$. Then $E^{X_1} X_{u2} = E^{S_1} X_{u2}$. Consequently $E^{X_1}\mathcal{U}X_{u1} = E^{X_1}\mathcal{U}(E^{X_1} X_{u2}) = E^{X_1}\mathcal{U}X_{u2} = E^{X_1}E^{S_2}\mathcal{U}X_{u2} \subset E^{X_1}X_{u2} = X_{u1}$ , which proves (2.16).

The equation (2.17) is immediate form the decomposition

$$X_2 = E^{X_2} X_{s1} \oplus (X_2 \cap (X_{s1})^{\perp}) \ ,$$

observing that $X_2 \cap (X_{s1})^{\perp} = X_{u2}$. (We have used that $X_2 \cap (X_1)^{\perp} = 0$ ).

Similar proof can be applied in the case when $\bar{S}_2 \subset \bar{S}_1$. □

**Remark:** Observe that $E^{X_2}\mathcal{U}_{|X_{u2}}$ is a quasiaffine transform of $E^{X_1}\mathcal{U}_{|X_{u1}}$ since

$$E^{X_1}(E^{X_2}\mathcal{U}_{|X_{u2}}) = (E^{X_1}\mathcal{U}_{|X_{u1}})E^{X_1}. \tag{2.18}$$

Similarly, $E^{X_1}\mathcal{U}^{-1}_{|X_{s1}}$ is a quasiaffine transform of $E^{X_2}\mathcal{U}^{-1}_{|X_{s2}}$.

As in Theorem 2.2 if the invariant subspaces are of full range then the operators are quasisimilar, and in the rational case – the minimal state spaces are finite dimensional – the quasisimilarity can be changed to similarity.

**Corollary 2.1** *There is a one-to-one correspondence between the $E^X\mathcal{U}$–invariant subspaces of $X$ and the $E^{X_+}\mathcal{U}$–invariant subspaces of $X_+$. Similarly, there is a one-to-one correspondence between the $E^X\mathcal{U}^{-1}$–invariant subspaces of $X$ and the $E^{X_-}\mathcal{U}^{-1}$–invariant subspaces of $X_-$.*

**Proof:** Since $X_+ \sim ((N^+)^\perp, H^+), X_- \sim (H^-, (N^-)^\perp)$ we can apply the previous proposition using that if $X \sim (S, \bar{S})$, then $H^- \subset S$ and $H^+ \subset \bar{S}$. □

## 3 Equations in the Finite Dimensional Case

Assume that $\Phi$ is a rational function. Consider a **minimal** Markovian splitting subspace $X$ together with a forward and a backward realization:

$$(\Sigma) \quad \begin{cases} x(k+1) = Ax(k) + Bw(k) \\ y(k) = Cx(k) + Dw(k) \end{cases} \tag{3.1}$$

$$(\bar{\Sigma}) \quad \begin{cases} \bar{x}(k) = A'\bar{x}(k+1) + \bar{B}\bar{w}(k) \\ y(k) = \bar{C}\bar{x}(k+1) + \bar{D}\bar{w}(k) \end{cases} \tag{3.2}$$

where $\bar{x}(k) = P^{-1}x(k)$, is the covariance matrix of $x(k)$.

We would like to construct acausal spectral factors with the state space $X$. In view of Lemma 2.1, we have to consider an $E^X\mathcal{U}$–invariant subspace of $X$. So without loss of generality, we can assume that

$$A = \begin{bmatrix} A_1 & A_{12} \\ 0 & A_2 \end{bmatrix}. \tag{3.3}$$

Partition the covariance matrix $P$ according to this, i.e. $P = \begin{bmatrix} P_1 & P_{12} \\ P_{21} & P_2 \end{bmatrix}$ and define

$$x_a(k) = \begin{bmatrix} x_{a1}(k) \\ x_{a2}(k) \end{bmatrix} = \begin{bmatrix} x_1(k) - P_{12}{P_2}^{-1}x_2(k) \\ {P_2}^{-1}x_2(k) \end{bmatrix}. \tag{3.4}$$

(Denote by $X_1(k)$ and $X_2(k)$ the subspaces generated by the coordinates of $x_{a1}(k)$ and $x_{a2}(k)$, respectively.) Then straightforward calculation gives that

- the projection of $x_{a1}(k+1)$ onto $X_1(k)$ is $A_1x_{a1}(k)$;

- the projection of $x_{a2}(k)$ onto $X_2(k+1)$ is $A_2{}'x_{a2}(k+1)$;
- and the projection of $y(k)$ onto $X_1(k) \vee X_2(k)$ is $C_1 x_{a1}(k) + \bar{C}_2 x_{a2}(k+1)$ .

Moreover, computing directly the cross-covariances, it can be proved that the sequence

$$\begin{bmatrix} x_{a1}(k+1) - A_1 x_{a1}(k) \\ x_{a2}(k) - A_2{}' x_{a2}(k+1) \\ y(k) - C_1 x_{a1}(k) - \bar{C}_2 x_{a2}(k+1) \end{bmatrix}$$

is an uncorrelated, although not necessarily normalized sequence. Normalizing it, we can write that

$$(\Sigma_a) \quad \begin{cases} x_{a1}(k+1) = A_1 x_{a1}(k) + \quad \cdot \quad w_a(k) \\ x_{a2}(k) = A_2' x_{a2}(k+1) + \quad \cdot \quad w_a(k) \\ y(k) = C_1 x_{a1}(k) + \bar{C}_2 x_{a2}(k+1) + \quad \cdot \quad w_a(k) \end{cases} \tag{3.5}$$

where $w_a(k), k \in \mathbb{Z}$, is a normalized, uncorrelated sequence. Then $x_{a1}$, and $x_{a2}$ are obviously Markovian processes. The poles of the corresponding spectral factor $W_a$ are given by the eigenvalues of $A_1$ – stable poles, and the reciprocals of the eigenvalues of $A_2'$ – antistable poles (defining $1/0 = \infty$, so possibly producing a pole at infinity).

If $A$ is nonsingular, then equation (3.5) can be written in the form of a forward realization as

$$(\Sigma_a) \quad \begin{cases} x_a(k+1) = \begin{bmatrix} A_1 & 0 \\ 0 & (A_2')^{-1} \end{bmatrix} x_a(k) + B_a w_a(k) \\ \\ y(k) = \left[ C_1, \bar{C}_2 (A_2')^{-1} \right] x_a(k) + D_a w_a(k) \end{cases} \tag{3.6}$$

Denote $A_a = \begin{bmatrix} A_1 & 0 \\ 0 & (A_2')^{-1} \end{bmatrix}$ , $C_a = \left[ C_1, \bar{C}_2 (A_2')^{-1} \right]$ .

**Remark:** Observe that $X \cap H^-(w_a) = X_s$ and $X \cap H^+(w_a) = X_u$, so the pole structure of $W_a$ is determined by the intersections of the state space $X$ and the past/future of the noise process $w_a$.

Proposition 2.3 and Corollary 2.1 give the possibility of choosing the coordinate system in each minimal Markovian splitting subspace in such a way that the same $A_a$ matrix describe the state transition matrix. Namely, if $X$ is a minimal Markovian splitting subspace, and $X_{u+}$ is an $E^{X_+}\mathcal{U}$–invariant subspace of $X_+$ , then $X_u = E^X X_{u+}, X_{u-} = E^{X_-} X_{u+}$ are invariant under the corresponding compressed shift operators. Let $X_{s-} = X_- \ominus X_{u-}, X_s = X \ominus X_u$ as usual. Choosing coordinate vectors in $X_{u+}$ and $X_{s-}$ and projecting them to $X$ gives the uniform choice of bases, because the corresponding compressed shift operators are similar.

**Time reversing:**

In the classical theory [14], [19] each state space is associated with two spectral factors (stable and antistable), the poles of which are *mirror images* of each other. Dealing with acausal spectral factors we may try to construct another spectral factor with poles flipped with respect to the unit circle. In other words a spectral factor for which the state transition matrix in the forward state equation of (3.5) is $A_2$, in the backward one is $A_1^{'}$, and in the observation equation $C_2, \bar{C}_1$ stand in place of $C_1, \bar{C}_2$. This can be clearly achieved if there exists another $E^X\mathcal{U}$ -invariant subspace $\bar{X}_u$ of $X$, which is *complementary* to $X_u$, or in other words if $A$ is similar to $\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$. In the sequel we shall refer to this situation, as the case when there exists a reversed system.

**Lemma 3.1** *Assume that $A$ is nonsingular. There exists a reversed system of $\Sigma_a$ if and only if there exists a symmetric solution of the Ljapunov equation*

$$P_a - A_a P_a A_a^{'} = B_a B_a^{'} \ . \tag{3.7}$$

**Proof:** Denote

$$\tilde{P} = \begin{bmatrix} P_1 - P_{12}P_2^{-1}P_{21} & P_{12}P_2^{-1} \\ P_2^{-1}P_{21} & -P_2^{-1} \end{bmatrix} \ . \tag{3.8}$$

From equation (3.6) direct computation gives that

$$\tilde{P} - \begin{bmatrix} A_1 & 0 \\ 0 & (A_2)^{'-1} \end{bmatrix} \tilde{P} \begin{bmatrix} (A_1)^{'} & 0 \\ 0 & (A_2)^{-1} \end{bmatrix}$$

$$- \begin{bmatrix} 0 & A_{12}(A_2)^{-1} \\ (A_2)^{'-1}A_{12}^{'} & 0 \end{bmatrix} = B_a B_a^{'}.$$

Now, if $A_{12} = 0$, then $P_a = \tilde{P}$ is a solution of (3.7).

Conversely, assume that $P_a = \begin{bmatrix} P_{a1} & P_{a12} \\ P_{a21} & P_{a2} \end{bmatrix}$ solves (3.7). Observe that $P_{a2} = -P_2^{-1}$. Define

$$\tilde{x}(k) = \begin{bmatrix} I & -P_{a12}P_{a2}^{-1} \\ 0 & P_{a2}^{-1} \end{bmatrix} x_a(k) \ .$$

In this case $\tilde{x}(k), k \in \mathbb{Z}$, is also a Markovian process, and

$$E^X \tilde{x}(1) = \begin{bmatrix} I & -P_{a12}P_{a2}^{-1} \\ 0 & P_{a2}^{-1} \end{bmatrix} \begin{bmatrix} A_1 & -B_{a1}B_{a2}^{'}P_{a2}^{-1} \\ 0 & P_{a2}A_2P_{a2}^{-1} \end{bmatrix} \begin{bmatrix} I & P_{a12} \\ 0 & P_{a2} \end{bmatrix} \tilde{x}(0)$$

$$= \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \tilde{x}(0),$$

because $A_1 P_{a12} - B_{a1} B_{a2}^{'} = P_{a12} A_2$. Thus $A$ is similar to $\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$. □

Observe that since we did not assume apriori the solvability condition of the Ljapunov equation which would imply also the uniqueness of the solution, i.e. the spectrum of $A_1$ and $A_2$ need not be disjoint, $\tilde{P}$ may not be the only solution of (3.7).

**Remark:** If the condition of Lemma 3.1 is fulfilled, i.e. (3.7) has a solution, then elementary calculation gives that the state vector of the reversed system $\bar{\Sigma}_a$ can be defined as

$$\bar{x}_a(k) = P_a^{-1} x_a(k) \ ,$$

where $P_a$ now denotes the solution given in (3.8), which is clearly nonsingular in view of the decomposition $\tilde{P} = \begin{bmatrix} P_1 & P_2 \\ 0 & -I \end{bmatrix} \begin{bmatrix} I & 0 \\ -P_2^{-1} P_{21} & P_2^{-1} \end{bmatrix}$. In this case we get that

$$(\bar{\Sigma}_a) \quad \begin{cases} \bar{x}_{a1}(k) = A_1^{'} \bar{x}_{a1}(k+1) + \quad \cdot \quad \bar{w}_a(k) \\ \bar{x}_{a2}(k+1) = A_2 \bar{x}_{a2}(k) + \quad \cdot \quad \bar{w}_a(k) \\ y(k) = \bar{C}_1 \bar{x}_{a1}(k+1) + C_2 \bar{x}_{a2}(k) + \quad \cdot \quad \bar{w}_a(k) \end{cases} \tag{3.9}$$

where $\bar{w}_a(k), k \in \mathbb{Z}$, is an uncorrelated sequence. The poles of the corresponding spectral factor $\bar{W}_a$ are determined by the reciprocals of the eigenvalues of $A_1^{'}$ – antistable poles, and by the eigenvalues of $A_2$ – stable poles.

Denote $\bar{C}_a = [\bar{C}_1, -C_2(A_2)^{-1}]$ . Straightforward computation leads to the system of equations

$$\begin{bmatrix} P_a - A_a P_a A_a^{'} & \bar{C}_a^{'} - A_a P_a C_a^{'} \\ \bar{C}_a - C_a P_a A_a^{'} & R(0) - (\bar{C}_2 A_2^{'-1} C_2^{'} + C_2 A_2^{-1} \bar{C}_2^{'}) - C_a P_a C_a^{'} \end{bmatrix} = \begin{bmatrix} B_a \\ D_a \end{bmatrix} [B_a^{'}, D_a^{'}] \geq 0 \tag{3.10}$$

which is almost identical to the system of equations used in the construction of weak realizations (cf. [7], [2]). We would like to emphasize again that here the matrix $A_a$ is not necessarily stable, so $P_a$ in general is *indefinite*. At the same time (3.10) has a *minimal* and *maximal* solution in $P_a$. This is immediate also from the following proposition.

**Proposition 3.1** *The transformation*

$$P = \begin{bmatrix} P_1 & P_{12} \\ P_{21} & P_2 \end{bmatrix} \longrightarrow P_a = \begin{bmatrix} P_1 - P_{12} P_2^{-1} P_{21} & P_{12} P_2^{-1} \\ P_2^{-1} P_{21} & -P_2^{-1} \end{bmatrix}$$

*is strictly monotone. (If $P \geq \tilde{P}$ then $P_a \geq \tilde{P}_a$ and if $P > \tilde{P}$ then $P_a > \tilde{P}_a$).*

**Proof:** Because of continuity it is enough to prove that $P > \tilde{P}$ implies that $P_a > \tilde{P}_a$. The assumption that $P > \tilde{P}$ is equivalent to $P_2 - \tilde{P}_2 > 0$ and $(P_1 - \tilde{P}_1) - (P_{12} - \tilde{P}_{12})(P_{22} - \tilde{P}_{22})^{-1}(P_{21} - \tilde{P}_{21}) > 0$.

Obviously $P_{a2} - \tilde{P}_{a2} = -P_2^{-1} + \tilde{P}_2^{-1} > 0$. On the other hand

$$
\begin{aligned}
&(P_{a1} - \tilde{P}_{a1}) - (P_{a12} - \tilde{P}_{a12})(P_{a22} - \tilde{P}_{a22})^{-1}(P_{a21} - \tilde{P}_{a21}) \\
&= (P_1 - P_{12}P_2^{-1}P_{21}) - (\tilde{P}_1 - \tilde{P}_{12}(\tilde{P}_2)^{-1}\tilde{P}_{21}) \\
&-(P_{12}P_2^{-1} - \tilde{P}_{12}\tilde{P}_2^{-1})(-P_2^{-1} + \tilde{P}_2^{-1})^{-1}(P_2^{-1}P_{21} - \tilde{P}_2^{-1}\tilde{P}_{21}) \\
&= (P_1 - \tilde{P}_1) - (P_{12} - \tilde{P}_{12})(P_{22} - \tilde{P}_{22})^{-1}(P_{21} - \tilde{P}_{21}) \ ,
\end{aligned}
$$

using that $(-P_2^{-1} + \tilde{P}_2^{-1})^{-1} = P_2^{-1}(P_2 - \tilde{P}_2)\tilde{P}_2^{-1}$. □

## 4 Zeros of Spectral Factors

In this section we prove that the zeros and the zero structure of a spectral factor depend only on the corresponding state space. Although there is a vast literature about various kinds of zeros of an input–output systems, we must start this section from the very beginning. Let us give a short explanation for this. The system zeros can be defined either using the Smith–McMillan form of a transfer function where the zeros of the corresponding numerator polynomials are the system zeros, or using the Rosenbrock matrix in which case the zeros are the points where it loses rank. But we want to consider the zero directions, too. Applying the Smith–McMillan form, we transform the zero directions so the analysis becomes more complicated. Using the Rosenbrock matrix to define the multiplicity of a zero as the defect in the rank of the Rosenbrock matrix is not satisfactory because in this way we do not get the right multiplicity. Every Jordan block may reduce the rank of the matrix only by 1, but the multiplicity of the corresponding eigenvalue is the dimension of the Jordan block. Unfortunately, this fact has been several times overlooked in the literature.

So we start with the definition of zeros in a way as we always think about zeros of a function - where its value and maybe its derivatives are zero. But since in the matrix case we have to take into consideration that not all elements are vanishing at a given point only some linear combinations and it may happen that at the same point there is also a pole, we have to define the zeros, zero directions in a more careful way.

Let $W_a(z)$ be an acausal spectral factor of the form

$$W_a(z) = D + C_1(zI - A_1)^{-1}B_1 + \bar{C}_2(z^{-1} - \bar{A}_2)^{-1}\bar{B}_2 \qquad (4.1)$$

where $A_1, \bar{A}_2$ are stable matrices and (4.1) defines a minimal realization of $W_a$. The vector-valued complex function $\phi(z)$ is called left zero function of $W_a$ at $z_0$, if it is analytic in a neighbourhood of $z_0$ and

$$\lim_{z\to z_0} \phi(z)W_a(z) = 0. \tag{4.2}$$

The order of the zero function is the largest positive integer $k$, for which

$$\lim_{z\to z_0} [(z-z_0)^{-k+1}\phi(z)W_a(z)] = 0 \tag{4.3}$$

(see Ball, Gohberg and Rodman ([3]). If $n$ is the McMillan degree of $W_a$, then it is enough to analyse the zero functions of order $k$ in the form

$$\sum_{i=0}^{n+k-1} a_i^{'}(z-z_0)^i$$

in the sense that a function $\phi$ is a left zero function if and only if the polynomial defined by the formula above using $a_i^{'} = \frac{1}{i!}\phi(z_0)^{(i)}$ as coefficients is a left zero function. The vectors $a_i^{'}, i = 1\dots k$, form a so-called left zero chain of $W(z)$ at $z_0$. Denote shortly by $M$ the matrix with rows $a_i^{'}, i = 1\dots k$ ($k$ is the order of the left zero function $\phi$.)

Let $\Lambda$ be the Jordan block of order $k$ determined by $z_0$.

$$\Lambda = \begin{bmatrix} z_0 & 0 & \dots & 0 \\ 1 & z_0 & 0\dots & 0 \\ & \ddots & \ddots & \\ & & 1 & z_0 \end{bmatrix}.$$

**Theorem 4.1** *The matrix $M$ determines a left zero chain of $W$ at $z_0$ if and only if there exist two matrices $\Pi_1, \Pi_2$ satisfying the matrix equation*

$$[\Pi_1 \ \Lambda\Pi_2 \ M]\begin{bmatrix} A_1 & 0 & B_1 \\ 0 & \bar{A}_2 & \bar{B}_2 \\ C_1 & \bar{C}_2 & D \end{bmatrix} = [\Lambda\Pi_1 \ \Pi_2 \ 0]\,. \tag{4.4}$$

**Proof:** By symmetry we may assume that $I - z_0\bar{A}_2$ is a regular matrix. Also assume that $z_0$ is different from $\infty$. The proof goes by induction. Let $\phi$ be a left zero function at $z_0$ . In view of (4.2),

$$a_0^{'}D + \lim_{z\to z_0}\phi(z)C_1(zI-A_1)^{-1}B_1 + a_0^{'}\bar{C}_2 z_0(I-z_0\bar{A}_2)^{-1}\bar{B}_2 = 0. \tag{4.5}$$

Denote $\eta_0^{'} = a_0^{'}\bar{C}_2(I-z_0\bar{A}_2)^{-1}$, $\xi_0(z)^{'} = \phi(z)C_1(zI-A_1)^{-1}$. We have

$$a_0^{'}\bar{C}_2 + z_0\eta_0^{'}\bar{A}_2 = \eta_0^{'} \tag{4.6}$$

$$\phi(z)C_1 + \xi_0(z)^{'} A_1 = z\xi_0(z)^{'}. \tag{4.7}$$

Equation (4.5) gives that $\lim_{z\to z_0} \xi_0(z)^{'} B_1$ exists. Multiplying (4.7) by $B_1$ from the right we get that $\lim_{z\to z_0} \xi_0(z)^{'} A_1 B_1$ exists. Now multiplying with $A_1 B_1$ we obtain that $\lim_{z\to z_0} \xi_0(z)^{'} A_1^2 B_1$ exists. Continuing this and using that $A_1, B_1$ is controllable we get that $\lim_{z\to z_0} \xi_0(z)^{'} = \xi_0^{'}$ exists. Obviously it solves the equation

$$\begin{aligned} a_0^{'} C_1 + \xi_0^{'} A_1 &= z_0 \xi_0^{'} \\ a_0^{'} D + \xi_0^{'} B_1 + z_0 \eta_0^{'} \bar{B}_2 &= 0; \end{aligned} \tag{4.8}$$

and (4.6) and (4.8) give (4.4) setting $\Pi_1 = [\xi_0^{'}], \Pi_2 = [\eta_0^{'}]$.

Consider the case when $k \geq 2$. Write $\phi$ in the form $\phi(z) = a_0^{'} + (z - z_0)\phi_1(z)$. Using the equations (4.6) (4.8), we obtain that

$$\begin{aligned} \phi(z)W(z) &= (z - z_0)[\phi_1(z)D + (\phi_1(z)C_1 - \xi_0^{'})(zI - A_1)^{-1} B_1 \\ &\quad + (\phi_1(z) z\bar{C}_2 + \eta_0^{'})(I - z\bar{A}_2)^{-1} \bar{B}_2]. \end{aligned} \tag{4.9}$$

Denoting $\eta_1^{'} = (a_1^{'} \bar{C}_2 + \eta_0^{'} \bar{A}_2)(I - z_0 \bar{A}_2)^{-1}$, $\xi_1(z)^{'} = (\phi_1(z) C_1 - \xi_0^{'})(zI - A_1)^{-1}$ we have

$$\begin{aligned} a_1^{'} \bar{C}_2 + (\eta_0^{'} + z_0 \eta_1^{'}) \bar{A}_2 &= \eta_1^{'} \\ \phi_1(z) C_1 + \xi_1(z)' A_1 &= z\xi_0(z)' + \xi_0^{'} \\ a_1^{'} D + \lim_{z\to z_0} \xi_1(z)' B_1 + (\eta_0^{'} + z_0 \eta_1^{'}) \bar{B}_2 &= 0. \end{aligned}$$

This gives that $\lim_{z\to z_0} \xi_1(z)' B_1$ exists. Multiplying the second equation again by $B_1, A_1 B_1, \ldots$, we get that $\lim_{z\to z_0} \xi_1(z)' A_1 B_1, \lim_{z\to z_0} \xi_1(z)' A_1^2 B_1$ $\ldots$ exist.

Using the controllability of $A_1, B_1$, we conclude that $\lim_{z\to z_0} \xi_1(z)' = \xi_1^{'}$ exists. Obviously it solves the equation

$$\begin{aligned} a_1^{'} C_1 + \xi_1^{'} A_1 &= z_0 \xi_1^{'} + \xi_0^{'} \\ a_1^{'} D + \xi_1^{'} B_1 + (\eta_0^{'} + z_0 \eta_1^{'}) \bar{B}_2 &= 0. \end{aligned}$$

Using the matrices $\Pi_1 = \begin{bmatrix} \xi_0^{'} \\ \xi_1^{'} \end{bmatrix}, \Pi_2 = \begin{bmatrix} \eta_0^{'} \\ \eta_1^{'} \end{bmatrix}$, we obtain the solution of (4.4) for $k = 2$. If $k \geq 3$ then we can repeat the argument given for $k = 2$. This leads to equation (4.4) in the general case.

Conversely, let us assume that equation (4.4) holds, and try to construct a left zero function of order at least $k$ at $z_0$. To this aim it is enough to determine $a_i^{'}, i = 0, \ldots n + k - 1$. Let the rows of the matrix $M$ define the first $k$ elements of this sequence. Together with this we are going to define two other sequences $\xi_i, i = 0 \ldots n + k - 1$, $\eta_i, i = 0 \ldots k - 1$ as follows. The first $k$ elements are given by the row vectors of the matrices $\Pi_1, \Pi_2$, respectively. The remaining elements of these sequences are defined successively as solutions of the equations

$$a_i^{'} C_1 - \xi_i^{'}(z_0 I - A_1) = \xi_{i-1}^{'} \tag{4.10}$$

$i = k \dots (n+k-1)$. (Let us observe that equation (4.4) imply that these equations are valid for $i = 0 \dots k-1$, too.) Similarly, the sequence $\eta_i, i = 0 \dots k-1$ satisfies the equation

$$a_i^{'} \bar{C}_2 - \eta_i^{'}(I - z_0 \bar{A}_2) = -\eta_{i-1}^{'} \bar{A}_2. \tag{4.11}$$

The observability of the pair $C_1, (z_0 I - A_1)$ imply that for any fixed vector $\xi_{i-1}^{'}$ (4.10) can be solved, giving the next $a_i{'}, \xi_i^{'}$ values.

In this case

$$\begin{aligned}
\phi(z)W(z) &= \sum_{i=1}^{n+k-1} a_i^{'}(z-z_0)^i D + \sum_{i=1}^{n+k-1} a_i^{'}(z-z_0)^i C_1 (zI - A_1)^{-1} B_1 \\
&\quad + \sum_{i=1}^{n+k-1} a_i^{'}(z-z_0)^i \bar{C}_2 z (I - z\bar{A}_2)^{-1} \bar{B}_2 \\
&= a_0^{'} D + \xi_0^{'} B_1 + z_0 \eta_0^{'} \bar{B}_2 \\
&+ \sum_{i=1}^{k-1} (z-z_0)^i (a_i^{'} D + \xi_i^{'} B_1 + (z_0 \eta_i^{'} + \eta_{i-1}^{'}) \bar{B}_2) \\
&+ \sum_{i=k}^{n+k-1} (z-z_0)^i (a_i^{'} D + \xi_i^{'} B_1) - \xi_0^{'} B_1 - z_0 \eta_0^{'} \bar{B}_2 \\
&- \sum_{i=1}^{n+k-1} (z-z_0)^i \xi_i^{'} B_1 - \sum_{i=1}^{k-1} (z-z_0)^i (z_0 \eta_i^{'} + \eta_{i-1}^{'}) \bar{B}_2 \\
&+ a_0^{'} C_1 (zI - A_1)^{-1} B_1 + a_0^{'} \bar{C}_2 z (I - z\bar{A}_2)^{-1} \bar{B}_2 \\
&+ \sum_{i=1}^{n+k-1} a_i^{'}(z-z_0)^i C_1 (zI - A_1)^{-1} B_1 \\
&+ \sum_{i=1}^{n+k-1} a_i^{'}(z-z_0)^i \bar{C}_2 z (I - z\bar{A}_2)^{-1} \bar{B}_2.
\end{aligned}$$

In view of (4.4), the first two elements are zero. Using the equations

$$a_0^{'} C_1 = \xi_0^{'}(z_0 I - A_1), \ a_0^{'} \bar{C}_2 = \eta_0^{'}(I - z_0 \bar{A}_2),$$

we obtain that

$$a_0^{'} C_1 (zI - A_1)^{-1} B_1 - \xi_0^{'} B_1 = -(z - z_0) \xi_0^{'} (zI - A_1)^{-1} B_1$$

$$a_0'\bar{C}_2 z(I - z\bar{A}_2)^{-1}\bar{B}_2 - z_0\eta_0'\bar{B}_2 = (z - z_0)\eta_0'(I - z\bar{A}_2)^{-1}\bar{B}_2.$$

In a similar way using equations (4.10) (4.11), we can eliminate the next elements in the sums. Continuing in this way we finally arrive at the equation

$$\begin{aligned}\phi(z)W(z) &= \sum_{i=k}^{n+k-1}(z - z_0)^i(a_i'D + \xi_i'B_1)\\ &\quad + (z - z_0)^{n+k}\xi_n'(zI - A_1)^{-1}B_1\\ &\quad + (z - z_0)^k\eta_{k-1}'(I - z\bar{A}_2)^{-1}\bar{B}_2\end{aligned}$$

proving (4.3) . □

**Remark:** It is easy to see from equation (3.5) that (4.4) is equivalent to the equation

$$-\Pi_1 x_{a1}(k+1) + \Pi_2 x_{a2}(k+1) = \Lambda(-\Pi_1 x_{a1}(k) + \Pi_2 x_{a2}(k)) + My(k)\ , \quad (4.12)$$

or denoting $\Pi = [-\Pi_1, \Pi_2]$, we get

$$\Pi x_a(k+1) = \Lambda\Pi x_a(k) + My(k)\ .$$

(Let us recall that $\Lambda$ denotes one single Jordan block.) The matrix $\Pi$ determines the zero directions at $z_0$ corresponding to a single Jordan block. In the case when we would like to describe *all* the zero directions *at a given point* $z_0$ we have to take into consideration that there may be several Jordan blocks with the same eigenvalue $z_0$.

The previous equation can be used for describing all the zero directions of a spectral factor corresponding to *all finite zeros*. Namely consider a maximal solution of this equation – not assuming that $\Lambda$ is a Jordan block – (maximal in the sense that the rank of $\Pi$ is maximal), and find the Jordan decomposition of the corresponding matrix $\Lambda$. Transforming the matrix $\Pi$ using the same transformation we get the corresponding zero directions. (This was investigated in Michaletzky [16] in the so-called regular case when there are no zeros at infinity and on the unit circle.)

**Remark:** In the case when $z_0 = \infty$ a similar argument can be used. We obtain the following equations. Let $\Lambda_0$ be the Jordan block of order $k$ determined by the eigenvalue 0. Instead of (4.4) we get

$$[\Lambda_0\Pi_1\ \Pi_2\ M]\begin{bmatrix} A_1 & 0 & B_1\\ 0 & \bar{A}_2 & \bar{B}_2\\ C_1 & \bar{C}_2 & D\end{bmatrix} = [\Pi_1\ \Lambda_0\Pi_2\ 0] \quad (4.13)$$

which is equivalent to

$$\Lambda_0\Pi x_a(k+1) = \Pi x_a(k) + My(k). \quad (4.14)$$

($\Lambda_0$ is the Jordan block defined by the eigenvalue 0.)

**Corollary 4.1** *$W(z), W_a(z), \bar{W}(z)$ have the same zeros with the same order and the same zero directions.*

This is immediate from the previous equation which shows that the zeros and zero directions are determined by the state space.

## 5 Comparison of Different Spectral Factors

In this section we return to the general – nonrational – case.

**Internal case:**

Let $X$ be a proper minimal Markovian splitting subspace determined by the subspaces $\mathcal{A}, \mathcal{B}$ and assume that $\mathcal{A} \oplus \mathcal{B} = H = H^- \vee H^+$. In this case $X \subset H^\Box = H \ominus (N^- \oplus N^+)$, and we have the following inclusions $N^- \subset \bar{S}^\perp \subset \mathcal{A} \subset S \subset (N^+)^\perp$. Denote the corresponding spectral factors by $\bar{W}_-, \bar{W}, W_a, W, W_+$.

**Lemma 5.1** *There are inner functions $U_1, V_1, U, V$ such that*

$$W_a = \bar{W}_- U_1, W_+ = W_a V_1, \tag{5.1}$$

$$W_a = \bar{W} U, W = W_a V \tag{5.2}$$

*and $U_1$ is a left inner divisor of $U_M = \bar{W}_-^{-1} W_+$, $U$ is a left inner divisor of $K = \bar{W}^{-1} W$.*

**Proof:** The proof of this statement is standard (see ([13])), but for the sake of readers' convenience we outline it. The function $U_1 = W_-^* W_a$ is a square, all–pass function and since $\bar{w}_-(-1) \in H^-(w_a), (N^- \subset \mathcal{A})$ it maps the function $I$ into an analytic function, so it is inner. The other equations can be proven similarly. $\Box$

**Time reversing – nonrational case:**

Let $\bar{X}_u$ be another $E^X\mathcal{U}$-invariant subspace of $X$ such that $X = X_u + \bar{X}_u$. Denote by $\bar{W}_a$ the corresponding spectral factor.

**Lemma 5.2** *Consider the decompositions $W_a = WV^*, \bar{W}_a = W\bar{V}^*, W_a = \bar{W}U^*, \bar{W}_a = \bar{W}\bar{U}^*$. Then*

*(i) $V$ and $\bar{U}$ are quasiequivalent,*

*(ii) $U$ and $\bar{V}$ are quasiequivalent.*

**Proof:** The structural function $K = \bar{W}^{-1}W$ of $X$ can be decomposed as $K = UV = \bar{U}\bar{V}$. The equations $X_u \cap \bar{X}_u = 0$ and $X_u \vee \bar{X}_u = X$ imply that $V$ and $\bar{V}$ are right coprime inner functions. Similarly, $U$ and $\bar{U}$ are left coprime. So, we obtain that $V$ and $\bar{U}$ (also $U$ and $\bar{V}$) are quasiequivalent. □

**Minimality in the internal case:**

In this part of this section we are going to characterize all the square acausal spectral factors in terms of left inner divisors of $\bar{W}_-^{-1}W_+$. Since every inner divisor is determined by an invariant subspace in Theorem 5.1 we give a geometric characterization of minimality. Using this characterization, we present another proof of the theorem given by Ferrante [8], and we show that under some assumptions the condition used in that theorem turns out to be a necessary and sufficient one.

Lemma (5.1) shows that if $W_a$ is an acausal square spectral factor corresponding to a minimal Markovian splitting subspace, then it is determined by an inner divisor $U_1$ of $U_M$ as $W_a = \bar{W}_- U_1$. In terms of subspaces, $\mathcal{X} = (N^+)^\perp \ominus \mathcal{A}$ is invariant under $E^{H^\Box}\mathcal{U}$. Conversely, any $E^{H^\Box}\mathcal{U}$ - invariant subspace $\mathcal{X}$ of $H^\Box$ determines an acausal spectral factor and also a Markovian splitting subspace $X$ by defining $\mathcal{B} = N^+ \oplus \mathcal{X}, \mathcal{A} = H \ominus \mathcal{B}$ and using the construction of Section 2. Denote $\mathcal{Y} = H^\Box \ominus \mathcal{X}$. Then $X_u = E^{\mathcal{X}}H^-, X_s = E^{\mathcal{Y}}H^+$.

**Proposition 5.1** *In the internal case*

$$\bar{S} = H^+ \vee \mathcal{X} \ ;$$

*also*

$$S = H^- \vee \mathcal{Y} \ .$$

**Proof:** $\mathcal{X} \oplus \mathcal{Y} = H^\Box$. $\mathcal{Y} = X_s \oplus (\mathcal{Y} \cap (H^+)^\perp$ Consequently, $\bar{S} \oplus (\mathcal{Y} \cap (H^+)^\perp) = N^+ \oplus H^\Box$. But $H^\Box \ominus (\mathcal{Y} \cap (H^+)^\perp) = \mathcal{X} \vee X_+$. This proves that $\bar{S} = H^+ \vee \mathcal{X}$. The other equation can be proved similarly. □

**Theorem 5.1** *$X$ is a minimal (internal) Markovian splitting subspace if and only if*

$$\mathcal{X} \vee X_+ = X_+ \vee (\mathcal{X} \cap (H^-)^\perp). \tag{5.3}$$

**Proof:** $X = X_u \oplus X_S$ is orthogonal to $(N^- \oplus N^+)$. In view of Theorem (4.10) of [14] the minimality of $X$ is equivalent to the condition

$$\bar{S} = H^+ \vee S^\perp. \tag{5.4}$$

Now $S^{\perp} = N^{+} \oplus (\mathcal{X} \cap (H^{-})^{\perp})$; Consequently. (5.4) is equivalent to

$$\bar{S} = N^{+} \oplus [X_{+} \vee (\mathcal{X} \cap (H^{-})^{\perp})]. \tag{5.5}$$

On the other hand, in the internal case

$$\bar{S} = H^{+} \vee \mathcal{X}. \tag{5.6}$$

Substituting (5.6) into (5.5) and taking the intersection of both sides with $H^{\square}$ we get the equation

$$\mathcal{X} \vee X_{+} = X_{+} \vee (\mathcal{X} \cap (H^{-})^{\perp}). \tag{5.7}$$

This proves the theorem. □

**Remark:** In the finite dimensional case (5.7) is obviously equivalent to

$$\mathcal{X} = (\mathcal{X} \cap Z_{+}) \vee (\mathcal{X} \cap X_{+}). \tag{5.8}$$

On the other hand, $\mathcal{X} \cap (H^{-})^{\perp}$ and $\mathcal{X} \cap X_{+}$ are both invariant under $E^{H^{\square}}\mathcal{U}$. $\mathcal{X} \cap X_{+} \subset X_{+}$ , so in the rational case the matrix version of the compressed shift operator is $A$ . $\mathcal{X} \cap Z_{+} \subset Z_{+}$ (error space), so again in the rational case it is connected with the zero matrix $,'$.

The minimality condition in the rational case is thoroughly investigated in the papers [10], [8] and [9] in the continuous time case. The final result with respect to these papers is that if $\bar{W}_{-}(s) = R^{1/2} + C(sI - A)^{-1}B$, and $,_{-} = A - BR^{-1/2}C$ is the zero matrix (in the continuous time case), then under the coercivity assumption the necessary and sufficient condition for that all left inner divisor $U_1$ of $U_M$ determine a minimal spectral factor is that $-A$ and $,_{-}'$ have no common eigenvalues. i.e. every invariant subspace of $\begin{bmatrix} -A & 0 \\ 0 & ,_{-}' \end{bmatrix}$ can be written as a direct sum of a $(-A)$–invariant and a $,_{-}'$–invariant subspace. This theorem is very much in the flavour of Theorem (5.1). In fact, assuming that there are no zeros on the unit circle, at the origin and at infinity, then (in the discrete time case) it is a direct consequence of the previous theorem since the matrix version of the operator $E^{H^{\square}}\mathcal{U}$ is $\begin{bmatrix} A & 0 \\ 0 & ,_{+}' \end{bmatrix}$.

Theorem 5.1 is not restricted to the rational case. It gives the possibility of analysing the minimality condition in the nonrational case in terms of inner functions. Denote

$$Q_{+} = W_{-}^{-1}W_{+}, K_{+} = \bar{W}_{+}^{-1}W_{+}, U = W_{a}^{-1}W_{+}.$$

Remark that $Q_{+}$, $K_{+}$ are always left coprime (see Lindquist and Picci [12], Lindquist and Pavon [12]). Also, denote by $q, k, u$ the minimal functions of the inner functions $Q_+, K_+$ and $U$ respectively. Applying the isometry which maps $(N^+)^\perp$ onto $H^2_p$ the subspaces $H^-, (H^+)^\perp, \mathcal{A}$ are mapped onto

$$H^- \to Q_+H^2_p, (H^+)^\perp \to K_+H^2_p, \mathcal{A} \to UH^2_p.$$

Thus the condition (5.3) is equivalent to

$$UH^2_p \supset (UH^2_p \vee Q_+H^2_p) \cap K_+H^2_p$$

or $U$ is a left inner divisor of $(U \wedge_L Q_+) \vee_L K_+$, where $\wedge_L$ denotes the greatest common left inner divisor, $\vee_L$ the smallest common left inner multiple of the corresponding inner functions.

The first part of the following theorem is essentially the same as Theorem (4.1) in Ferrante ([8]). Here we present a different proof of it based on Theorem (5.1).

**Theorem 5.2** *(i) If the minimal functions $q, k$ are coprime then every $E^{H^\square}\mathcal{U}$ – invariant subspace $\mathcal{X}$ of $H^\square$ determines a minimal Markovian splitting subspace,*

*(ii) If $Q_+$ and $K_+$ are strongly left coprime and every $E^{H^\square}\mathcal{U}$ – invariant subspace $\mathcal{X}$ of $H^\square$ determines a minimal Markovian splitting subspace then the minimal functions $q, k$ are coprime.*

**Proof:**

(i) Assume that $q, k$ are coprime. Since $\mathcal{X} \subset H^\square = X_+ \vee Z_+$, i.e.

$$UH^2_p \supset Q_+H^2_p \cap K_+H^2_p, \tag{5.9}$$

we get that $u$ is a common inner divisor of $q$ nd $k$. This implies a coprime factorization of $u$ in the form $u = u_q u_k$, where $u_q \mid q$, $u_k \mid k$.

Define

$$M_\rho = \{f \in H^2_p \mid u_k f \in UH^2_p\}.$$

According to the Beurling-Lax Theorem the shift invariant subspace $M_\rho$ can be written as $M_\rho = RH^2_p$, where $R$ is an inner function. We show that

$$RH^2_p \supset Q_+H^2_p \vee UH^2_p.$$

Obviously $UH^2_p \subset M_\rho$. On the other hand, if $l \geq 0$ then

$$z^l u_q(u_k Q_+ H^2_p) \subset UH^2_p,$$

and

$$z^l k(u_k Q_+ H^2_p) \subset K_+H^2_p \cap Q_+H^2_p \subset UH^2_p,$$

so $Q_+H^2_p \subset M_\rho$ using the coprimeness of $u_q$ and $k$.

In order to show that condition (5.9) holds it is enough to prove that if $f \perp UH_p^2$, then $f \perp RH_p^2 \cap K_+H_p^2$. Now, if $g \in RH_p^2 \cap K_+H_p^2$, then

$$z^l u_k g \in UH_p^2, z^l qg \in qK_+H_p^2 \subset K_+H_p^2 \cap Q_+H_p^2 \subset UH_p^2,$$

for all $l \geq 0$. But $u_k$ and $q$ are coprime implying that $f \perp g$.

(ii) Now assume that $Q_+$ and $K_+$ are strongly left coprime which implies that $H^\square = X_+ + Z_+$ (i.e. every element in $H^\square$ can be written in a unique way as a sum of elements from $X_+$ and $Z_+$) and their minimal functions $q$ and $k$ are not coprime. So $q = pq_1, k = pk_1$ where $p, k_1, q_1$ are inner functions, $p$ is not identically constant. The factorization of the minimal functions imply that $Q_+$ and $K_+$ have left inner divisors $Q_1 \mid Q_+$ , $K_1 \mid K_+$ with the same minimal function $p$ (see Theorem III-4-3 in Fuhrmann [11]). In other words the largest invariant factors of $Q_1, K_1$ coincide (Theorem II-15-10 [11]). In terms of subspaces this implies that there are $E^{H^\square}\mathcal{U}$ – invariant subspaces $X_0 \subset X_+, Z_0 \subset Z_+$ such that $E^{H^\square}\mathcal{U}_{|X_0}, E^{H^\square}\mathcal{U}_{|Z_0}$ are quasisimilar, so there exists a quasiinvertible transformation $T : X_0 \to Z_0$ such that

$$TE^{H^\square}\mathcal{U}_{|X_0} = E^{H^\square}\mathcal{U}_{|Z_0}T.$$

Define $\mathcal{X} = \{1/2(x + Tx) | x \in X_0\}$. Obviously, it is $E^{H^\square}\mathcal{U}$ –invariant and – because the projection of $H^\square$ onto $X_+$ parallel with $Z_+$ is continuous – $\mathcal{X} \cap (H^-)^\perp = \emptyset, \mathcal{X} \cap X_+ = \emptyset$ so condition (5.3) does not hold. □

**Remark:** Under the coprimality condition of $q$ and $k$ of the previous Theorem it can be proved that $M_\rho = UH_p^2 \vee Q_+H_p^2$. Namely, if $f \perp UH_p^2 \vee Q_+H_p^2, g \in M_\rho$ then $z^l u_k g \in UH_p^2$ and $z^l q_+ g \in Q_+H_p^2$ for all $l \geq 0$, and $u_k, q$ are coprime, implying that $g \perp f$, i.e. $M_\rho \subset UH_p^2 \vee Q_+H_p^2$.

**Remark:** Again the same condition imply that every $E^{H^\square}\mathcal{U}$ –invariant subspace $\mathcal{X}$ of $H^\square$ can be written in the form of (5.8). Defining $M_\pi = \{f \in H_p^2 \mid u_q f \in UH_p^2\}$ in a similar way as in the proof of the theorem it can be checked that $M_\pi = UH_p^2 \vee K_+H_p^2$. Now, if $f \in M_\rho \cap M_\pi$ and $g \perp UH_p^2$, then multiplying $f$ with $z^l u_q$ and $z^l u_k$ $(l \geq 0)$, we obtain that $f \perp g$ implying that $UH_p^2 \supset M_\rho \cap M_\pi$, i.e. $\mathcal{X} \subset (\mathcal{X} \cap X_+) \vee (\mathcal{X} \cap Z_+)$.

**Remark:** Let us observe that in the proof of sufficiency we have used only that there is no common inner divisor of $u, q, k$.

**Acknowledgement:** The authors would like to thank an anonymous referee for a number of helpful comments. Without them this paper would not be the same as it is now.

## References


[1] B. D. O. Anderson. The inverse problem of stationary covariance generation, *J. Statistical Physics* **1** (1969), 133–147.

[2] B. D. O. Anderson. Algebraic properties of minimal degree spectral factors, *Automatica* **9** (1973), 491–500.

[3] J. A. Ball, I. Gohberg and L. Rodman. Realization and interpolation of rational matrix functions, *Operator Theory, Advances and Applications* **33** (1988), 1-72.

[4] S. Bittanti, A. J. Laub and J. C. Willems. *The Riccati Equation.* Berlin: Springer-Verlag, 1991.

[5] P. E. Caines and D. Delchamps. Splitting subspaces, spectral factorization and the positive real equation: structural features of the stochastic realization problem, *Proc. 1980 IEEE Conf. Decision and Control* (1980), 358–362.

[6] P. E. Caines. *Linear Stochastic Systems.* Paris: Dunod, 1979.

[7] P. Faurre, M. Clerget, and F. Germain. *Opérateurs Rationnels Positifs.* New York: Wiley, 1988.

[8] A. Ferrante. A parametrization of the minimal square spectral factors of a nonrational spectral density, *Journal of Math. Systems, Estim. & Control* **4(4)** (1994), 489-492.

[9] A. Ferrante. A parametrization of minimal stochastic realizations, in preparation, 1993.

[10] A. Ferrante, Gy. Michaletzky and M. Pavon. Parametrization of all minimal square spectral factors, *Systems & Control Letters* **21** (1993), 249–254.

[11] P. Fuhrmann. *Linear Operators and Systems in Hilbert Space.* New York: McGraw Hill, 1981.

[12] A. Lindquist and M. Pavon. On the structure of state-space models for discrete-time stochastic vector processes, *IEEE, Trans. Aut. Control* **23** (1984), 418-432.

[13] A. Lindquist and G. Picci. Realization theory for multivariate stationary Gaussian processes, *SIAM J. Control and Optim.* **23** (1985), 809-857.

[14] A. Lindquist and G. Picci. A geometric approach to modelling and estimation of linear stochastic systems, *Journal of Mathematical Systems, Estimation and Control* **1** (1990), 241–333.

[15] K. Mårtensson. On the matrix Riccati equation, *Inform. Sci.* **3** (1971), 17–49.

[16] Gy. Michaletzky. Zeros of (non-square) spectral factors and canonical correlations, *Proc. 11th IFAC Triennial World Congress, Tallinn, Estonia* (1990), 167-172; also in preprint collection: (1990), 221–226.

[17] B. Moore and E. Nordgren. On quasi-equivalence and quasi-similarity, *Acta Math. Sci.* **34** (1973), 311-316.

[18] G. Picci and S. Pinzoni. Acausal models of stationary processes, *Proceedings of the First European Control Conference*, July 1991.

[19] G. Ruckebusch. *Théorie géometrique de la representation Markovienne.* Ph.D. Thesis, Univ. Paris, 1980.

[20] J. C. Willems. Least squares stationary optimal control and the algebraic Riccati equation, *IEEE Trans. Autom. Control* **16** (1971), 621–634.

Department of Probability Theory and Statistics, Eötvös Loránd University, Budapest, Hungary; also at Computer and Automation Institute, Hungarian Academy of Science, Budapest, Hungary

Dipartimento di elettronica e informatica, Universitá di Padova, Padova, Italy

Communicated by A. Lindquist