Journal of Mathematical Systems, Estimation, and Control
Vol. 5, No. 4, 1995, pp. 1–26

Global Solutions for Differential/Algebraic Systems and Implications for Lyapunov Direct Stability Methods*

R.E. Mahony† I.M. Mareels†

Abstract

A commonly used mathematical model for the dynamic behaviour of power systems is that of a differential/algebraic system. Such a system consists of a mixture of differential equations and algebraic constraints. In this paper the behaviour of solutions to such systems which display certain jump discontinuities are investigated. To analyse stability properties of these solutions Lyapunov theory is used, which has the advantage that solutions of the system need not be uniquely defined to obtain strong results.

Key words: differential/algebraic systems, power systems, direct Lyapunov methods

AMS Subject Classifications: 34A09, 34D20, 58F14

1 Introduction

A number of recent failures of large scale power networks throughout the world has prompted increased research effort to better understand the dynamics of stressed power systems. System failures appear to result from a gradual weakening of the system followed by a rapid collapse of system integrity. Studying short time system dynamics can provide an understanding of the state of a power system network immediately prior to collapse.

A common mathematical model for power system networks involves the use of differential/algebraic systems, (DA-systems). Structure preserving

*Received October 26, 1993; received in final form March 30, 1994. Summary appeared in Volume 5, Number 4, 1995.

†The authors wish to acknowledge the funding of the activities of the Cooperative Research Centre for Robust and Adaptive Systems by the Australian Commonwealth Government under the Cooperative Research Centres Program.

models of power systems are of this form [2, 7, 15, 18]. In such models the combination of the power flow balance equations at nodes without generators, along with the classical machine model at nodes with generators, provide a coupled set of nonlinear equations of the form

$$\begin{aligned} \dot{x} &= f(x,y) \\ 0 &= g(x,y) \end{aligned}$$

where $x \in \mathbf{R}^{n_x}$, $y \in \mathbf{R}^{n_y}$ and $n_x + n_y = n$. The function g is known as the algebraic constraint or the constraint function, and defines the set of allowable states in $\mathbf{R}^n$. The function f is known as the dynamics of the DA-system and follows from the machine dynamics. If dynamics for the load dependence are known these can also be included in structure preserving models [15, 26]. Additional dynamics such as transformer tap settings are not explicitly modelled in structure preserving models of power systems, however, such considerations can be incorporated into practical power system models once the short-time behaviour is properly understood [10, 16].

Of particular interest in the analysis of power systems is the transient stability analysis of equilibrium points [20, 25]. It is also of interest to identify points at which system collapse is likely and the local transient domains of attraction for these points. A suitable mathematical tool for the analysis of such problems is Lyapunov theory [8, 11, 14]. One can also use singular perturbations models of the power system to obtain similar results [3, 4]. Singular perturbation techniques yield considerable insight into the changes (and bifurcations) in DA-system behaviour at singular points subjected to perturbations of both the dynamics and the algebraic constraint. Recent work in this area is presented by Venkatasubramanian et al. [24].

In this paper we consider a structure preserving model of a power system given in the form of a differential/algebraic system. We discuss the theoretical questions of existence and uniqueness of solutions to such systems and identify points at which such questions are difficult to answer. We propose a definition of "global" solutions to DA-systems which allow for jump discontinuities occurring at certain specific points. Such a definition allows solutions to be defined on longer time intervals than was previously possible, however, it also allows for non-unique solutions to the DA-system. To deal with the theoretical difficulties of analysing non-unique behaviour we utilise Lyapunov theory to obtain stability and convergence results.

The paper is divided into seven sections including the introduction. In Section 2 we briefly present a structure preserving model of a power system in the form of a DA-system. In Section 3 we define local solutions to DA-systems and review the standard theory available for such solutions. Section 4 discusses the concept of global solutions to DA-systems and proposes a definition of a solution to a DA-system that may contain jump discontinuities. Section 5 provides the necessary definitions for Section 6 where a number of Lyapunov style theorems are proved for global solutions of DA-systems. Section 7 provides a conclusion.

2 A Simple Power System Network Model

In this section a simple structure preserving power system model is presented based on the work of Hill and Hiskens [9, 14]. For simplicity we use the classical machine model [1] where a synchronous machine is modelled as a constant voltage in series with a transient reactance. Also real and reactive loads are modelled as functions of voltage only (i.e. no load damping). It is possible, however, to include load damping in structure preserving power system models [14], if it is deemed necessary.

Consider a network of n_0 buses connected by lossless transmission lines. At m $(1 \leq m \leq n_0)$ of these buses there are generators. Label the buses without generators $i = 1, \ldots, n_0 - m$. We augment the network with m fictitious buses corresponding to the generator internal buses and label these $i = n_0 + 1, \ldots, n_0 + m$. Thus, the total number of buses in the system is $n_1 := n_0 + m$. By assumption the network is lossless and the bus admittance matrix is $\mathbf{j}B$, ($\mathbf{j}^2 = -1$, $B \in \mathbf{R}^{n_1 \times n_1}$) a $n_1 \times n_1$ pure imaginary matrix.

Denote the complex voltage at the i'th bus as the (time varying) phasor $V_i = |V_i| \angle \delta_i$, where δ_i is the bus phase angle with respect to a synchronously rotating reference frame. The bus phase angle variables used are

$$\alpha \in \mathbf{R}^{n_1 - 1}, \;\; \alpha_i := \delta_i - \delta_{n_1}, \;\; i = 1, \ldots n_1 - 1.$$

Thus the bus phase angles are measured with respect to the rotation of the n_1'th generator. Since no load damping is assumed one need only consider frequency deviation for the internal generator buses,

$$\omega \in \mathbf{R}^m, \;\; \omega_i := \frac{d}{dt}\delta_i, \;\; i = n_0 + 1, \ldots n_1.$$

Observe that the time derivative $\frac{d}{dt}\alpha_i = \omega_i - \omega_{n_1}$, $i = n_0 + 1, \ldots, n_1 - 1$. The voltage magnitudes in the system are represented as an n_1-vector

$$|V| \in \mathbf{R}^{n_1}, \;\; |V|_i = \begin{cases} |V_i| & i = 1, \ldots, n_0 \\ E_i & i = n_0 + 1, \ldots, n_1 \end{cases}$$

where the generator internal voltages $E_i > 0$, $i = n_0 + 1, \ldots, n_1$, are constant by assumption.

The total real and reactive power leaving the i'th bus via transmission lines are

$$\begin{aligned} P_b(\alpha,|V|)_i &= \sum_{j=1}^{n_1} |V|_i|V|_j B_{ij}\sin(\alpha_i-\alpha_j), i=1,\dots,n_1, \quad \text{(real power)} \\ Q_b(\alpha,|V|)_i &= \sum_{j=1}^{n_1} |V|_i|V|_j B_{ij}\cos(\alpha_i-\alpha_j), i=1,\dots,n_1, \\ & \quad \text{(reactive power.)} \end{aligned}$$

The real and reactive load at buses $1,\dots,n_1$ are modelled as differentiable non-linear functions of voltage

$$\begin{aligned} (P_d)_i &= (P_d^0)_i \;\; i=1,\dots,n_0, \quad \text{(real load)} \\ (Q_d)_i &= Q_d(|V|)_i, \;\; i=1,\dots,n_0, \quad \text{(reactive load.)} \end{aligned}$$

The assumption that the active power load is voltage independent is somewhat restrictive. Structure preserving DA-system models for voltage dependent loads can easily be constructed, however, suitable energy functions for such power networks are still under investigation [6, 16]. For networks of the form considered in the sequel, energy functions (for Lyapunov stability analysis) are well understood [8, 18].

The power system dynamics are modelled by combining the classic machine swing dynamics (equations (1) and (2) below) at buses $n_0+1,\dots,n_1$ with the power balance equations (equations (3) and (4) below) at nodes $1,\dots,n_0$.

$$\begin{aligned} \frac{d}{dt}\alpha_i &= \omega_i-\omega_{n_1}, \;\; i=n_0+1,\dots,n_1-1, &(1)\\ M_i\frac{d}{dt}\omega_i + D_i\omega_i + P_b(\alpha,|V|)_i &= (P_M)_i, \;\; i=n_0+1,\dots,n_1, &(2)\\ P_b(\alpha,|V|)_i &= -(P_d^0)_i, \;\; i=1,\dots,n_0, &(3)\\ Q_b(\alpha,|V|)_i &= -Q_d(|V|)_i, \;\; i=1,\dots,n_0, &(4) \end{aligned}$$

where $M_i>0$, $i=n_0+1,\dots,n_1$ is the i'th generator inertia constant, $D_i\geq 0$, $i=n_0+1,\dots,n_1$ is the i'th generator damping constant and $(P_M)_i>0$ is the mechanical power input into the i'th generator.

In Hill, Hiskens and Mareels [9] it is shown that by employing a simple change of variables one may assume

$$\sum_{i=n_0+1}^{n_1}(P_M)_i - \sum_{i=1}^{n_0}(P_d^0)_i = 0,$$

without loss of generality. This condition is of use when developing energy functions for the network.

The equilibrium points of (1)-(4) are given by the full solutions of the power balance equations [5]. To see this, sum (2) over $i = n_0 + 1, \ldots, n_1$ and add the result to the sum of (3) over $i = 1, \ldots, n_0$ to give

$$\sum_{j=n_0+1}^{n_1} M_j \frac{d}{dt}\omega_j + \sum_{j=n_0+1}^{n_1} D_i \omega_i = 0$$

since $\sum_{i=1}^{n_1} P_b(\alpha, |V|)_i = 0$. Setting $\frac{d}{dt}\alpha_i = 0$ in (1) gives $\omega_i = \omega_n$, $i = n_0 + 1, \ldots, n_1 - 1$ which substituted into the above relation along with $\frac{d}{dt}\omega_j = 0$ gives

$$\omega_n \sum_{i=n_0+1}^{n_1} D_i = 0.$$

Assuming at least one non-zero damping term D_i gives $\omega_i = 0$, $i = n_0 + 1, \ldots, n_1$ which substituted into (2) gives

$$P_b(\alpha, |V|)_i = (P_M)_i, \quad i = n_0 + 1, \ldots, n_1.$$

Should all the damping terms D_i be zero then this equation is obtained directly by setting $\frac{d}{dt}\omega_i = 0$ in (2). This equation along with (3) and (4) are the power balance equations for the power system.

The variables of the system are

$$\begin{aligned} x &= (\alpha_{n_0+1}, \ldots, \alpha_{n_1-1}, \omega_{n_0+1}, \ldots, \omega_{n_1}) \in \mathbf{R}^{2m-1}, \\ y &= (\alpha_1, \ldots, \alpha_{n_0}, |V|_1, \ldots, |V|_{n_0}) \in \mathbf{R}^{2n_0}. \end{aligned}$$

Let $n_x = 2m - 1$, $n_y = 2n_0$, $n = n_x + n_y = 2n_1 - 1$ then $(x, y) \in \mathbf{R}^n = \mathbf{R}^{n_x} \times \mathbf{R}^{n_y}$ is the state vector for the structure preserving power system model. The partial state $x \in \mathbf{R}^{n_x}$ is known as the dynamic variable while $y = \mathbf{R}^{n_y}$ are the dependent or algebraic variables. To simplify notation in the sequel equations (1), (2), (3) and (4) are combined into a pair of equations

$$\begin{aligned} \dot{x} &= f(x, y) && (5) \\ 0 &= g(x, y), && (6) \end{aligned}$$

which is the standard form for a differential/algebraic system. The vector field $f : \mathbf{R}^n \to \mathbf{R}^{n_x}$ is a Lipschitz continuous function, i.e. for any $(x, y) \in \mathbf{R}^n$ there exists $\epsilon > 0$ and $k(x, y, \epsilon) \in (0, \infty)$ such that for any $(x', y') \in \mathbf{R}^n$ with $||(x, y) - (x', y')|| < \epsilon$ then

$$||f(x, y) - f(x', y')|| \leq k(x, y, \epsilon)||(x, y) - (x', y')||,$$

where $|| \cdot ||$ is the standard Euclidean 2-norm. The function $g : \mathbf{R}^n \to \mathbf{R}^{n_y}$ is differentiable with Lipschitz continuous derivative.

Remark 2.1 In the model discussed above there has been no attempt to model parasitic (or fast) dynamics in the dependent y variables. DA-systems following from a singular perturbation model, for example

$$\begin{aligned} \dot{x} &= f(x,y) \\ \epsilon\dot{y} &= g(x,y) \end{aligned}$$

for ϵ arbitrarily small, have slightly different properties from the models considered in the sequel. In particular, sections of the zero level set $g(x,y)=0$ may be unstable to perturbations in y. For the purposes of this paper it is assumed that the y dynamics of the zero level set are almost always stable, the only exception being points at which the geometry of the zero level set forces instability. We believe that it should be relatively simple task to combine the theory developed in the sequel with more general models where knowledge of the parasitic dynamics is available. □

For a more complete discussion of structure preserving models, (including practical considerations of transformer tap settings, static voltage compensators, voltage dependent load models, energy function derivations, etc.) the reader is referred to references [2, 5, 6, 7, 8, 10, 14, 15, 16, 18, 22, 24, 26].

3 Local Solutions

In this section the classical notions of solutions to DA-systems are reviewed. The algebraic constraint plays an important role in the study of DA-systems and a number results are given describing the zero level set of the algebraic constraint.

Definition 3.1 *Let* $(x_0,y_0)\in\mathbf{R}^n$ *satisfy* $g(x_0,y_0)=0$*. A* local *solution of the DA-system (5), (6) with initial condition* (x_0,y_0)*, defined on the time interval* $t\in[0,T(x_0,y_0))$*, is a function* $(x(t),y(t)):[0,T(x_0,y_0))\rightarrow \mathbf{R}^{n_x}\times\mathbf{R}^{n_y}$ *such that:*

i) $(x(0),y(0))=(x_0,y_0)$.

ii) $g(x(t),y(t))=0$ *for all* $t\in[0,T(x_0,y_0))$.

iii) The map $t\mapsto(x(t),y(t))\in\mathbf{R}^n$ *is continuous.*

iv) Equation (5) is satisfied in the integral sense

$$x(t)=\int_0^t f(x(\tau),y(\tau))d\tau + x_0, \quad t\in[0,T(x_0,y_0)).$$

Remark 3.1 Observe that the definition requires solutions to exist forward in time only. □

In the vicinity of a point $(x_0, y_0) \in \mathbf{R}^n$, such that $g(x_0, y_0) = 0$ and where the matrix partial derivative

$$D_y g(x_0, y_0) = \frac{\partial g}{\partial y}(x_0, y_0) \in \mathbf{R}^{n_y \times n_y}$$

is non-singular a unique local solution to (5), (6) exists, [11]. Indeed, the implicit function theorem guarantees the existence of a unique differentiable function $u : \mathbf{R}^{n_x} \to \mathbf{R}^{n_y}$ such that $g(x, u(x)) = 0$ for all x in the vicinity of x_0. Substituting $y = u(x)$ into (5) gives the ordinary differential equation (O.D.E.)

$$\dot{x} = f(x, u(x)) \tag{7}$$

whose solutions are local solutions of the DA-system. We will term this differential equation the *induced O.D.E. lift* of the DA-system. From the implicit function theorem it is easily verified that solving the induced O.D.E. lift is equivalent to solving the ordinary differential equation

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} f(x,y) \\ -(D_y g(x,y))^{-1} \cdot D_x g(x,y) \cdot f(x,y) \end{pmatrix}, \tag{8}$$

in regions where $\det(D_y g) \neq 0$. In particular, it is not necessary to determine the implicit relation $y = u(x)$ to compute solutions of the DA-system using the O.D.E. lift. Solutions of (8), however, provide no insight into the behaviour of the DA-system at points where $\det(D_y g(x,y)) = 0$.

An understanding of the geometric and topological structure of the zero level set $g(x, y) = 0$, of the algebraic constraint, provides important insight into the behaviour of the differential/algebraic system (5), (6).

Definition 3.2 *Let M be the zero level set of the function g given by (6)*

$$M = \{(x, y) \in \mathbf{R}^n \mid g(x, y) = 0\}. \tag{9}$$

In particular, M is the preimage of a closed set $\{0\}$ via a continuous function g and hence is closed in the standard Euclidean topology on $\mathbf{R}^n$.

Definition 3.3 *A point $(x, y) \in M$ is termed regular if rank[1] $Dg(x, y)$ is n_y. The point (x, y) is termed singular if rank $Dg(x, y)$ is strictly less than n_y.*

[1] The full derivative of g with respect to (x, y) is denoted

$$Dg(x,y) = \left(\frac{\partial g}{\partial x}(x,y) \vdots \frac{\partial g}{\partial y}(x,y) \right) \in \mathbf{R}^{n_y \times n}.$$

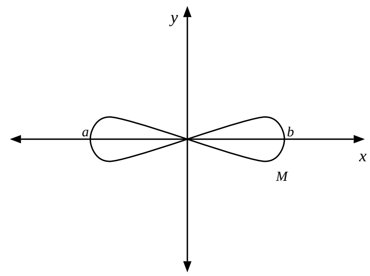

Figure 1: The constraint set $M = \{(x,y) \in \mathbf{R}^2 \mid 4x^2(1-x^2) - y^2 = 0 \}$

Lemma 3.1 *The set of singular points in M*

$$\mathcal{S} = \{(x,y) \in M \mid \text{rank} Dg(x,y) < n_y\}$$

is closed as a subset of $\mathbf{R}^n$ and closed as a subset of M in the subspace topology.

Proof: A useful characterisation of rank degeneracy for $Dg(x,y)$ is that $\det(C(x,y)) = 0$ where $C(x,y) \in \mathbf{R}^{n_y \times n_y}$ is any combination of n_y columns of $Dg(x,y)$. However, for a fixed choice of columns $\det(C(x,y))$ is a continuous function $\mathbf{R}^n \to \mathbf{R}$. Thus the set $\mathcal{S}$ can also be characterised as the intersection of M with the zero sets of $\det(C(x,y))$, for each choice of n_y columns. But each of these sets is the preimage of the closed set $\{0\}$ via a continuous function and hence is closed. The finite intersection of these sets is also closed. ∎

The rank condition for a regular point can also be thought of as requiring that the tangent map $Dg(x,y) : T_{(x,y)}\mathbf{R}^n \to T_{(x,y)}\mathbf{R}^{n_y}$ is a surjection. Thus, at a regular point $(x,y) \in M$, g is locally a submersion, and consequently in the vicinity of (x,y), M is a submanifold of $\mathbf{R}^n$ [12, pg. 22]. Observe that $\mathbf{R}^n - \mathcal{S}$ is an open subset of $\mathbf{R}^n$ and consequently is a submanifold.

Definition 3.4 *Denote the set of all regular points in M by*

$$M_r = \{(x,y) \in M \mid \text{rank} Dg(x,y) = n_y\}.$$

The set $M_r \subset M - \mathcal{S}$ is a submanifold of $\mathbf{R}^n - \mathcal{S}$ [12, pg. 22] and by composition of the submanifold charts is a submanifold of $\mathbf{R}^n$.

There is no submanifold structure of M at singular points. For example, the zero level set of the function $g(x,y) = 4x^2(1-x^2) - y^2$. The point (0,0) is a singular point of this function, see Figure 1.

As was seen in (8) the inverse of the matrix $D_y g(x,y)$ plays a crucial role in defining the induced O.D.E. lift of a DA-system. If $Dg(x,y)$ is rank deficient then $D_y g(x,y)$ is singular. The geometry of the constraint set M at a regular point for which $\det D_y g(x,y) = 0$ is not generically that of a singular point, take for example points a and b in Figure 1. The following definition differentiates between such points.

Definition 3.5 *A point* $(x,y) \in M$ *is termed* degenerate *if* $\det(D_y g(x,y)) = 0$ *and is termed* non-degenerate *if* $\det(D_y g(x,y)) \neq 0$. *A regular point* $(x,y) \in M$*, Definition 3.3, for which* $\det(D_y g(x,y)) = 0$ *is termed a* regular degenerate *point.*

Observe that any singular point is degenerate but that the converse is not true. Thus, in Figure 1 the points a, b and (0,0) are degenerate points. The points a and b are regular degenerate points while point (0,0) is singular. All other points in the set M are non-degenerate points.

The set M inherits the subspace topology from $\mathbf{R}^n$. Moreover, M is a metric space when equipped with the induced metric

$$d((x_1,y_1),(x_2,y_2)) = ||(x_1,y_1) - (x_2,y_2)||$$

where $(x_1,y_1),(x_2,y_2) \in M$ and $||\cdot||$ is the standard 2-norm in $\mathbf{R}^n$. Obviously the metric topology corresponds to the subspace topology. Since M_r, the set of all regular points in M, is a submanifold of $\mathbf{R}^n$ then the induced manifold topology is equivalent to the subspace topology on $M_r \subset \mathbf{R}^n$. Similarly the differential structure on M_r is induced by the submanifold charts. The set M is not a manifold and cannot be given a differential structure.

4 Existence of Global Solutions

In this section a definition for *global* solutions to differential/algebraic systems is proposed which allows for certain jump discontinuities. The motivation for the approach lies in the observation that the presence of regular degenerate points in DA-systems can naturally induce jump discontinuities into the solutions [17, 21]. The association of regular degenerate points (non-causal points in [14, 16]) and points of voltage collapse in differential/algebraic power system models further strengthens the argument for considering such solutions. To provide an intuitive feel for jump discontinuities in solutions of DA-systems consider the following simple example.

Example Consider the following DA-system

$$\dot{x} = f(x,y) = \frac{2}{h(x)} \tag{10}$$

$$0 = g(x,y) = (x^2 + y^4 - 1)(y - 2), \tag{11}$$

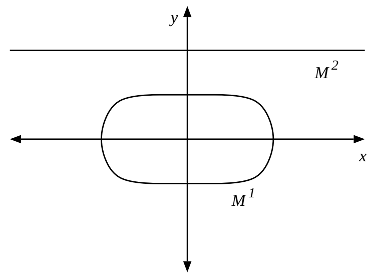

Figure 2: The constraint sets M^1 and M^2.

where the piecewise linear function $h : \mathbf{R} \rightarrow \mathbf{R}$ is given by

$$h(x) = \begin{cases} 1 & x < \frac{1}{4} \\ -2x + 3/2 & \frac{1}{4} \leq x < \frac{1}{2} \\ x & \frac{1}{2} \leq x \end{cases} \tag{12}$$

The function $h(x)$ is Lipschitz continuous and strictly bounded away from zero. It is included only as an artifact to ensure that $D_x g(x,y) f(x,y)$ does not depend on x at the point (1,0) and serves to simplify the expressions obtained below. The zero level set of the function g for this DA-system is the union of the oval $x^2 + y^4 = 1$ and the straight line $y = 2$, see Figure 2. These two sets do not intersect and are both manifolds, though the line $y = 2$ is not compact. We will denote the level set $x^2 + y^4 = 1$ as M^1 and the line $y = 2$ as M^2 where the full level set of g is $M = M^1 \cup M^2$. By inspection, the point (1,0) is a regular degenerate point of the DA-system.

Consider a small open set $N \subset M^1$ (in the subspace topology) around the point (1,0) such that for $(x, y) \in N$ then $x > \frac{1}{2}$ and $\mid y \mid < 1$. Observe that the vector field

$$-(D_y g(x,y))^{-1} D_x g(x,y) f(x,y) = -\frac{1}{y^3}$$

in N, since $f(x,y) = \frac{2}{x}$ for $x > \frac{1}{2}$. The induced O.D.E. lift of the DA-system is then

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \frac{2}{x} \\ -\frac{1}{y^3} \end{pmatrix}. \tag{13}$$

Consider any initial condition $(x_0, y_0) \in N$ such that $(x_0, y_0) \neq (1,0)$, then a solution of (13) is

$$\begin{aligned} x(t) &= \sqrt{1 - y(t)^4} \\ y(t) &= \mathrm{sgn}(y_0)(y_0^4 - 4t)^{\frac{1}{4}} \end{aligned}$$

for $t \in [0, \frac{y_0^4}{4})$ and where $\mathrm{sgn}(y_0)$ is the sign of y_0. Note that for any initial condition, then

$$\lim_{t \to \frac{y_0^4}{4}} (x(t), y(t)) = (1, 0),$$

however, since $f(1,0) \neq 0$ then $(1,0)$ is not an equilibrium point, and thus no local solution, with initial condition $(x_0, y_0) \in \Omega_M$, can exist beyond time $\frac{y_0^4}{4}$.

To define a global solution of a DA-system it is not sensible to consider any piecewise continuous map $t \mapsto (x(t), y(t))$ which satisfies the algebraic constraint and the dynamic constraint (in an integral sense) as a possible solution. Rather, we assume that jump behaviour will only occur when the system fails to have a local solution given by the induced O.D.E. (8). By imposing this requirement it is implicitly assumed that the parasitic dynamics are stable on M except possibly at degenerate points where $\det D_y g(x,y) = 0$. This issue was discussed in Remark 2. The geometry of degenerate points need not necessarily force discontinuous jump behaviour in the solution of a DA-system and the limiting dynamics of a DA-system in the vicinity of degenerate points are used to provide a means of determining where discontinuous jumps may be observed.

Definition 4.1 *Let $(x_0, y_0) \in \mathbf{R}^n$ satisfy $g(x_0, y_0) = 0$. A* global *solution of the DA-system (5), (6) with initial condition (x_0, y_0), defined on the time interval $t \in [0, T(x_0, y_0))$, is a function $(x(t), y(t)) : [0, T(x_0, y_0)) \to \mathbf{R}^{n_x} \times \mathbf{R}^{n_y}$ such that:*

i) The partial map $t \mapsto y(t) \in \mathbf{R}^{n_y}$ is piecewise continuous with discontinuities occurring at times t_ι, where $\iota \in \mathcal{I}$ and $\mathcal{I}$ is some general index set.

ii) The partial map $t \mapsto x(t) \in \mathbf{R}^{n_x}$ is continuous and piecewise differentiable with a finite number of discontinuities in its derivative occurring at times t_ι, where $\iota \in \mathcal{I}$.

iii) The curve $(x(t), y(t))$ is a solution to the DA-system in the sense that $g(x(t), y(t)) = 0$ for all $t \in [0, T(x_0, y_0))$ and

$$x(t) = \int_0^t f(x(\tau), y(\tau)) d\tau + x(0), \quad t \in [0, T(x_0, y_0)).$$

iv) For each $\iota \in \mathcal{I}$ there exists a degenerate point $(x_\iota, y_\iota) \in M$ with

$$\limsup_{(x,y) \to (x_\iota, y_\iota)} ||(D_y g(x,y))^{-1} D_x g(x,y) f(x,y)||_2 = \infty,$$

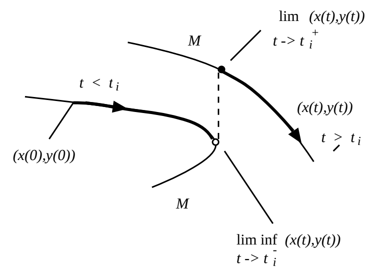

Figure 3: An example of a discontinuous jump of the nature described in Definition 4.1. In this example the discontinuous jump occurs at time t_i and the lim inf operation could be replaced by a standard limit if desired.

for $(x, y) \in M$, *where* $||\cdot||_2$ *is the induced matrix norm, and*

$$\liminf_{t \to t_\iota^-} ||(x(t), y(t)) - (x_\iota, y_\iota)|| = 0,$$

where $t \to t_\iota^-$ *indicates* $t < t_\iota$.

v) The limit from above at time t_ι, *of the state* $(x(t), y(t))$ *exists and is equal to*

$$\lim_{t \to t_\iota^+} (x(t), y(t)) = (x(t_\iota), y(t_\iota)),$$

where $(x(t_\iota), y(t_\iota)) \in M$ *is the initial condition of a local solution to the DA-system, defined on some time interval* $[t_\iota, T(x(t_\iota), y(t_\iota)))$. *The global solution will correspond to a local solution, in the vicinity of* $(x(t_\iota), y(t_\iota))$, *at least on an open time interval* $(t_\iota, t_\iota + \delta)$, $\delta \leq T(x(t_\iota), y(t_\iota))$.

Remark 4.1 Observe that uniqueness will not usually be a property of solutions satisfying Definition 4.1. Indeed, it is expected any given local solution that converges to a degenerate point may be extended to several global solutions by any one of a number of different, and equally valid, discontinuous jumps. □

Remark 4.2 An important property of Definition 4.1 is that for every time $t^* \in [0, T(x_0, y_0))$ then there exists $\delta > 0$ such that the global solution $(x(t; x_0), y(t; y_0))$ corresponds to a local solution on the time interval $t \in [t^*, t^* + \delta))$. □

Remark 4.3 Note that only the asymptotic behaviour in the vicinity of a degenerate point is considered in the criterion for jump behaviour. In particular, we do not believe that examining the algebraic equation

$$D_x g(x^*, y^*) \cdot f(x^*, y^*) + D_y g(x^*, y^*) \cdot q = 0, \;\; q \in \mathbf{R}^{n_y},$$

at a degenerate point (x^*, y^*) will provide any useful indication of the systems behaviour. Observe, however, that if $D_x g(x^*, y^*) \cdot f(x, y)$ has a component lying in the null space of $D_y g(x^*, y^*)$ then

$$\limsup_{(x,y)\to(x^*,y^*)} ||(D_y g(x,y))^{-1} D_x g(x,y) f(x,y)||_2 = \infty,$$

where the limit is taken for (x, y) regular non-degenerate. Thus, the only degenerate points with a neighbourhood in which $\dot{y}$ remains bounded are those for which $D_x g(x^*, y^*) \cdot f(x, y)$ lies completely in the column space of $D_y g(x^*, y^*)$. It is easily verified that this is a non-generic situation (the property fails for arbitrarily small perturbations of f and g). □

Point i) allows discontinuities in the y coordinates. Point ii) guarantees the continuity of the x coordinate but allows jumps in the derivative $\dot{x}$ as indeed must occur if y jumps and $f(x, y)$ depends explicitly on y. Point iii) simply ensures that the curve $(x(t), y(t))$ is a solution of the DA-system.

Point iv) states that jumps may only occur at degenerate points for which the vector field $-(D_y g(x,y))^{-1} D_x g(x,y) f(x,y)$ becomes unbounded. The characterisation of such points as potential discontinuities in the solution follows from a heuristic argument based on the observation that the derivative $\dot{y}$ of the induced O.D.E. lift goes to infinity at such points, an idea suggested by Zaborszky [26]. In the case where $\dot{y}$ remains bounded in the vicinity of a degenerate point then a local solution is certainly defined and there appears to be no reason to consider jump behaviour. Of course, since bounded behaviour in $\dot{y}$ near a degenerate point is a non-generic property (cf. Remark 4) then in a practical situation one would tend to view the presence of any degenerate point in the solution as a potential jump discontinuity. It is possible that by fully understanding the limiting behaviour of a solution at degenerate points a better idea of global behaviour could be obtained. However, until a better understanding of practical behaviour at such points is obtained, we suggest that the definition remains open to any finite jump in the variable y. Point v) requires that the global solution continue as a local solution, at least for some short time period, immediately after the jump.

Remark 4.4 In power system models it may be practical to consider further restrictions on possible discontinuities in the solution. For example, allowing voltage magnitudes to jump discontinuously but requiring that the bus phase angles remain continuous. Definition 4.1, however, is deliberately general to provide a clearer perspective of the theoretical issues. □

Example Consider the following global solution to (10), (11) for some initial condition $(x_0, y_0) \neq (1,0)$, $x_0 > \frac{1}{2}$, $|y_0| < 1$.

$$x(t) = \begin{cases} \sqrt{1-y(t)^4} & t \in [0, \frac{y_0^4}{4}) \\ 1 & t = \frac{y_0^4}{4} \\ \sqrt{4t+1-y_0^4} & t \in (\frac{y_0^4}{4}, \infty) \end{cases}$$

$$y(t) = \begin{cases} \mathrm{sgn}(y_0)(y_0^4 - 4t)^{\frac{1}{4}} & t \in [0, \frac{y_0^4}{4}) \\ 2 & t = \frac{y_0^4}{4} \\ 2 & t \in (\frac{y_0^4}{4}, \infty) \end{cases}$$

Thus the global solution to the DA-system is in the form of a piecewise continuous function $t \mapsto (x(t), y(t))$ in which the discontinuities lie in the y variables.

Remark 4.5 In the case where the state of a DA-system converges to a regular degenerate point for which jump behaviour is predicted and for which there is no point in the algebraic constraint set to which the solution may jump, a global solution to the DA-system cannot exist past the time at which the state reaches the regular degenerate point. □

Remark 4.6 A regular degenerate point need not always be associated with discontinuous solutions. If the vector field

$$-(D_y g(x,y))^{-1} D_x g(x,y) f(x,y)$$

remains integrable in the vicinity of such a point then the integral

$$y(t) = -\int_0^t (D_y g(x,y))^{-1} D_x g(x,y) f(x,y) + y_0$$

exists and defines a local solution. Indeed, boundedness of $\dot{y}$ is not necessary for the existence of a local solution. An interesting example is the global behaviour of the DA-system [23]

$$\begin{aligned} \dot{x_1} &= 1 - x_1 \\ \dot{x_2} &= 2 - x_2 \\ 0 &= x_2 - x_1 y - y^3 \end{aligned}$$

In this case the solution passing through $(0,0,0)$ must remain a local solution (there are no valid discontinuous jumps available), however the y dynamics

$$\dot{y} = \frac{2 - x_2 + y - x_1 y}{3y^2 + x_1}$$

certainly become unbounded at $(0,0,0)$. □

5 Stability Definitions

In this section the definition of stability and attractivity in the context of differential algebraic systems is considered. The definition of *structural stability* proposed is analogous to that defined for classical dynamical systems [13].

Definition 5.1 *Consider the DA-system (5), (6). We call a point* $(x^*, y^*) \in \mathbf{R}^n$ *an* equilibrium *point if both* $g(x^*, y^*) = 0$ *and* $f(x^*, y^*) = 0$.

Definition 5.2 Stability *[11] Consider the DA-system (5), (6) with an equilibrium point* (x^*, y^*). *Let* $(x(t; x_0), y(t; y_0))$ *denote a solution of the DA-system with initial conditions* (x_0, y_0) *satisfying* $g(x_0, y_0) = 0$. *The point* (x^*, y^*) *is* stable *if for all* $\epsilon > 0$ *there exists a positive number* $\delta > 0$ *such that for any* $(x_0, y_0) \in M$ *with* $||(x_0, y_0) - (x^*, y^*)|| < \delta$ *then* $||(x(t; x_0), y(t; y_0)) - (x^*, y^*)|| < \epsilon$ *for all* $t > 0$. *The point* (x^*, y^*) *is* asymptotically stable *if it is stable and there exists a positive number* $\delta > 0$ *such that for all* (x_0, y_0) *with* $||(x_0, y_0) - (x^*, y^*)|| < \delta$ *then* $(x(t; x_0), y(t; y_0))$ *converges to* (x^*, y^*) *as* $t \to \infty$.

To provide practical stability results it is necessary to consider structural stability of equilibrium points, where the qualitative behaviour of the DA-system at a given equilibrium point is preserved for small perturbations of the DA-system. The following definition is based on the development in Section 1, Chapter 16 [13]. The definition is not exactly the definition of structural stability [13, pg. 312], however, in a classical O.D.E. setting the two concepts are equivalent [13, Theorem 1], at least for asymptotically stable equilibria.

Definition 5.3 Structural Stability. *Consider the DA-system (5), (6) and let* (x_0^*, y_0^*) *be an equilibrium point of this system. Consider any* $\mathcal{C}^1$ *perturbations* $f_\mu(x, y)$ *and* $g_\mu(x, y)$ *of* f *and* g *of magnitude less than* μ *in the* $\mathcal{C}^1$ *infinity norm, i.e. one has the inequalities*

$$\begin{aligned} \sup_{(x,y)\in\mathbf{R}^n} ||f_\mu(x, y) - f(x, y)|| &\leq \mu, \\ \sup_{(x,y)\in\mathbf{R}^n} ||Df_\mu(x, y) - Df(x, y)||_2 &\leq \mu, \end{aligned}$$

and similarly for g_μ. *The equilibrium point* (x_0^*, y_0^*) *is* structurally *stable if:*

i) (x_0^*, y_0^*) *is an asymptotically stable equilibrium point of the DA-system (5), (6).*

ii) For any $\epsilon > 0$, there exists a $\beta > 0$ such that any DA-system defined by perturbed functions $f_\mu(x,y)$ and $g_\mu(x,y)$ for any $\mu \leq \beta$ has an asymptotically stable equilibrium point (x_μ^*, y_μ^*) such that $||(x_\mu^*, y_\mu^*) - (x_0^*, y_0^*)|| \leq \epsilon$.

Intuitively, one expects that an equilibrium point will be structurally stable if and only if all the eigenvalues of its linearization have negative real parts. For a classical O.D.E. this is a standard result [13, Theorem 2, pg. 305], however, for a DA-system the situation is not so simple. Before one may consider the structural stability of the induced O.D.E. $\dot{x} = f(x, u(x))$ (cf. equation (7)) it is necessary to show that the existence of an equilibrium point is a structural property of the system. This condition fails at degenerate points. Considering only non-degenerate points one can rely on classical theory as long as the equilibrium point of $\dot{x} = f(x, u(x))$ is hyperbolic.

Theorem 5.1 *Consider the DA-system (5), (6) with (x^*, y^*) a non-degenerate (Definition 3.5) equilibrium point. The point (x^*, y^*) is structurally stable if and only if the linearization of $\dot{x} = f(x, u(x))$ at the point (x^*, y^*),*

$$\dot{\xi} = \left(D_x f(x^*, y^*) - D_y f(x^*, y^*) D_y g(x^*, y^*)^{-1} D_x g(x^*, y^*)\right) \cdot (\xi - x^*),$$

has eigenvalues with strictly negative real part.

Proof: Only a brief sketch of the proof is provided. Consider the combined vector function $(f(x,y), g(x,y)) : \mathbf{R}^n \to \mathbf{R}^n$ and observe that its derivative $D_{(x,y)}(f(x,y), g(x,y))$ can be written

$$D \begin{pmatrix} f(x^*, y^*) \\ g(x^*, y^*) \end{pmatrix} \begin{pmatrix} I_{n_x} & 0 \\ -D_y g(x^*, y^*)^{-1} D_x g(x^*, y^*) & I_{n_y} \end{pmatrix} = \begin{pmatrix} A(x^*, y^*) & D_y f(x^*, y^*) \\ 0 & D_y g(x^*, y^*) \end{pmatrix},$$

where $A(x^*, y^*)$ is

$$A(x^*, y^*) = D_x f(x^*, y^*) - D_y f(x^*, y^*) D_y g(x^*, y^*)^{-1} D_x g(x^*, y^*), \quad (14)$$

the matrix appearing in the linearization of $\dot{x} = f(x, u(x))$ at (x^*, y^*). By assumption $A(x^*, y^*)$ is full rank (all eigenvalues have strictly negative real parts) and consequently $D_{(x,y)}(f(x,y), g(x,y))$ is non-singular.

Consider any differentiable one parameter perturbation $f_\mu(x,y)$, $g_\mu(x,y)$ (where $f_0(x,y) = f(x,y)$ and $g_0 = g(x,y)$) of $f(x,y)$, $g(x,y)$. Since $D_{(x,y)}(f(x,y), g(x,y))$ is non-singular one can use the implicit function theorem to find continuous functions

$$\begin{aligned} (x,y) &= (x(\mu), y(\mu)) \\ (f(x(\mu), y(\mu)), g(x(\mu).y(\mu))) &= (f(x^*, y^*), g(x^*, y^*)) = (0,0) \end{aligned}$$

in a neighbourhood of (x^*, y^*). This proves that the existence of a hyperbolic equilibrium point is robust to small perturbations.

To complete the proof it is sufficient to apply classical theory [13, Theorem 2, pg. 305] to the induced O.D.E. $\dot{x} = f(x, u(x))$, around the point $(x(\mu), u(x(\mu)))$. ■

As mentioned above, the requirement that the point be non-degenerate is necessary, however, convergence of DA-system solutions to degenerate points lies at the heart of the discontinuous jump behaviour discussed in Section 4. To study such points we use the concept of attractivity, which is modified slightly to allow for non-uniqueness and possible finite time existence of solutions.

Definition 5.4 Attractivity *[19, Pg. 8] Consider the DA-system (5), (6) along with a point $(x^*, y^*) \in M$. The point (x^*, y^*) is* attractive *if there exists an open neighbourhood $N \subseteq M$ of (x^*, y^*) such that; for each $(x_0, y_0) \in N$ and every global solution $(x(t; x_0), y(t; y_0)) \in N$ of the DA-system that exists and remains in N for a maximal time interval $t \in [0, T(x_0, y_0))$, and each $\epsilon > 0$ there exists $\sigma(x_0, y_0) \in (0, T(x_0, y_0))$ such that*

$$||(x(t; x_0), y(t; y_0)) - (x^*, y^*)|| \leq \epsilon$$

for all $t \in [\sigma(x_0, y_0), T(x_0, y_0))$.

Observe that the global solution may continue to exists in M after time $T(x_0, y_0)$, however, such a solution must not remain in N. In particular, regular degenerate points responsible for discontinuous jump behaviour may well be attractive but certainly are not stable.

Remark 5.1 An equivalent definition of asymptotic stability [19, pg. 10] is a point (x^*, y^*) which is both stable (Definition 5.2) and attractive (Definition 5.4). □

6 Lyapunov Stability Results

In this section the role of Lyapunov theory in stability analysis of global solutions of DA-systems is considered. A major advantage of Lyapunov theory in such analysis lies in its applicability to systems with non-unique solutions.

Definition 6.1 *[19, pg. 12] A function $a : \mathbf{R}_+ \to \mathbf{R}_+$ is said to be of* class $\mathcal{K}$ *if it is continuous, strictly increasing and $a(0) = 0$.*

Definition 6.2 *[19, pg. 345] Let $(x(t;x_0), y(t;y_0))$ be a global solution of (5), (6) with initial conditions (x_0, y_0). Let $V : M \to \mathbf{R}$ be a Lipschitz continuous function. The Dini derivative of $V(x(t;x_0), y(t;y_0))$ with respect to time is*

$$D_t^+ V(x(t;x_0), y(t;y_0)) = \limsup_{h \to 0^+} \frac{V(x(t+h;x_0), y(t+h;y_0)) - V(x(t;x_0), y(t;y_0))}{h}, \quad (15)$$

where this limit exists.

In particular, if $(x(t;x_0), y(t;y_0))$ is a local solution (Definition 3.1) of (5), (6) for some time interval $[t, t+\delta)$ where $\delta > 0$ then the Dini derivative $D_t^+ V(x(t;x_0), y(t;y_0))$ at time t is well defined. Indeed, if $(x(t;x_0), y(t;y_0))$ is a global solution of (5), (6) defined on some time interval $[0, T(x_0, y_0))$ then the Dini derivative (15) is well defined for all $t \in [0, T(x_0, y_0))$. We use the following lemma.

Lemma 6.1 *[19, pg. 349] Let $V : [t_1, t_2] \to \mathbf{R}$ be a Lipschitz continuous function for which there exists $\epsilon > 0$ such that for any $t \in (t_1, t_2)$*

$$D_t^+ V(t) \leq -\epsilon,$$

then

$$V(t_2) \leq V(t_1) - \epsilon(t_2 - t_1).$$

A simple corollary of Lemma 6.1 is that if $D_t^+(V(t)) \leq 0$ for all $t \in (t_1, t_2)$ then $V(t)$ is monotonic non-increasing. Similarly if $D_t^+(V(t)) \geq 0$ then $V(t)$ is monotonic non-decreasing [19, pg. 347].

Theorem 6.1 *Let $(x^*, y^*) \in M$ and let $U \subseteq M$ be some open set in M (in the subspace topology) such that $U - \{(x^*, y^*)\}$ contains no equilibrium points. Let (x_0, y_0) be some initial condition in $U - \{(x^*, y^*)\}$ and denote a solution of the DA-system (5), (6) with this initial condition as $(x(t;x_0), y(t;y_0))$. Define $T^*(x_0, y_0)$ (possibly infinite) to be the infinum over all possible global solutions, $(x(t;x_0), y(t;y_0))$, of the maximum time $T(x_0, y_0)$ for which each solution $(x(t;x_0), y(t;y_0)) \in U$ exists and remains in U.*

Assume there exists a class $\mathcal{K}$ function $a : \mathbf{R}_+ \to \mathbf{R}_+$ such that

$$T^*(x_0, y_0) \geq a(||(x_0, y_0) - (x^*, y^*)||).$$

Assume that there exists a Lipschitz continuous "Lyapunov function" $V : U \to \mathbf{R}$ with $V(x^*, y^*) = 0$. Furthermore, assume there exist two functions $b, d : \mathbf{R}_+ \to \mathbf{R}_+$ of class $\mathcal{K}$ such that V satisfies the following conditions:

i) For all $(x, y) \in U$ *then*

$$b(||(x,y) - (x^*, y^*)||) \le V(x,y).$$

ii) The Dini derivative of $V(x(t;x_0), y(t;y_0))$ *with respect to time satisfies*

$$D_t^+ V(x(t;x_0), y(t;y_0)) \le -d(||(x(t;x_0), y(t;y_0)) - (x^*, y^*)||).$$

iii) If a discontinuous jump occurs in the solution at time t_ι *then*

$$\liminf_{t \to t_\iota^-} V(x(t;x_0), y(t;y_0)) \ge \lim_{t \to t_\iota^+} V(x(t;x_0), y(t;y_0)).$$

Then the point (x^*, y^*) *is attractive (Definition 5.4). If furthermore, the point* (x^*, y^*) *is a non-degenerate equilibrium point, then it is asymptotically stable.*

Remark 6.1 Observe that Assumption iii) constrains discontinuous jump behaviour of global solutions in the same manner that the standard Lyapunov monotonicity assumption (Assumption ii)) constrains continuous solutions. It is necessary to consider an assumption of this form when applying Lyapunov theory to discontinuous solutions of dynamical systems [21, pg. 146]. In applying Theorem 6.1 one would aim to guarantee (by prior analysis and heuristic knowledge of the system) that any global solution of the DA-system satisfies the theorem's assumptions on the set of interest. Thus, the results do not require that the actual solutions of the DA-system be computed. □

Proof: Define $c : \mathbf{R}_+ \to \mathbf{R}_+$ via

$$c(\delta) = \sup_{||(x,y)-(x^*,y^*)|| \le \delta} V(x,y), \quad (x,y) \in U.$$

Thus, c is a class $\mathcal{K}$ function with the property $V(x,y) \le c(||(x,y) - (x^*, y^*)||)$ on U. For any $\epsilon > 0$, we exploit the class $\mathcal{K}$ properties of c to choose $\eta > 0$ such that $c(\eta) < b(\epsilon)$ and $\alpha \in (0, \epsilon)$ such that $c(\alpha) < d(\eta)a(\eta)$. Let $(x_0, y_0) \in V^{-1}((0, \alpha)) \cap U$ be any initial condition in the intersection of the inverse image of $(0, \alpha)$ and U. By assumption, for $t \in [0, T^*(x_0, y_0))$ then $(x(t;x_0), y(t;y_0)) \in U$ and conditions ii) and iii) now yield

$$(x(t;x_0), y(t;y_0)) \in V^{-1}((0,\alpha)) \cap U, \text{ for all } t \in [0, T^*(x_0, y_0)).$$

Now choose $\sigma(x_0, y_0) \in (\frac{c(\alpha)}{d(\eta)}, a(\eta))$ and observe that $\sigma(x_0, y_0) < a(\eta) \le T^*(x_0, y_0)$. It follows that $||(x(t;x_0), y(t;y_0)) - (x^*, y^*)||$ cannot be larger than η for every $t \in [0, \sigma(x_0, y_0)]$, since if this were the case

$$D_t^+ V(x(t;x_0), y(t;y_0)) \le -d(\eta)$$

for all $t \in [0, \sigma(x_0, y_0)]$ and Lemma 6.1 along with property iii) gives

$$0 \leq V(x(t_1; x_0), y(t_1; y_0)) \leq c(\alpha) - d(\eta)\sigma(x_0, y_0),$$

which contradicts the choice of $\sigma(x_0, y_0)$. Thus, there exists a $t_1 \in [0, \sigma(x_0, y_0)]$ such that

$$c(||(x(t_1; x_0), y(t_1; y_0)) - (x^*, y^*)||) \leq c(\eta) < b(\epsilon).$$

Since V is decreasing along solutions one obtains for $t \in [t_1, T^*(x_0, y_0))$

$$\begin{aligned} b(||(x(t; x_0), y(t; y_0)) - (x^*, y^*)||) &\leq V(x(t; x_0), y(t; y_0)) \\ &\leq V(x(t_1; x_0), y(t_1; y_0)) \\ &\leq c(||x(t_1; x_0), y(t_1; y_0) - (x^*, y^*)||) \\ &< b(\epsilon). \end{aligned}$$

This ensures that

$$||(x(t; x_0), y(t; y_0)) - (x^*, y^*)|| < \epsilon$$

for all $t \in [t_1, T^*(x_0, y_0))$ and proves that (x^*, y^*) is an attractive point, Definition 5.4.

If (x^*, y^*) is a non-degenerate equilibrium point then choose $N \subset U$ an open neighbourhood of (x^*, y^*) to contain no singular or degenerate points. Choose $\alpha \in (0, \epsilon)$ such that $\left(V^{-1}((0, \alpha)) \cap U\right) \subseteq N$ and $c(\alpha) < d(\eta)a(\eta)$. Consequently, any solution $(x(t; x_0), y(t; y_0))$ with $(x_0, y_0) \in V^{-1}((0, \alpha)) \cap U$ remains in N and the induced O.D.E. lift guarantees infinite time existence of solutions ($T^*(x_0, y_0) = \infty$). It follows directly that the point (x^*, y^*) is stable. Choose $\sigma(x_0, y_0) > \frac{c(\alpha)}{d(\eta)}$ and observe that the above argument gives that (x^*, y^*) is also attractive. Consequently, (x^*, y^*) is asymptotically stable. ■

Remark 6.2 The existence assumption (i.e. $T^*(x_0, y_0) > 0$ for $(x_0, y_0) \neq (x^*, y^*)$) in Theorem 6.1 is vital to the above argument. This should not come as any surprise since Lyapunov theory always comes with the implicit assumption that the solutions of the system considered exist. For classical Lipschitz continuous O.D.E. theory the existence of a Lyapunov function can be used to prove infinite time existence of solutions [19, pg. 25], however, for DA-systems the assumption must be made explicit. Observe that for an attractive non-degenerate equilibrium point of a DA-system one does indeed obtain infinite time existence in the sense of classical O.D.E. theory. The exact nature of the existence assumption need not necessarily be that given in the theorem statement. Other existence assumptions (perhaps motivated by physical insight) may be directly available and provide a more intuitive result. We believe, however, that all basic existence assumptions will be formally equivalent to the assumption made in Theorem 6.1. □

Remark 6.3 The result does not require uniqueness of solutions. To check structural stability one must still compute the eigenvalues of the linearization of the induced O.D.E. lift. □

Remark 6.4 The metric structure of M has been exploited throughout the above proof, however, the induced Euclidean metric is not necessarily the best to use for Lyapunov results. Consider the metric defined by taking the infinum of the length of all curves lying wholly in M connecting the two points of interest. The advantage of such a metric is that the DA-system evolves on M not $\mathbf{R}^n$ and it should be easier to find suitable functions a, b, d of class $\mathcal{K}$ when the distance measure used is more representative of the geometry of the problem. □

There are several aspects of Theorem 6.1 which should be commented upon. The theorem can be used to determine the attractive nature of any point, regular or degenerate. Similarly the set $U - \{(x^*, y^*)\}$ may contain singular and degenerate points at which discontinuous jumps and non-uniqueness of solutions is possible. This freedom causes the main technical difficulties in the proof. Firstly, it is necessary to consider Dini derivatives and Lipschitz continuous Lyapunov functions since the results must be valid at singular points where M has no differentiable structure and the classical derivative is not well defined. We believe that this is a considerable advantage since most methods of choosing Lyapunov functions for power systems are valid only on path connected regular submanifolds of M [8]. Such functions can now be pasted together along singular boundaries (preserving Lipschitz continuity) to provide global Lyapunov functions. Secondly, some form of existence assumption is necessary. Without such an assumption the presence of regular degenerate points (x^*, y^*) may cause the global solution to jump outside the set U. In this aspect the theory differs from traditional Lyapunov arguments for O.D.E.'s where existence of a Lyapunov function ensures boundedness and consequently infinite time existence of solutions [19, Theorem 6.2]. Thirdly, the best general result possible is a proof of attractivity since the solution may cease to exist at the point (x^*, y^*). In the case where (x^*, y^*) is non-degenerate the existence assumptions can be dropped and the result extends to asymptotic stability.

Example 6.1 Consider the DA-system (cf. [23])

$$\begin{aligned} \dot{x_1} &= 1 - x_1 \\ \dot{x_2} &= 2 - x_2 \\ 0 &= x_2 - x_1 y - y^3. \end{aligned}$$

Expressing x_2 explicitly as a function of x_1 and y one obtains a cubic function $x_2 = x_1 y + y^3$ which is monotonic increasing in y for $x_1 > 0$ and has two turning points for $x_1 < 0$. It is easily verified that the non-monotonic nature of the constraint in the region $x_1 < 0$ will certainly induce discontinuous jump behaviour in the solution. However, the fact that the constraint is a cubic in y ensures that for any choice of x_1 and x_2 there will always be at least one value of y which satisfies the constraint. As a consequence it is easily verified that global solutions to the DA-system exist for all time. In the notation of Theorem 6.1 one has that the set $U = \{(x_1, x_2, y) \mid x_2 = x_1 y + y^3\}$ and that $T^*(x_1(0), x_2(0), y(0)) = \infty$ for all initial conditions $(x_1(0), x_2(0), y(0)) \in U$.

Consider the Lyapunov function $V(x_1, x_2, y) := (1 - x_1)^2 + (2 - x_2)^2$. Along solutions of the DA-system one has

$$\begin{aligned} \frac{d}{dt} V(x_1(t), x_2(t), y(t)) &= -2(1 - x_1(t))\dot{x}_1(t) - 2(1 - x_2(t))\dot{x}_2(t) \\ &= -2V(x_1(t), x_2(t), y(t)). \end{aligned}$$

By inspection V satisfies Assumptions i) and ii) of Theorem 6.1. Moreover, since V does not depend on y then Assumption iii) will also be satisfied by any jump discontinuities the solution may display. It follows that Theorem 6.1 applies and one concludes that the point $(x_1, x_2, y) = (1, 2, 1)$ is an attractive point of the DA-system. Moreover, since $(1, 2, 1)$ is non-degenerate then in fact it is globally asymptotically stable. □

It is important to consider extensions of Theorem 6.1 which consider the basin of attraction of the point (x^*, y^*).

Corollary 6.1 *Let $(x^*, y^*) \in M$ be an attractive point for the DA-system (5), (6) with its basin of attraction containing an open neighbourhood N of (x^*, y^*). Let $U \subseteq M$ be some open set in M such that $U - \{(x^*, y^*)\}$ contains no equilibrium points. Let (x_0, y_0) be some initial condition in $U - \{(x^*, y^*)\}$ and denote a solution of the DA-system (5), (6) with this initial condition as $(x(t; x_0), y(t; y_0))$. Define $T^*(x_0, y_0)$ (possibly infinite) to be the infinum over all possible global solutions, $(x(t; x_0), y(t; y_0))$, of the maximum time $T(x_0, y_0)$ for which each solution $(x(t; x_0), y(t; y_0)) \in U$ is well defined.*

Assume that there exists a Lipschitz continuous "Lyapunov function" $V : U \to \mathbf{R}$ with $V(x^*, y^*) = 0$ and $V(x, y) \geq 0$ for all $(x, y) \in U$. Furthermore, assume there exists real numbers $\epsilon_1, \epsilon_2 > 0$ such that

i) For all $(x(t; x_0), y(t; y_0)) \in (U - N)$ then

$$V(x(t; x_0), y(t; y_0)) \geq \epsilon_1$$

ii) The Dini derivative of $V(x(t;x_0), y(t;y_0))$ *with respect to time satisfies*

$$D_t^+ V(x(t;x_0), y(t;y_0)) \leq -\epsilon_2.$$

iii) If a discontinuous jump occurs in the solution at time t_ι *then*

$$\liminf_{t \to t_\iota^-} V(x(t;x_0), y(t;y_0)) \geq \lim_{t \to t_\iota^+} V(x(t;x_0), y(t;y_0)).$$

The basin of attraction of (x^*, y^*) *will contain the set*

$$\{(x_0, y_0) \in U \mid V(x_0, y_0) < \epsilon_1 + \epsilon_2 T^*(x_0, y_0)\}.$$

Proof: Consider any initial condition $(x_0, y_0) \in U$. If $V(x_0, y_0) < \epsilon_1 + \epsilon_2 T^*(x_0, y_0)$ then choose $\sigma(x_0, y_0) \in (\frac{V(x_0,y_0)-\epsilon_1}{\epsilon_2}, T^*(x_0, y_0))$. It follows that there exists a time $t_1 \in (0, \sigma(x_0, y_0)$ such that $(x(t_1;x_0), y(t_1;y_0)) \in N$ otherwise Lemma 6.1 gives

$$0 \leq V(x(\sigma(x_0,y_0);x_0), y(\sigma(x_0,y_0);y_0)) \leq V(x_0,y_0) - \epsilon_2 \sigma(x_0,y_0),$$

which contradicts $V(x,y) \geq 0$. Since, N is contained in the attractive basin of (x^*, y^*) the result follows. ■

7 Conclusion

The main contribution of this paper is the analysis of discontinuous jump behaviour in the solutions of differential/algebraic systems. The framework in which this theory has been presented is deliberately general to provide a clear perspective of the technical difficulties that need to be addressed. There are of course practical considerations that must be addressed before jump discontinuities could be considered in the modelling of power system behaviour. The applicability of Lyapunov theory to deal with "global" solutions to a differential/algebraic system, however, is an encouraging reason to pursue this approach. Other approaches to modelling the degenerate behaviour of power systems should not be ignored. In particular, singular perturbation theory may offer a better understanding of the manner in which perturbations will effect the algebraic constraint function, and consequently, the dynamics of the differential/algebraic system. Recent work in this area has been undertaken by Venkatasubramanian et al. [24].

Acknowledgements

The authors would would like to thank the staff of Unite Auto, Catholic University Leuven, Louvain-la-Neuve for creating a stimulating environment to work in. Also we would like to thank Ed Henrich, Ian Hiskens, Robert Davy and an anonymous referee for their comments and David Hill for his helpful cooperation.

References

[1] P. M. Anderson and A. A. Fouad. *Power System Control and Stability.* Ames, Iowa: Iowa State University Press, 1977.

[2] A. R. Bergen and D. J. Hill. A structure preserving model for power system stability analysis, *IEEE Transactions on Power Apparatus and Systems* **PAS-100(1)** (1981), 25–33.

[3] I. Dobson. Towards a theory of voltage collapse in electric power systems, *Systems and Control Letters* **13** (1989), 253–262.

[4] L. Fekih-Ahmed and C. Hsiao-Dong. Stability regions of systems governed by differential-algebraic equations. private communication.

[5] D. J. Hill. On the equilibria of power systems with non-linear loads, *IEEE Transactions on Circuits and Systems* **CAS-36(11)** (1989), 1458–1463.

[6] D. J. Hill. Nonlinear dynamic load models with recovery for voltage stability studies, *IEEE Transactions of Power Systems* **8(1)** (1993), 166–176.

[7] D. J. Hill and A. R. Bergen. Stability analysis of multimachine power networks with linear frequency dependent loads, *IEEE Transactions on Circuits and Systems* **CAS-29(12)** (1982), 840–848.

[8] D. J. Hill and C. N. Chong. Lyapunov functions of Lur'e-Postinikov form for structure preserving models of power systems, *Automatica* **25(3)** (1989), 453–460.

[9] D. J. Hill, I. A. Hiskens, and I. M. Y. Mareels. Stability theory of differential/algebraic models of power systems. In *Special Issue of Sadhana, Academy Journal in Engineering Sciences* (M. A. Pai, ed.) **18** (1993), 731–748.

[10] D. J. Hill, P-A. Lof, and G. Andersson. Analysis of long-term voltage stability, In *Proceedings 10th Power Systems Computation Conference*, pp. 1252–1259, Graz, Austria, 1993.

[11] D. J. Hill and I. M. Y. Mareels. Stability theory for differential/algebraic systems with application to power systems, *IEEE Transactions on Circuits and Systems* **CAS-37(11)** (1990), 1416–1423.

[12] M. W. Hirsch. *Differential Topology.* Number 33 in Graduate Texts in Mathematics. New York: Springer-Verlag, 1976.

[13] M. W. Hirsch and S. Smale. *Differential Equations, Dynamical Systems and Linear Algebra*, volume 60 of *Pure and Applied Mathematics*. New York: Academic Press, 1974.

[14] I. A. Hiskens. *Energy Functions, Transient Stability and Voltage Behaviour in Power Systems*. PhD thesis, Department of Electrical Engineering and Computer Science, University of Newcastle, 1990.

[15] I. A. Hiskens and D. J. Hill. Energy functions, transient stability and voltage behaviour in power systems with nonlinear loads, *IEEE Transaction on Power Systems* **PWRS-4(4)** (1989), 1525–1533.

[16] I. A. Hiskens and D. J. Hill. Dynamic failure modes in the collapse of power systems. In *Proceedings of IEEE PES Winter Meeting*, 1993.

[17] H. G. Kwatny, A. K. Pasrija, and L. Y. Bahar. Static bifurcations in electric power networks: Loss of steady-state stability and voltage collapse, *IEEE Transactions on Circuits and Systems* **CAS-33(10)** (1986), 981–991.

[18] N. Narasimhamurthi and M. T. Musavi. A generalized energy function for transient stability analysis of power systems, *IEEE Transactions Circuits and Systems* **CAS-31** (1984), 637–645.

[19] N. Rouche, P. Habets, and M. Laloy. *Stability Theory by Lyapunov's Direct Method*, volume 22 of *Applied Mathematical Sciences*. New York: Springer-Verlag, 1977.

[20] P. W. Sauer and M. A. Pai. *Modelling and Simulation of Multimachine Power Dynamics*, volume 43 of *Advances in Control and Dynamic Systems*. New York: Academic Press, 1991.

[21] F. Takens. Constrained equations; a study of implicit differential equations and their discontinuous solutions. In *Structural Stability, the Theory of Catastrophes and Applications in the Sciences*, volume 525 of *Lecture Notes in Mathematics*, (P. Hilton, ed.). New York, U.S.A., 1976. Springer-Verlag.

[22] N. A. Tsolas, A. Arapostathis, and P. P. Varaiya. A structure preserving energy function for power system transient stability analysis, *IEEE Transactions on Circuits Systems* **CAS-32(10)** (1985), 1041–1049.

[23] V. Venkatasubramanian, H. Schättler, and J. Zaborszky. A stability theory of differential algebraic systems such as the power system. In *Proceedings of the IEEE International Symposium on Circuits and Systems*, pages 2517–2520, San Diego, USA, 1992.

[24] V. Venkatasubramanian, H. Schättler, and J. Zaborszky. Voltage dynamics: Study of a generator with voltage control, transmission, and matched MW load, *IEEE Transactions on Automatic Control* **37(11)** (1992), 1717–1733.

[25] V. Venkatasubramanian, H. Schättler, and J. Zaborszky. A stability theory of large differential algebraic systems: A taxonomy, Technical Report SSM 9201 - Part I, Washington University, Saint Louis, Missouri 63130, 1993.

[26] J. Zaborszky. Some basic issues in voltage stability and viability. In *Proceedings of the International Workshop on Bulk Power System Voltage Phenomena: Voltage Stability and Security*, Missouri, 1988.

Department of Systems Engineering, Research School of Physical Sciences and Engineering, Australian National University, A.C.T., 0200

Department of Systems Engineering, Research School of Physical Sciences and Engineering, Australian National University, GPO Box 4, Canberra ACT 2601

Communicated by David Hill