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Approximation for an Integro-Partial Differential Equation with Strongly Singular Kernel*

Richard H. Fabiano

Abstract

We consider an approximation scheme for integro-partial differential equations which arise in the theory of linear viscoelasticity. This scheme is based on a modification (to account for the singular kernel) of certain averaging type approximation methods for delay equations. We use this scheme to investigate the effects of a history parameter (the delay length) on the behavior of the eigenvalues, and to consider the numerical solution of an optimal control problem.

Key words: integro-differential equation, singular kernel

AMS Subject Classifications: 45K05, 93C20

1 Introduction

Among the many problems that arise in the study of control and stabilization of large flexible space structures is the problem of including internal damping in the mathematical model for the motion of the structure. Many feasible models of internal damping have been proposed in the literature (see, for example, [2], [5], [15], [16]). One such model arises in the theory of linear viscoelasticity, and leads to the following type of integro-partial differential equation: ([2], [4], [8])

$$\frac{\partial^2 u}{\partial t^2}(t,x) + \frac{\partial^2}{\partial x^2}[Eu_{xx}(t,x) + \int_{-r}^{0} a(\theta)u_{xxt}(t+\theta,x)\,d\theta] = f(t,x), \quad (1.1)$$

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for $0 < x < l$, $t \geq 0$. Here, E is a stiffness parameter, $f(t,x)$ is an applied force, and $u(t,x)$ represents the displacement at time t and position x along a long thin Euler-Bernoulli beam with time hysteresis damping. We discuss the function $a(\theta)$ below. Appropriate boundary conditions and initial data have been omitted.

Our approach to the study of (1.1) is to observe that it can be reformulated as a special case of the following integro-differential equation:

$$\ddot{u}(t) + A[Eu(t) + \int_{-r}^{0} a(\theta)\dot{u}(t+\theta)\,d\theta] = f(t). \tag{1.2}$$

This equation evolves in a Hilbert space H (which can be taken to be $H = L^2(0,l)$ when one views (1.2) as a reformulation of (1.1)). We take (1.2) as the starting point of our analysis. It is assumed that A is a positive definite, self-adjoint unbounded operator on H, and $f(t)$ is a locally integrable H-valued function. The stiffness constant E and the delay length r are positive constants. We assume that $a(\theta) = \int_{-r}^{\theta} g(\xi)\,d\xi$ and refer to g as the history kernel. The nature of the singularity of g at $\theta = 0$ determines whether we say that (1.2) has a weakly or strongly singular kernel. More specifically, weakly singular kernels satisfy

$$\begin{aligned}
&\text{(a)} \qquad g(\theta) > 0 \quad \text{for} \quad \theta \in (-r,0),\\
&\text{(b)} \qquad g(\theta) \in L^1(-r,0),\\
&\text{(c)} \qquad g \in H^1(-r,-\alpha) \quad \text{for all } \alpha > 0, \text{ and} \quad g'(\theta) \geq 0 \quad \text{for } \theta \in (-r,0).
\end{aligned} \tag{1.3}$$

This class includes the function $g(\theta) = e^{\gamma\theta}/|\theta|^p$ for $0 \leq p < 1$ and $\gamma > 0$. Strongly singular kernels satisfy

$$\begin{aligned}
&\text{(a)} \qquad a(-r) = 0, \text{ and } a(\theta) \in L^1(-r,0) \cap H^2(-r,-\alpha) \text{ for all } \alpha > 0,\\
&\text{(b)} \qquad g(\theta) = a'(\theta) > 0 \text{ for } \theta \in (-r,0)\\
&\text{(c)} \qquad g'(\theta) \geq 0 \text{ on } (-r,0).
\end{aligned} \tag{1.4}$$

Please see [9] and [10] for further discussion. In this paper, we make the following assumption on g:

$$g(\theta) = \frac{\bar{g}(\theta)}{|\theta|^p}, \qquad 1 \leq p < 2, \tag{1.5}$$

where $\bar{g}(\theta)$ is continuous, strictly positive, and has the property that $g(\theta)$ is nondecreasing. Thus g is a strongly singular kernel.

The rest of the paper is organized as follows. In section 2, we introduce the spaces and operators necessary to reformulate (1.2) as an abstract

Cauchy problem, and we describe an approximation scheme for the problem. Well-posedness and convergence results will be stated. In section 3, we will discuss two numerical experiments at length. In these experiments, we describe numerical investigations (with numerical methods based upon the approximation scheme developed in section 2) of two unresolved theoretical issues.

The first issue involves a 'gap' between the finite delay case ($r < \infty$) and the infinite delay case ($r = \infty$). For example, consider a scalar version of (1.2) in the sense that A is a positive constant instead of a positive definite operator. This leads to

$$\ddot{u}(t) + A[Eu(t) + \int_{-r}^{0} a(\theta)\dot{u}(t+\theta)\, d\theta] = f(t) \tag{1.6}$$

evolving on $\mathbb{R}$. We may consider (1.6) for either finite or infinite delay r. Well-posedness for both cases has been shown in [9]. However, the approximation scheme described in this paper is only valid for finite r. One reason for interest in the case $r = \infty$ is the connection between (1.6) and a class of Volterra integro-differential equations frequently seen in the literature (see [13], [6] for example). The change of variables $s = t + \theta$ in (1.6) leads to

$$\ddot{u}(t) + A[Eu(t) + \int_{0}^{t} \hat{a}(t-s)\dot{u}(s)\, ds] = \hat{f}(t) \tag{1.7}$$

where $\hat{a}(\theta) = a(-\theta)$ and $\hat{f}(t) = f(t) + \int_{-\infty}^{0} \hat{a}(t-s)\dot{u}(s)\, ds$. The relationship between (1.6) for $r < \infty$ and (1.6) for $r = \infty$ is, to the authors knowledge, not well understood. This gap in the theory will be investigated numerically by considering the dependence of the eigenvalues of (1.6) on the delay length r.

The second unresolved theoretical issue is related to questions of convergence for finite dimensional approximations of a linear quadratic regulator (LQR) problem for (1.1). There are several results in the distributed parameter control literature (see for example [11], [12]) giving sufficient conditions which guarantee that solutions of a finite dimensional LQR problem will converge to solutions of an infinite dimensional LQR problem. These conditions on the approximation scheme typically require, among other things, convergence of the approximating open loop semigroups, convergence of the approximating adjoint semigroups, and uniform stabilizability and detectability of the approximating system operators. The approximation scheme described in section 2 gives convergence of the approximating open loop semigroups, but the question of convergence for the adjoint semigroup and the question of uniform stabilizability and detectability are still under investigation. Nonetheless, in the second experiment discussed in section 3, we have implemented this approximation scheme to investigate numerically the convergence of approximating functional gains for an LQR problem for a viscoelastic beam.

2 Abstract Formulation and Approximation Method

In discussing a state space formulation for (1.2), we follow the development in [8],[9],[10] which was motivated by ideas found in [7] and [17]. To proceed, observe that since A is positive definite and self-adjoint, it has a positive definite, self-adjoint square root. We can define a Hilbert space V by $V = \text{dom}A^{\frac{1}{2}}$, with inner product defined by $\langle u, v\rangle_V = \langle A^{\frac{1}{2}}u, A^{\frac{1}{2}}v\rangle_H$. Since $A^{\frac{1}{2}}$ is positive definite, V is equivalent to the Hilbert space obtained by equipping $\text{dom}A^{\frac{1}{2}}$ with the graph norm. It follows that V is densely and continuously imbedded in H. Define a symmetric sesquilinear form σ on V by $\sigma(u, v) = \langle u, v\rangle_V$ for all $u, v \in V$. Then σ satisfies

$$|\sigma(u,v)| \leq c_1 |u|_V |v|_V \quad \text{for } u, v \in V,$$
$$\sigma(u,u) \geq c_2 |u|_V^2 \quad \text{for } u \in V,$$

where $c_1, c_2 > 0$. This leads to a standard Gelfand triple framework. That is, by identifying H with its dual, we have $V \subset H \subset V^*$. An alternative way to view the relationship between A and σ is to observe that σ defines a bounded linear operator $\bar{A}: \text{dom}\bar{A} = V \rightarrow V^*$ by $\langle \bar{A}u, v\rangle_{V^* \times V} = \sigma(u, v)$. It follows from the Lax-Milgram theorem that range $\bar{A} = V^*$. One can view A as a restriction of $\bar{A}$. That is, dom $A = \{u \in V : \bar{A}u \in H\}$. Next, let $W = L_g^2(-r, 0; V)$ be the Hilbert space of all weighted square integrable functions with values in V, equipped with the norm

$$|w|_W^2 = \int_{-r}^{0} g(\theta)|w(\theta)|_V^2 \, d\theta.$$

Let Z denote the Hilbert space $Z = V \times H \times W$ equipped with the norm

$$|(u, v, w)|_Z^2 = E|u|_V^2 + |v|_H^2 + |w|_W^2,$$

and compatible inner product

$$\langle (u, v, w), (\phi, \psi, \gamma)\rangle_Z = E\langle u, \phi\rangle_V + \langle v, \psi\rangle_H + \int_{-r}^{0} g(\theta)\langle w(\theta), \gamma(\theta)\rangle_V \, d\theta.$$

If we introduce the state function $z(t) = (u(t), \dot{u}(t), u(t) - u(t + \theta))$, then observe that (1.2) can be reformulated as (see [9] for more details):

$$\dot{z}(t) = \mathcal{A}z(t) + (0, f(t), 0). \tag{2.1}$$

Here the operator $\mathcal{A}$ is defined on the domain

$$\operatorname{dom}\mathcal{A} = \left\{ \begin{pmatrix} u \\ v \\ w \end{pmatrix} \in Z : \begin{array}{l} w \text{ is locally absolutely continuous,} \\ \dfrac{dw}{d\theta} + v \in W, \quad v \in V, \quad w(0) = 0 \\ Eu + \displaystyle\int_{-r}^{0} g(\theta)w(\theta)\,d\theta \in \operatorname{dom} A \end{array} \right\},$$

by

$$\mathcal{A}\,(u,v,w) = (v,\, -A[Eu + \int_{-r}^{0} g(\theta)w(\theta)\,d\theta],\, \frac{dw}{d\theta} + v).$$

The following well-posedness result is proved in [9].

Theorem 2.1 *The operator $\mathcal{A}$ is the infinitesimal generator of a strongly continuous semigroup of contractions $T(t)$ on Z.*

Next we introduce the operator $\mathcal{A}_1$ which will be shown to be the adjoint of the operator $\mathcal{A}$. To this end define $\mathcal{A}_1$ on the domain

$$\operatorname{dom}\mathcal{A}_1 = \left\{ \begin{pmatrix} u \\ v \\ w \end{pmatrix} \in Z : \begin{array}{l} w \text{ is locally absolutely continuous,} \\ \dfrac{1}{g}\dfrac{d}{d\theta}(gw) + v \in W, \quad v \in V, \quad w(-r) = 0 \\ Eu + \displaystyle\int_{-r}^{0} g(\theta)w(\theta)\,d\theta \in \operatorname{dom} A \end{array} \right\},$$

by

$$\mathcal{A}_1(u,v,w) = (-v,\, A[Eu + \int_{-r}^{0} g(\theta)w(\theta)\,d\theta],\, -\frac{1}{g}\frac{d}{d\theta}(gw) - v).$$

Theorem 2.2 *The operator $\mathcal{A}_1$ is the infinitesimal generator of a strongly continuous semigroup of contractions $T_1(t)$ on Z.*

Proof: We proceed by first showing that $\mathcal{A}_1$ is dissipative, and then showing that $(\lambda I - \mathcal{A}_1)\operatorname{dom}\mathcal{A}_1 = Z$ for some $\lambda > 0$. It then follows from theorems 1.4.5 and 1.4.6 of [14] that $\mathcal{A}_1$ has dense domain, and we may conclude from the Lumer-Phillips theorem that $\mathcal{A}_1$ is the infinitesimal generator of a contraction semigroup. Before proceeding, note that if $(u,v,w) \in \operatorname{dom}\mathcal{A}_1$, then w is locally absolutely continuous, and hence $w(0) = 0$ since $w \in W$ and $g(\theta)$ has a nonintegrable singularity at $\theta = 0$. Hence the same argument used in the proof of Theorem 2.1 in [9] shows that if $(u,v,w) \in \operatorname{dom}\mathcal{A}_1$, then

$\int_{-r}^{0} g(\theta)|w(\theta)|_{V}\, d\theta$ exists. Hence the integrals in the following dissipativeness argument exist. To show that $\mathcal{A}_1$ is dissipative, let $(u, v, w) \in \mathrm{dom}\mathcal{A}_1$. Then

$$\begin{aligned}
\mathrm{Re}\langle \mathcal{A}_1(u,v,w),(u,v,w)\rangle_{Z} &= -\mathrm{Re}\{E\langle v,u\rangle_{V} + \langle Eu + \int_{-r}^{0} g(\theta)w(\theta)\,d\theta,\ v\rangle_{V} \\
&\qquad + \int_{-r}^{0} g(\theta)\langle -\frac{1}{g}\frac{d}{d\theta}(gw) - v,\ w(\theta)\rangle_{V}\ d\theta\} \\
&= -\mathrm{Re}\int_{-r}^{0} \langle \frac{d}{d\theta}(gw), w(\theta)\rangle_{V}\ d\theta \\
&= -\mathrm{Re}\int_{-r}^{0} g(\theta)\langle \frac{dw}{d\theta}, w\rangle_{V}\ d\theta - \int_{-r}^{0} \dot{g}(\theta)|w|_V^2\,d\theta.
\end{aligned}$$

For $\epsilon > 0$, consider

$$\begin{aligned}
-\mathrm{Re}\int_{-r}^{-\epsilon} g(\theta)\langle \frac{dw}{d\theta}, w(\theta)\rangle_{V}\ d\theta &= -\frac{1}{2}\int_{-r}^{-\epsilon} g(\theta)\frac{d}{d\theta}|w(\theta)|_V^2\,d\theta \\
&= -\frac{1}{2}g(-\epsilon)|w(-\epsilon)|_V^2 + \frac{1}{2}\int_{-r}^{-\epsilon} \dot{g}(\theta)|w|_V^2\,d\theta \\
&\le \frac{1}{2}\int_{-r}^{-\epsilon} \dot{g}(\theta)|w(\theta)|_V^2\,d\theta.
\end{aligned}$$

Letting $\epsilon \to 0^+$, we see that $\mathrm{Re}\langle \mathcal{A}_1(u,v,w),\,(u,v,w)\rangle_{Z} \le 0$. It remains to be shown that $(\lambda I - \mathcal{A}_1)\mathrm{dom}\,\mathcal{A}_1 = Z$ for some $\lambda > 0$. To this end, let $(\phi, \psi, h) \in Z$, and consider the equation $(\lambda I - \mathcal{A}_1)(u,v,w) = (\phi,\psi,h)$. This equation may be written as

$$\lambda u + v = \phi, \tag{2.2}$$

$$\lambda v - A[Eu + \int_{-r}^{0} g(\theta)w(\theta)\,d\theta] = \psi, \tag{2.3}$$

$$\lambda w + \frac{1}{g}\frac{d}{d\theta}(gw) + v = h. \tag{2.4}$$

We introduce the following sesquilinear form μ on V:

$$\mu(u,y) = \lambda^2\langle u,y\rangle_{H} + [E + \int_{-r}^{0}\int_{-r}^{\theta} g(\xi)e^{\lambda(\xi-\theta)}\lambda\,d\xi\,d\theta]\sigma(u,y) \quad \text{for } u,y \in V.$$

There are positive constants k_1, k_2 so that

$$|\mu(u,y)| \le k_1|u|_{V}\,|y|_{V}$$

for all $u, y \in V$, and

$$\mu(u,u) \geq \lambda^2 |u|_H^2 + k_2 |u|_V^2 \geq k_2 |u|_V^2$$

for all $u \in V$. Thus, from the Lax-Milgram Theorem, there is a unique solution $u \in V$ to the equation

$$-\mu(u,y) = \langle \psi - \lambda\phi + A \int_{-r}^{0} \int_{-r}^{\theta} g(\xi) e^{\lambda(\xi-\theta)} (h(\xi) - \phi)\, d\xi\, d\theta,\, y \rangle_{V^* \times V} \quad (2.5)$$

for all $y \in V$. For this u, set $v = \phi - \lambda u$, and clearly (2.2) is satisfied. Also set $w(\theta) = \frac{1}{g(\theta)} \int_{-r}^{\theta} g(\xi) e^{\lambda(\xi-\theta)} [h(\xi) - v]\, d\xi$. It is easy to check that (2.4) is satisfied by these choices of v and w. Finally, (2.3) holds because of (2.5). Clearly $(u, v, w) \in Z$, and it remains to verify that $(u, v, w) \in \text{dom}\mathcal{A}_1$. We observe that $v \in V$ and w is locally absolutely continuous. Also, $w(-r) = 0$ and $\frac{1}{g(\theta)} \frac{d}{d\theta}(gw) + v = h(\theta) - \lambda w(\theta) \in W$. Finally, note that

$$\begin{aligned}
&\sigma(Eu + \int_{-r}^{0} g(\theta) w(\theta)\, d\theta, y) \\
&\quad = \sigma(Eu + \int_{-r}^{0} \int_{-r}^{\theta} g(\xi) e^{\lambda(\xi-\theta)} [h(\xi) - \phi + \lambda u] d\xi d\theta, y) \\
&\quad = \mu(u,y) - \lambda^2 \langle u, y \rangle_{V^* \times V} + \sigma(\int_{-r}^{0} \int_{-r}^{\theta} g(\xi) e^{\lambda(\xi-\theta)} [h(\xi) - \phi] d\xi d\theta, y) \\
&\quad = \langle \lambda\phi - \psi - \lambda^2 u, y \rangle_{V^* \times V}.
\end{aligned}$$

Thus it follows that $\bar{A}(Eu + \int_{-r}^{0} g(\theta) w(\theta)\, d\theta) = \lambda\phi - \psi - \lambda^2 u$ (recall that σ defines $\bar{A} : V \to V^*$). Since $\lambda\phi - \psi - \lambda^2 u \in H$, we conclude that $Eu + \int_{-r}^{0} g(\theta) w(\theta)\, d\theta \in \text{dom}A$, and the proof is complete.

Theorem 2.3 *The operator $\mathcal{A}_1$ is the adjoint of $\mathcal{A}$.*
Proof: Let $(u, v, w) \in \text{dom}\mathcal{A}$ and $(u_1, v_1, w_1) \in \text{dom}\mathcal{A}_1$. Then

$$\begin{aligned}
\langle \mathcal{A}(u,v,w), (u_1, v_1, w_1) \rangle_Z &= E\langle v, u_1 \rangle_V + \langle -A[Eu + \int_{-r}^{0} g(\theta) w(\theta)\, d\theta], v_1 \rangle_H \\
&\quad + \int_{-r}^{0} g(\theta) \langle [\frac{dw}{d\theta} + v], w_1(\theta) \rangle_V \, d\theta \\
&= E\langle u, -v_1 \rangle_V + \langle v, Eu_1 + \int_{-r}^{0} g(\theta) w_1(\theta)\, d\theta \rangle_V \\
&\quad + \int_{-r}^{0} g(\theta) \langle w(\theta), [-\frac{1}{g} \frac{d}{d\theta}(g w_1) - v_1] \rangle_V \, d\theta \\
&= \langle (u,v,w), \mathcal{A}_1(u_1, v_1, w_1) \rangle_Z.
\end{aligned}$$

Hence, $\mathcal{A}_1 \subseteq \mathcal{A}^*$. But $\mathcal{A}_1$ is a maximal dissipative operator, so $\mathcal{A}_1 = \mathcal{A}^*$ and the result follows.

Next, we give a brief description of the semidiscrete approximation scheme developed in [10]. To discretize the spatial variable, let V^N be any sequence of finite dimensional subspaces of V satisfying the following approximation condition: for any $\phi \in V$, there exists a sequence $\phi^N \in V^N$ such that $|\phi^N - \phi|_V \to 0$ as $N \to \infty$. Let P_H^N and P_V^N denote (respectively) the orthogonal projections of H and V onto V^N. The above condition guarantees that $P_V^N \to I$ and $P_H^N \to I$ (strong operator convergence). To finish the spatial discretization, for each N define the operator $A^N : V^N \to V^N$ by

$$\langle A^N x, y\rangle_H = \sigma(x,y), \qquad \text{for all } x, y \in V^N.$$

The discretization of the delay variable involves a modification of the averaging scheme for delay equations introduced in [1]. First, define a partition of $[-r,0]$ by $\theta_j^M = \frac{-jr}{M}, j = 0, 1, \ldots, M$. Then, for $i = 2, \ldots, M$, define basis functions $E_i^M(\theta)$ by

$$E_i^M(\theta) = \begin{cases} 1, & \text{if } \theta_i^M \le \theta \le \theta_{i-1}^M \\ 0, & \text{elsewhere.} \end{cases}$$

In addition, define

$$E_1^M(\theta) = \begin{cases} -\frac{M}{r}\theta, & \text{if } \theta_1^M \le \theta \le 0 \\ 0, & \text{elsewhere.} \end{cases}$$

Set $W^{N,M} = \{w \in W : \ w = \sum_{i=1}^M a_i E_i^M, \ a_i \in V^N\}$, and set $Z^{N,M} = V^N \times V^N \times W^{N,M}$. The finite dimensional approximation of the operator $\mathcal{A}$ is the operator $\mathcal{A}^{N,M} : Z^{N,M} \to Z^{N,M}$ defined as follows.
For $z^{N,M} = (u, v, \sum_{i=1}^M a_i E_i^M)$,

$$\mathcal{A}^{N,M} z^{N,M} = \begin{pmatrix} v \\ -A^N[Eu + \int_{-r}^0 g(\theta) \sum_{i=1}^M a_i E_i^M \, d\theta] \\ \frac{M}{r}\sum_{i=2}^M (a_{i-1} - a_i) E_i^M - 2\frac{M}{r} a_1 E_1^M + f^M(\theta) v \end{pmatrix}.$$

Here $f^M(\theta)$ is given by

$$f^M(\theta) = \sum_{i=2}^M E_i^M + \alpha^M E_1^M, \qquad \alpha^M = \frac{-\int_{-r/M}^0 \theta g(\theta)\, d\theta}{\frac{M}{r}\int_{-r/M}^0 \theta^2 g(\theta)\, d\theta}.$$

The term $f^M(\theta)v$ in the third component of $\mathcal{A}^{N,M}$ is an approximation of the term v in the third component of $\mathcal{A}$. Thus $f^M(\theta)$ is an approximation of the constant function 1 defined on the interval $[-r, 0]$. The formula given for $f^M(\theta)$ is derived by 'formally' projecting the constant function 1 ('formally' because the constant function 1 is not in W) onto $W^{N,M}$ according to the projection equation

$$\int_{-r}^{0} g(\theta) \cdot 1 \cdot w^{N,M} \, d\theta = \int_{-r}^{0} g(\theta) \cdot f^M \theta \cdot w^{N,M} \, d\theta \quad \forall w^{N,M} \in W.$$

The result is the following approximation for equation (2.1):

$$\dot{z}^{N,M}(t) = \mathcal{A}^{N,M} z^{N,M}(t) + (0, P_H^N f(t), 0). \tag{2.5}$$

This approximation scheme is used for the numerical experiments discussed in the next section. We have the following convergence result (see [10]).

Theorem 2.4 *For all $z \in Z$, $T^{N,M}(t)P^{N,M}z \to T(t)z$ as $N, M \to \infty$, uniformly on bounded t-intervals, where $T^{N,M}(t) = e^{\mathcal{A}^{N,M}t}$.*

This semigroup theoretic result provides for convergence of solutions of (2.5) to the solution of (2.1).

3 Numerical Experiments

As discussed in section 1, in this section we present two examples, each involving the use of numerical experiments to investigate certain unresolved theoretical issues.

Example 1 We consider again the following scalar integro-differential equation (1.6) discussed in section 1:

$$\ddot{u}(t) + A[Eu(t) + \int_{-r}^{0} a(\theta)\dot{u}(t+\theta) \, d\theta] = f(t). \tag{3.1}$$

This equation evolves on $\mathbb{R}$, so the state spaces become $W = L_g^2(-r, 0; \mathbb{R})$ and $Z = \mathbb{R}^2 \times W$, with norm on Z given by $|(u, v, w)|_Z^2 = EA|u|^2 + |v|^2 + A \int_{-r}^{0} g(\theta) w^2(\theta) \, d\theta$. We will investigate the eigenvalues of this equation (that is, the eigenvalues of $\mathcal{A}$) for the case $r = \infty$. Then we will apply the approximation scheme outlined in section 2 to compute the eigenvalues of $\mathcal{A}^{N,M}$ for various finite values of r, and compare the results.

Thus, let us consider (3.1) with $r = \infty$. Well-posedness for this case has been shown in [9]. For the history kernel we consider

$$g(\theta) = \frac{\bar{g}(\theta)}{(-\theta)^{3/2}}, \quad \text{where} \quad \bar{g}(\theta) = \alpha e^{\beta\theta}(\frac{1}{2} - \beta\theta). \tag{3.2}$$

The positive constants α and β can be specified later. Such exponentially decaying kernels are frequently seen in the literature ([6], [7], [13]), and are thought to be a reasonable model for viscoelastic materials with 'fading memory' behavior. We have chosen $\bar{g}(\theta)$ as indicated in order to facilitate the computation of the eigenvalues of $\mathcal{A}$. In particular, if λ is an eigenvalue of the operator $\mathcal{A}$, then

$$\lambda u - v = 0 \tag{3.3}$$

$$\lambda v + A[Eu + \int_{-\infty}^{0} g(\theta)w(\theta)\, d\theta] = 0 \tag{3.4}$$

$$\lambda w - \frac{d}{d\theta}(w + \theta v) = 0 \tag{3.5}$$

for some $(u, v, w) \in \operatorname{dom} \mathcal{A}$. Solving (3.5) for w yields $w(\theta) = \frac{1}{\lambda}(1 - e^{\lambda\theta})v$. Equation (3.3) implies that $v = \lambda u$ so that $w(\theta) = (1 - e^{\lambda\theta})u$. Plugging in to (3.4) yields $\lambda^2 u + A[Eu + \int_{-\infty}^{0} g(\theta)(1 - e^{\lambda\theta})u\, d\theta] = 0$. Observe that $\frac{d}{d\theta}\frac{\alpha e^{\beta\theta}}{(-\theta)^{1/2}} = g(\theta)$ (in fact, this is the reason for the choice of $\bar{g}(\theta)$ in the form given above). An integration by parts yields

$$\int_{-\infty}^{0} g(\theta)(1 - e^{\lambda\theta})\, d\theta = \int_{-\infty}^{0} \frac{\alpha e^{\lambda\theta}}{(-\theta)^{\frac{1}{2}}} \lambda e^{\beta\theta} d\theta. \tag{3.6}$$

Thus if λ is an eigenvalue of $\mathcal{A}$, then λ is a root of the characteristic equation

$$\lambda^2 + A[E + \lambda\alpha \int_{-\infty}^{0} \frac{e^{(\lambda+\beta)\theta}}{(-\theta)^{\frac{1}{2}}} d\theta] = 0. \tag{3.7}$$

The integral in (3.7) can be written as

$$\int_{-\infty}^{0} \frac{e^{(\lambda+\beta)\theta}}{(-\theta)^{\frac{1}{2}}} d\theta = \lambda \int_{0}^{\infty} e^{-\lambda t} t^{-\frac{1}{2}} e^{-\beta t}\, dt. \tag{3.8}$$

Viewing λ as a transform variable, this last integral is the Laplace transform of $t^{-1/2}e^{-\beta t}$, and it follows that

$$\int_{0}^{\infty} e^{-\lambda t} t^{-\frac{1}{2}} e^{-\beta t}\, dt = \sqrt{\frac{\pi}{\lambda + \beta}} \quad \text{for } \operatorname{Re} \lambda > -\beta. \tag{3.9}$$

Thus, if λ is an eigenvalue of $\mathcal{A}$, then λ is a root of the characteristic equation

$$\lambda^2 + A[E + \frac{\alpha\sqrt{\pi}\lambda}{\sqrt{\lambda+\beta}}] = 0. \tag{3.10}$$

Further, in order for the integrals in (3.6)-(3.9) to exist it is necessary that $\mathrm{Re}\,\lambda > -\beta$ (so that $e^{\lambda\theta}g(\theta)$ is integrable at $-\infty$). There are no purely imaginary solutions of (3.10). Also, if we consider the real function $f(x) = x^2 + AE + \frac{A\alpha\sqrt{\pi}x}{\sqrt{x+\beta}}$, a straightforward calculus argument shows that f has exactly one root on the interval $(-\beta, \infty)$, and it is negative. Thus, (3.10) has one and only one real root λ_1, and $-\beta < \lambda_1 < 0$. Next observe that any solution of (3.10) is also a solution of

$$\lambda^5 + \beta\lambda^4 + 2AE\lambda^3 + (2\beta AE - A^2\alpha^2\pi)\lambda^2 + A^2E^2\lambda + \beta A^2E^2 = 0 \tag{3.11}$$

There are at most 5 roots of (3.11) and nonreal roots appear as conjugate pairs. Thus (3.11) can be written as

$$(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \bar{\lambda}_2)(\lambda - \lambda_3)(\lambda - \bar{\lambda}_3) = 0 \tag{3.12}$$

where $\lambda_2 = a + bi$ and $\lambda_3 = c + di$. Expanding (3.12) and comparing the constant and λ coefficients with (3.11) gives

$$2\lambda_1 a(c^2 + d^2) + (a^2 + b^2)(c^2 + d^2) + 2\lambda_1 c(a^2 + b^2) = A^2E^2$$
$$-\lambda_1(a^2 + b^2)(c^2 + d^2) = \beta A^2E^2.$$

Multiplying the first equation by $-\lambda_1$ and simplifying yields

$$-2\lambda_1^2[a(c^2 + d^2) + c(a^2 + b^2)] = -(\beta + \lambda_1)A^2E^2.$$

This implies that $\mathrm{Re}\,\lambda_2$ and $\mathrm{Re}\,\lambda_3$ cannot both be negative. Thus (3.10) has one negative real root and at most two roots (a conjugate pair) with negative real part. But, since $\mathcal{A}$ is dissipative, its eigenvalues have negative real part. We conclude that $\mathcal{A}$ has at most 3 eigenvalues - one is real and negative, and two have negative real part (a conjugate pair).

In the numerical experiment which we report on here, we took $A = 12.36236$, $E = 40$, $\alpha = 0.10$, and $\beta = 0.50$. (The values for A and E were chosen to resemble data for the first vibration mode of a cantilevered viscoelastic beam). For $r = \infty$, we used (3.10) to compute the three eigenvalues of $\mathcal{A}$: $\lambda_1 = -0.49999509632$, and a conjugate pair $\lambda_\pm = -0.1660964694 \pm 22.39966735i$. We then computed the eigenvalues of $\mathcal{A}^M$ (since (3.1) is scalar, there is no spatial discretization, so there is no index N) for several values of M and r. In Figure 1, we show the eigenvalues of $\mathcal{A}^M$ for $M = 300$ and $r = 8, 16, 32, 48$.

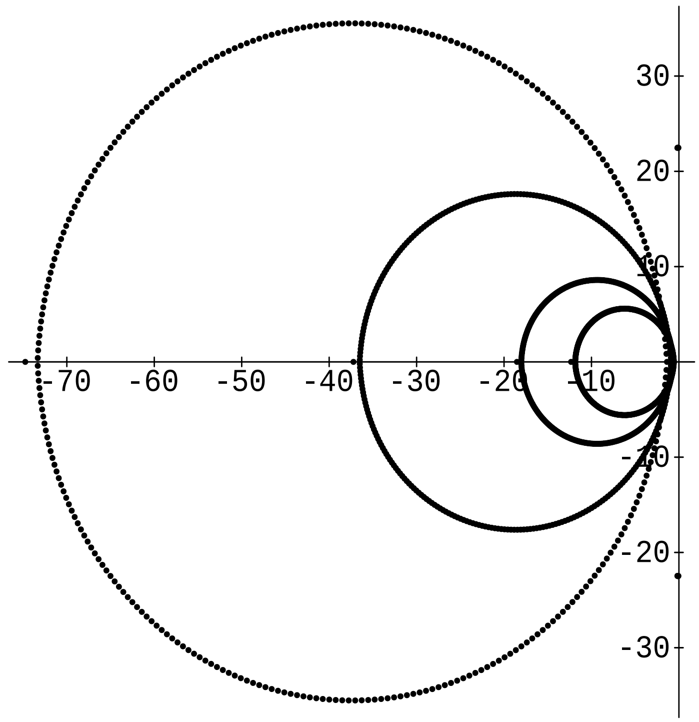

Figure 1: Eigenvalues of $\mathcal{A}^M$

For each M, $\mathcal{A}^M$ has $M+2$ eigenvalues. Two of these are a conjugate pair $\lambda_\pm^M$ which are converging to the eigenvalues $\lambda_\pm$ of $\mathcal{A}$. In fact, the convergence for these two eigenvalues is quite good, but due to scale it is not possible to distinguish these eigenvalues in Figure 1 (they appear near ± 22 on the imaginary axis). The purpose of Figure 1 is to exhibit the interesting behavior of the remaining M eigenvalues of $\mathcal{A}^M$. Of these M eigenvalues, we observe numerically that two are real and $M-2$ are complex conjugate pairs which lie on a circle. As r increases, these circles are becoming smaller, and thus these eigenvalues are 'converging as $r \to \infty$' to the remaining real eigenvalue λ_1 of $\mathcal{A}$. Of course, this is only a numerically observed convergence behavior.

Next, for each M, r we let $\lambda_1^{M,r}$ denote the real eigenvalue of $\mathcal{A}^M$ nearest to the origin. In Table 1 we list $\lambda_1^{M,r}$ for several values of M and r. Again we observe a convergence behavior, as $M, r \to \infty$, to the value λ_1.

We experimented with various values of M, r, α, and β, and this example is indicative of the general behavior which we observed. This appears to be a positive indication for the possible application of our approximation scheme to infinite delay and Volterra equations as mentioned above.

r	M	$\lambda_1^{M,r}$
4	200	-2.362
8	400	-1.407
16	800	-0.930
32	1600	-0.705
48	3200	-0.632
64	6400	-0.597
128	16000	-0.545

Table 1 - Eigenvalues of $\mathcal{A}^M$

Example 2 In the next example, we consider an optimal control problem for a viscoelastic beam. As discussed in section 1, we present only numerical results and do not address issues of uniform stabilizability and detectability or convergence of the approximating adjoint semigroups. The optimal control problem which we consider is to choose a control function $\eta^*(t) \in L^2(0,\infty)$ which minimizes

$$J(\eta) = \int_0^\infty \left\{ |z(t)|_Z^2 + |\eta(t)|^2 \right\} dt \tag{3.13}$$

subject to dynamics governed by

$$\dot{z}(t) = \mathcal{A}z(t) + B\eta(t). \tag{3.14}$$

In this example, $\mathcal{A}$ is the same as in (2.1) with $r = 1$, $E = 40$, and $g(\theta) = \frac{0.1e^{5\theta}}{(-\theta)^{3/2}}$. The spaces V and H are given by $V = \{u \in H^2(0,1) : u(0) = u'(0) = 0\}$ and $H = L^2(0,1)$. The operator A is defined from the bilinear form $\sigma : V \times V \to \mathbb{C}$ given by $\sigma(u,v) = \langle u'', v'' \rangle_H$. The operator $B : \mathbb{R} \to Z$ is defined by $B\eta = (0, b(x)\eta, 0)$ and $b(x) = x$. For this choice of data, equation (3.14) is an abstract formulation of (1.1) with $f(t,x) = b(x)\eta(t)$, and models the motion of a cantilevered viscoelastic beam of length 1. Although the issue of exponential stabilizability of (3.14) is unresolved, it can be said (see [12]) that if an optimal control η^* exists then it will be given in feedback form by $\eta(t) = -Kz(t)$, where $K : Z \to \mathbb{R}$ is defined in the usual way in terms of a solution to an algebraic operator

Riccati equation on Z. Since K is a bounded linear functional on Z it may be represented by

$$Kz = \langle k_1, \phi \rangle_V + \langle k_2, \psi \rangle_H + \langle k_3, w \rangle_W \tag{3.15}$$

for $z = (\phi, \psi, w) \in Z$. Here k_1, k_2, and k_3 are the optimal functional gains, and in our numerical experiments we use our approximation scheme to compute approximate (sub-optimal) gains $k_1^{N,M}$, $k_2^{N,M}$, and $k_3^{N,M}$. Again, the issue of convergence is at present unresolved for this approximation scheme, but our preliminary numerical results indicate "experimental" convergence. In Figure 2 we plot the approximating functional gains $k_2^{N,M}$ for $N = 5$ and $M = 8, 16, 32, 64$.

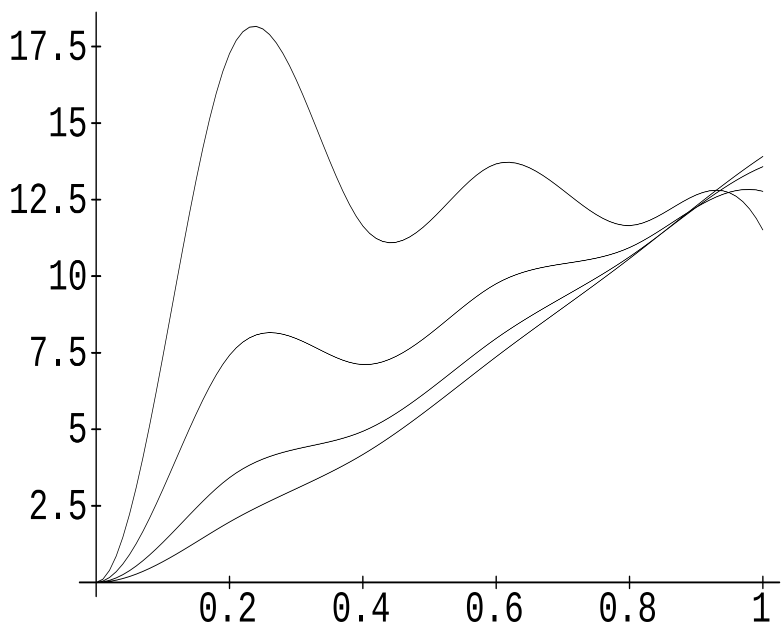

Figure 2: Gain $k_2^{5,M}$ for $M = 8, 16, 32, 64$

Even better convergence was observed for $k_1^{N,M}$. In Figures 3-6 we plot the gains $k_3^{N,M}(s, x)$ for $N = 5$ and $M = 16, 32, 64, 100$. These results are representative of several numerical experiments with various values of N and M.

We may conclude that these numerical results, although preliminary, indicate the usefulness of our approximation scheme for applications involving strongly singular kernels. We are still investigating several unresolved theoretical issues. In addition, we are investigating the implementation of a nonuniform mesh for the case of a strongly singular kernel.

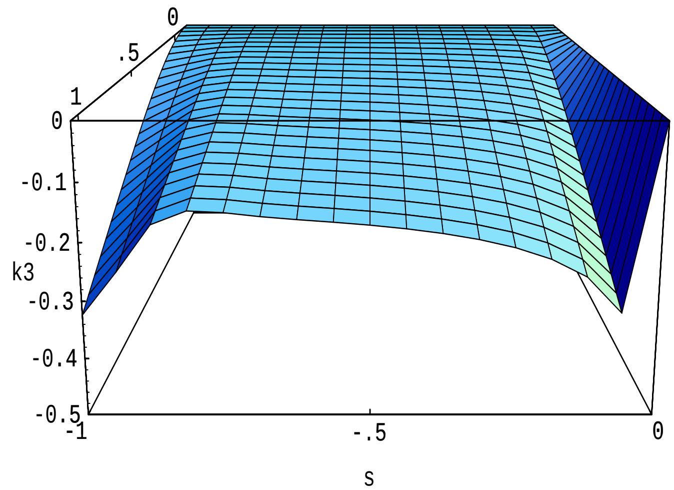

Figure 3: Gain $k_3^{5,16}$

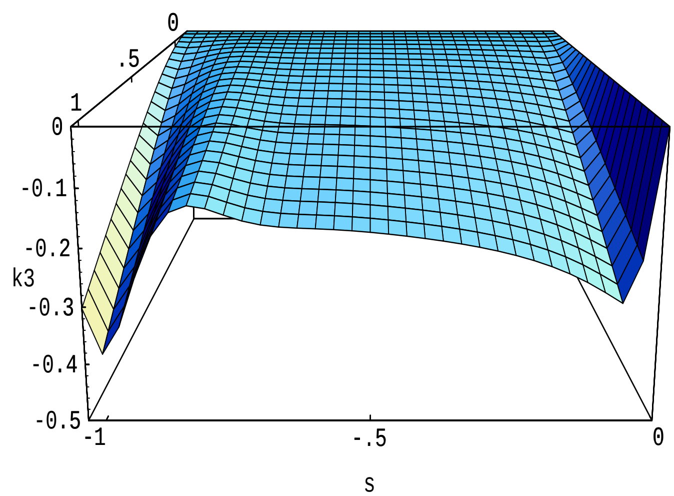

Figure 4: Gain $k_3^{5,32}$

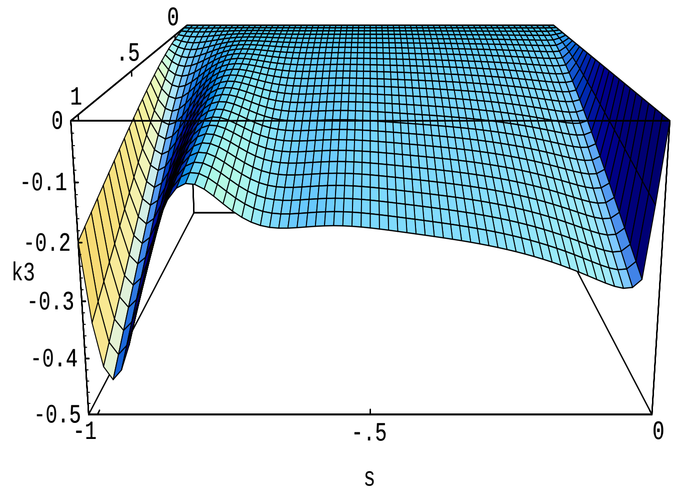

Figure 5: Gain $k_3^{5,64}$

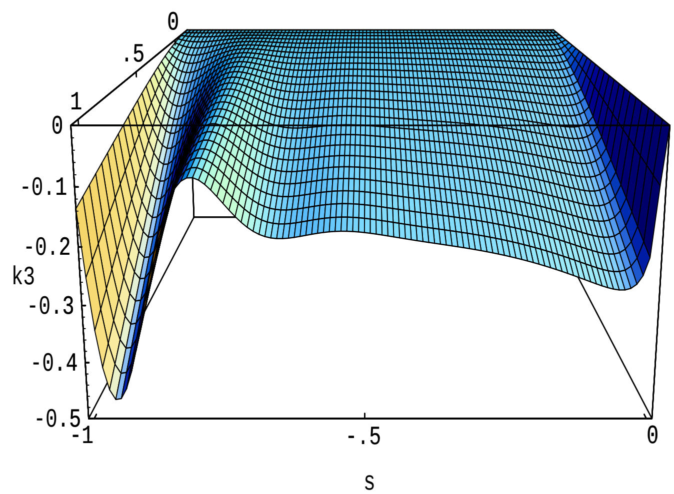

Figure 6: Gain $k_3^{5,100}$

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Department of Mathematics, Texas A&M University, College Station, TX 77843

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