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Spectral Approach to Well-Posedness and Stability Analysis of Hybrid Feedback Systems*

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Abstract

The modern method of analysis of the distributed parameter systems relies on the transformation of the dynamical model to an abstract differential equation on an appropriately chosen Banach or, if possible, Hilbert space. A linear dynamical model in the form of the first order abstract differential equation is considered to be well-posed if its right-hand side generates a strongly continuous semigroup. Similarly, a dynamical model in the form of the second order abstract differential equation is well-posed if its right-hand side generates a strongly continuous cosine family of operators.

Unfortunately, the presence of a feedback leads to serious complications or even excludes a direct verification of assumptions of the Hille-Phillips-Yosida and/or the Sova-Fattorini Theorems. The class of operators which are similar to a normal discrete operator on a Hilbert space describes a wide variety of linear operators. In the present paper two groups of similarity criteria for a given hybrid closed-loop system operator are given. The criteria of the first group are based on some perturbation results, and of the second, on the application of Shkalikov's theory of the Sturm-Liouville eigenproblems with a spectral parameter in the boundary conditions.

The results are applied to RLCG-transmission lines, a model of an elastic robot arm and a class of neutral systems.

Key words: Riesz bases, distributed parameter systems, stability

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1 Abstract Differential Equations

Abstract Differential Equations with Right-hand Side being a Linear Operator which is Similar to a Normal One

The mathematical models of systems involving such physical phenomena as diffusion, wave propagation as well as information and transport delays engage the partial and/or functional differential equations and integral operators. Particular examples can be found in the mathematical description of diffusion of heat, electric charges, molecules participating in chemical reactions, genetic characters, pathogenic viruses, oscillations of overhead high-voltage transmission lines, lifting ropes, antenna masts, deformations of shafts, beams and mechanical constructions, oscillations of robot elastic arms, propagation of electromagnetic waves in transmission lines, wave-guides, oscillations of quantum generators, etc. Such systems are called distributed parameter systems, as opposed to lumped parameter systems described by ordinary differential equations.

Feedback is an essential feature of many distributed parameter systems in automatic control, electronics (nonlinear oscillation generators), chemistry (reactors with recycles), mechanical engineering (stabilizers and dampers of mechanical construction) and must be taken into account in the analysis.

The modern method of analysis of the distributed parameter systems relies on the transformation of the dynamical model to an abstract differential equation on an appropriately chosen Banach or, if possible, Hilbert space.

The first order abstract differential equation has the form

$$\dot{u}(t) = Au(t), \quad u(0) = u_0 \in H \tag{1.1}$$

where H denotes a real Hilbert space with scalar product $\langle \cdot, \cdot \rangle$, $A : (D(A) \subset H) \longrightarrow H$ is an unbounded linear operator.

The family $\{T(t)\}_{t \geq 0} \subset \mathbf{L}(H)$ will be called a C_0-*semigroup* if $T(0) = I$, $T(s+t) = T(t)T(s) \;\; \forall t, s \geq 0$; $T(t)u \longrightarrow u$ as $t \longrightarrow 0+ \quad \forall u \in H$.

If additionally, the mapping $t \longrightarrow T(t)u$ is an analytic function on $(0, \infty)$ for any fixed $u \in H$, then we say that A *generates an analytic semigroup* on H. If both A and $-A$ generate C_0-semigroups then we say that A generates a C_0-*group* on H.

The following conditions are equivalent:

(i) A is a linear, closed densely defined operator and for any $\mathrm{u}_0 \in D(A)$, $T > 0$ there exists a unique classical solution

$$u \in \mathrm{C}^1([0, T], H) \cap \mathrm{C}([0, T], D_A)$$

of problem (1.1), where D_A denotes the Banach space $D(A)$ equipped with the norm $\|u\|_A = \|u\| + \|Au\|_A$;

(ii) A generates a C_0-semigroup $\{T(t)\}_{t\geq 0}$ on H.

If the above conditions are satisfied then the function $u(t) = T(t)u_0$, where $u_0 \in H$, is called a *weak solution* of (1.1).

The second order abstract differential equation has the form

$$\ddot{u}(t) = Au(t), \quad u(0) = u_0 \in H, \quad \dot{u}(0) = u_1 \in H \tag{1.2}$$

where H denotes a real Hilbert space with scalar product $\langle \cdot, \cdot \rangle$, and $A : (D(A) \subset H) \longrightarrow H$, is generally, an unbounded linear operator.

The family $\{C(t)\}_{t\in\mathbf{R}} \subset \mathbf{L}(H)$, such that $C(0) = I$, $C(s+t)+C(t-s) = = 2C(t)C(s) \ \ \forall t, s \in \mathbf{R}$ and the functions $\mathbf{R} \ni t \longmapsto C(t)u$ is continuous for any fixed $u \in H$ is called a *strongly continuous cosine family of operators* on H.

The following conditions are equivalent:

(i) A is a linear, closed densely defined operator and for any $u_0 \in D(A)$, $T > 0$ there exists a unique classical solution $u \in \mathrm{C}^2([0,T], H) \cap \mathrm{C}([0,T], D_A)$ of the problem (1.2);

(ii) A generates a strongly continuous cosine family $\{C(t)\}_{t\in\mathbf{R}}$ on H.

If the above conditions are satisfied then the function $u(t) = \int_0^t C(s)u_1 ds + C(t)u_0$, where $u_0, u_1 \in H$, is called a *weak solution* of (1.2).

The concept of semigroup is a formal extension of the definition of the exponential scalar function $\mathbf{C} \ni \lambda \longmapsto e^{t\lambda}\,(t \geq 0)$, to an argument being an unbounded linear operator A, while the strongly continuous cosine family of operators is a similar extension of the scalar entire function $\mathbf{C} \in \lambda \longmapsto \cosh(t\sqrt{\lambda})$ $(t \in \mathbf{R})$. This justifies the notation $T(t) = e^{tA}(t \geq 0)$, $C(t) = \cosh tA^{1/2}(t \in \mathbf{R})$. The fundamental results of the semigroup theory as the Hille-Phillips-Yosida theorem - see [29, Corollary 3.8,p.12] and the Sova-Fattorini [6, Theorem 5.1,p.37] theorem determine those classes of linear unbounded operators on a general Banach space for which such extensions are possible. To verify the assumptions of the above theorems one should estimate the norm $\|(\lambda I - A)^{-n}\|$ of the n-th power of the resolvent of A on appropriate subsets of $\mathbf{C}$ (observe that for the semigroup generator A, $(\lambda I - A)^{-1}u_0 = \int_0^\infty e^{-t\lambda}T(t)u_0 dt$ is the Laplace transform of a weak solution of (1.1)). This is a difficult task especially for an operator A describing a feedback system with boundary control and/or boundary observation.

Let $A : (D(A) \subset H) \longrightarrow H$ be a closed, densely defined linear operator on a Hilbert space H. $D(A^*) := \{v \in H :$ there exists (a unique) $h_v \in H$ such that $\langle Au, v\rangle = \langle u, h_v\rangle$ for all $u \in D(A)\}$ is the domain of the *adjoint operator* $A^* : (D(A^*) \subset H) \longrightarrow H$ with respect to A, defined as $A^*v := h_v$, $v \in D(A^*)$. A is called *normal* if

$$D(A) = D(A^*), \quad AA^* = A^*A \ . \tag{1.3}$$

It follows from the spectral theorem for normal operators that

(i) The resolvent of A satisfy an estimate

$$\left\|(\lambda I \Leftrightarrow A)^{-n}\right\| = [\mathrm{dist}(\lambda, \sigma(A))]^{-n} \tag{1.4}$$

where $\lambda \in \mathbf{C} \setminus \sigma(A), n \in \mathbf{N}$ and $\sigma(A)$ denotes the spectrum of A;

(ii) For any Borel, function f bounded on $\sigma(A)$, the formula

$$\langle f(A)u, v\rangle = \int_{\sigma(A)} f(\lambda) d\langle E(\lambda)u, v\rangle \quad \forall u, v \in H \tag{1.5}$$

determines an operator $f(A) \in \mathbf{L}(H)$. Here $E(\lambda)$ is the unique (by the spectral theorem) spectral resolution of identity. If, additionally, A has a compact resolvent (A is a *discrete* operator) then (1.5) takes an equivalent form

$$f(A)u = \sum_{i=1}^{\infty} f(\lambda_i)\langle u, e_i\rangle e_i \ , \tag{1.6}$$

where $\{e_i\}_{i=1}^{\infty}$ is the orthonormal system of eigenvectors of A, corresponding to the eigenvalues of A denoted by $\{\lambda_i\}_{i=1}^{\infty}$, $Ae_i = \lambda_i e_i$.

The result **(i)** requires an explanation. If $\lambda \in \mathbf{C} \setminus \sigma(A)$ then applying the result from [44, Theorem 7.34(b), p.217] we get $\left\|(\lambda I \Leftrightarrow A)^{-1}\right\| = [\mathrm{dist}(\lambda, \sigma(A))]^{-1}$. Moreover, from [44, Corollary, p.126] we know that the resolvent $(\lambda I \Leftrightarrow A)^{-1}$ is also normal. Hence $\|(\lambda I \Leftrightarrow A)^{-n}\| = \|(\lambda I \Leftrightarrow A)\|^{-n}$ $= [\mathrm{dist}(\lambda, \sigma(A))]^{-n}$ - see [44, Theorem 5.44, p.127] or [12, problem 162]. The results **(ii)** are known as the *functional calculus for normal operators.*

An operator $A : (D(A) \subset H) \Leftrightarrow\!\to H$ is *similar* to a normal operator N, if there exists an isomorphism $S \in \mathbf{L}(H)$ such that $S^{-1}AS = N$. The similarity relation does not change the spectrum of operators.

Putting: $f(\lambda) = e^{t\lambda}$ (for semigroup $t \geq 0$ and $t \in \mathbf{R}$ for group), $f(\lambda) = (\mu \Leftrightarrow \lambda)^{-1}$ (for an analytic semigroup, $\mu \in S_{b,\theta}$) and $f(\lambda) = \cosh(t\sqrt{\lambda})$ (for a strongly cosine family of operators, $t \in \mathbf{R}$), in **(ii)** we obtain, respectively statements **(a)**, **(b)**, **(c)** and **(d)** of the following theorem.

Theorem 1.1 *If A is similar to a normal operator then*

*(**a**) A generates a C_0-semigroup* $\Longleftrightarrow$ $\sup\{Re\lambda : \lambda \in \sigma(A)\} < \infty$

*(**b**) A generates a C_0-group* $\Longleftrightarrow$
$\Leftrightarrow\infty < \inf\{Re\lambda : \lambda \in \sigma(A)\}, \quad \sup\{Re\lambda : \lambda \in \sigma(A)\} < \infty$

*(**c**) A generates an analytic semigroup* $\Longleftrightarrow$
$\exists\ b \in \mathbf{R},\ \theta \in (\frac{\pi}{2}, \pi) : \quad S_{b,\theta} = \{\lambda \in \mathbf{C} : |\arg(\lambda \Leftrightarrow b)| \leq \theta, \lambda \neq b\} \subset \mathbf{C} \setminus \sigma(A)$

*(**d**) A generates a strongly cosine family of operators* $\Longleftrightarrow$
$\exists\ \omega \in \mathbf{R};\ \sigma(A) \subset \{\lambda \in \mathbf{C}:\ Re\lambda \le \omega^2 \Leftrightarrow \frac{1}{4\omega^2} Im^2\lambda\}$.

Frequently, in the analysis of finite-dimensional dynamics it is enough to consider the state matrices of simple structure (matrices with linear divisors exclusively or, equivalently, with a diagonal Jordan form). The class of such matrices is identical with the class of matrices which are similar to normal ones.

B. Nagy pointed out (see [6]) that if A generates a uniformly bounded C_0-group (i.e. there exists $M \ge 0$ such that $\|T(t)\| \le M \quad \forall t \in \mathbf{R}$) then $\imath A$ is similar to a self-adjoint operator and a similarity isomorphism S can be found in the class of self-adjoint, positive definite operators. This is a partial inverse of the claim (**b**).

Recall that a system $\{f_i\}_{i\in I}$ is a *Riesz basis* in a Hilbert space H if there exists a linear, bounded operator S mapping H onto itself and an orthonormal basis $\{e_i\}_{i\in I}$ of H such that $f_i = Se_i \quad \forall i \in I$. An operator $A : (D(A) \subset H) \Leftrightarrow\!\!\rightarrow H$ with a compact resolvent is similar to a normal operator iff A possesses a system of eigenvectors forming a Riesz basis of H. This follows immediately from the fact that an operator with a compact resolvent is normal iff it possesses a system of eigenvectors forming an orthonormal basis of H. For the proof of necessity see [18, pp.260-263 and pp.276-277], [38, pp.250-255] or, less explicitly [44, Theorem 7.2, p.167]. Sufficiency can be deduced from [44, Theorem 7.2, p.167].

Remark 1.1 There are operators which are not similar to normal ones but still satisfy an estimate analogous to (1.4). This is the case for hyponormal operators (a densely defined operator $A :\ (D(A) \subset H) \Leftrightarrow\!\!\rightarrow H$ is called *hyponormal* if $D(A) \subset D(A^*)$ and $\|Au\| \ge \|A^*u\| \quad \forall u \in D(A)$). As an example of a hyponormal operator one may take the generator of a right-shift semigroup on $\mathrm{L}^2(0,\infty)$. In [17] this observation is employed to show that the statements of Theorem 1.1 remain true for hyponormal operators. Let us recall, however, that for operators with a compact resolvent the notions of normality and hyponormality are equivalent.

A very important feature of the spectral approach to the problem of well-posedness of systems (1.1) and (1.2) is the possibility of collecting essential information by the examination of the spectral properties of A, which makes considerations simpler than with other analytical tools. This enables one to investigate a wide class of infinite-dimensional systems by elementary methods available also for engineers. As an example we shall consider the stability problem of the system (1.1).

The most commonly used concepts of asymptotic stability of the system (1.1) are:

- $\langle v, T(t)u\rangle \longrightarrow 0$ as $t \to \infty$ $\forall u, v \in H$ (*weak asymptotic stability*, w–(**AS**))
- $T(t)u \longrightarrow 0$ as $t \to \infty$ $\forall u \in H$ (*strong asymptotic stability*, s–(**AS**))
- $\|T(t)\| \longrightarrow 0$ as $t \to \infty$ (*uniform asymptotic stability*) $\Longleftrightarrow$
- $\exists M \geq 1, \exists \alpha > 0 : \|T(t)\| \leq Me^{-\alpha t}$ $\forall t \geq 0$ (*exponential stability*, (**EXS**)).

The following implications hold: (**EXS**) $\Longrightarrow$ s–(**AS**) $\Longrightarrow$ w–(**AS**). For *eventually compact* semigroups (i.e. there exists $t_0 > 0$ such that $T(t)$ is a compact operator on H for all $t \geq t_0$) all the above concepts are equivalent. In particular, this is the case if dim $H < \infty$. For the semigroup whose infinitesimal generator has a compact resolvent we have: s–(**AS**) $\Longleftrightarrow$ w–(**AS**).

To derive practically checkable criteria of (**EXS**), it is of great importance to characterize the notion of (**EXS**) in terms of the spectrum of semigroup generator. Prüss [30], Huang[14] and Weiss [45] have proved that the following conditions are equivalent:

(i) (**EXS**)

(ii) $\lambda \longmapsto (\lambda I - A)^{-1}$ is an analytic function on the open right complex halfplane and bounded on the closed right complex halfplane

(iii) $\lambda \longmapsto (\lambda I - A)^{-1}$ is a bounded function on $\imath\mathbf{R}$ and $\sigma(A)$ lies in the open left complex halfplane.

Only an incomplete spectral characterization of the notion of s–(**AS**) is known. The next theorem follows from the functional calculus for normal operators and the diagram obtained in [11, p.88].

Theorem 1.2 *Let A be an operator which is similar to a normal one. Then:*

(**a**) *A generates a uniformly bounded semigroup* $\Leftrightarrow \sup\{Re\lambda : \lambda \in \sigma(A)\} \leq 0$

(**b**) *A generates an* (**EXS**) *semigroup* $\Longleftrightarrow \sup\{Re\lambda : \lambda \in \sigma(A)\} < 0$

(**c**) *Under the additional assumption that A has a compact resolvent we have: A generates a s–*(**AS**) *semigroup* $\Longleftrightarrow$ *$\sigma(A)$ is contained in the left open complex half-plane.*

Remark 1.2 The last statement appears also in [15, Corollary 2.5/(i), p.319].

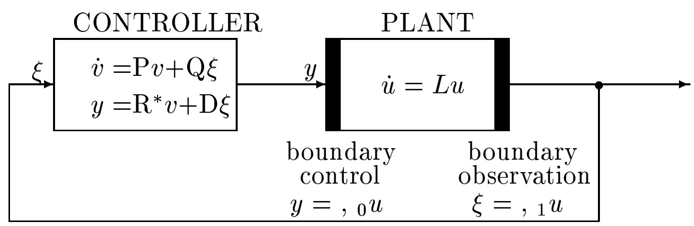

Figure 1: The feedback control system

Remark 1.3 Levan [24] has proved that if A is normal then A is *strictly dissipative* (i.e. $Re\langle Af, f\rangle \leq 0 \ \ \forall f \in D(A)$ with equality only for $f = 0$) iff the semigroup generated by A is s–(**AS**). However, his results are not explicitly expressed by the spectrum of A.

2 Hybrid Feedback Operators

Let us consider a feedback control system consisting of a distributed parameter plant with boundary observation and boundary control worked out by a finite-dimension controller (e.g. conventional controller), depicted in Fig.1.

Here P$\in$ $\mathbf{L}(\mathbf{R}^n)$, Q$\in$ $\mathbf{L}(\mathbf{R}^r, \mathbf{R}^n)$, R$\in$ $\mathbf{L}(\mathbf{R}^m, \mathbf{R}^n)$, D$\in$ $\mathbf{L}$ $(\mathbf{R}^r, \mathbf{R}^m)$, H is a Hilbert space; $L : (D(L) \subset H) \Leftrightarrow\!\!\rightarrow H$, is a linear closed operator with domain $D(L) \subset D(,_0)$, $D(L) \subset D(,_1)$ where $,_0, ,_1$ are some boundary operators, e.g. Dirichlet or Neumann trace operators. The closed-loop system is naturally described on the space $X = \mathbf{C}^n \oplus H$ by a *hybrid linear operator*

$$A \begin{bmatrix} v \\ u \end{bmatrix} = \begin{bmatrix} \mathrm{P}v + \mathrm{Q},_1 u \\ Lu \end{bmatrix} \ ,$$

$$D(A) = \left\{ \begin{bmatrix} v \\ u \end{bmatrix} \in X : \ u \in D(L), \ \ \mathrm{R}^* v + \mathrm{D},_1 u = ,_0 u \right\} \ . \tag{2.1}$$

The problem is to recognize whether a closed-loop system operator A generates a strongly continuous semigroup on X. As we know from Theorem 1.1, the spectral approach is an effective tool for establishing the well-posedness of the feedback system (i.e. generation of a semigroup by the closed-loop system operator) if we can prove that the operator describing the closed-loop system is a discrete operator, similar to a normal one.

A critical survey of the existing criteria for a given ordinary differential operator to a system of eigenvectors which forms a Riesz basis, done from a viewpoint of practical applications, is present in [11, Chapter I and references therein]. In particular, from that survey we know that the so-called *strict regularity of the boundary conditions* decides about the existence of a Riesz basis of eigenvectors. There are some criteria based on determinants which allow to check strict regularity of given boundary conditions in a simple way, provided that a spectral parameter does not enter these conditions. The case of a spectral parameter appearing in the boundary conditions is much more involved.

The sequel of this paper is devoted to continuation of a discussion of that problem initiated in [11].

The eigenproblem for A takes the form

$$\left\{ \begin{array}{rcl} (\lambda \mathrm{I} \Leftrightarrow \mathrm{P})v & = & \mathrm{Q},{}_1 u \\ Lu & = & \lambda u, u \in D(L) \\ \mathrm{R}^* v + \mathrm{D},{}_1 u & = & ,{}_0 u \end{array} \right\} \tag{2.2}$$

and for $\lambda \notin \sigma(\mathrm{P})$ (2.2) reduces to

$$\left\{ \begin{array}{rcl} Lu & = & \lambda u, \quad u \in D(L) \\ \mathrm{W}(\lambda),{}_1 u & = & ,{}_0 u, \\ \mathrm{W}(\lambda) & = & \mathrm{D} + \mathrm{R}^*(\lambda \mathrm{I} \Leftrightarrow \mathrm{P})^{-1}\mathrm{Q} \end{array} \right\} . \tag{2.3}$$

The spectral parameter λ rationally enters the transfer function W in (2.3), but after the multiplication of both sides of the boundary condition by the characteristic polynomial $\det(\lambda \mathrm{I} \Leftrightarrow \mathrm{P})$ of the matrix P, it enters the boundary condition polynomially.

In some cases the existence of a system of eigenvectors of A forming a Riesz basis can be proved by the use of *perturbation methods*. The essence of the perturbation methods is to characterize the admissible classes of perturbations under which the property of having a Riesz basis of eigenvectors is stable with respect to a perturbation from that class. Usually, an unperturbed operator corresponds to an open-loop system, but sometimes it may describe its conservative part. The perturbation expresses the feedback or the dissipative part of the closed-loop system operator.

For other classes of system (2.3), one may seek a similarity isomorphism between A and the *linearizing operator* for (2.3). For this last operator some criteria guaranteeing the existence of a Riesz basis of eigenvectors are known.

Proving the existence of Riesz bases by both direct and perturbation approaches is in practice limited to problems with one-dimensional spatial variable. There are only a few incomplete results on the problem of Riesz bases in multidimesional case and we shall not invoke them here.

3 Perturbation Methods

A typical application of perturbation methods to control theory problems relies on the treatment of a feedback as a perturbation of the open-loop system operator. Our approach to the problem of similarity of the operator (2.3) to a normal operator will be different and we shall regard A as the result of perturbing the operator G,

$$G\begin{bmatrix} v \\ u \end{bmatrix} = \begin{bmatrix} \mathrm{P}v \\ Lu \end{bmatrix}, \quad D(G) = D(A) \tag{3.1}$$

by a finite-rank operator $\begin{bmatrix} v \\ u \end{bmatrix} \Leftrightarrow\!\to \begin{bmatrix} \mathrm{Q},\,_1 u \\ 0 \end{bmatrix}$. Notice that G does not correspond to an open-loop system operator. Usually in applications the observation operator $,\,_1$ is G-bounded. Lasiecka & Triggiani [22] pointed out that the property of having a Riesz basis of eigenvectors is not stable even with respect to a one-dimension perturbation of a discrete normal operator. To be more precise, if $(D(N) \subset H) \Leftrightarrow\!\to H$ is a discrete normal operator acting on a Hilbert space H then not for all $a,\ b \in H$; a $\notin D(N^*)$ the operator $N + ba^*N$ $(ba^*Nu = b\langle Nu, a\rangle,\ u \in D(N))$ has a Riesz basis of eigenvectors (i.e.is not similar to a normal discrete operator under one-dimensional perturbation). This difficulty causes that there is only a limited number of results which can be used to solve the question of similarity of A to a normal operator, provided that G is similar to a normal operator. This problem has been investigated in [22, Theorem 3.1, p.71], [2, Theorem 3.2, p.50] and [1, Theorem 2.3, p.1428-1429]. This last result seems to have the simplest possible set of assumptions to be effectively verified.

Theorem 3.1 (Baskakov-Katsaran) *Let $A : (D(A) \subset H) \Leftrightarrow\!\to H$ be a linear operator in Hilbert space H, with the following properties*

(**i**) *A is invertible with a compact inverse and $\{\lambda \in \mathbf{C} : \phi_1 \le |arg\lambda| \le \phi_2,\ \phi_1 < \phi_2\} \subset \mathbf{C} \setminus \sigma(A)$;*

(**ii**) *$\sigma(A) = \{\lambda_j\}_{j\in J}$ $(i \ne j \Longleftrightarrow \lambda_i \ne \lambda_j)$ contains only a finite number of nonsingle eigenvalues and for some $\nu \in [0,1)$ we have*

$$\sup_{i\ne j} \frac{|\lambda_i|}{|\lambda_j|^{1-\nu}} \frac{1}{|\lambda_i \Leftrightarrow \lambda_j|} < \infty, \quad \sup_{i \ne j} |\lambda_i|^{\nu} \frac{1}{|\lambda_i \Leftrightarrow \lambda_j|} < \infty \ ; \tag{3.2}$$

(**iii**) *generalized eigenvectors of A form a Riesz basis in H (clearly only finitely many of them are not eigenvectors).*

If T is a Hilbert-Schmidt operator and TA^{v} is A-bounded (A^{ν} denotes a fractional power of A) then the operator $\tilde{A} = A + TA^{\nu}$ has the spectrum

$$\{\tilde{\lambda}_j\}_{j\in J} = \sigma(\tilde{A}) \quad (i \neq j \Longleftrightarrow \tilde{\lambda}_i \neq \tilde{\lambda}_j), \quad \left\{ \frac{\left|\lambda_j \Leftrightarrow \tilde{\lambda}_j\right|}{|\lambda_j|^{\nu}} \right\}_{j\in J} \in \ell^{\,2}(J)$$

only a finite number of eigenvalues are nonsingle, and the corresponding system of generalized eigenvectors also forms a Riesz basis in H (again only finitely many of generalized eigenvectors of $\tilde{A}$ are not eigenvectors).

In fact, the generalized eigenvectors of $\tilde{A}$ form the so-called *Bari basis* of H and this is why the eigenvalues of A and $\tilde{A}$ are asymptotically equal.

The next result corresponds in a way to the limit case $\nu = 0$ of Theorem 3.1, but now the perturbation is assumed to be a closed operator.

Theorem 3.2 (Katsnel'son-Shkalikov) *Let H be a Hilbert space and $L : (D(L) \subset H) \Leftrightarrow\!\to H$ be a linear operator similar to a normal discrete operator. Suppose also that there exists $p \in (0,1]$ such that*

*(**i**) $\limsup_{n\to\infty} n\,|\mu_n|^{-p} < \infty$, where $\{\mu_n\}_{n\in\mathbf{N}} = \sigma(L)$, $|\mu_1| \leq |\mu_2| \leq \dots$,*

*(**ii**) $\sigma(L) \subset \{\lambda \in \mathbf{C} : |Im\lambda| \leq h\,|\lambda \Leftrightarrow c|^{(p-1)/p}\}$ for some $h \geq 0$ and $c \in \mathbf{C}$.*

If $T : (D(T) \subset H) \Leftrightarrow\!\to H$ is a linear closed operator such that $D(L) \subset D(T)$, $L^{(p-1)/2}TL^{(p-1)/2}$ has an extension to an operator from $\mathbf{L}(H)$ and all eigenvalues of $A = L + T$ are simple and asymptotically separated then A is also similar to a normal discrete operator.

The above theorem was proved initially under the assumption that L is self-adjoint (in this case (**ii**) is trivially satisfied) - see [19, Theorem 3.1, p.47]. The generalization above is taken from [39, p.236].

4 Shkalikov's Theory

The theory concerns the Sturm-Liouville boundary-value problems, containing a spectral parameter in the boundary conditions,

$$\ell(y,\lambda) = y^{(n)} + p_1(x,\lambda)y^{(n-1)} + \cdots + p_n(x,\lambda)y = 0 \quad (4.1)$$

$$U_j(y,\lambda) = \sum_{k=0}^{n-1} a_{jk}(\lambda)y^{(k)}(0) + b_{jk}(\lambda)y^{(k)}(1) = 0, \quad j = 1,2,\dots,n \quad (4.2)$$

where $p_s(x,\lambda) = \sum_{\nu=0}^{s} p_{\nu s}(x)\lambda^{\nu}$; $p_{ss}(x) = const$, $s = 1,2,\dots,n$; $a_{jk}(\lambda)$, $b_{jk}(\lambda)$ - are arbitrary polynomials of the spectral parameter λ.

Definition 4.1 *A nonnegative integer* κ_j *is said to be the* order of the boundary condition $U_j(y,\lambda)$ *of the form (4.2) if the linear form* $U_j(y,\lambda)$ *contains the terms* $\lambda^\nu y^{(k)}(0)$ *or* $\lambda^\nu y^{(k)}(1)$ *for* $\nu + k = \kappa_j$ *and it does not contain such terms for* $\nu + k > \kappa_j$. $\kappa = \kappa_1 + \kappa_2 + \ldots + \kappa_n$ *is then called the* total order *of the boundary conditions (4.2). If any* n *boundary conditions equivalent to (4.2), i.e. obtained from (4.2) by taking linear combinations, have the total order not less than* κ *then we say that the boundary conditions (4.2) are* normalized.

For further considerations we assume without loss of generality that the boundary conditions (4.2) are normalized and that they are arranged in the decreasing orders, to be precise: $\kappa_1 \geq \kappa_2 \geq \ldots \kappa_n$.

Assume also that $p_{\nu s} \in \mathrm{W}_1^r(0,1)$, $r \geq 0$ and the characteristic polynomial of the problem (4.1), (4.2)

$$\omega^n + p_{11}\omega^{n-1} + \cdots + p_{nn} = 0 \tag{4.3}$$

has only simple roots: $\omega_1, \omega_2, \ldots, \omega_n$.

Remark 4.1 Without loss of generality we may assume that $0 \leq \nu \leq s \Leftrightarrow 1$. This implies $r + (\nu \Leftrightarrow s + 1) \leq r$, and $p_{\nu s} \in \mathrm{W}_1^{r-s+\nu+1}(0,1) \cap \mathrm{L}^1(0,1)$.

Under the above assumptions the complex plane $\mathbf{C}$ can be decomposed into $2h$, $h \leq n$ sectors $S_1, S_2, \ldots, S_{2h}$ and in each sector (4.1) has the fundamental system of solutions of the following asymptotic form as $|\lambda| \to \infty$ (the theory of Birkhoff and Tamarkin),

$$y_k^{(s-1)}(x,\lambda) = \omega_k^{s-k}\lambda^{s-1}\exp(\omega_k\lambda x)\left[\sum_{\nu=0}^{r}\lambda^{-\nu}\eta_{ks\nu}(x) + O(\lambda^{-r-1})\right] \tag{4.4}$$

$k, s = 1, 2, \ldots, n$; $r \geq 0$, r - is arbitrary and fixed; $\eta_{ks\nu} \in \mathrm{W}_1^{r-\nu+1}(0,1)$, $\nu = 0, 1, \ldots, r$, η_{ks0} does not depend on s, and $\eta_{ks\nu}$ does not depend on the choice of a sector.

Let $\mu_{J_k} = \sum_{\alpha \in J_k}\omega_\alpha$, where $J_k (k = 1, 2, \ldots, n)$ denotes a k-element subset of $\{1, 2, \ldots, n\}$; for $k = 0$ we put $\mu_{J_0} = 0$. Let us consider the set of all complex numbers μ_{J_k} which can be obtained by variating over all possible selections of J_k (in this way we get nothing more than the set of all possible sums which can be created from the set of complex numbers $\omega_1, \omega_2, \ldots, \omega_n$). Let $\mathcal{M}$ be the smallest convex polygon containing all points μ_{J_k}. It may happen that $\mathcal{M}$ is an interval.

Further, we consider the characteristic determinant

$$\Delta(\lambda) = \det[U_j(y_k,\lambda)]_{j,k=1,2,\ldots,n} \tag{4.5}$$

with functions y_k defined in sectors $S_1, S_2, \ldots, S_{2h}$ by (4.4). This determinant may be expressed as

$$\Delta(\lambda) = \lambda^{\kappa} \sum_{J_k} [F^{J_k}]_r \exp(\lambda \mu_{J_k}) \ , \tag{4.6}$$

$$\left[F^{J_k}\right]_r = F_0^{J_k} + \lambda^{-1} F_1^{J_k} + \cdots + \lambda^{-r} F_r^{J_k} + O(\lambda^{-r-1}) \ . \tag{4.7}$$

Definition 4.2 *The problem (4.1), (4.2) is said to be* regular *if the numbers $F_0^{J_k}$ in the resolutions of $[F^{J_k}]_0$, corresponding to the vertexes of $\mathcal{M}$ are nonzero. The problem (4.1), (4.2)* is strictly regular *if it is regular and additionally, the zeros of $\Delta(\lambda)$ are asymptotically simple and isolated one from another.*

In what follows, we assume without loss of generality that $p_{nn} = 1$ and for simplicity of notation we represent (4.1) in the form

$$\ell(y, \lambda) = \ell_0(y) + \lambda \ell_1(y) + \lambda^2 \ell_2(y) + \cdots + \lambda^{n-1} \ell_{n-1}(y) + \lambda^n y = 0 \ . \tag{4.8}$$

For any fixed $r \geq 0$ let us denote:

$$W_2^r = \mathrm{W}_2^{n-1+r}(0,1) \oplus \mathrm{W}_2^{n-2+r}(0,1) \oplus \cdots \oplus \mathrm{W}_2^{r}(0,1) \ (n \text{ components})$$

and define an operator

$$W_2^r \ni \tilde{v} = \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{n-2} \\ v_{n-1} \end{bmatrix} \Leftrightarrow\!\!\rightarrow \mathrm{H}\tilde{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n-1} \\ \Leftrightarrow \sum_{\nu=0}^{n-1} \ell_\nu(v_\nu) \end{bmatrix}$$

where $v_0 = y$, $v_1 = \lambda v_0, \ldots, v_{n-1} = \lambda v_{n-2}$ and hence, $\mathrm{H}^\nu \tilde{v} \in W_2^{r-\nu}$ (ν-th power of H, $\nu = 0, 1, 2, \ldots$. In (4.2) we make substitutions according to the rule

$$\lambda^\nu y^{(k)}(0 \text{ or } 1) = \left\{ \begin{array}{ll} (\mathrm{H}^\nu \tilde{v})_0^{(k)}(0 \text{ or } 1), & \nu + k < n + r \\ \lambda^{\nu+k-n-r+1} (\mathrm{H}^{n+r-k-1} \tilde{v})_0^{(k)}(0 \text{ or } 1), & \nu + k \geq n + r \end{array} \right\} \tag{4.9}$$

where the subscript 0 means that we take the first component of an appropriate vector. As a result of these substitutions we represent the boundary conditions in a form

$$\tilde{U}_j(\tilde{v}, \lambda) = \sum_{i=0}^{\nu_j(r)} \lambda^i U_j^i(\tilde{v}), \quad 1 \leq j \leq n \tag{4.10}$$

where now the functionals U_j^i do not depend on λ. Next, we make the following partition of indices $\nu_j(r)$:

$$\nu_1(r) \geq \nu_2(r) \geq \dots \nu_q(r) > 0 = \nu_{q+1}(r) = \dots \nu_n(r) \ .$$

Consider the space $W_{2,U}^r \oplus \mathbf{C}^{N_r}$ where

$$W_{2,U}^r = \{\tilde{v} \in W_2^r : \tilde{U}_j(\mathrm{H}^k\tilde{v}, \lambda) = \tilde{U}_j(\mathrm{H}^k\tilde{v}) = 0 \quad \text{for} \quad 0 \leq k \leq n + r \Leftrightarrow 2$$

$$\text{and all boundary conditions of order } \leq n + r \Leftrightarrow k \Leftrightarrow 2\} \tag{4.11}$$

$$N_r = \sum_{j=1}^{q} \nu_j(r) \tag{4.12}$$

(if all $\nu_j(r)$ are zero then $N_r = 0$). Let us define an operator

$$H_r : (D(H_r) \subset W_{2,U}^r \oplus \mathbf{C}^{N_r} \Leftrightarrow\!\!\to W_{2,U}^r \oplus \mathbf{C}^{N_r} \ ,$$

$$H_r \begin{bmatrix} \tilde{v} \\ \boxed{\begin{array}{c} U_1^{\nu_1(r)} \\ z_{12} \\ \cdots \\ z_{1(\nu_1(r)-1)} \\ z_{1\nu_1(r)} \end{array}} \\ \cdots \\ \text{similar blocks} \\ \text{of variables} \\ \text{for successive} \\ \text{numbers } \nu_j(r), \\ j = 2, 3, \dots, q \end{bmatrix} = \begin{bmatrix} \mathrm{H}\tilde{v} \\ \boxed{\begin{array}{c} z_{12} \Leftrightarrow U_1^{\nu_1(r)-1}(\tilde{v}) \\ z_{13} \Leftrightarrow U_1^{\nu_1(r)-2}(\tilde{v}) \\ \cdots \\ z_{1\nu_1(r)} \Leftrightarrow U_1^1(\tilde{v}) \\ \Leftrightarrow U_1^0(\tilde{v}) \end{array}} \\ \cdots \\ \text{similar blocks} \\ \text{of variables} \\ \text{for successive} \\ \text{numbers } \nu_j(r), \\ j = 2, 3, \dots, q \end{bmatrix}$$

$$D(H_r) = \{(\tilde{v}, U_1^{\nu_1(r)}, z_{12}, \dots z_{1\nu_1(r)}, \dots U_q^{\nu_q(r)}, z_{q2}, \dots, z_{q\nu_q(r)}) \ :$$

$$\tilde{v} \in W_{2,U}^{r+1}, \quad z_{j\nu} \in \mathbf{C}, \ 2 \leq \nu \leq \nu_j(r), \quad 1 \leq j \leq q\} \ . \tag{4.13}$$

(4.13) will be called Shkalikov's linearization of the problem (4.1), (4.2) because the eigenvalue problem for H_r in this space $W_{2,U}^r \oplus \mathbf{C}^{N_r}$ reduces clearly to (4.1), (4.2).

Theorem 4.1 (Shkalikov [40]) *Let the above assumptions hold and, additionally, let the boundary conditions be strictly regular. Under these assumptions:*

(**i**) *There exists a system of generalized eigenvectors (only finitely many of them are not eigenvectors) of the operator (4.13) which forms a Riesz basis in* $W^r_{2,U} \oplus \mathbf{C}^{N_r}$.

(**ii**) *A necessary and sufficient condition for the existence of a system of generalized eigenvectors of the operator (4.13) which forms a Riesz basis in* $W^r_{2,U}$, *is that all boundary conditions should have the order* $\leq n + r \Leftrightarrow 1$ *(the case of* $N_r = 0$*). If such a system of generalized eigenvectors exists, then only a finite number of them are not eigenvectors.*

Sargsian generalized Shkalikov's theory to a multidimensional boundary-value problem of the order with special boundary conditions,

$$\left\{ \begin{array}{ccc} y'(x) + \mathrm{P}(x)y(x) & = & \lambda y(x), \quad 0 \leq x \leq 1 \\ (\mathrm{A}_0 + \lambda \mathrm{A}_1)y(0) + (\mathrm{B}_0 + \lambda \mathrm{B}_1)y(1) & = & 0 \end{array} \right\} \quad (4.14)$$

where $\mathrm{A}_0, \mathrm{A}_1, \mathrm{B}_0, \mathrm{B}_1 \in \mathbf{L}(\mathbf{R}^n)$, $\mathrm{P}(x) \in \mathbf{L}(\mathbf{R}^n)$ for a fixed $x \in [0,1]$. Let V_i denote the i-th row of the boundary conditions.

Definition 4.3 *We say that* V_i *is of the* first order *if the parameter* λ *enters this row. Otherwise, we say that* V_i *is of the* null order. *The sum of orders of all rows creating boundary conditions is called the* total order *of boundary conditions. The boundary conditions are* normalized *if any n rows, equivalent to the given, i.e. obtained from the given by taking linear combinations, have the same total order.*

In what follows, we assume that the boundary conditions in (4.14) are normalized. Without loss of generality one may assume that the first s rows have the null order.

Let

$$W^{2,k} = \mathrm{W}^{2,k} \oplus \mathrm{W}^{2,k} \oplus \mathrm{W}^{2,k} \oplus \cdots \oplus \mathrm{W}^{2,k}$$

denote n copies of the standard Sobolev $\mathrm{W}^{2,k}(0,1)$ space.

$$\mathrm{w}^1_{2,V} := \{ y \in W^{1,2} : \quad V_i(y) = 0, \quad i = 1,2,3,\ldots,s \}$$

$$\mathrm{w}^2_{2,V} := \{ y \in W^{2,2} : \mathrm{A}_0 y(0) + \mathrm{A}_1[y'(0) + \mathrm{P}(0)y(0)] + \mathrm{B}_0 y(1) + \\ + \mathrm{B}_1[y'(1) + \mathrm{P}(1)y(1)] = 0, \quad V_i(y' + \mathrm{P}y) = 0, \quad i = 1,2,3,\ldots,s \} \ .$$

The problem (4.14) can be regarded as an eigenvalue problem in the space $\mathrm{w}^1_{2,V}$ for the operator

$$Hy = y' + \mathrm{P}(x)y, \quad D(H) = \mathrm{w}^2_{2,V} \ . \quad (4.15)$$

The characteristic function of this problem can be written in the form $\Delta(\lambda) = \sum_{k=0}^n p_k(\lambda) e^{k\lambda}$ where $p_k(\lambda)$ is a polynomial.

Definition 4.4 *If* $\deg p_0(\lambda) = \deg p_n(\lambda)$ *then (4.14) is called a* regular *eigenvalue problem. If* $p_0(\lambda) \neq 0$, $p_n(\lambda) \neq 0$ *then (4.14) is called an* almost regular *eigenvalue problem.*

Sargsian [35, Theorem 2, p.5] and [34, Theorem 4.1, p.14] obtained the following result

Theorem 4.2 (Sargsian) *If the boundary value problem (4.14) is regular, all eigenvalues of* H *are simple and* $\inf\{|\lambda \Leftrightarrow \mu|: \lambda,\ \mu \in \sigma(H),\ \lambda \neq \mu\} > 0$ *then* H *has a system of eigenvectors which forms a Riesz basis in* $w^1_{2,V}$. *If the boundary problem is almost regular, all eigenvalues of* H *are simple and* $\inf\{|\lambda \Leftrightarrow \mu|: \lambda,\ \mu \in \sigma(H),\ \lambda \neq \mu\} > 0$ *then the system eigenvectors of* H *form a complete system in* $w^1_{2,V}$ *for which there exists a biorthogonal system.*

To apply the above results to problem (2.3) one should look for an isomorphism under which Shkalikov's linearization is similar to the operator A.

5 Examples of Applications

5.1 Example 1: RLCG transmission line

Consider an RLCG transmission line with a proportional feedback depicted in Fig.2.

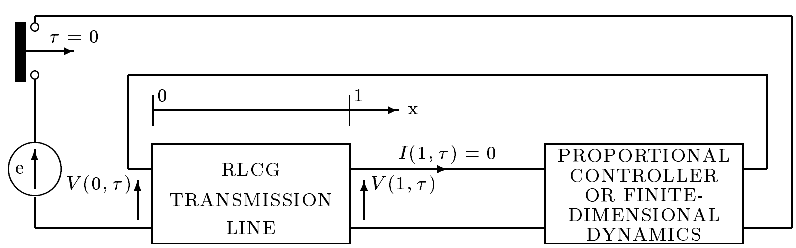

Figure 2: The control system with RLCG transmission line

The closed-loop system is governed by the equations

$$\left\{ \begin{array}{lcll} \mathrm{L}\frac{\partial I(x,\tau)}{\partial \tau} & = & \Leftrightarrow\frac{\partial V(x,\tau)}{\partial x} \Leftrightarrow \mathrm{R}I(x,\tau), & 0 \le x \le 1, \quad \tau \ge 0 \\ \mathrm{C}\frac{\partial V(x,\tau)}{\partial \tau} & = & \Leftrightarrow\frac{\partial I(x,\tau)}{\partial x} \Leftrightarrow \mathrm{G}V(x,\tau), & 0 \le x \le 1, \quad \tau \ge 0 \\ I(1,\tau) & = & 0, & \qquad\qquad\quad \tau \ge 0 \\ V(0,\tau) & = & \mathrm{K}V(1,\tau) + \mathrm{e}, & \qquad\qquad\quad \tau \ge 0 \\ V(x,0) & = & 0, & 0 \le x \le 1 \\ I(x,0) & = & 0, & 0 \le x \le 1 \end{array} \right\} . \tag{5.1}$$

5.1.1 The case of RC transmission line ($\mathbf{L} = 0,\ \mathbf{G} = 0$)

Eliminating $I(x,\tau)$ and substituting $u(x,t) = V(x,\mathrm{RC}t) \Leftrightarrow \mathrm{e}/(1\text{-}\mathrm{K})$ we get

$$\left\{ \begin{array}{lcll} \frac{\partial u(x,t)}{\partial t} & = & \frac{\partial^2 u(x,t)}{\partial x^2}, & 0 \le x \le 1, \quad t \ge 0 \\ \frac{\partial u(x,t)}{\partial x} & = & 0, & \qquad\qquad\quad t \ge 0 \\ u(0,t) & = & \mathrm{K}u(1,t), & \qquad\qquad\quad t \ge 0 \\ u(x,0) & = & \mathrm{e}/(\mathrm{K}\text{-}1), & 0 \le x \le 1 \end{array} \right\} . \tag{5.2}$$

In $H = \mathrm{L}^2(0,1)$ with standard scalar product

$$\langle u_1, u_2 \rangle = \int_0^1 u_1(x)\overline{u_2(x)}dx$$

we can rewrite (5.2) into an abstract form (1.1) with

$$Au = u'', \quad D(A) = \{u \in \mathrm{H}^2(0,1) : \quad u'(1) = 0,\ u(0) = \mathrm{K}u(1)\} \tag{5.3}$$

$$u_0 = \mathrm{e}/(\mathrm{K}\text{-}1) \quad (u_0 \notin D(A)) \ . \tag{5.4}$$

The eigenproblem for A, $Au = \lambda u$, $u \in D(A)$, $u \neq 0$ reduces to the two-point boundary value problem

$$\left\{ \begin{array}{lcll} u''(x) & = & \lambda u(x), & 0 \le x \le 1 \\ u'(1) & = & 0 & \\ u(0) & = & \mathrm{K}u(1) & \end{array} \right\} . \tag{5.5}$$

Assuming a solution in the form $u(x) = \mathrm{C}_1 e^{-\sqrt{\lambda}x} + \mathrm{C}_2 e^{\sqrt{\lambda}x}$, we obtain

$$\lambda_n \quad = \quad \left\{ \begin{array}{lrcr} \left[\ln^2\Delta \Leftrightarrow (2n\pi + \pi)^2\right] + 2\imath(2n\pi + \pi)\ln\Delta, & \mathrm{K} & < & \Leftrightarrow 1 \\ \Leftrightarrow(\phi + 2n\pi)^2, & |\mathrm{K}| & \le & 1 \\ \left[\ln^2\Delta \Leftrightarrow 4n^2\pi^2\right] + 4n\pi\imath\ln\Delta, & \mathrm{K} & > & 1 \end{array} \right\},$$

$$n \in \mathbf{Z} \tag{5.6}$$

where $\Delta = |\mathrm{K}| + (\mathrm{K}^2 - 1)^{1/2}$, $|\mathrm{K}| > 1$; $\phi = \arccos \mathrm{K}$, $|\mathrm{K}| \le 1$.

Thus, if $|\mathrm{K}| \le 1$ the point spectrum of A is located on the negative real semiaxis, but if $|\mathrm{K}| > 1$ it is located on the parabola

$$Re\lambda = \ln^2 \Delta - \frac{1}{4 \ln^2 \Delta} Im^2 \lambda \ .$$

All eigenvalues are single for $|\mathrm{K}| \neq 1$ and double for $|\mathrm{K}| = 1$, except for 0 which is single if $\mathrm{K} = 1$.

Now, we are going to prove

Lemma 5.1 *If $|K| \neq 1$ then the system $\{u_n\}_{n \in \mathbf{Z}}$ of eigenvectors of A, corresponding to eigenvalues λ_n,*

$$u_n(x) = 2s \cosh[\mu_n(1 - x)], \quad 0 \le x \le 1, \quad n \in \mathbf{Z} \tag{5.7}$$

where

$$\mu_n = \left\{ \begin{array}{lccc} \imath(\phi + 2n\pi), & |K| & < & 1 \\ \ln \Delta + 2n\pi\imath, & K & > & 1 \\ \ln \Delta + (\pi + 2n\pi)\imath, & K & < & -1 \end{array} \right\} \tag{5.8}$$

and

$$s = e^{\mu_n} = \left\{ \begin{array}{rccc} e^{\imath\phi}, & |K| & < & 1 \\ \Delta, & K & > & 1 \\ -\Delta, & K & < & -1 \end{array} \right\} \tag{5.9}$$

forms a Riesz basis in $H = L^2(0,1)$.

Proof: Observe that $\mu_n^2 = \lambda_n$, $n \in \mathbf{Z}$ and

$$u_n(x) = \varepsilon_n(x) + s\varepsilon_n(1 - x) \tag{5.10}$$

where

$$\varepsilon_n(x) = \exp(\mu_n x) = g(x) \cdot \exp(2n\pi x\imath) \ . \tag{5.11}$$

$$g(x) = \left\{ \begin{array}{lccc} e^{\imath x\phi}, & |\mathrm{K}| & < & 1 \\ e^{x \ln \Delta}, & \mathrm{K} & > & 1 \\ e^{\imath\pi x} e^{x \ln \Delta}, & \mathrm{K} & < & -1 \end{array} \right\} \ . \tag{5.12}$$

In virtue of (5.11), (5.12), the operator M of multiplication by g transforms the classical Fourier basis $\{\exp(2n\pi\imath(\cdot))\}_{n \in \mathbf{Z}}$ in $H = \mathrm{L}^2(0,1)$ into the system of exponentials $\{\varepsilon_n\}_{n \in \mathbf{Z}}$. Next, the operator

$$(L\varepsilon)(x) = \varepsilon(x) + s\varepsilon(1 - x) \tag{5.13}$$

transforms the system of exponentials $\{\varepsilon_n\}_{n \in \mathbf{Z}}$ into $\{u_n\}_{n \in \mathbf{Z}}$. Since M and L are linear and bounded jointly with their inverses, the definition of Riesz basis implies that the system (5.7) is such a basis.

The operator A is similar to a normal discrete operator with spectrum located on the negative real semiaxis or on the parabola having the branches directed to the left. However, this implies that A generates a strongly cosine operator family and hence an analytic semigroup. This semigroup is (**EXS**), equivalently (**AS**), iff

$$\Leftrightarrow \cosh \pi < \mathrm{K} < 1 \ . \tag{5.14}$$

The resolvent of A has the form

$$((\lambda I \Leftrightarrow A)^{-1}v)(x) = \frac{\Leftrightarrow \mathrm{K}}{\cosh\sqrt{\lambda} \Leftrightarrow \mathrm{K}} \int_0^1 \mathbf{1}(y \Leftrightarrow x) \frac{\sinh\sqrt{\lambda}(x \Leftrightarrow y)}{\sqrt{\lambda}} v(y)dy +$$

$$+ \frac{1}{\cosh\sqrt{\lambda} \Leftrightarrow \mathrm{K}} \int_0^1 \left\{ \begin{array}{ll} \frac{\sinh\sqrt{\lambda}x \cosh\sqrt{\lambda}(1-y)}{\sqrt{\lambda}}, & x < y \\ \frac{\sinh\sqrt{\lambda}y \cosh\sqrt{\lambda}(1-x)}{\sqrt{\lambda}}, & x > y \end{array} \right\} v(y)dy, \quad v \in H \ . \tag{5.15}$$

To examine the solvability of the initial-value problem (5.3), (5.4) and to establish the exponential decay of solutions without the knowledge that A has a system of eigenvectors forming a Riesz basis in H, one should estimate precisely $\|(\lambda I \Leftrightarrow A)^{-n}\|$ on the real axis for $n = 2, 3, \ldots$ and on the right complex halfplane for $n = 1$, which is a difficult task.

5.1.2 RC transmission line steered by a SISO dynamical controller

If in the above example we replace a proportional controller by a SISO finite-dimensional dynamical system then the closed-loop system is governed by the equations

$$\left\{ \begin{array}{rcll} \dot{v}(t) & = & \mathrm{F}v(t) + u(1,t)\mathrm{g}, & t \geq 0 \\ \frac{\partial u(x,t)}{\partial t} & = & \frac{\partial^2 u(x,t)}{\partial x^2}, & 0 \leq x \leq 1, \quad t \geq 0 \\ \frac{\partial u(1,t)}{\partial x} & = & 0, & t \geq 0 \\ \mathrm{h}^* v(t) + \mathrm{d}u(1,t) & = & u(0,t), & t \geq 0 \end{array} \right\}$$

where $\mathrm{F} \in \mathbf{L}(\mathbf{R}^n)$; g,h $\in \mathbf{R}^n$, d $\in \mathbf{R}$. Here we neglect the initial conditions as they are immaterial. In the space $X = \mathbf{C}^n \oplus \mathrm{L}^2(0,1)$ we may write down the dynamics in the form (1.1) with the right-hand side

$$A \begin{bmatrix} v \\ u \end{bmatrix} = \begin{bmatrix} \mathrm{F}v + u(1)\mathrm{g} \\ u'' \end{bmatrix} ,$$

$$D(A) = \left\{ \begin{bmatrix} v \\ u \end{bmatrix} \in X : \ u \in \mathrm{H}^2(0,1), \ u'(1) = 0, \ u(0) = \mathrm{h}^* v + \mathrm{d}u(1) \right\} \ . \tag{5.16}$$

Comparing (2.1) and (5.16) we may find that in the discussed example $H = \mathrm{L}^2(0,1)$; $Lu = u''$, $D(L) = \{u \in \mathrm{H}^2(0,1) : u'(1) = 0\}$; $\mathrm{P} = \mathrm{F}$, $\mathrm{Q} = \mathrm{g}$, $\mathrm{R} = \mathrm{h}$, $\mathrm{D} = \mathrm{d}$ ($r = m = 1$) and the boundary control and observation operators are the Dirichlet trace operators,

$$,_0 u = u(0), \quad ,_1 u = u(1) \ .$$

Thus, the reduced eigenvalue problem (2.3) takes the form

$$\left\{ \begin{array}{l} u''(x) = \lambda u(x), \quad u \in \mathrm{H}^2(0,1) \\ u'(1) = 0 \\ u(1)[h^* \ \mathrm{adj}(\lambda \mathrm{I} \Leftrightarrow \mathrm{F})\mathrm{g} + \mathrm{d}\det(\lambda \mathrm{I} \Leftrightarrow \mathrm{F})] = u(0)\det(\lambda \mathrm{I} \Leftrightarrow \mathrm{F}) \end{array} \right\} \ . \tag{5.17}$$

The polynomials in λ appearing in the last equation can be easily identified as the numerator and denominator of the controller transfer function, respectively. The plant is of parabolic type and thus the term $\lambda^2 u(x)$ does not appear in the right-hand side of the first equation of (5.17), which is needed for the applicability of Theorem 4.1. We try to apply Theorem 3.1 and to do this we represent the operator (5.16) as a perturbation of the operator

$$G \begin{bmatrix} v \\ u \end{bmatrix} = \begin{bmatrix} \mathrm{F}v \\ u'' \end{bmatrix}, \quad D(G) = D(A) \ . \tag{5.18}$$

The perturbation has a form $\begin{bmatrix} v \\ u \end{bmatrix} \Leftrightarrow\!\rightarrow \begin{bmatrix} \mathrm{g} \\ 0 \end{bmatrix} u(1)$. Under the assumption that G^{-1} exists, or equivalently F^{-1} exists and $\mathrm{d} \neq 1$, we have

$$u(1) = \left\langle G \begin{bmatrix} v \\ u \end{bmatrix}, \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \right\rangle_X = \left\langle \begin{bmatrix} \mathrm{F}v \\ u'' \end{bmatrix}, \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \right\rangle_X =$$
$$= v^* \mathrm{F}^* \alpha + \int_0^1 u''(x)\beta(x)dx \quad \forall \begin{bmatrix} v \\ u \end{bmatrix} \in D(G) \ .$$

The integration-by-parts yields

$$\alpha = \Leftrightarrow \frac{1}{\mathrm{d} \Leftrightarrow 1} (\mathrm{F}^*)^{-1} \mathrm{h}, \quad \beta(x) = \frac{x}{\mathrm{d} \Leftrightarrow 1}, \quad 0 \le x \le 1 \ .$$

Hence, the operator (5.16) can be written as

$$A \begin{bmatrix} v \\ u \end{bmatrix} = G \begin{bmatrix} v \\ u \end{bmatrix} + \begin{bmatrix} \mathrm{g} \\ 0 \end{bmatrix} \left\langle G \begin{bmatrix} v \\ u \end{bmatrix}, \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \right\rangle_X \ . \tag{5.19}$$

Observe that $\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \notin D(G^*)$ where G^* is the adjoint operator with respect to G,

$$G^* \begin{bmatrix} v \\ u \end{bmatrix} = \begin{bmatrix} \mathrm{F}^* v + u'(0)\mathrm{h} \\ u'' \end{bmatrix} \ ,$$

$$D(G^*) = \left\{ \begin{bmatrix} v \\ u \end{bmatrix} \in X: \quad u \in \mathrm{H}^2(0,1), \quad u(0)=0, \quad \mathrm{d}u'(0) = u'(1) \right\} .$$

The perturbation therefore is G-bounded. G^{-1} belongs to $\mathbf{B}_\infty(X)$, the class of compact linear operators on X, which can be established by calculating G^{-1} explicitly. The spectrum of G is a union of the spectrum of the matrix F and the operator (5.3), describing the RC-transmission line steered only by a proportional controller,

$$\left\{ \begin{array}{lll} \phi''(x) & = & \lambda\phi(x), \quad \phi \in \mathrm{H}^2(0,1) \\ \phi'(1) & = & 0 \\ \phi(0) & = & \mathrm{d}\phi(1) \end{array} \right\} . \tag{5.20}$$

The eigenproblem (5.20) was discussed in details previously. As we know for $|\mathrm{d}| \neq 1$ it has a sequence of simple eigenvalues $\{\lambda_n\}_{n\in\mathbf{Z}}$ and the corresponding system of eigenvectors $\{\phi_n\}_{n\in\mathbf{Z}}$ forms a Riesz basis in $\mathrm{L}^2(0,1)$. This means that $\sigma(G)\setminus\sigma(\mathrm{F}) = \{\lambda_n\}_{n\in\mathbf{Z}}$ and the system of eigenvectors of G, $\left\{ \begin{bmatrix} 0 \\ \phi_n \end{bmatrix} \right\}_{n\in\mathbf{Z}}$ forming a Riesz basis of the subspace $\{0\}\oplus\mathrm{L}^2(0,1)$ in X, corresponds to the sequence $\{\lambda_n\}_{n\in\mathbf{Z}}$. It is clear that completing this system by the n-tuple of generalized eigenvectors of G, corresponding to $\sigma(\mathrm{F})$ one obtains a Riesz basis in the whole space X (this is justified by the result of [23, Theorem 3.6, p.323] according to which codimension of a subspace spanned by the generalized eigenvectors of G, $G^{-1} \in \mathbf{B}_\infty(X)$, is either 0 or ∞; thus if the completing n-tuple would not exist then the codimension would be exactly n). From the formulae for $\{\lambda_n\}_{n\in\mathbf{Z}}$ which are already known we deduce that (3.2) holds for $\nu = \frac{1}{2}$ (see also [1, p.1431]). Now, it follows from (5.19) and Theorem 3.1 that A has a system of generalized eigenvectors forming a Riesz basis in X, provided that $\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \in D\left[(G^*)^{1/2}\right]$. This last condition holds iff

$$\sum_{n\in\mathbf{Z}} |\lambda_n| \left| \left\langle \begin{bmatrix} 0 \\ \phi_n \end{bmatrix}, \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \right\rangle \right|^2 = \sum_{n\in\mathbf{Z}} \frac{1}{|\lambda_n|} \left| \left\langle G \begin{bmatrix} 0 \\ \phi_n \end{bmatrix}, \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \right\rangle \right|^2 =$$

$$= \sum_{n\in\mathbf{Z}} \frac{1}{|\lambda_n|} |\phi_n(1)|^2 < \infty$$

(a contribution from the spectrum of the matrix F is not essential as it gives only a finite number of components and therefore is neglected). The convergence of last series follows from the previously examined properties of the systems $\{\lambda_n\}_{n\in\mathbf{Z}}$, $\{\phi_n\}_{n\in\mathbf{Z}}$. The result we have just proved can be formulated as

Lemma 5.2 *If $|d| \neq 1$, then there exists a system of generalized eigenvectors of A forming a Riesz basis in X and only a finitely many of them are not eigenvectors. The asymptotic eigenvalues of A are equal to eigenvalues of the problem (5.20), corresponding to the system with a proportional controller.*

Remark 5.1 Formally, we have proved Lemma 5.2 under an additional assumption that $\det \mathrm{F} \neq 0$. This assumption is not essential as δv can be added to and subtracted from the $\mathbf{C}^n$- component of A with $\delta \neq 0$ chosen such that $\det(\delta \mathrm{I} \Leftrightarrow \mathrm{F}) \neq 0$, and $\mathrm{F} \Leftrightarrow \delta \mathrm{I}$ can play the role of F; δv is counted into the perturbation and the vector $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ is modified appropriately.

Corollary 5.1 *In virtue of Lemma 5.2 the spectrum of A lies in a region bounded by a parabola which guarantees that A generates a strongly continuous cosine family of operators on X and hence an analytic semigroup.*

5.1.3 The case of RLCG transmission line (LC > 0)

If $H = \mathrm{L}^2(0,1) \oplus \mathrm{L}^2(0,1)$, the dynamic equations (5.1) can be rewritten in the abstract form (1.1) with

$$A \begin{bmatrix} i \\ v \end{bmatrix} = \begin{bmatrix} \Leftrightarrow \frac{1}{\mathrm{L}} v' \Leftrightarrow \frac{\mathrm{R}}{\mathrm{L}} i \\ \Leftrightarrow \frac{1}{\mathrm{C}} i' \Leftrightarrow \frac{\mathrm{G}}{\mathrm{C}} v \end{bmatrix} ,$$

$$D(A) = \left\{ \begin{bmatrix} i \\ v \end{bmatrix} \in \mathrm{H}^1(0,1) \oplus \mathrm{H}^1(0,1) : \quad i(1) = 0, \quad v(0) = \mathrm{K} v(1) \right\} . \tag{5.21}$$

For simplicity we introduce

$$\alpha = \frac{1}{2}\left(\frac{\mathrm{R}}{\mathrm{L}} + \frac{\mathrm{G}}{\mathrm{C}}\right), \quad \beta = \frac{1}{2}\left(\frac{\mathrm{R}}{\mathrm{L}} \Leftrightarrow \frac{\mathrm{G}}{\mathrm{C}}\right); \quad z = \sqrt{\frac{\mathrm{L}}{\mathrm{C}}}, \quad r = \sqrt{\mathrm{LC}}$$

and now we are able to represent A in the particular form

$$A = \frac{1}{r} A_0 + P$$

where

$$A_0 \begin{bmatrix} i \\ v \end{bmatrix} = \begin{bmatrix} 0 & \Leftrightarrow z^{-1} \\ \Leftrightarrow z & 0 \end{bmatrix} \begin{bmatrix} i' \\ v' \end{bmatrix}, \quad D(A_0) = D(A) \tag{5.22}$$

$$P \begin{bmatrix} i \\ v \end{bmatrix} = \begin{bmatrix} \Leftrightarrow \alpha \Leftrightarrow \beta & 0 \\ 0 & \Leftrightarrow \alpha + \beta \end{bmatrix} \begin{bmatrix} i \\ v \end{bmatrix}, \quad P \in \mathbf{L}(H) \ . \tag{5.23}$$

The operator $\frac{1}{r}A_0$ describes the lossless transmission line, and P describes dissipation.

Observe that A_0^2 can be decomposed into the current operator,

$$A_i i = i'', \quad D(A_i) = \{i \in \mathrm{H}^2(0,1): \quad i(1) = 0, \quad i'(0) = \mathrm{K}i'(1)\} \tag{5.24}$$

and the voltage operator,

$$A_v v = v'', \quad D(A_v) = \{v \in \mathrm{H}^2(0,1): \quad v'(1) = 0, \quad v(0) = \mathrm{K}v(1)\} \ . \tag{5.25}$$

The operator (5.25) is equal to the operator (5.3) which we discussed in the RC-transmission line case while (5.24), after replacing x by $1 \Leftrightarrow x$ is its adjoint. Thus, the system of eigenvectors of A_0,

$$\left\{ \begin{bmatrix} i_n^{\pm} \\ v_n^{\pm} \end{bmatrix} \right\}_{n\in\mathbf{Z}}, \begin{bmatrix} i_n^{\pm}(x) \\ v_n^{\pm}(x) \end{bmatrix} = \begin{bmatrix} 2s\sinh\left[\mu_n(1 \Leftrightarrow x)\right] \\ \pm 2sz\cosh\left[\mu_n(1 \Leftrightarrow x)\right] \end{bmatrix}, \quad 0 \le x \le 1 \ ,$$

where s is defined by (5.9), corresponding to eigenvalues $\pm\mu_n$, with μ_n given by (5.8), may be represented in the form

$$\left\{ \begin{bmatrix} i_n^{+} \\ v_n^{+} \end{bmatrix}, \begin{bmatrix} i_n^{-} \\ v_n^{-} \end{bmatrix} \right\} = \begin{bmatrix} N & 0 \\ 0 & L \end{bmatrix} \begin{bmatrix} 1 & 1 \\ z & \Leftrightarrow z \end{bmatrix} \left\{ \begin{bmatrix} \varepsilon_n \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \varepsilon_n \end{bmatrix} \right\} \tag{5.26}$$

where $\{\varepsilon_n\}_{n\in\mathbf{Z}}$ is the system of exponentials (5.11), (5.12), L is defined by (5.13) and

$$(N\varepsilon)(x) = \Leftrightarrow \varepsilon(x) + s\varepsilon(1 \Leftrightarrow x) \ . \tag{5.27}$$

However, we know from Lemma 5.1 that for $|\mathrm{K}| \neq 1$, $\{\varepsilon_n\}_{n\in\mathbf{Z}}$ forms a Riesz basis in $\mathrm{L}^2(0,1)$, hence $\left\{ \begin{bmatrix} \varepsilon_n \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \varepsilon_n \end{bmatrix} \right\}_{n\in\mathbf{Z}}$ is a Riesz basis in H and since L, N are linear and bounded jointly with their inverses then by (5.26), $\frac{1}{r}A_0$ has a Riesz basis of eigenvectors. Moreover, its eigenvalues $\{\frac{1}{\mathrm{r}}\mu_\mathrm{n}\}_{\mathrm{n}\in\mathbf{Z}}$ lie in a strip parallel to $\imath\mathbf{R}$. Now, one can apply the perturbation Theorem 3.2 to establish that if all eigenvalues $\lambda_n^{\pm}$ of A, corresponding to the two solutions of the equation

$$\mu_n^2 = (\mathrm{R} + \lambda\mathrm{L})(\mathrm{G} + \lambda\mathrm{C}) = r^2\left[(\lambda+\alpha)^2 \Leftrightarrow \beta^2\right] \tag{5.28}$$

are single then the operator (5.21) has the system $\left\{ \begin{bmatrix} i_n^{\pm} \\ v_n^{\pm} \end{bmatrix} \right\}_{n\in\mathbf{Z}}$ of eigenvectors forming a Riesz basis of H, e.g.

$$\begin{bmatrix} i_n^{\pm}(x) \\ v_n^{\pm}(x) \end{bmatrix} = \begin{bmatrix} 2s\sinh\left[\mu_n(1 \Leftrightarrow x)\right] \\ 2s(\mathrm{R} + \lambda_n^{\pm}\mathrm{L})^{1/2}(\mathrm{G} + \lambda_n^{\pm}\mathrm{C})^{-1/2}\cosh\left[\mu_n(1 \Leftrightarrow x)\right] \end{bmatrix}, \quad 0 \le x \le 1. \tag{5.29}$$

A detailed analysis of (5.28) shows that all eigenvalues are single iff $|\mathrm{K}| > 1$ or $|\mathrm{K}| < 1$, $((\arccos \mathrm{K} + 2n\pi)/r)^2 \neq \beta^2$. Though the perturbation P changes the eigenvalues of $\frac{1}{r}A_0$ into the eigenvalues of A, they are still located in a strip parallel to $\imath\mathbf{R}$. A is similar to a normal discrete operator with its spectrum located in a vertical strip, and so A generates a group on H. Under the reasonable assumption: $2\alpha = \left(\frac{\mathrm{R}}{\mathrm{L}} + \frac{\mathrm{G}}{\mathrm{C}}\right) > 0$, this group is (**EXS**), equivalently (**AS**) iff

$$\Leftrightarrow \cosh \sqrt{\frac{(\mathrm{LG}+\mathrm{RC})^2(\pi^2+\mathrm{RG})}{(\mathrm{LG}+\mathrm{RC})^2+4\pi^2\mathrm{LC}}} < \mathrm{K} < \cosh\sqrt{\mathrm{RG}} \ . \qquad (5.30)$$

5.1.4 RLCG transmission line steered by a SISO dynamic controller (LC ≥ 0)

The closed-loop system is described in the state space $X = \mathbf{C}^n \oplus \mathrm{L}^2(0,1) \oplus \mathrm{L}^2(0,1)$ by a linear operator

$$A \begin{bmatrix} y \\ i \\ u \end{bmatrix} = \begin{bmatrix} \mathrm{F}y & + & \mathrm{g}u(1) \\ \Leftrightarrow\frac{1}{\mathrm{L}}u' & \Leftrightarrow & \frac{\mathrm{R}}{\mathrm{L}}i \\ \Leftrightarrow\frac{1}{\mathrm{C}}i' & \Leftrightarrow & \frac{\mathrm{G}}{\mathrm{C}}u \end{bmatrix} ,$$

$$D(A) = \left\{ \begin{bmatrix} y \\ i \\ u \end{bmatrix} \in X : \ i, u \in \mathrm{W}^{1,2}(0,1), i(1) = 0, u(0) = \mathrm{d}u(1) + \mathrm{h}^* y \right\} . \qquad (5.31)$$

On the one hand it has the form (2.1), with $H = \mathrm{L}^2(0,1) \oplus \mathrm{L}^2(0,1)$,

$$L \begin{bmatrix} i \\ u \end{bmatrix} = \begin{bmatrix} \Leftrightarrow\frac{1}{\mathrm{L}}u' \Leftrightarrow \frac{\mathrm{R}}{\mathrm{L}}i \\ \Leftrightarrow\frac{1}{\mathrm{C}}i' \Leftrightarrow \frac{\mathrm{G}}{\mathrm{C}}u \end{bmatrix} ,$$

$$D(L) = \left\{ \begin{bmatrix} i \\ u \end{bmatrix} \in H : \ i, u \in \mathrm{W}^{1,2}(0,1), \ i(1) = 0 \right\} ;$$

$$\mathrm{P} = \mathrm{F}, \ \mathrm{Q} = \mathrm{g}, \ \mathrm{R} = \mathrm{h}, \ \mathrm{D} = \mathrm{d} \ (r = m = 1);$$

$$,_0 \begin{bmatrix} i \\ u \end{bmatrix} = u(0), \ \ ,_1 \begin{bmatrix} i \\ u \end{bmatrix} = u(1)$$

but on the other hand A can be represented in a perturbed form

$$A \begin{bmatrix} y \\ i \\ u \end{bmatrix} = G \begin{bmatrix} y \\ i \\ u \end{bmatrix} + \begin{bmatrix} \mathrm{g} \\ 0 \\ 0 \end{bmatrix} \left\langle G \begin{bmatrix} y \\ i \\ u \end{bmatrix}, \begin{bmatrix} a \\ b \\ c \end{bmatrix} \right\rangle, \quad \text{where}$$

$$G\begin{bmatrix} y \\ i \\ u \end{bmatrix} = \begin{bmatrix} \mathrm{F}y \\ -\frac{1}{\mathrm{L}}u' - \frac{\mathrm{R}}{\mathrm{L}}i \\ -\frac{1}{\mathrm{C}}i' - \frac{\mathrm{G}}{\mathrm{C}}u \end{bmatrix}, \quad D(G) = D(A) \ ;$$

$a = -\frac{1}{\mathrm{L}}b(0)(\mathrm{F}^*)^{-1}\mathrm{h}$, a, b and c are solutions of the two boundary-value problems

$$\left\{ \begin{array}{ll} b''(x) = \mathrm{RG}b(x), & 0 \le x \le 1 \\ b'(0) = 0 & \\ \mathrm{d}b(0) - b(1) = \mathrm{L} & \end{array} \right\}, \quad \left\{ \begin{array}{ll} c''(x) = \mathrm{RG}c(x), & 0 \le x \le 1 \\ c(0) = 0 & \\ \mathrm{d}c'(0) - c'(1) = \mathrm{RC} & \end{array} \right\}$$

provided that G^{-1} exists.

Unfortunately, Theorem 3.1 cannot be applied. To explain this recall that from the examination of the eigenproblem for (5.21) follows that G, has a Riesz basis of generalized eigenvectors for $|\mathrm{d}| \neq 1$ (only finitely many of them are not eigenvectors). But (3.2) holds only for $\nu = 0$, and since $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \notin D(G^*)$ the perturbation does not belong to $\mathbf{B}^2(X)$, the class of Hilbert-Schmidt linear operators.

A positive answer to the existence problem of a Riesz basis formed by the eigenvectors of the operator (5.31) can be derived in other way, which we will now show.

1^o (5.31) can be written in a different perturbed form. The role of perturbation plays now the "dissipative" part of A,

$$A\begin{bmatrix} y \\ u \\ i \end{bmatrix} = G\begin{bmatrix} y \\ u \\ i \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{\mathrm{G}}{\mathrm{C}}u \\ -\frac{\mathrm{R}}{\mathrm{L}}i \end{bmatrix},$$

$$G\begin{bmatrix} y \\ u \\ i \end{bmatrix} = \begin{bmatrix} \mathrm{F}y + \mathrm{g}u(1) \\ -\frac{1}{\mathrm{C}}i' \\ -\frac{1}{\mathrm{L}}u' \end{bmatrix}, \quad D(G) = D(A) \qquad (5.32)$$

and without loss of generality we may assume that G^{-1} exists and belongs to $\mathbf{L}(X)$.

2^o G^n is an isomorphism of D_{G^n}, $\mathrm{D}_{G^n} = (D(G^n), \langle\cdot,\cdot\rangle_n)$,

$$\left\langle \begin{bmatrix} y_1 \\ u_1 \\ i_1 \end{bmatrix}, \begin{bmatrix} y_2 \\ u_2 \\ i_2 \end{bmatrix} \right\rangle_n = \left\langle G^n \begin{bmatrix} y_1 \\ u_1 \\ i_1 \end{bmatrix}, G^n \begin{bmatrix} y_2 \\ u_2 \\ i_2 \end{bmatrix} \right\rangle_X,$$

onto X and thus G has a Riesz basis of generalized eigenvectors in X iff $G_{|D(G^{n+1})}$, i.e. the part of G in D_{G^n} has a Riesz basis of generalized eigenvectors in D_{G^n}.

3^o The space D_{G^n} is formed by elements $\begin{bmatrix} y \\ u \\ i \end{bmatrix}, y \in \mathbf{C}^n;\ u, i \in \mathrm{W}^{n,2}(0,1)$ for which n pairs of equalities hold:

$$\left\{ \begin{array}{rcl} \mathrm{h}^* y & = & Lu \\ i(1) & = & 0; \\ \mathrm{h}^*\mathrm{F} y & = & \Leftrightarrow \mathrm{h}^*\mathrm{g} M u + L(\Leftrightarrow i'/\mathrm{C}) \\ u'(1) & = & 0; \\ \mathrm{h}^*\mathrm{F}^2 y & = & \Leftrightarrow \mathrm{h}^*\mathrm{Fg} M u \Leftrightarrow \mathrm{h}^*\mathrm{g} M(\Leftrightarrow i'/\mathrm{C}) + L(u''/\mathrm{LC}) \\ i''(1) & = & 0; \\ \mathrm{h}^*\mathrm{F}^3 y & = & \Leftrightarrow \mathrm{h}^*\mathrm{F}^2\mathrm{g} M u \Leftrightarrow \mathrm{h}^*\mathrm{Fg} M(\Leftrightarrow i'/\mathrm{C}) \Leftrightarrow \mathrm{h}^*\mathrm{g} M(u''/\mathrm{LC}) + \\ & & + L(\Leftrightarrow i'''/\mathrm{LC}^2) \\ u'''(1) & = & 0; \\ \ldots & & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array} \right\} \tag{5.33}$$

where $L\phi = \phi(0) \Leftrightarrow \mathrm{d}\phi(1)$, $M\phi = \phi(1)$. If the pair $(\mathrm{F}, \mathrm{h}^*)$ is *observable* then y can be uniquely determined from (5.33). To be more precise,

$$Oy = \Delta_1 u + \Delta_2 i \ , \tag{5.34}$$

where $O^* = \left[\mathrm{h}, \mathrm{F}^*\mathrm{h}, (\mathrm{F}^*)^2\mathrm{h}, \ldots, (\mathrm{F}^*)^{n-1}\mathrm{h}\right]$ is *the Kalman matrix of observability*; $\Delta_1, \Delta_2 \in \mathbf{L}(\mathrm{W}^{1,2}(0,1); \mathbf{C})$, and according to (5.33)

$$\Delta_1 \phi = \begin{bmatrix} L\phi \\ \Leftrightarrow \mathrm{h}^*\mathrm{g} M\phi \\ \Leftrightarrow \mathrm{h}^*\mathrm{Fg} M\phi + L(\frac{1}{\mathrm{LC}}\phi'') \\ \Leftrightarrow \mathrm{h}^*\mathrm{F}^2\mathrm{g} M\phi \Leftrightarrow \mathrm{h}^*\mathrm{g} M(\frac{1}{\mathrm{LC}}\phi'') \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{bmatrix} ,$$

$$\Delta_2 \phi = \begin{bmatrix} 0 \\ L(\Leftrightarrow \frac{1}{\mathrm{C}}\phi') \\ \Leftrightarrow \mathrm{h}^*\mathrm{g} M(\Leftrightarrow \frac{1}{\mathrm{C}}\phi') \\ \Leftrightarrow \mathrm{h}^*\mathrm{Fg} M(\Leftrightarrow \frac{1}{\mathrm{C}}\phi') + L(\Leftrightarrow \frac{1}{\mathrm{LC}}\frac{1}{\mathrm{C}}\phi''') \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{bmatrix} .$$

Hence

$$y = \delta_1 u + \delta_2 i; \quad \delta_1 \phi = O^{-1}\Delta_1 \phi, \quad \delta_2 \phi = O^{-1}\Delta_2 \phi \tag{5.35}$$

and finally any vector from D_{G^n} is of the form $\begin{bmatrix} \delta_1 u + \delta_2 i \\ u \\ i \end{bmatrix}$, where $i, u \in \mathrm{W}^{n,2}(0,1)$ and satisfies an appropriate number of conditions (5.33).

4^o To get an idea what is the form of $G_{|D(G^{n+1})}$, the part of G in D_{G^n}, we should add a successive, (n+1)-th pair of equalities to (5.33). The added pair of relationships can be written also in a form in which y does not appear as we may eliminate y, with the aid of (5.35), from the equation determining $\mathrm{h}^*\mathrm{F}^n y$. Moreover, an equivalent form of the equation determining $\mathrm{h}^*\mathrm{F}^n y$ may be obtained by expressing $\mathrm{h}^*\mathrm{F}^n y$ in terms of $\mathrm{h}^*\mathrm{g}$, $\mathrm{h}^*\mathrm{F}^n\mathrm{g}, \ldots, \mathrm{h}^*\mathrm{F}^{n-1}\mathrm{g}$ (here we use the Cayley-Hamilton Theorem) and employing directly (5.33). Taking the last n pairs of equalities into account, we get from the extended in such a way system (5.33), an identity

$$O\mathrm{F}y + O\mathrm{g}Mu = \Delta_1\left(\Leftrightarrow\frac{1}{\mathrm{C}}i'\right) + \Delta_2\left(\Leftrightarrow\frac{1}{\mathrm{L}}u'\right) \quad . \tag{5.36}$$

By (5.36) $G_{|D(G^{n+1})}$ can be expressed in D_{G^n} as

$$\begin{bmatrix} 0 & \delta_2(\Leftrightarrow\frac{1}{\mathrm{L}}\frac{d}{dx}) & \delta_1(\Leftrightarrow\frac{1}{\mathrm{C}}\frac{d}{dx}) \\ 0 & 0 & \Leftrightarrow\frac{1}{\mathrm{C}}\frac{d}{dx} \\ 0 & \Leftrightarrow\frac{1}{\mathrm{L}}\frac{d}{dx} & 0 \end{bmatrix} \begin{bmatrix} \delta_1 u + \delta_2 i \\ i \\ u \end{bmatrix} .$$

5^o Now, it can be proved that the spaces: D_{G^n} and $\{0\}\oplus W_{2,U}^{n-1}$ are isomorphic and under the natural isomorphism between them, the operator $G_{|D(G^{n+1})}$ is similar to $0\oplus H_{n-1}$, where $W_{2,U}^{n-1}$ denotes Shkalikov's space and H_{n-1} is Shkalikov's operator defined in $W_{2,U}^{n-1}$, corresponding to a linearization of the boundary-value problem with spectral parameter λ in the boundary conditions,

$$\left\{ \begin{array}{l} u''(x) = \mathrm{LC}\lambda^2 u(x), \qquad (\mathrm{LC} > 0) \\ \mathrm{M}(\lambda)[u(0) \Leftrightarrow \mathrm{d}u(1)] \Leftrightarrow \mathrm{L}(\lambda)u(1) = 0 \\ u'(1) = 0 \end{array} \right\} \tag{5.37}$$

where $M(\lambda) = \det(\lambda\mathrm{I} \Leftrightarrow \mathrm{F})$, $\mathrm{L}(\lambda) = \mathrm{h}^*\mathrm{adj}[\lambda\mathrm{I} \Leftrightarrow \mathrm{F}]\mathrm{g}$ ($\mathrm{d} + \frac{\mathrm{L}(\lambda)}{\mathrm{M}(\lambda)}$ expresses the controller transfer function).

The natural isomorphism is given by

$$\underbrace{\begin{bmatrix} 0 \\ v_0 \\ v_1 \end{bmatrix}}_{\in\{0\}\oplus W_{2,U}^{n-1}} = \begin{bmatrix} I & -\delta_1 & -\delta_2 \\ 0 & I & 0 \\ 0 & 0 & -\frac{1}{\mathrm{C}}\frac{d}{dx} \end{bmatrix} \underbrace{\begin{bmatrix} \delta_1 u + \delta_2 i \\ u \\ i \end{bmatrix}}_{\in \mathrm{D}_{G^n}} \tag{5.38}$$

with an inverse

$$\begin{bmatrix} \delta_1 u + \delta_2 i \\ u \\ i \end{bmatrix} = \begin{bmatrix} I & \delta_1 & \delta_2(\mathrm{C}\int_x^1 (\cdot) ds) \\ 0 & I & 0 \\ 0 & 0 & \mathrm{C}\int_x^1 (\cdot) ds \end{bmatrix} \begin{bmatrix} 0 \\ v_0 \\ v_1 \end{bmatrix} .$$

The similarity relation can be written in terms of the operator matrices

$$\begin{bmatrix} I & -\delta_1 & -\delta_2 \\ 0 & I & 0 \\ 0 & 0 & -\frac{1}{\mathrm{C}}\frac{d}{dx} \end{bmatrix} \begin{bmatrix} 0 & \delta_2(-\frac{1}{\mathrm{L}}\frac{d}{dx}) & \delta_1(-\frac{1}{\mathrm{C}}\frac{d}{dx}) \\ 0 & 0 & -\frac{1}{\mathrm{C}}\frac{d}{dx} \\ 0 & -\frac{1}{\mathrm{L}}\frac{d}{dx} & 0 \end{bmatrix} \cdot$$

$$\cdot \begin{bmatrix} I & \delta_1 & \delta_2(\mathrm{C}\int_x^1 (\cdot) ds) \\ 0 & I & 0 \\ 0 & 0 & \mathrm{C}\int_x^1 (\cdot) ds \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & I \\ 0 & \frac{1}{\mathrm{LC}}\frac{d^2}{dx^2} & 0 \end{bmatrix}$$

and is determined up to some evident correspondence between boundary conditions. Hence, the part $G_{|D(G^{n+1})}$ of G in D_{G^n} has a Riesz basis formed by generalized eigenvectors iff for Shkalikov's operator H_{n-1} defined in Shkalikov's space $W_{2,U}^{n-1}$ the same property holds. But by Theorem 4.1 this occurs if the boundary conditions (5.37) are strictly regular. Observe that (5.37) can be obtained directly from the eigenproblem for G by eliminating y and i. The fundamental system of solution of (5.37) is built of two exponential functions $x \mapsto e^{-\lambda\sqrt{\mathrm{LC}}x}, e^{\lambda\sqrt{\mathrm{LC}}x}$ and according to (4.5) the characteristic equation of the problem (5.37) has the form

$$\Delta(\lambda) = 2\lambda\sqrt{\mathrm{LC}}\{\mathrm{M}(\lambda)\cosh\lambda\sqrt{\mathrm{LC}} - [\mathrm{L}(\lambda) + \mathrm{dM}(\lambda)]\} = 0 \ . \tag{5.39}$$

The regularity of boundary conditions easily follows from (5.39). Asymptotic distribution of zeros of an entire function Δ can be derived by dividing both sides of (5.39) by $\mathrm{M}(\lambda)$ ($\mathrm{M}(\lambda) \neq 0$ if $|\lambda| > \|\mathrm{F}\|$). Since $\lim_{|\lambda|\to\infty} \frac{\mathrm{L}(\lambda)}{\mathrm{M}(\lambda)} \mapsto 0$, the asymptotic zeros agree with the roots of the equation $\cosh \lambda\sqrt{\mathrm{LC}} = \mathrm{d}$, and thus for $|\mathrm{d}| \neq 1$ they are simple. Finally, the boundary conditions (5.37) are strongly regular, provided that $|\mathrm{d}| \neq 1$.

6^o The whole discussion of spectral properties of G can be summarized as

Lemma 5.3 *If the pair (F,h^*) is observable and $|d| \neq 1$, then G has a Riesz basis of generalized eigenvectors in X and only finitely many of them are not eigenvectors. The spectrum $\sigma(G)$ is asymptotically equal to the spectrum of an unperturbed operator related to proportional controller only.*

7^o Similarly as it was done for an entire function Δ we may point out that the spectrum of A is asymptotically equal to the spectrum of the perturbed operator related to a purely proportional controller. Such an operator has been already examined. Those investigations, Lemma 5.3 and Theorem 3.2 lead to

Lemma 5.4 *If the pair (F,h^*) is observable and $|d| \neq 1$, then the operator A has a Riesz basis of generalized eigenvectors in X and only finitely many of them are not eigenvectors. Moreover, A generates a linear C_0-group on X.*

5.2 Example 2: Stabilization system of an elastic robot arm

A stabilization system of an elastic robot arm has been discussed in [8]. Its dynamical model is governed by the system of equations

$$\left\{\begin{array}{rcl} \dot{y}(t) & = & \Leftrightarrow kay(t) + k\mathrm{EI}w_{xx}(0,t) \\ w_t(x,t) & = & v(x,t) \\ v_t(x,t) & = & \Leftrightarrow \frac{\mathrm{EI}}{\mathrm{m}} w_{xxxx}(x,t) \end{array}\right\}, \quad 0 \leq x \leq 1, \quad t \geq 0 \tag{5.40}$$

with the boundary conditions

$$w(0,t) = 0, \quad w_{xx}(1,t) = 0, \quad w_{xxx}(1,t) = 0, \quad w_x(0,t) = y(t); \quad t \geq 0 \ . \tag{5.41}$$

The last two equations of (5.40) correspond to the Euler-Bernoulli model of an elastic beam, while the first equation constitutes the dynamic equation of a stabilizing controller acting in a feedback loop with boundary observation and boundary control.

Taking $n = m = r = 1$, $H = \mathrm{L}^2(0,1) \oplus \mathrm{L}^2(0,1)$ we may represent the above dynamical model in the abstract form (2.1) with

$$A \begin{bmatrix} y \\ w \\ v \end{bmatrix} = \begin{bmatrix} \Leftrightarrow kay + k\mathrm{EI}w''(0) \\ v \\ \Leftrightarrow \frac{\mathrm{EI}}{\mathrm{m}} w'''' \end{bmatrix}, \quad D(A) = \{ \begin{bmatrix} y \\ w \\ v \end{bmatrix} \in \mathbf{C} \oplus H :$$

$$w \in \mathrm{H}^4(0,1), \quad w(0) = 0, \quad w''(1) = 0, \quad w'''(1) = 0, \quad w'(0) = y\} \ ;$$

$$\mathrm{P} = [\Leftrightarrow ka], \quad \mathrm{Q} = [k\mathrm{EI}], \quad \mathrm{R} = [1], \quad \mathrm{D} = [0]; \quad L \begin{bmatrix} w \\ v \end{bmatrix} = \begin{bmatrix} v \\ \Leftrightarrow \frac{\mathrm{EI}}{\mathrm{m}} w'''' \end{bmatrix} \ ,$$

$$D(L) = \left\{ \begin{bmatrix} w \\ v \end{bmatrix} \in H : \ w \in \mathrm{H}^4(0,1), \ w(0) = 0, w''(1) = 0, \ w'''(1) = 0 \right\} \ ;$$

$$,_0 \begin{bmatrix} w \\ v \end{bmatrix} = w'(0), \quad ,_1 \begin{bmatrix} w \\ v \end{bmatrix} = w''(0) \ .$$

Any system of eigenvectors of A cannot form a Riesz basis in the space $X = \mathbf{C} \oplus H = \mathbf{C} \oplus \mathrm{L}^2(0,1) \oplus \mathrm{L}^2(0,1)$. Indeed, if it would be the case then the component systems: $\{w_n\}_{n=1}^{\infty}, \{v_n\}_{n=1}^{\infty}$ would form Riesz bases in $\mathrm{L}^2(0,1)$. Hence these systems would be uniformly bounded with respect to $n \in \mathbf{N}$ (i.e. *quasinormalized*) and since $v_n = \lambda_n w_n$, the sequence of eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ would be bounded. This contradicts the fact that A has countably many eigenvalues and is unbounded. The use of our spectral approach to solve the question of well-posedness of the system under investigation is therefore impossible. However, following [8], we may seek a realization of an abstract dynamical model on a smaller space, choosing

$$H = \mathrm{H}_0^2(0,1) \oplus \mathrm{L}^2(0,1), \quad \mathrm{H}_0^2(0,1) = \{u \in \mathrm{H}^2(0,1) : \ u(0) = 0\}$$

as a candidate for the state space. The realization of an abstract dynamical model on this space is

$$\frac{d}{dt} \begin{bmatrix} w \\ v \end{bmatrix} = \tilde{A} \begin{bmatrix} w \\ v \end{bmatrix}, \quad \tilde{A} \begin{bmatrix} w \\ v \end{bmatrix} = \begin{bmatrix} v \\ \Leftrightarrow \frac{\mathrm{EI}}{\mathrm{m}} w'''' \end{bmatrix} \ ,$$

$$D(\tilde{A}) = \{ \begin{bmatrix} w \\ v \end{bmatrix} \in H : \ w \in \mathrm{H}^4(0,1) \cap \mathrm{H}_0^2(0,1),$$

$$w(0) = 0, \quad w''(1) = 0, \quad w'''(1) = 0, \quad v'(0) = \Leftrightarrow kaw'(0) + k\mathrm{EI}w''(0)\} \ . \tag{5.42}$$

Observe that an equation describing the feedback appears in the definition of the domain of $\tilde{A}$ and thus we cannot represent $\tilde{A}$ in the form: "the main part + perturbation, related to the feedback". This makes it impossible to apply the perturbation methods to solve the question whether $\tilde{A}$ generates a linear C_0-semigroup on H. Gnedin [8] has found a scalar product in H, equivalent to the natural one, with respect to which $\tilde{A}$ is *maximally dissipative*. This means that $\tilde{A}$ generates a linear C_0-semigroup of contractions on H. Applying the Prüss-Huang-Weiss criterion, Gnedin has pointed out that this semigroup is (**EXS**) iff $k > 0$.

Now, we show how the results of [8] can be derived and even strengthened by a direct spectral analysis based on Shkalikov's theory. The eigenproblem for $\tilde{A}$ takes the form

$$\left\{ \begin{array}{rcl} v & = & \lambda w \\ \Leftrightarrow\frac{\mathrm{EI}}{\mathrm{m}} w'''' & = & \lambda v \\ v'(0) & = & k\mathrm{EI}w''(0) \Leftrightarrow k a w'(0) \\ w(0) & = & 0 \\ w''(1) & = & 0 \\ w'''(1) & = & 0 \end{array} \right\} \tag{5.43}$$

and leads to the following Sturm-Liouville problem with a spectral parameter entering the boundary conditions,

$$\left\{ \begin{array}{rcl} \frac{\mathrm{EI}}{\mathrm{m}} w'''' + \lambda^2 w(x) & = & 0, \quad 0 \leq x \leq 1 \\ (\lambda + ka)w'(0) & = & k\mathrm{EI}w''(0) \\ w'''(1) & = & 0 \\ w''(1) & = & 0 \\ w(0) & = & 0 \end{array} \right\} . \tag{5.44}$$

Substituting $\lambda = \rho^2 \sqrt{\frac{\mathrm{EI}}{\mathrm{m}}}$, $\tilde{a} = a/(\mathrm{EI})$, $\mathrm{K} = k\sqrt{\mathrm{mEI}}$ we get from (5.44),

$$\left\{ \begin{array}{rcl} w'''' + \rho^4 w(x) & = & 0, \qquad 0 \leq x \leq 1 \\ (\rho^2 + \mathrm{K}\tilde{a})w'(0) & = & \mathrm{K}w''(0) \\ w'''(1) & = & 0 \\ w''(1) & = & 0 \\ w(0) & = & 0 \end{array} \right\} . \tag{5.45}$$

The orders of successive boundary conditions in (5.45) are $\kappa_1 = 3$, $\kappa_2 = 3$, $\kappa_3 = 2$, $\kappa_4 = 0$, respectively, while coefficients $p_{\nu s}$ in the first equation of (5.45) are constant. The characteristic polynomial of (5.45) is $\omega^4 + 1 = (\omega^2 \Leftrightarrow \sqrt{2}\omega + 1)(\omega^2 + \sqrt{2}\omega + 1) = 0$.

The construction of the smallest convex polygon $\mathcal{M}$, containing all possible sums of the roots $\omega_1, \omega_2, \omega_3, \omega_4$ of the characteristic polynomial is depicted in Fig.3. Assuming the solution of the first equation of (5.45) to be

$$w(x) = C_1 e^{\rho\omega_1 x} + C_2 e^{\rho\omega_2 x} + C_3 e^{\rho\omega_3 x} + C_4 e^{\rho\omega_4 x}$$

and putting w into the remaining equations of the system (5.45) we get

$$\begin{bmatrix} f(\rho,\omega_1) & f(\rho,\omega_2) & f(\rho,\omega_3) & f(\rho,\omega_4) \\ \rho^3\omega_1^3 e^{\rho\omega_1} & \rho^3\omega_2^3 e^{\rho\omega_2} & \rho^3\omega_3^3 e^{\rho\omega_3} & \rho^3\omega_4^3 e^{\rho\omega_4} \\ \rho^2\omega_1^2 e^{\rho\omega_1} & \rho^2\omega_2^2 e^{\rho\omega_2} & \rho^2\omega_3^2 e^{\rho\omega_3} & \rho^2\omega_4^2 e^{\rho\omega_4} \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

where $f(\rho,\omega) = \omega\rho(\rho^2 + \mathrm{K}\tilde{a}) \Leftrightarrow \mathrm{K}\rho^2\omega^2$. The above homogeneous, linear system has a nonzero solution iff ρ is a root of the characteristic determinant

$$\Leftrightarrow 2\rho^6\{e^{\imath\rho\sqrt{2}}\left[\rho^2 + \mathrm{K}\tilde{a} + \sqrt{2}\imath\mathrm{K}\rho\right] + e^{-\rho\sqrt{2}}\left[\rho^2 + \mathrm{K}\tilde{a} \Leftrightarrow \sqrt{2}\mathrm{K}\rho\right] + \\ + e^{-\imath\rho\sqrt{2}}\left[\rho^2 + \mathrm{K}\tilde{a} \Leftrightarrow \sqrt{2}\imath\mathrm{K}\rho\right] + e^{\rho\sqrt{2}}\left[\rho^2 + \mathrm{K}\tilde{a} + \sqrt{2}\mathrm{K}\rho\right] + 4(\rho^2 + \mathrm{K}\tilde{a})\}$$

which can be represented as

$$\Leftrightarrow 2\rho^8\{e^{\imath\rho\sqrt{2}}\left[1 + \sqrt{2}\imath\mathrm{K}\rho^{-1} + \mathrm{K}\tilde{a}\rho^{-2}\right] + e^{-\rho\sqrt{2}}\left[1 \Leftrightarrow \sqrt{2}\mathrm{K}\rho^{-1} + \mathrm{K}\tilde{a}\rho^{-2}\right] + \\ + e^{-\imath\rho\sqrt{2}}\left[1 \Leftrightarrow \sqrt{2}\imath\mathrm{K}\rho^{-1} + \mathrm{K}\tilde{a}\rho^{-2}\right] + e^{\rho\sqrt{2}}\left[1 + \sqrt{2}\mathrm{K}\rho^{-1} + \mathrm{K}\tilde{a}\rho^{-2}\right] + \\ + 4(1 + \mathrm{K}\tilde{a}\rho^{-2})\} \ .$$

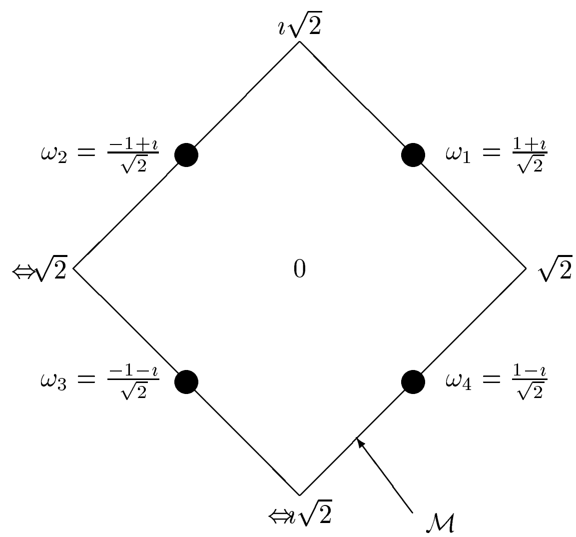

Figure 3: The construction of the polygon $\mathcal{M}$

Since for large $|\rho|$, 1 is the dominating term of each expression in the square brackets, the boundary conditions (5.45) are regular. Substituting $\eta = \frac{1-\imath}{\sqrt{2}}\rho$, the analysis of asymptotic form of the spectrum can be repeated - see [8, Theorem 3; it states that $Re\lambda_n \Leftrightarrow\!\!\to \Leftrightarrow 2k\sqrt{\mathrm{mEI}}$ and $|\lambda_n| \approx \pi^2(\frac{n+1}{2})^2\sqrt{\frac{\mathrm{EI}}{\mathrm{m}}}$ as $n \Leftrightarrow\!\!\to \infty$, $\{\lambda_n\}_{n\in\mathbf{N}} = \sigma(\tilde{A})$)], from which we deduce that the boundary conditions are in fact strictly regular. Now, by Theorem 4.1 a realization of $\tilde{A}$ on a suitable space has a system of eigenvectors forming a Riesz basis. To determine both the state space and the particular realization of $\tilde{A}$ we take $r = 0$. The orders of all boundary conditions are not greater than $n + r \Leftrightarrow 1 = 3$. The second part of Theorem 4.1 applies ($N_0 = 0$). The state space is therefore $W^0_{2,U}$, while the operator H takes a form

$$\mathrm{H}\tilde{v} = \mathrm{H}\begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \Leftrightarrow v_0'''' \end{bmatrix} .$$

According to the rule (4.9), the term $\rho^2 w'(0)$, for which $\nu = 2$, $k = 1$ should be replaced by $(\mathrm{H}^2\tilde{v})^{(1)}_0 = v_2'(0)$ as $\nu + k = 3 < n + r = 4$. As a result the first boundary condition now is

$$v_2'(0) + \tilde{a}\mathrm{K}v_0'(0) \Leftrightarrow \mathrm{K}v_0''(0) = 0$$

while the remaining ones do not change essentially,

$$v_0'''(1) = 0, \quad v_0''(1) = 0, \quad v_0(0) = 0 \ .$$

An exact form of Shkalikov's space can be determined to be

$$W^0_{2,U} = \{\tilde{v} \in W^0_2 : \ \ U_j(\mathrm{H}^k\tilde{v}) = 0 \quad \text{for} \quad 0 \le k \le n + r \Leftrightarrow 2 = 2$$
$$\text{and all boundary conditions of order } \ \le 2 \Leftrightarrow k\} =$$
$$= \{\tilde{v} = \begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathrm{W}^3_2(0,1) \oplus \mathrm{W}^2_2(0,1) \oplus \mathrm{W}^1_2(0,1) \oplus \mathrm{L}^2(0,1) : \ v_0''(1) = 0 \ ,$$
$$v_0(0) = 0 \ \ (k = 0); \ \ v_1(0) = 0 \ \ (k = 1); \ \ v_2(0) = 0 \ \ (k = 2)\} \ .$$

By (4.13) the Shkalikov operator H_0 takes the form (now $r = 1$)

$$H_0\begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix} = \mathrm{H}\begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \Leftrightarrow v_0'''' \end{bmatrix}, \quad D(H_0) = W^1_{2,U} =$$
$$= \{\tilde{v} = \begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathrm{W}^4_2(0,1) \oplus \mathrm{W}^3_2(0,1) \oplus \mathrm{W}^2_2(0,1) \oplus \mathrm{W}^1_2(0,1) :$$

$$U_j(\mathrm{H}^k\tilde{v}) = 0 \quad \text{for} \quad 0 \le k \le n + r \Leftrightarrow 2 = 3 \quad \text{and}$$

$$\text{all boundary conditions of order} \le n + r \Leftrightarrow k \Leftrightarrow 2 = 3 \Leftrightarrow k\} =$$

$$= \{\tilde{v} = \begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathrm{W}_2^4(0,1) \oplus \mathrm{W}_2^3(0,1) \oplus \mathrm{W}_2^2(0,1) \oplus \mathrm{W}_2^1(0,1) \quad :$$

$$v_2'(0) + \tilde{a}\mathrm{K}v_0'(0) \Leftrightarrow \mathrm{K}v_0''(0) = 0, \; v_0'''(1) = 0,$$

$$\boxed{v_0''(1) = 0}\,, \; \boxed{v_0(0) = 0} \;\; (k = 0) \;\; ;$$

$$v_1''(1) = 0, \; \boxed{v_1(0) = 0} \; (k = 1); \boxed{v_2(0) = 0} \; (k = 2); \; v_3(0) = 0 \;\; (k = 3)\} \;\; .$$

The boundary conditions marked by frames also enter the definition of state space.

The relationship between (5.44) and (5.45) suggests a consideration of the operator H_0^2,

$$H_0^2 \begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_2 \\ v_3 \\ \Leftrightarrow v_0'''' \\ \Leftrightarrow v_1'''' \end{bmatrix} =$$

$$= H_0 \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \Leftrightarrow v_0'''' \end{bmatrix} \tag{5.46}$$

$$W_{2,U}^0 \supset D(H_0^2) = \{\tilde{v} = \begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix} \in D(H_0) : H_0\tilde{v} \in D(H_0)\} =$$

$$= \{\tilde{v} \in \mathrm{W}_2^5(0,1) \oplus \mathrm{W}_2^4(0,1) \oplus \mathrm{W}_2^3(0,1) \oplus \mathrm{W}_2^2(0,1) : \text{the components of}$$

$$\tilde{v} \text{ satisfy those boundary conditions which enter the definition of } D(H_0)$$

$$\text{and the following additional boundary conditions:}$$

$$v_3'(0) + \tilde{a}\mathrm{K}v_1'(0) \Leftrightarrow \mathrm{K}v_1''(0) = 0, \quad v_1'''(1) = 0, \quad v_1''(1) = 0, \quad v_1(0) = 0 \;\; ;$$

$$v_2''(1) = 0, \quad v_2(0) = 0; \quad v_3(0) = 0; \quad v_0''''(0) = 0\} \;\; . \tag{5.47}$$

Now, it is not difficult to see that two groups of variables can be extracted from $\tilde{v}$-the first includes the variable v_0 and v_2, the other - variables v_1 and v_3. This corresponds to a decomposition of H_0^2 into the direct sum of two operators. One of them, connected with the group of relationships including only the variables v_1, v_3 is marked by frames in (5.46). To be more precise, we have defined the operator

$$H_{0R}^2 \begin{bmatrix} W \\ V \end{bmatrix} = \begin{bmatrix} V \\ \Leftrightarrow W'''' \end{bmatrix},$$

$$D(H_{0R}^2) = \{ \begin{bmatrix} W \\ V \end{bmatrix} \in H : \ W \in \mathrm{H}^4(0,1) \cap \mathrm{H}_0^2(0,1),$$

$$W(0) = 0, \quad W''(1) = 0, \quad W'''(1) = 0, \quad V'(0) + \mathrm{K}\tilde{a}W'(0) = \mathrm{K}W''(0)\} \ .$$

Since by Theorem 4.1 H_{0R}^2 has a system of generalized eigenvectors (only finitely many of them are not eigenvectors) which forms a Riesz basis in $H = \mathrm{H}_0^2(0,1) \oplus \mathrm{L}^2(0,1)$ and $S^{-1}H_{0R}^2 S = \sqrt{\frac{\mathrm{EI}}{\mathrm{m}}}\tilde{A}$, where

$$\begin{bmatrix} W \\ V \end{bmatrix} = S \begin{bmatrix} w \\ v \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{\mathrm{EI}}{\mathrm{m}}} w \\ v \end{bmatrix} \ ,$$

the same holds for $\tilde{A}$. Recall that $\imath\eta^2 = \rho^2 = \lambda\sqrt{\frac{\mathrm{m}}{\mathrm{EI}}}$ and in virtue of the above mentioned result of [8, Theorem 3], the spectrum of $\tilde{A}$ is located in a vertical strip. $\tilde{A}$ can be decomposed into the direct sum of two operators - one acts in a finite-dimensional subspace spanned by generalized eigenvectors corresponding to nonsimple eigenvalues, the second acts on a complementary subspace spanned by eigenvectors of $\tilde{A}$. Clearly, the finite-dimensional component generates a C_0-group. By Theorem 1.1 the infinite-dimensional part generates a C_0-group as well and thus $\tilde{A}$ generates a C_0-group on H. Note that Gnedin obtained only the generation of a C_0-semigroup.

For $k > 0$ we have $\sup\{Re\lambda : \lambda \in \sigma(\tilde{A})\} < 0$, and thus this semigroup is (**EXS**) in the topology induced by the norm of H. We have established this result without the use of the Prüss-Huang-Weiss criterion.

5.3 Example 3: Multidimensional neutral system

In this subsection we will examine spectral properties of the operator

$$A \begin{bmatrix} v \\ \psi \end{bmatrix} = \begin{bmatrix} \mathrm{A}v + (\mathrm{AC} + \mathrm{B})\psi(\Leftrightarrow r) \\ \psi' \end{bmatrix}, \quad D(A) = \{ \begin{bmatrix} v \\ \psi \end{bmatrix} \in \mathbf{M}^2 =$$

$$\mathbf{C}^n \oplus \mathrm{L}^2(\Leftrightarrow r, 0; \mathbf{C}^n) : \quad \psi \in \mathrm{W}^{1,2}(\Leftrightarrow r, 0; \mathbf{C}^n), \quad v = \psi(0) \Leftrightarrow \mathrm{C}\psi(\Leftrightarrow r)\} \ . \tag{5.48}$$

Our investigations are motivated by the fact that A arises in the problem of building up an abstract differential equation on $\mathbf{M}^2$ related to the delay-differential system of neutral type,

$$\left\{\begin{array}{lcll} \dot{v}(t) & = & \mathrm{A}v(t) + (\mathrm{AC}+\mathrm{B})x(t \Leftrightarrow r), & t \geq 0 \\ v(t) & = & x(t) \Leftrightarrow \mathrm{C}x(t \Leftrightarrow r), & t \geq 0 \\ v(0) & = & v_0 & \\ x(\theta) & = & \phi(\theta), & \Leftrightarrow r \leq \theta \leq 0 \end{array}\right\}$$

-see [10, Section 6.2.5].

If $0 \notin \sigma(A)$ then A is an isomorphism of D_A onto $\mathbf{M}^2$. Thus the existence of a Riesz basis of eigenvectors of A in $\mathbf{M}^2$ is equivalent to the existence of a Riesz basis of eigenvectors of $A_{|D(A^2)}$ in D_A, where D_A denotes the Hilbert space $(D(A), \langle\cdot,\cdot\rangle_A)$, $\langle a, b\rangle_A = \langle a, b\rangle_{\mathbf{M}^2} + \langle Aa, Ab\rangle_{\mathbf{M}^2}$. Notice that any element of D_A has the form:

$$\left[\begin{array}{c} L\psi \\ \psi \end{array}\right] = T\left[\begin{array}{c} 0 \\ \psi \end{array}\right], \quad T = \left[\begin{array}{cc} I & L \\ 0 & I \end{array}\right], \quad L\psi = \psi(0) \Leftrightarrow \mathrm{C}\psi(\Leftrightarrow r) \ .$$

The operator matrix T defines an isomorphism of D_A onto $\{0\}\oplus \mathrm{W}^{1,2}(\Leftrightarrow r, 0; \mathbf{C}^n)$. Now

$$T^{-1}A_{|D(A^2)}T\left[\begin{array}{c} 0 \\ \psi \end{array}\right] = \left[\begin{array}{cc} I & \Leftrightarrow L \\ 0 & I \end{array}\right]\left[\begin{array}{cc} 0 & L\frac{d}{dx} \\ 0 & \frac{d}{dx} \end{array}\right]\left[\begin{array}{cc} I & L \\ 0 & I \end{array}\right]\left[\begin{array}{c} 0 \\ \psi \end{array}\right] =$$

$$= \left[\begin{array}{cc} 0 & 0 \\ 0 & \tilde{A} \end{array}\right]\left[\begin{array}{c} 0 \\ \psi \end{array}\right] \ \forall \psi \in D(\tilde{A}) \subset \mathrm{W}^{1,2}(\Leftrightarrow r, 0; \mathbf{C}^n) \Longleftrightarrow \left[\begin{array}{c} L\psi \\ \psi \end{array}\right] \in D(A^2) \ ,$$

where

$$\tilde{A}\psi = \psi' \ ,$$

$$D(\tilde{A}) = \{\psi \in \mathrm{W}^{2,2}(\Leftrightarrow r, 0; \mathbf{C}^n) : \mathrm{A}\psi(0) + \mathrm{B}\psi(\Leftrightarrow r) = \psi'(0) \Leftrightarrow \mathrm{C}\psi'(\Leftrightarrow r)\} \ . \tag{5.49}$$

This means that A has a system of eigenvectors which forms a Riesz basis in $\mathbf{M}^2$ if and only if $\tilde{A}$ has a system of eigenvectors which forms a Riesz basis in $\mathrm{W}^{1,2}(\Leftrightarrow r, 0; \mathbf{C}^n)$.

The operator (5.48) is a particular form of (2.1) with $X = \mathbf{M}^2$, $H = \mathrm{L}^2(\Leftrightarrow r, 0; \mathbf{C}^n)$; $L\psi = \psi'$, $D(L) = \mathrm{W}^{1,2}(\Leftrightarrow r, 0; \mathbf{C}^n)$; $\mathrm{P} = \mathrm{A}$, $\mathrm{Q} = \mathrm{AC}+\mathrm{B}$, R=I, D=C; , $_0\psi = \psi(0)$, , $_1\psi = \psi(\Leftrightarrow r)$. (2.2) reduces to

$$\left\{\begin{array}{rcl} \psi' & = & \lambda\psi, \quad \psi \in \mathrm{W}^{1,2}(\Leftrightarrow r, 0; \mathbf{C}^n) \\ (\lambda\mathrm{C}+\mathrm{B})\psi(\Leftrightarrow r) & = & (\lambda\mathrm{I} \Leftrightarrow \mathrm{A})\psi(0) \end{array}\right\} \tag{5.50}$$

i.e. to (4.14) with $\mathrm{P}(x) = 0$, $\Leftrightarrow r \leq x \leq 0$; $\mathrm{A}_0 = \mathrm{B}$, $\mathrm{A}_1 = \mathrm{C}$, $\mathrm{B}_0 = \mathrm{A}$, $\mathrm{B}_1 = \Leftrightarrow \mathrm{I}$. All boundary conditions are of the first order and (4.15) is identical with the eigenproblem for $\tilde{A}$. Since the characteristic function is $\Delta(\lambda) = \det(\lambda \mathrm{I} - \lambda e^{-\lambda r}\mathrm{C} - \mathrm{A} - e^{-\lambda r}\mathrm{B})$, we have in Definition 4.4: $p_0(\lambda) = (-1)^n \det(\lambda \mathrm{I} - \mathrm{A})$, $\deg p_0(\lambda) = n$; $p_1(\lambda) = \det(\lambda \mathrm{C} + \mathrm{B})$, $\deg p_n(\lambda) = n$, provided that $\det \mathrm{C} \neq 0$. In virtue of Theorem 4.2, $\tilde{A}$ has a system of eigenvectors which forms a Riesz basis in $\mathrm{W}^{1,2}(-r, 0; \mathbf{C}^n)$ if $\det \mathrm{C} \neq 0$, all eigenvalues of $\tilde{A}$ are simple and $\inf\{|\lambda - \mu| : \lambda,\ \mu \in \sigma(A),\ \lambda \neq \mu\}$. The assumption $0 \notin \sigma(A)$ is not essential. Thus we have proved

Lemma 5.5 *If* $\det C \neq 0$, *all eigenvalues of* A *are simple and* $\inf\{|\lambda - \mu|: \lambda,\ \mu \in \sigma(A),\ \lambda \neq \mu\} > 0$ *then there exists a system of eigenvectors forming a Riesz basis in* $\mathbf{M}^2$.

For comparison, let us try to use Theorem 3.1. To do this we represent (5.48) in a particular form

$$A \begin{bmatrix} v \\ \psi \end{bmatrix} = G \begin{bmatrix} v \\ \psi \end{bmatrix} + \sum_{j=1}^{n} \begin{bmatrix} e_j \\ 0 \end{bmatrix} \left\langle \begin{bmatrix} p_j \\ q_j \end{bmatrix}, G \begin{bmatrix} v \\ \psi \end{bmatrix} \right\rangle_{\mathbf{M}^2} ,$$

where

$$G \begin{bmatrix} v \\ \psi \end{bmatrix} = \begin{bmatrix} \mathrm{A}v \\ \psi' \end{bmatrix} ,$$

$\{e_j\}_{j=1}^n$ is a Cartesian basis in $\mathbf{C}^n$; $\{\beta_j^T\}_{j=1}^n$ denotes the sequence of rows of the matrix AC $+$ B; $\{p_j\}_{j=1}^n$, $p_j = (\mathrm{A}^*)^{-1}(\mathrm{I} - \mathrm{C}^*)^{-1}\beta_j \in \mathbf{C}^n$; $\{q_j\}_{j=1}^n$ denotes the sequence of constant functions defined on $[-r, 0]$ with their values $(\mathrm{C}^* - \mathrm{I})^{-1}\beta_j \in \mathbf{C}^n$. We assume that G is invertible, which is equivalent to two requirement: $\det \mathrm{A} \neq 0$, $1 \notin \sigma(\mathrm{C})$. Moreover, $G^{-1} \in \mathbf{B}_\infty(\mathbf{M}^2)$. The spectrum of G is a union of the spectrum of the matrix A and the operator

$$Q\psi = \psi', \quad D(Q) = \{\psi \in \mathrm{W}^{1,2}(-r, 0; \mathbf{C}^n) : \psi(0) - \mathrm{C}\psi(-r) = 0\} .$$

If all elementary divisors of C are linear then its modal matrix S is nonsingular as it is constructed of n linearly-independent eigenvectors of C. Under the isomorphism induced in $\mathrm{L}^2(-r, 0; \mathbf{C}^n)$ by the multiplication operator $\mathrm{S}\phi(\theta) = \psi(\theta)$, Q is similar to

$$R\phi = \phi', \quad D(R) = \{\phi \in \mathrm{W}^{1,2}(-r, 0; \mathbf{C}^n) \ :$$
$$\phi(0) - \mathrm{diag}\{\mu_1, \mu_2, \ldots, \mu_n\}\phi(-r) = 0\}, \quad \{\mu_j\}_{j=1}^n = \sigma(\mathrm{C}) .$$

Suppose that $\det \mathrm{C} \neq 0$. Then, any scalar eigenproblem to which the eigenproblem for R can be decomposed, generates a Riesz basis of exponentials $\{\exp(\lambda_{j,k}(\cdot))\}_{k \in \mathbf{Z}}$ in $\mathrm{L}^2(-r, 0)$, where $\lambda_{j,k} = \frac{1}{r}[\ln|\mu_j| + \imath(\arg\mu_j + 2k\pi)]$, $k \in \mathbf{Z}$. This follows from the spectral properties of the RC-transmission line steered only by a proportional controller. Now,

$$\{\exp(\lambda_{j,k}(\cdot))e_j\}_{j=1,2,\cdots,n;k\in\mathbf{Z}} \quad \text{and} \quad \{\exp(\lambda_{j,k}(\cdot))\mathrm{S}e_j\}$$

are the systems of eigenvectors of Q and R, respectively, which form Riesz bases in $\mathrm{L}^2(\Leftrightarrow r, 0; \mathbf{C})$. After completing the first system by the n-tuple of generalized eigenvectors of A we get a Riesz basis in $\mathbf{M}^2$ of generalized eigenvectors of G and only finitely many of then are not eigenvectors. However, the condition (3.2) is not satisfied for any $\nu \in (0,1)$, while for $\nu = 0$ the perturbation does not belong to $\mathbf{B}_2(\mathbf{M}^2)$ except for the trivial case AC+B= 0, when $G = A$. Thus if AC+B$\neq$ 0, Theorem 3.1 cannot be applied.

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