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Error Estimates for Distributed Parameter Identification in Linear Elliptic Equations*

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Abstract

The identification problem of a functional coefficient in an elliptic equation is considered. For this purpose methods are introduced, which combine a modified equation error and the well-known output least squares methods. Estimates of the rate of convergence for the proposed approach are proved, when the equation is discretized with the finite element method. The work is concluded with some numerical results.

Key words: parameter identification, finite element method, estimates of the rate of convergence

AMS Subject Classifications: 65N30, 49N50, 35J25

1 Introduction

In this article we consider the homogeneous, elliptic boundary value problem

$$\begin{aligned} -\nabla \cdot (b(x)\,\nabla u(x)) &= f(x) \quad \text{in } \Omega\,, \\ u\big|_{\Gamma_0} = \frac{\partial u}{\partial n}\Big|_{\Gamma_1} &= 0\,, \end{aligned} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \leq 3$, with smooth boundary $\partial\Omega = \overline{\Gamma_0} \cup \overline{\Gamma_1}$, where Γ_0 and Γ_1 are open disjoint subsets of $\partial\Omega$. If $\partial\Omega = \overline{\Gamma_1}$, we include the standard compatibility and uniqueness conditions $\int_\Omega f\,dx = \int_\Omega u\,dx = 0$ to (1.1). This equation can describe many physical phenomena, for example, the flow of a water through an aquifer ([1] and articles therein). In this case, u represents pressure within an aquifer Ω, b is the transmissivity of the rock and f is a source or sink term. A direct problem related to (1.1) would consist of finding the unknown solution u, when we know both functions b and f. In this article we are interested in the

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inverse problem arising from (1.1): Having some knowledge of the solution u, recover the parameter b.

Equation (1.1) can be viewed as a first order PDE in the unknown function b :

$$-\nabla b \cdot \nabla u - b\,\Delta u = f\,. \tag{1.2}$$

Evidently, this equation becomes singular, when $\nabla u = 0$. If ∇u vanish on some open set, then (1.2) provides no information about the behavior of b on this set. This suggests that we should either assume some conditions for $\nabla u, \Delta u$, such that (1.2) can be solved uniquely as is done in the papers [2], [3]. Otherwise ∇u should appear as a weight to the final error estimate between the true solution and the calculated one. This will be the case with our estimates.

We assume that we have a distributed observation of the solution u, and we use the output least squares method to transform the identification problem of b to a minimization problem. The main idea of this work is to include an extra term to the least squares cost functional, which takes into account the underlying equation (1.1). This approach is similar to that used in [3], but avoids the use of an intermediate variable in the optimization. The same kind of formulation for the identification problem is also behind the so-called augmented Lagrangian technique, which is presented in [1] on page 264 but without any estimates of the rate of convergence. A good review of the existing methods for the parameter identification problems can be found in [1].

In practice we can usually measure observations at some points of the domain Ω, i.e., we have a discrete observation of the form $u(x_i)$, $i = 0, ..., n$. After interpolating this point data we can get a distributed observation with some interpolation and measurement errors.

This paper is organized as follows. In Section 2 we recall some approximation results and inequalities needed in the analysis of the identification problem. In Section 3 we formulate the identification problem as an optimal control problem by introducing a cost functional, which is to be minimized in the computational procedure. This is followed by the main results of this work, estimates of the rate of convergence in this identification process. We point out that in Theorems 3.2 and 3.3 we obtain, in $1d$ case, an optimal order estimate for the rate of convergence. Finally, in Section 4 we present some numerical results.

2 Notations and Preliminaries

The standard notations for Sobolev spaces and associated norms will be used. We will not include the domain Ω in the spaces and norms, since we assume it to be always fixed. We use $(\cdot, \cdot)$ to denote the L^2-inner product on Ω. We regard $C, \widetilde{C}$ as generic constants, which may vary in different contexts, but are always independent of h.

In order to define the finite element spaces let $\mathcal{T}_h, 0 < h < 1$, be a family of triangulations of $\bar{\Omega}$. If the boundary of Ω is curved, we use triangles with one edge replaced by the curved segment of the boundary ([6]). We assume that the family $\mathcal{T}_h$ is regular and quasi-uniform. For fixed integers $r \geq 1, l \geq 0$, we define a finite element space as

$$S^r_{h,l} = \left\{ v \mid v \in C^{l-1}(\Omega),\ \ v|_T \in P_r \ \forall\, T \in \mathcal{T}_h \right\}, \tag{2.1}$$

where P_r is the space of polynomials of degree less or equal to r and $C^{-1}(\Omega)$ is interpreted as $L^2(\Omega)$. By $S^{r,0}_{h,l}$ we denote the subspace of $S^r_{h,l}$ of functions, which vanish on $\Gamma_0 \subset \partial\Omega$. By the results in [4] we know that for all $v \in W^{m,p}(\Omega)$ there is (an interpolant) $v_h \in S^r_{h,l}$ such that

$$\|v - v_h\|_{k,p} \leq C\, h^{m-k} \|v\|_{m,p} \text{ for } \ 0 \leq k \leq l,\ k \leq m \leq r+1,\ 1 \leq p \leq \infty\,. \tag{2.2}$$

Also, these spaces satisfy the following inverse inequalities

$$\|v_h\|_{1,p} \leq C\, h^{-1} \|v_h\|_{0,p} \quad \forall v_h \in S^r_{h,l},\ 1 \leq p \leq \infty \tag{2.3}$$

and

$$\|v_h\|_\infty \leq C\, h^{-\frac{n}{2}} \|v_h\|_0 \quad \forall v_h \in S^r_{h,l}\,. \tag{2.4}$$

We will use constantly the following trigonometric inequality: Let $a, b \in \mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$. Then, for $\alpha \in (0,1)$ it holds

$$a\, b \leq \frac{1}{4\,\alpha} a^2 + \alpha b^2\,. \tag{2.5}$$

By using (2.5) with $\alpha = \frac{1}{2}$ it is easy to prove that for $v_1, ..., v_m \in X$ we have an estimate

$$\|v_1 + ... + v_m\|^2_X \leq m\,(\|v_1\|^2_X + ... + \|v_m\|^2_X)\,. \tag{2.6}$$

Let $\widetilde{H}^1 = \widetilde{H}^1(\Omega) \subset H^1(\Omega)$ be a subspace. We denote by $\widetilde{H}^{-1} = \widetilde{H}^{-1}(\Omega)$ the dual space $\left(\widetilde{H}^1(\Omega)\right)^*$ equipped with the natural norm

$$\|v\|_{-1} = \sup_{\psi \in \widetilde{H}^1(\Omega)} \frac{|(v,\psi)|}{\|\psi\|_1}\,. \tag{2.7}$$

A direct consequence of this definition for $v \in \widetilde{H}^{-1}(\Omega)$ and $\psi \in \widetilde{H}^1(\Omega)$ is an inequality

$$|(v,\psi)| \leq \|v\|_{-1} \|\psi\|_1\,. \tag{2.8}$$

3 The Identification Problem and Error Estimates

In this section we formulate our method for the identification of the unknown coefficient in (1.1). This is followed by estimates of the rate of convergence for the proposed method. Let $z(x) \in H^{r+2}$ be a distributed L^2-observation of the state u with an observation error (i.e., how close this function is to the actual solution of (1.1)) of the form

$$\|u - z\|_0 \leq \varepsilon \,. \tag{3.1}$$

Recall that the weak formulation of equation (1.1) reads as

$$(b\,\nabla u, \nabla v) = (f, v) \quad \forall v \in \widetilde{H}^1 \,, \tag{3.2}$$

where

$$\widetilde{H}^1 = \left\{ v \in H^1 \mid \, v|_{\Gamma_0} = 0 \right\} \,. \tag{3.3}$$

The discretization of (3.2) with finite element Galerkin method is then defined as:

$$\text{find } u_h \in U_h \text{ s.t. } (b\,\nabla u_h, \nabla v_h) = (f, v_h) \quad \forall v_h \in U_h \,. \tag{3.4}$$

Here $U_h \subset \widetilde{H}^1$ is a suitable discrete space.

Let us now introduce those finite dimensional spaces, which are needed to define the computational procedure. We need altogether four different discretization spaces: U_h for the solution u, B_h for the parameter b, Z_h for the observation z and finally F_h, which will be used to discretize the right-hand side f. Following the definitions of Section 2 we assume that these spaces are:

$$\begin{aligned} U_h &= S_{h,2}^{r+1,0} \,, \\ B_h &= S_{h,1}^{r} \,, \\ Z_h &= S_{h,0}^{r+1,0} \,, \\ F_h &= S_{h,0}^{r-1} \,. \end{aligned} \tag{3.5}$$

We see that all spaces in (3.5) correspond to the same triangulation of the domain Ω. This is just to simplify the things to come. It is not at all necessary or obligatory to have same grids (i.e. same h) for all spaces.

In the computations we try to find a minimizer for a cost functional

$$\begin{aligned} J(b_h) &= \int_\Omega |u_h(b_h) - z_h|^2 \, dx + h^4 \int_\Omega |\nabla \cdot (b_h \, \nabla u_h(b_h)) + f_h|^2 \, dx \\ &= \|u_h(b_h) - z_h\|_0^2 + h^4 \, \|\nabla \cdot (b_h \, \nabla u_h(b_h)) + f_h\|_0^2 \,. \end{aligned} \tag{3.6}$$

Here $u_h(b_h)$ is the solution of equation (3.4), which corresponds to a given parameter $b_h \in B_h$, z_h is the interpolant of z in Z_h and f_h the interpolant of f in F_h. We notice that (3.6) can be computed exactly by applying a suitable quadrature formula, because all functions there are piecewise polynomials and the approximation of $\partial\Omega$ is assumed to be exact. The weight h^4 in front of the second term is for balancing the different amount of differentiation in two terms.

As we can see, the cost functional (3.6) consists of two parts: The first part represents the usual least squares formulation with L^2-observation, while the second part takes into account the actual state equation (1.1). Thus, the second term is of the same form as in the so-called equation error method, which is introduced in [1] on page 253. The difference between the method in [1] and our formulation is, that in [1] they substitute the observation z directly into the operator $\nabla \cdot (b_h \nabla z)$. The disadvantage of such an approach is the requirement for differentiation of error corrupted data z. The method of this paper avoids this potential difficulty.

Now we are ready to define the actual identification problem:

$$\text{find } b_h \in M_h : \; J(b_h) \leq J(\tilde{b}_h) \quad \forall \, \tilde{b}_h \in M_h \,, \tag{3.7}$$

where

$$M = \{ b \mid 0 < \lambda_1 \leq b \leq \lambda_2 < \infty \text{ a.e. in } \Omega \,, \|\nabla b\|_0 \leq \mu < \infty \} \tag{3.8}$$

is the set for admissible parameters, $\lambda_1, \lambda_2, \mu \in \mathbb{R}$ are given constants and $M_h = M \cap B_h$.

Let b_h be the minimizer of (3.7) and let $w_h = u_h(b_h)$ be the solution of (3.4), which is calculated with this parameter. Concerning the smoothness of the functions in (1.1), we assume that $u(x) \in \widetilde{H}^1 \cap H^{r+2} \cap W^{2,\infty}$, $b(x) \in H^{r+1} \cap W^{1,\infty}$ and $f(x) \in H^r$, where $r \geq \frac{n}{2}$.

Between the true solution u of (1.1) and the discrete solution u_h, which is calculated from (3.4) with the true parameter b, we have by (2.2) and the regularity of our functions a standard error estimate

$$\|u - u_h\|_0 + h \, \|u - u_h\|_1 \leq C \, h^{r+2} \|u\|_{r+2} \,. \tag{3.9}$$

This can be found in [4], and in the case of a curved boundary $\partial\Omega$ it can be shown as in [6], when the discretization points on the boundary are appropriately chosen and the homogeneous Dirichlet condition is realized in the discretization points on $\Gamma_0 \subset \partial\Omega$.

Lemma 3.1 *Let θ_h be the L^2-projection of b into B_h and $u_h(\theta_h)$ the solution of equation* (3.4)*, which corresponds to this parameter. Moreover, assume that the true parameter satisfies*

$$\begin{aligned} \lambda_1 + \delta \quad &< \quad b(x) < \lambda_2 - \delta \quad \forall x \in \Omega \,, \\ \|\nabla b\|_0 \quad &\leq \quad \mu - \delta \end{aligned} \tag{3.10}$$

for some $\delta > 0$. Then, for h small enough $\theta_h \in M_h$, and we have

$$\sum_{k=0}^{2} h^k \, \|u_h(\theta_h) - u\|_k \le C \, h^{r+2} \, .$$

Proof: From (2.2) and the regularity of b we know that

$$\|b - \theta_h\|_k \le C \, h^{r+1-k} \, \|b\|_{r+1} \text{ for } 0 \le k \le 1 \, . \tag{3.11}$$

Moreover, from (2.2), (2.3), (2.4) and [5] we get, when ϕ_h is the interpolant of b in B_h and $r \ge \frac{n}{2}$, that

$$\begin{aligned} \|b - \theta_h\|_{1,\infty} &\le \|b - \phi_h\|_{1,\infty} + C \, h^{-(\frac{n}{2}+1)} \|\phi_h - \theta_h\|_0 \\ &\le C \, \|b\|_{1,\infty} + C \, h^{-(\frac{n}{2}+1)} (\|\phi_h - b\|_0 + \|b - \theta_h\|_0) \\ &\le C \, (\|b\|_{1,\infty} + \|b\|_{r+1}) \le \widetilde{C} \, , \\ \|b - \theta_h\|_{\infty} &\le \|b - \phi_h\|_{\infty} + C \, h^{-\frac{n}{2}} (\|\phi_h - b\|_0 + \|b - \theta_h\|_0) \\ &\le C \, h \, (\|b\|_{1,\infty} + \|b\|_{r+1}) \le C \, h \, . \end{aligned} \tag{3.12}$$

Hence, we can assume that $\|\theta_h\|_{1,\infty}$ is uniformly bounded and that there exists h_0, such that for all $h < h_0$, θ_h satisfies the bounds

$$\begin{aligned} &\lambda_1 \le \theta_h \le \lambda_2 \, , \\ &\|\nabla \theta_h\|_0 \le \mu \end{aligned} \tag{3.13}$$

as a consequence of (3.10) - (3.12). This implies that $\theta_h \in M_h$ for h small enough.

It follows from (3.4) that $u_h(\theta_h)$ is the solution of

$$(\theta_h \, \nabla u_h(\theta_h), \nabla v_h) = (f, v_h) \quad \forall v_h \in U_h \, . \tag{3.14}$$

A combination of (3.4) and (3.14) leads to a formula

$$\begin{aligned} (\theta_h \, \nabla (u_h(\theta_h) - u_h), \nabla v_h) &= (f, v_h) - (\theta_h \, \nabla u_h, \nabla v_h) \\ &= (b \, \nabla u_h, \nabla v_h) - (\theta_h \, \nabla u_h, \nabla v_h) \\ &= ((b - \theta_h) \, \nabla u_h, \nabla v_h) \, . \end{aligned} \tag{3.15}$$

By choosing $v_h = u_h(\theta_h) - u_h$ in (3.15), using (3.13) and (3.11) we deduce that

$$\begin{aligned} \lambda_1 \, \|\nabla (u_h(\theta_h) - u_h)\|_0^2 &\le |(\theta_h \, \nabla (u_h(\theta_h) - u_h), \nabla (u_h(\theta_h) - u_h))| \\ &\le C \|u_h\|_{1,\infty} \, \|b - \theta_h\|_0 \, \|\nabla (u_h(\theta_h) - u_h)\|_0 \\ &\le C \, h^{r+1} \, \|\nabla (u_h(\theta_h) - u_h)\|_0 \, . \end{aligned} \tag{3.16}$$

The boundedness of $\|u_h\|_{1,\infty}$ can be proved with the technique used in (3.12). Hence, (3.16) and the triangle inequality together with (3.9) proves the H^1-estimate.

Next we prove the L^2-estimate by using duality. Let us first define a function ψ as a solution of

$$\begin{aligned} -\nabla\cdot(b\,\nabla\psi) &= u_h(\theta_h)-u\,, \\ \psi|_{,\,0} = \frac{\partial\psi}{\partial n}\bigg|_{,\,1} &= 0\,. \end{aligned} \tag{3.17}$$

By the standard regularity results

$$\|\psi\|_2 \le C\,\|u_h(\theta_h)-u\|_0 \le \tilde{C}\,. \tag{3.18}$$

By integration by parts we have, in view of the boundary conditions, when ψ_h is the interpolant of ψ in U_h :

$$\begin{aligned} \|u_h(\theta_h)-u\|_0^2 &= (b\,\nabla\psi, \nabla(u_h(\theta_h)-u)) \\ &= ((b-\theta_h)\,\nabla\psi, \nabla(u_h(\theta_h)-u)) \\ &\quad +(\theta_h\,\nabla(\psi-\psi_h), \nabla(u_h(\theta_h)-u)) \\ &\quad +(\theta_h\,\nabla\psi_h, \nabla(u_h(\theta_h)-u))\,. \end{aligned} \tag{3.19}$$

Next we manipulate the last term in (3.19) in the same way as in (3.15):

$$\begin{aligned} (\theta_h\,\nabla\psi_h, \nabla(u_h(\theta_h)-u)) &= (f,\psi_h)-(\theta_h\,\nabla u, \nabla\psi_h) \\ &= (b\,\nabla u, \nabla\psi_h)-(\theta_h\,\nabla u, \nabla\psi_h) \\ &= ((b-\theta_h)\,\nabla u, \nabla\psi_h) \\ &= ((b-\theta_h)\,\nabla u, \nabla(\psi_h-\psi)) \\ &\quad +(b-\theta_h, \nabla u\cdot\nabla\psi)\,. \end{aligned} \tag{3.20}$$

By the definition of L^2-projection we have

$$(b-\theta_h,\phi_h)=0, \quad \forall\phi_h\in B_h\,. \tag{3.21}$$

Thus, by taking ϕ_h as the interpolant of $\nabla u\cdot\nabla\psi$ in B_h, we obtain by the regularity of u,ψ and the results (3.11), (2.2) for the last term in (3.20)

$$\begin{aligned} (b-\theta_h, \nabla u\cdot\nabla\psi) &= (b-\theta_h, \nabla u\cdot\nabla\psi-\phi_h) \\ &\le C\,\|b-\theta_h\|_0\,h\,\|\nabla u\cdot\nabla\psi\|_1 \\ &\le C\,h^{r+2}\,\|u\|_{2,\infty}\,\|\psi\|_2 \\ &\le C\,h^{r+2}\,\|\psi\|_2\,. \end{aligned} \tag{3.22}$$

Finally, a combination of (3.12), (3.18) - (3.22) and (2.2) gives

$$
\begin{aligned}
\|u_h(\theta_h) - u\|_0^2 &\leq C\,(\|b-\theta_h\|_\infty\,\|\nabla(u_h(\theta_h)-u)\|_0\,\|\nabla\psi\|_0 \\
&\quad +\|\theta_h\|_\infty\,\|\nabla(\psi-\psi_h)\|_0\,\|\nabla(u_h(\theta_h)-u)\|_0 \\
&\quad +\|\nabla u\|_\infty\,\|b-\theta_h\|_0\,\|\nabla(\psi_h-\psi)\|_0 + h^{r+2}\,\|\psi\|_2) \\
&\leq C\,h^{r+2}\,\|\psi\|_2 \\
&\leq C\,h^{r+2}\,\|u_h(\theta_h)-u\|_0\,.
\end{aligned}
\tag{3.23}
$$

This proves the L^2-estimate.

To this end, let χ be the interpolant of u in U_h. By using the inverse inequality (2.3) and the estimate (2.2) we deduce

$$
\begin{aligned}
\|u_h(\theta_h) - u\|_2 &\leq C\,h^{-1}\,\|u_h(\theta_h)-\chi\|_1 + \|u-\chi\|_2 \\
&\leq C\,h^{-1}\,(\|u_h(\theta_h)-u\|_1 + \|u-\chi\|_1) + C\,h^r \\
&\leq C\,h^r\,.
\end{aligned}
\tag{3.24}
$$

This completes the proof.

Lemma 3.2 *Between the solution* u *of* (1.1) *and the solution* $w_h = u_h(b_h)$, *which corresponds to the minimizer* b_h *of* (3.7) *we have, for* h *small enough, estimates*

$$
\begin{aligned}
\|w_h - u\|_0 &\leq C\,(h^{r+2}+\varepsilon)\,, \\
\|\nabla\cdot(b_h\,\nabla w_h) - \nabla\cdot(b\,\nabla u)\|_0 &\leq C\,(h^r + h^{-2}\,\varepsilon)\,.
\end{aligned}
$$

Proof: Because b_h is the minimizer of (3.7) and because for h small enough also $\theta_h \in M_h$, we have $J(b_h) \leq J(\theta_h)$. By (3.6) this means

$$
\begin{aligned}
&\|w_h - z_h\|_0^2 + h^4\,\|\nabla\cdot(b_h\,\nabla w_h) + f_h\|_0^2 \\
&\leq \|u_h(\theta_h) - z_h\|_0^2 + h^4\,\|\nabla\cdot(\theta_h\,\nabla u_h(\theta_h)) + f_h\|_0^2 \\
&= I_1 + h^4\,I_2\,,
\end{aligned}
\tag{3.25}
$$

where we have denoted $I_1 = \|u_h(\theta_h) - z_h\|_0^2$ and $I_2 = \|\nabla\cdot(\theta_h\,\nabla u_h(\theta_h)) + f_h\|_0^2$. For I_1 we have, using (2.6), Lemma 3.1, (3.1) and the regularity of z together with (2.2):

$$
\begin{aligned}
I_1 &\leq 3\,(\|u_h(\theta_h)-u\|_0^2 + \|u-z\|_0^2 + \|z-z_h\|_0^2) \\
&\leq C\,(h^{2(r+2)} + \varepsilon^2)\,.
\end{aligned}
\tag{3.26}
$$

Similarly, adding and subtracting suitable terms to I_2, we get using (2.6), (1.1) and the regularity of f, θ_h and u:

$$\begin{aligned}
I_2 &\leq 2\,(\|\nabla\cdot(\theta_h\,\nabla u_h(\theta_h)) + f\|_0^2 + \|f_h - f\|_0^2) \\
&\leq 2\,\|\nabla\cdot(\theta_h\,\nabla u_h(\theta_h)) - \nabla\cdot(b\,\nabla u)\|_0^2 + Ch^{2r} \\
&\leq 4\,(\|\nabla\cdot(\theta_h\,\nabla(u_h(\theta_h) - u))\|_0^2 + \|\nabla\cdot((\theta_h - b)\,\nabla u)\|_0^2) + Ch^{2r} \\
&\leq C\,(\|u_h(\theta_h) - u\|_2^2 + \|\theta_h - b\|_1^2 + h^{2r})\,.
\end{aligned} \tag{3.27}$$

Lemma 3.1 bounds the first term and (3.11) the second term in (3.27) with $O(h^{2r})$. This proves that I_2 is also of order $O(h^{2r})$.

A combination of (3.25) - (3.27) gives us the following estimates

$$\begin{aligned}
\|w_h - z_h\|_0 &\leq C\,(h^{r+2} + \varepsilon)\,, \\
\|\nabla\cdot(b_h\,\nabla w_h) + f_h\|_0 &\leq C\,(h^r + h^{-2}\,\varepsilon)\,.
\end{aligned} \tag{3.28}$$

From the first estimate we get, using once again triangle inequality

$$\begin{aligned}
\|w_h - u\|_0 &\leq \|w_h - z_h\|_0 + \|z_h - z\|_0 + \|z - u\|_0 \\
&\leq C\,(h^{r+2} + \varepsilon)\,.
\end{aligned} \tag{3.29}$$

Similarly, it follows from (3.28) that

$$\begin{aligned}
&\|\nabla\cdot(b_h\,\nabla w_h) - \nabla\cdot(b\,\nabla u)\|_0 \\
&= \|\nabla\cdot(b_h\,\nabla w_h) + f\|_0 \\
&\leq C\,(\|\nabla\cdot(b_h\,\nabla w_h) + f_h\|_0 + \|f - f_h\|_0) \\
&\leq C\,(h^r + h^{-2}\,\varepsilon)\,,
\end{aligned} \tag{3.30}$$

which ends the proof.

Theorem 3.1 *For h small enough the calculated parameter b_h and the original parameter b satisfy an error estimate*

$$\int_\Omega |b - b_h||\nabla u|^2\,dx \leq C\left(h^r + h^{-2}\,\varepsilon\right)\,.$$

Proof: The following equation between b, u and b_h, w_h is valid in $L^2(\Omega)$

$$\begin{aligned}
-\nabla\cdot((b - b_h)\,\nabla u) &= -\nabla\cdot(b\,\nabla u) + \nabla\cdot(b_h\,\nabla w_h) \\
&\quad -\nabla\cdot(b_h\,\nabla(w_h - u))\,.
\end{aligned} \tag{3.31}$$

Now we proceed with the technique introduced in [1] on page 243. Let us first define two disjoint subsets of Ω, such that $R_1 = \{x \in \Omega : b(x) - b_h(x) \geq 0\}$ and $R_2 = \Omega - R_1$. Let us also define a function $\psi \in L^\infty(\Omega)$ by taking $\psi = 1$ in R_1 and $\psi = -1$ in R_2. Now, by taking the L^2-inner product of (3.31) with $\psi u \in L^\infty$ we get

$$\begin{aligned}
-(\nabla\cdot(|b - b_h|\,\nabla u), u) &= (\nabla\cdot(b_h\,\nabla w_h) - \nabla\cdot(b\,\nabla u), \psi u) \\
&\quad -(\nabla\cdot(b_h\,\nabla(w_h - u), \psi u)\,.
\end{aligned} \tag{3.32}$$

Since b and b_h are both bounded in H^1, this implies that $x \to |b - b_h|$ is an element of H^1, and therefore $|b - b_h|\, \nabla u \in H^1$ as a consequence of $u \in W^{2,\infty}$. So, by integration by parts on the left-hand side we find from (3.32)

$$\int_\Omega |b-b_h|\, |\nabla u|^2\, dx \le C\, (\|\nabla\cdot(b_h\, \nabla w_h) - \nabla\cdot(b\, \nabla u)\|_0 + \|w_h - u\|_2)\,. \tag{3.33}$$

This is a consequence of a calculation

$$\begin{aligned}(\nabla\cdot(b_h\, \nabla(w_h-u), \psi u) &= (b_h\, \Delta(w_h-u), \psi u) + (\nabla b_h \cdot \nabla(w_h-u), \psi u) \\ &\le C\, \|\psi u\|_\infty\, (\|b_h\|_0\, \|\Delta(w_h-u)\|_0 \\ &\quad + \|\nabla b_h\|_0\, \|\nabla(w_h-u)\|_0) \\ &\le C(\lambda_2, \mu)\, \|w_h - u\|_2\,,\end{aligned} \tag{3.34}$$

which is true because $b_h \in M_h$. From Lemma 3.2 we know that the first term in (3.33) satisfies the result of the theorem. For the second term we get from (2.2) and Lemma 3.2, by using the inverse inequality (2.3)

$$\begin{aligned}\|w_h - u\|_2 &\le C\, (h^{-2}\, \|w_h - \chi\|_0 + \|\chi - u\|_2) \\ &\le C\, \left(h^{-2}\, (\|w_h - u\|_0 + \|u - \chi\|_0) + h^r\right) \\ &\le C\, \left(h^{-2}\, (h^{r+2} + \varepsilon) + h^r\right) \\ &\le C\, \left(h^r + h^{-2}\, \varepsilon\right),\end{aligned} \tag{3.35}$$

where χ is now the interpolant of u in U_h. A combination of (3.33) and (3.35) proves the result.

Next we will introduce better estimates for the case $n = 1$. In this case the domain Ω reduces to an interval $I = (a, b)$. We assume that at least on one end of the interval we have a Neumann condition $u'(a) = 0$ or $u'(b) = 0$. We change the cost functional (3.6) to

$$\widetilde{J}(b_h) = \|u_h(b_h) - z_h\|_0^2 + h^2\, \|(b_h\, u_h'(b_h))' + f\|_{-1}^2\,, \tag{3.36}$$

where $'$ denotes the differentiation with respect to x-variable and the second norm is realized in the dual space $\widetilde{H}^{-1}$ of the test function space $\widetilde{H}^1$.

Theorem 3.2 *For $n = 1$ we have, for h small enough, an error estimate*

$$\|(b - b_h)\, u'\|_0 \le C\, (h^{r+1} + h^{-1}\, \varepsilon)\,,$$

which holds, if $b_h \in M_h$ is the minimizer of (3.36).

Proof: A weak form of equation (3.31) reads as

$$((b-b_h)\, u', v') = ((b_h\, w_h')' - (b\, u')', v) + (b_h\, (w_h - u)', v') \quad \forall v \in \widetilde{H}^1\,. \tag{3.37}$$

Because $\widetilde{H}^1$ is now either the whole space H^1 or its subspace of the form

$$\widetilde{H}^1 = \left\{ v \in H^1 \mid v(a) = 0 \text{ or } v(b) = 0 \right\} , \tag{3.38}$$

we can define a test function $v \in \widetilde{H}^1$ as a solution of a boundary value problem

$$\begin{aligned} &v'(x) = [(b - b_h)\, u'](x), \quad x \in I \,, \\ &v(a) = 0 \text{ or } v(b) = 0 \,. \end{aligned} \tag{3.39}$$

So, by using this v in (3.37), applying the Poincare inequality and inequality (2.8) we get

$$\begin{aligned} &\|(b - b_h)\, u'\|_0^2 \\ &\le C\ (\|b_h\, w_h')' - (b\, u')'\|_{-1} + \|b_h\,(w_h - u)'\|_0)\ \|v'\|_0 \\ &\le C\ (\|b_h\, w_h')' - (b\, u')'\|_{-1} + \|(w_h - u)'\|_0)\ \|(b - b_h)\, u'\|_0 \,. \end{aligned} \tag{3.40}$$

A direct calculation shows that for the dual norm $\widetilde{H}^{-1}$ we have an inequality

$$\|(a\, g')'\|_{-1} \le \|a\, g'\|_0 \,, \tag{3.41}$$

when g satisfies the boundary conditions in (1.1). Hence, as in (3.25) - (3.27) we get, for the cost functional (3.36), when using the inequality (3.41)

$$\begin{aligned} &\|w_h - z_h\|_0^2 + h^2\, \|(b_h\, w_h')' - (b\, u')'\|_{-1}^2 \\ &= \|w_h - z_h\|_0^2 + h^2\, \|(b_h\, w_h')' + f\|_{-1}^2 \\ &\le \|u_h(\theta_h) - z_h\|_0^2 + h^2\, \|(\theta_h\, u_h'(\theta_h))' + f\|_{-1}^2 \\ &\le C\, h^2\, (\|(\theta_h\, u_h'(\theta_h))' + f\|_{-1}^2 + h^{2(r+1)} + h^{-2}\, \varepsilon) \\ &\le C\, h^2\, (\|(\theta_h\, (u_h(\theta_h) - u)')'\|_{-1}^2 + \|((\theta_h - b)\, u')'\|_{-1}^2 + h^{2(r+1)} + h^{-2}\, \varepsilon) \\ &\le C\, h^2\, (\|u_h(\theta_h) - u\|_1^2 + \|\theta_h - b\|_0^2 + h^{2(r+1)} + h^{-2}\varepsilon) \\ &\le C\, h^2\, (h^{2(r+1)} + h^{-2}\varepsilon) \,. \end{aligned} \tag{3.42}$$

As in (3.28) - (3.30) this gives us the estimates

$$\begin{aligned} \|w_h - u\|_0 \quad &\le \quad C\,(h^{r+2} + \varepsilon) \,, \\ \|(b_h\, w_h')' - (b\, u')'\|_{-1} \quad &\le \quad C\,(h^{r+1} + h^{-1}\, \varepsilon) \,. \end{aligned} \tag{3.43}$$

From the inverse inequality (2.3), (3.9) and (3.43) we then deduce

$$\begin{aligned} \|(w_h - u)'\|_0 \quad &\le \quad C\,(h^{-1}\|w_h - u_h\|_0 + \|u - u_h\|_1) \\ &\le \quad C\,\left(h^{-1}(\|w_h - u\|_0 + \|u - u_h\|_0) + h^{r+1}\right) \\ &\le \quad C\,(h^{r+1} + h^{-1}\varepsilon) \,. \end{aligned} \tag{3.44}$$

A combination of (3.40) - (3.44) proves the result.

Remark 3.1 Because Theorem 3.2 needs only H^1-estimate for $u_h(\theta_h)-u$, it might be enough to take $U_h = S_{h,1}^{r+1,0}$. Notice that with this discrete space the homogeneous Neumann condition is not satisfied exactly for w_h. However, it is reasonable to expect that the convergence of w_h' to zero on the boundary is of same order as in the previous estimate.

In the next theorem we show that a similar result as in Theorem 3.2 can be proved in $1d$ also for the cost functional (3.6)

Theorem 3.3 *For $n = 1$ we have, for h small enough, an error estimate*

$$\|(b - b_h)\, u'\|_0 \leq C\,(h^{r+1} + h^{-1}\,\varepsilon)\,,$$

which holds, if b_h is the minimizer of (3.6).

Proof: Like in Theorem 3.2 we see that

$$\|(b - b_h)\, u'\|_0 \leq C\,(\|(b_h\, w_h')' - (b\, u')'\|_{-1} + \|(w_h - u)'\|_0)\,. \tag{3.45}$$

The second term in (3.45) is treated in Theorem 3.2, (3.44). From the results of Lemma 3.2 we know that for the cost functional (3.6) we have an estimate

$$\|(b_h\, w_h')' - (b\, u')'\|_0 \leq C\,(h^r + h^{-2}\,\varepsilon)\,. \tag{3.46}$$

Moreover, we know that b_h, w_h satisfy an equation

$$(b_h\, w_h', v_h') = (f, v_h) \quad \forall v_h \in U_h\,. \tag{3.47}$$

Because $b_h \in B_h \subset H^1$ and $w_h \in U_h \subset H^2$, we have $b_h\, w_h' \in H^1$ by the standard regularity results in $1d$. Therefore, we can integrate by parts in (3.47), which becomes to the form

$$-\,((b_h\, w_h')', v_h) = (f, v_h) \quad \forall v_h \in U_h\,. \tag{3.48}$$

Hence, this combined with (1.1) gives

$$((b_h\, w_h')' - (b\, u')', v_h) = 0 \quad \forall v_h \in U_h\,. \tag{3.49}$$

From the definition of $\widetilde{H}^{-1}$-norm we then deduce by using (3.49), (3.46) and taking ϕ_h as the interpolant of ϕ in U_h

$$\begin{aligned}
\|(b_h\, w_h')' - (b\, u')'\|_{-1} &= \sup_{\phi \in \widetilde{H}^1} \frac{|((b_h\, w_h')' - (b\, u')', \phi)|}{\|\phi\|_1} \\
&= \sup_{\phi \in \widetilde{H}^1} \frac{|((b_h\, w_h')' - (b\, u')', \phi - \phi_h)|}{\|\phi\|_1} \\
&\leq C\, h \|(b_h\, w_h')' - (b\, u')'\|_0 \\
&\leq C\,(h^{r+1} + h^{-1}\,\varepsilon)\,.
\end{aligned} \tag{3.50}$$

This proves the result.

Remark 3.2 From the proof of Theorem 3.2 we see that this kind of technique (i.e., the way of choosing the test function v in (3.39)) can not be applied, when the equation (1.1) is given with the homogeneous Dirichlet boundary conditions. However, we believe that the estimates in Theorems 3.2 and 3.3 are valid, for $n = 1$, with all kind of combinations of homogeneous boundary conditions. This general statement is verified in the next section, where we have computed some numerical examples with the Dirichlet conditions as well.

Corollary 3.1 *If the observation z is assumed to approximate u in H_0^1 instead of L^2 with an observation error*

$$\|\nabla(u - z)\|_0 \leq \varepsilon \, . \tag{3.51}$$

Then for a cost functional

$$J(b_h) = \|\nabla(u_h(b_h) - z_h)\|_0^2 + h^2 \, \|\nabla \cdot (b_h \, \nabla u_h(b_h)) + f_h\|_0^2 \tag{3.52}$$

an error estimate

$$\int_\Omega |b - b_h| |\nabla u|^2 \, dx \leq C \left(h^r + h^{-1} \, \varepsilon\right) \tag{3.53}$$

is valid for $n = 2, 3$ and estimate

$$\|(b - b_h) \, u'\|_0 \leq C \, (h^{r+1} + \varepsilon) \tag{3.54}$$

is true for $n = 1$ with the assumptions previously made.

For $n = 1$ a minimization of

$$\widetilde{J}(b_h) = \|(u_h(b_h) - z_h)'\|_0^2 + \|(b_h \, u_h'(b_h))' + f\|_{-1}^2 \tag{3.55}$$

leads to an estimate

$$\|(b - b_h) \, u'\|_0 \leq C \, (h^{r+1} + \varepsilon) \tag{3.56}$$

with same remarks as before.

Remark 3.3 The estimates of Theorems 3.1, 3.2, 3.3 and Corollary 3.1 are also valid for an equation

$$\begin{array}{rcl} -\nabla \cdot (b(x) \, \nabla u(x)) + a(x) \, u(x) & = & f(x) \quad \text{in } \Omega \, , \\ u\big|_{, \, 0} = \frac{\partial u}{\partial n}\big|_{, \, 1} & = & 0 \, , \end{array} \tag{3.57}$$

when $a(x)$ is a given, nonnegative function in L^∞.

4 Numerical Examples

Now we introduce some numerical experiments, which have been made with the methods of Section 3 for $n = 1, 2$. We restrict ourselves to the standard domain $\Omega = [0, 1]$ or $\Omega = [0, 1] \times [0, 1]$.

From the previous section we know that we should guarantee the boundedness of ∇b_h in L^2 with respect to a given constant μ. In order to realize this nonlinear constraint we use an external penalty formulation. This means that in the actual computations we minimize functionals

$$J(b_h) = \|u_h(b_h) - z_h\|_0^2 + h^4 \, \|\nabla \cdot (b_h \, \nabla u_h(b_h)) + f_h\|_0^2 + \frac{1}{\nu} \, \max\{0, \|\nabla b_h\|_0 - \mu\}^2 \tag{4.1}$$

for $n = 1, 2$, and

$$\widetilde{J}(b_h) = \|u_h(b_h) - z_h\|_0^2 + h^2 \, \|(b_h \, u_h'(b_h))' + f\|_{-1}^2 + \frac{1}{\nu} \, \max\{0, \|b_h'\|_0 - \mu\}^2 \tag{4.2}$$

for $n = 1$ with a suitable chosen penalty parameter ν.

In the cost functional (4.2) we need $\widetilde{H}^{-1}$-norm of the term $(b_h \, u_h'(b_h))' + f$. This can be obtained by first calculating function ψ as the solution of

$$\begin{aligned} -\psi'' &= (b_h \, u_h'(b_h))' + f \,, \\ \psi|_{\Gamma_0} &= \left.\frac{\partial \psi}{\partial n}\right|_{\Gamma_1} = 0 \,, \end{aligned} \tag{4.3}$$

(with the condition $\int_\Omega \psi \, dx = 0$, if $\partial\Omega = \Gamma_1$). Then, a simple calculation shows that $\|\psi'\|_0$ is completely equivalent with the desired H^{-1}-norm. A finite element analogue of equation (4.3) is used in the computations.

Example 4.1 *First we compute a one dimensional example with mixed boundary conditions* $u(0) = u'(1) = 0$. *We have* $u(x) = \sin(\pi x)^2$ *and* $b(x) = \exp(-x) + 1$. *Cost functional* (4.2) *is minimized with the E04UCF-routine from NAG-library with double precision. As initial value we set* $b_h(x) = 3$, *and we take* $\mu = 25$, $\nu = 0.01$. *We do not have any observation error, i.e.,* $z = u$. *We use second order Lagrange basis for discrete functions* u_h, z_h *and piecewise linear approximation for* b_h. *Notice that the result of Theorem* 3.2 *predicts* $O(h^2)$ *convergence between* b *and* b_h, *when taking into account Remark* 3.1.

h	L^2-error	L^2-err / h^2
1/6	$3.58\cdot10^{-1}$	12.888
1/9	$1.83\cdot10^{-1}$	14.841
1/12	$1.08\cdot10^{-1}$	15.527
1/15	$1.66\cdot10^{-2}$	3.727
1/18	$8.45\cdot10^{-3}$	2.737
1/21	$4.56\cdot10^{-3}$	2.011
1/24	$2.74\cdot10^{-3}$	1.577
1/27	$1.79\cdot10^{-3}$	1.305
1/30	$1.23\cdot10^{-3}$	1.106
1/33	$1.04\cdot10^{-3}$	1.131

Table 4.1: Weighted L^2-error between b and b_h in Example 4.1 with different values of h. In the last column L^2-error divided by h^2.

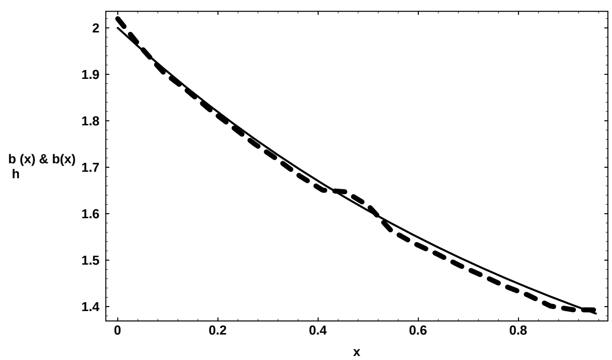

Figure 4.1: True and computed parameter in Example 4.1 with $h = \frac{1}{21}$.

Remark 4.1 *Figure* 4.1 *shows that the maximum error between* b *and* b_h *lies near the points* $0, \frac{1}{2}, 1$ *as expected, because* $u'(0) = u'(\frac{1}{2}) = u'(1) = 0$.

Example 4.2 *Same as Example* 4.1*, but this time we include also an observation error to the computations. We assume that* $z = u + c\sin(4\pi x)$*, where the constant* c *defines the distributed error* ε *between* u *and* z*. Here we take* $\varepsilon = \frac{1}{100}$*. In this example we expect, due to Theorem* 3.2*, an error of the form* $O(h^2 + \frac{\varepsilon}{h})$.

h	L^2-error	L^2-er$/(h^2+\frac{\varepsilon}{h})$
1/6	$4.24 \cdot 10^{-1}$	4.834
1/9	$4.31 \cdot 10^{-1}$	4.214
1/12	$2.83 \cdot 10^{-1}$	2.226
1/15	$2.38 \cdot 10^{-1}$	1.539
1/18	$2.12 \cdot 10^{-1}$	1.158
1/21	$2.16 \cdot 10^{-1}$	1.017
1/24	$2.12 \cdot 10^{-1}$	0.878
1/27	$2.12 \cdot 10^{-1}$	0.780
1/30	$2.12 \cdot 10^{-1}$	0.704
1/33	$2.12 \cdot 10^{-1}$	0.639

Table 4.2: Weighted L^2-error between b and b_h in Example 4.2 with different values of h. In the last column L^2-error divided by $h^2 + \frac{\varepsilon}{h}$.

Example 4.3 *In this example we try to verify the result of Theorem* 3.3. *We take* $u(x) = \sin(\pi x)^2$, $b(x) = \cos(2\pi x) + 2$ *and minimize cost functional* (4.1) *with initial condition* $b_h = 4$ *without an observation error by using same values for* μ, ν *as in the previous examples. The discrete space* U_h *consists of third order Hermite polynomials,* Z_h *is constructed with second order and* B_h *and* F_h *with first order Lagrange polynomials, respectively.*

h	L^2-error	L^2-err $/h^2$
1/9	$2.03 \cdot 10^{-1}$	16.414
1/12	$9.48 \cdot 10^{-2}$	13.651
1/15	$4.85 \cdot 10^{-2}$	10.908
1/18	$2.81 \cdot 10^{-2}$	9.090
1/21	$1.72 \cdot 10^{-2}$	7.573
1/24	$1.15 \cdot 10^{-2}$	6.649
1/27	$7.98 \cdot 10^{-3}$	5.816
1/30	$3.12 \cdot 10^{-3}$	2.809
1/33	$2.60 \cdot 10^{-3}$	2.832
1/36	$1.89 \cdot 10^{-3}$	2.448
1/39	$1.87 \cdot 10^{-3}$	2.845

Table 4.3: Weighted L^2-error between b and b_h in Example 4.3.

As we mentioned in Remark 3.2, the better convergence estimates should be also valid with homogeneous Dirichlet boundary conditions. The next two examples illustrate this situation.

Example 4.4 *Otherwise the same as Example* 4.1, *but now computed with homogeneous Dirichlet boundary conditions.*

h	L^2-error	L^2-err / h^2
1/6	$3.10\cdot10^{-1}$	11.163
1/9	$1.64\cdot10^{-1}$	13.286
1/12	$9.31\cdot10^{-2}$	13.411
1/15	$1.49\cdot10^{-2}$	3.369
1/18	$7.74\cdot10^{-3}$	2.506
1/21	$4.20\cdot10^{-3}$	1.852
1/24	$2.61\cdot10^{-3}$	1.505
1/27	$1.77\cdot10^{-3}$	1.292
1/30	$1.26\cdot10^{-3}$	1.135
1/33	$9.14\cdot10^{-4}$	0.995

Table 4.4: Weighted L^2-error between b and b_h in Example 4.4.

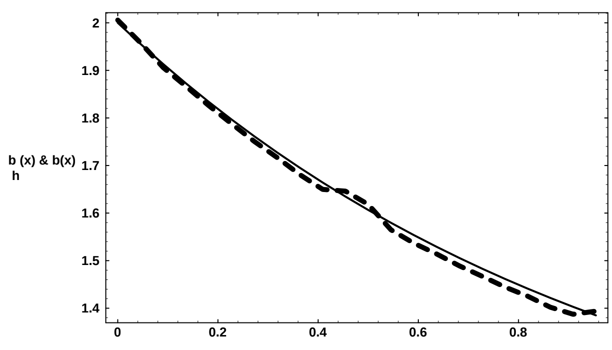

Figure 4.2: True and computed parameter in Example 4.4 with $h = \frac{1}{21}$.

Example 4.5 *Example* 4.3 *computed with homogeneous Dirichlet boundary conditions.*

h	L^2-error	L^2-err $/h^2$
1/9	$2.33 \cdot 10^{-1}$	18.904
1/12	$1.15 \cdot 10^{-1}$	16.561
1/15	$6.12 \cdot 10^{-2}$	13.774
1/18	$3.64 \cdot 10^{-2}$	11.783
1/21	$2.33 \cdot 10^{-2}$	10.290
1/24	$1.58 \cdot 10^{-2}$	9.090
1/27	$1.10 \cdot 10^{-2}$	8.134
1/30	$7.93 \cdot 10^{-3}$	7.234
1/33	$5.72 \cdot 10^{-3}$	6.381
1/36	$4.15 \cdot 10^{-3}$	5.423
1/39	$2.91 \cdot 10^{-3}$	4.423

Table 4.5: Weighted L^2-error between b and b_h in Example 4.5.

Example 4.6 *Two dimensional example using Dirichlet boundary conditions with the cost functional* (4.1). $u(x,y) = \sin(\pi x)\sin(\pi y)$ *and* $b(x,y) = x^4y^4 + 1$ *without an observation error. Initially* $b_h(x) = 4$. *The basis for functions* b_h, z_h, f_h *is taken as a tensor product of 1d second order Lagrange polynomials and third order Hermite polynomials are used for* U_h. *Due to Theorem* 3.1 *we expect* $O(h^2)$ *convergence.*

h	L^1-error	L^1-err $/h^2$
1/3	$1.53 \cdot 10^{-1}$	1.381
1/4	$1.34 \cdot 10^{-1}$	2.152
1/5	$4.49 \cdot 10^{-2}$	1.121
1/6	$6.50 \cdot 10^{-2}$	2.338
1/7	$2.13 \cdot 10^{-2}$	1.042
1/8	$1.60 \cdot 10^{-2}$	1.025

Table 4.6: Weighted L^1-error between b and b_h in Example 4.6.

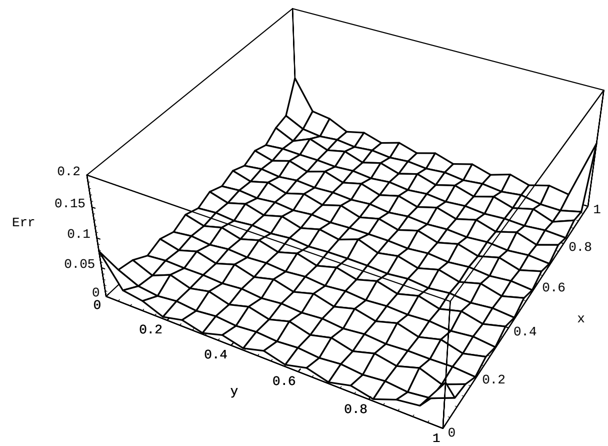

Figure 4.3: Error function in Example 4.6 with $h = \frac{1}{8}$.

Remark 4.2 *From Figure* 4.3 *it can be seen that the maximum error for the computed parameter is in the corners of the domain, where the gradient of* u *vanishes.*

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