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# Viscosity Solutions of Hamilton–Jacobi Equations Arising in Nonlinear $H_\infty$–Control*

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**Abstract**

This note extends the necessary conditions for the solvability of a nonlinear $H_\infty$-control problem obtained in previous work of the authors to the case where the required energy function on the state space is not assumed to be smooth. This leads to consideration of viscosity subsolutions of the Hamilton-Jacobi equations associated with the nonlinear bounded real lemma and $H_\infty$-control.

## Introduction

The nonlinear $H_\infty$ control problem (i.e. the problem of selecting a stabilizing state or measurement feedback subject to an $L_2$-gain constraint for the closed loop input-output map) recently has been intensively studied (see [1, 2, 19, 14, 15, 21, 22, 23]). All these works involve the investigation of a Hamilton-Jacobi type equation which is satisfied by the *storage* or *energy* function associated with the closed loop system if this function happens to be smooth. However in general this storage function need not be smooth and hence does not correspond to a solution of the Hamilton-Jacobi equation in the classical sense. M. Crandall and P.L. Lions [5] have discovered a notion of weak, or so-called "viscosity" solution (or "subsolution") for a Hamilton-Jacobi equation which has since been applied in various optimal control and differential game contexts (see e.g. [20, 8]), including applications to systems governed by partial differential equations. Most of these

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problems are time-varying and finite-horizon. The $H_\infty$-problem in the nonlinear context is just beginning to attract attention. While it is closely related to older work in differential game theory, there are differences; the $H_\infty$ setting emphasizes the time-invariant, infinite horizon context where stability is a major issue. For example, in the $H_\infty$-theory solutions of an algebraic Riccati equation are uniquely determined by a stabilizing side condition whereas in the time-varying finite horizon context solutions of a differential Riccati equation are uniquely specified by an initial condition. Another major distinctive feature of $H_\infty$ control is its emphasis is on measurement feedback, a feature not treated in traditional differential games. The recent book [1] treats the $H_\infty$-control problem (including some results for the measurement feedback problem) from the game theory point of view; these authors use dynamic programming ideas for the smooth case and convert to a Hamiltonian rather than viscosity solution formulation to handle some nonsmooth cases. A recent treatment of the role of viscosity solutions in both deterministic and stochastic control problems is [10].

The purpose of this paper is to extend one of the existing studies on the nonlinear $H_\infty$-control problem (specifically the necessity analysis of the authors in [2] on the measurement feedback problem) to the case of a nonsmooth storage function. In this paper we deal only with the $L_2$-gain condition in the formulation of the $H_\infty$-control problem; the internal stability side condition is a separate issue which under appropriate conditions can be handled in the same way as in the smooth case (see [14, 15]). As preparation and to keep the paper self-contained, we also develop the ideas in detail for the state feedback problem. We invite others with more expertise in nonsmooth analysis to improve these results in due time.

The first section of the paper discusses the role of a storage or energy function for a $\gamma$-gain stable nonlinear system as originally set down in [24] and [12]. We recall the recent result of James [18] that a storage function for an finite-gain stable system is necessarily a viscosity subsolution for a related Hamilton-Jacobi equation and conversely. Here we add the result that the available storage function under certain conditions is actually a viscosity solution of the Hamilton-Jacobi equation. Related results are well known in various control and differential game contexts and in fact were a motivation for the introduction of viscosity solutions (see [20, 8, 11, 9]). The second section obtains the existence of a nonnegative-valued viscosity subsolution of an appropriate Hamilton-Jacobi equation as a necessary condition for the existence of a solution of the $H_\infty$-control state feedback problem. The last section extends the analysis (to the extent possible) to the measurement feedback case; there the role of viscosity subsolutions to analogues of the two Riccati equations playing a prominent role in the well known solution of the linear case (see [7]) is developed.

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## 1 The Nonlinear Bounded Real Lemma

Consider a nonlinear input-output system $(\Sigma)$ assumed to be affine with respect to the input signal $w$

$$\begin{aligned} \dot{x} &= A(x) + B(x)w \\ z &= C(x) + D(x)w. \end{aligned} \tag{1.1}$$

Here we take the state vector $x(t)$ to have values in the state space $\mathcal{X}$ which we take to be $\mathbf{R^N}$, the input vector $w(t)$ has values in the input space $\mathcal{W} = \mathbf{R^{n_w}}$ and the output vector $z(t)$ to have values in the output space $\mathcal{Z} = \mathbf{R^{n_z}}$. We assume that $A(x), B(x), C(x)$ and $D(x)$ are continuously differentiable matrix functions (i.e., are in the class $C^1$) such that

$$A(0) = 0, \;\; C(0) = 0$$

(so 0 in $\mathbf{R^N}$ is an equilibrium point if $w$ is taken equal to 0). We assume that the unique solution $x(t)$ of the differential equation in $(\Sigma)$ exists for all time $t > 0$ for any initial condition $x(0)$ and input $w \in L_{2,e}^{n_w}$ (inputs $w$ with values in $\mathbf{R^{n_w}}$ which are norm square-integrable on any finite subinterval of $[0, \infty)$).

We are interested in studying such systems $(\Sigma)$ having $L_2$-gain at most $\gamma$ (for some prescribed $\gamma > 0$); by this we mean that the output $z$ associated with any input $w \in L_{2,e}^{n_w}$ satisfies

$$\int_0^T \|z(t)\|^2 dt \leq \gamma^2 \int_0^T \|w(t)\|^2 dt \tag{1.2}$$

for all $T < \infty$ if we take $x(0) = 0$. As developed in [24, 12], a state space mechanism for the validity of (1.2) is the existence of an energy or storage function $\varphi : X \to \mathbf{R}^+ = \{\mathbf{r} \in \mathbf{R} : \mathbf{r} \geq \mathbf{0}\}$ with $\varphi(0) = 0$ such that the energy balance inequality

$$\varphi(x(t_2)) - \varphi(x(t_1)) \leq \int_{t_1}^{t_2} \{\gamma^2 \|w(t)\|^2 - \|z(t)\|^2\} dt \tag{$EB$}$$

holds over all paths $(w(t), x(t), z(t))$ of the system $\Sigma$. Indeed, when specializing (EB) to $t_1 = 0$ with $x(0) = 0$, then the fact that $0 \leq \varphi(x(t))$ and $0 = e(0) = \varphi(x(0))$ in (EB) gives

$$\begin{aligned} 0 &\leq \varphi(x(T)) \\ &\leq \int_0^T \{\gamma^2 \|w(t)\|^2 - \|z(t)\|^2\} dt \end{aligned}$$

from which (1.2) follows.

Conversely, if the system $\Sigma$ has $L_2$-gain at most $\gamma$ and if the system is *completely reachable* in the sense that there exists a control input to drive the state vector $x(t)$ from $x(0) = 0$ to any prescribed state $x(T) = x_0$ in finite time $T$, then (see [12]) storage functions exists; two such are the *required storage function* $\varphi_r$ given by

$$\varphi_r(x) = \inf_{\substack{w \in L_{2e}^{n_w}, t_0 \leq t_1 \\ x(t_0)=0, x(t_1)=x}} \int_{t_0}^{t_1} (\gamma^2 w(t)^T w(t) - z(t)^T z(t)) dt \tag{1.3}$$

and the *available storage function* $\varphi_a$ given by

$$\varphi_a(x) = - \inf_{\substack{w \in L_{2e}^{n_w}, t_2 \geq t_1 \\ x(t_1)=x}} \int_{t_1}^{t_2} (\gamma^2 w(t)^T w(t) - z(t)^T z(t)) dt. \tag{1.4}$$

Moreover, any other storage function $\varphi$ must satisfy

$$\varphi_a(x) \leq \varphi(x) \leq \varphi_r(x).$$

While it is well known that storage functions need not be smooth even if the original data of the system (1.1) is smooth, the following simple argument guarantees continuity under a controllability hypothesis. As a matter of notation let us denote by $x(t) = x(t, t_0; x_0, w)$ and $z(t) = z(t, t_0; x_0, w)$ the solution of (1.1) with initial condition $x(t_0) = x_0$ and input $w(t)$ over the interval $[t_0, t]$. Let us say that the dynamical system (1.1) is *locally uniformly controllable* if , for each $x_1$ in $\mathbf{R}^\mathbf{N}$ there exists a $\delta > 0$ and a continuous function $\alpha : [0, \delta) \to \mathbf{R}^+$ with $\alpha(0) = 0$ such that, for any any state $x_2$ in $\mathbf{R}^\mathbf{N}$ with $\|x_2 - x_1\| < \delta$, there exists finite times $t_1 < t_2$ and an $L_2$ input signal $w$ defined over the time interval $[t_1, t_2]$ such that $x(t_2, t_1; x_1, w) = x_2$ and

$$\|w\|_{L_2^{n_w}([t_1, t_2])} \leq \alpha(\|x_2 - x_1\|).$$

**Proposition 1.1.** *Assume that the system (1.1) is locally uniformly controllable and that $\varphi : \mathbf{R}^\mathbf{N} \to \mathbf{R}^+$ is a storage function for (1.1). Then $\varphi$ is continuous.*

**Proof:** Suppose that $\varphi$ has a discontinuity at $x_0 \in \mathbf{R}^\mathbf{N}$. Then there is a sequence of states $x_n$ with $lim_{n\to\infty}\, x_n = x_0$ such that $|\varphi(x_n) - \varphi(x_0)| \geq \epsilon$ for some $\epsilon > 0$. By choosing a subsequence if necessary we may suppose that $\varphi(x_n) - \varphi(x_0)$ has a fixed sign. As a first case suppose that $\varphi(x_n) - \varphi(x_0) \geq \epsilon$ for all $n$. Use the local uniform controllability hypothesis to choose, for all $n$ sufficiently large, a control $w_n$ on an interval $[t_0, t_n]$

with $x(t_n, t_0; x_0, w_n) = x_n$ and $\|w_n\|_{L_2^{n_w}([t_0,t_n])} \leq \alpha(\|x_n - x_0\|)$ where $\alpha : [0, \delta) \to \mathbf{R}^+$ is continuous with $\alpha(0) = 0$. In particular, we see that $\|w_n\|_{L_2^{n_w}([t_0,t_n])}$ tends to zero as $n$ tends to infinity. On the other hand the energy balance inequality (EB) and the construction of the sequence $\{x_n\}$ gives

$$\begin{aligned} 0 < \epsilon &\leq \varphi(x_n) - \varphi(x_0) \\ &\leq \int_{t_0}^{t_n} [\|w_n(t)\|^2 - \|z_n(t)\|^2] dt \\ &\leq \|w_n\|_{L_2^{n_w}([0,t_n])} \end{aligned}$$

which leads us to a contradiction. The case where $\varphi(x_0) - \varphi(x_n) \geq \epsilon$ can be handled in a similar way by choosing a control $w_n$ on intervals $[t_n, t_0]$ $(t_n < t_0)$ with $w(t_0, t_n; x_n, w_n) = x_0$.

We henceforth will consider only continuous storage functions $\varphi$. If $\varphi$ is a smooth storage function, then the energy balance inequality (EB) can be written in infinitesimal form

$$\nabla\varphi(x) \cdot [A(x) + B(x)] + (C(x) + D(x)w)^T (C(x) + D(x)w) - \gamma^2 w^T w \leq 0.$$

As this is required to hold for all $w \in \mathcal{W}$ we get that $\varphi$ is a solution of an infinitesimal form of the energy balance inequality, namely the Hamilton-Jacobi inequality

$$H(x, \nabla\varphi(x)) \leq 0 \qquad (EB')$$

where

$$\begin{aligned} H(x,p) = \max_w \{ & p^T [A(x) + B(x)w] \\ & + [C(x) + D(x)w]^T [C(x) + D(x)w] - \gamma^2 w^T w \}. \end{aligned} \qquad (1.5)$$

If $\Delta(x) := \gamma^2 I - D(x)^T D(x)$ is (strictly) positive definite for all $x$, then $H(x,p)$ can be written explicitly as

$$\begin{aligned} H(x,p) = & p[A(x) + B(x)\Delta(x)^{-1} D(x)^T C(x)] + \frac{1}{4} p^T B(x)\Delta(x)^{-1} B(x)^T p \\ & + C(x)^T [I + D(x)\Delta(x)^{-1} D(x)^T] C(x) \end{aligned}$$

but this representation is not essential for the analysis to follow.

However it can easily happen that a storage function $\varphi$ is not smooth. Nevertheless, as was shown in [18], any such storage function $\varphi$ satisfies the infinitesimal inequality (EB$'$) in the generalized viscosity sense introduced by Crandall and Lions (see [5, 6, 3, L, 4]). The result from [18] is stated more generally where more general supply rates and not necessarily continuous storage functions are allowed; we formulate the result only for the situation considered here.

To state the result we first recall the notion of viscosity solution. If $\varphi$ is a continuous function defined on an open set $\mathcal{O}$ in $\mathbf{R^N}$ with values in $\mathbf{R}$ and $x_0 \in \mathcal{O}$, the *superdifferential* of $\varphi$ at $x_0$ is the set, denoted by $D^+\varphi(x_0)$, of all $p_0 \in \mathbf{R^N}$ such that

$$\limsup_{x \to x_0} (\varphi(x) - \varphi(x_0) - p_0^T(x - x_0))\|x - x_0\|^{-1} \leq 0 \qquad (1.6)$$

Similarly the *subdifferential* of $\varphi$ at $x_0$ is the set, denoted by $D^-\varphi(x_0)$, of all $p_0 \in \mathbf{R^N}$ such that

$$\liminf_{x \to x_0} (\varphi(x) - \varphi(x_0) - p_0^T(x - x_0))\|x - x_0\|^{-1} \geq 0.$$

We say that the function $\varphi$ is a *viscosity subsolution* of

$$H(x, \nabla\varphi(x)) = 0 \qquad (1.7)$$

in $\mathcal{O}$ if

$$H(x,p) \leq 0 \text{ for all } x \in \mathcal{O} \text{ and for all } p \in \mathcal{D}^+\varphi(\S). \qquad (1.8)$$

Similarly $\varphi$ is a *viscosity supersolution* of (1.7) if

$$H(x,p) \geq 0 \qquad (1.9)$$

for all $x \in \mathcal{O}$ and all $p \in D^-\varphi(x)$. If $\varphi$ is both a viscosity subsolution and a viscosity supersolution, we say that $\varphi$ is a viscosity solution.

**Theorem 1.2 (see [18]).** *Suppose that $\varphi : \mathbf{R^N} \to \mathbf{R}^+$ is continuous. Then $\varphi$ is a storage function for the system (1.1) if and only if $\varphi$ is a viscosity supersolution of*

$$-H(x, \nabla\varphi(x)) = 0$$

*where $H$ is given by (1.5).*

**Remark 1.1.** By reversing the direction of time in the proof of Theorem 3.1 from [18], one can also show that $\varphi : \mathbf{R^N} \to \mathbf{R}^+$ is a storage function for the system (1.1) if and only if $\varphi$ is a viscosity subsolution of

$$H(x, \nabla\varphi(x)) = 0$$

with $H$ as in (1.5). Thus, in this context at least, the inequality

$$H(x,p) \leq 0$$

holding for all $p \in D^+\varphi(x)$ for each $x$ is equivalent to the same inequality holding for all $p \in D^-\varphi(x)$ for each $x$.

The extremal storage functions $\varphi_r$ and $\varphi_a$ are rather special. We shall show that under certain conditions $\varphi_a$ is a viscosity solution of

$$-H(x, \varphi_a(x), \nabla\varphi_a(x)) = 0$$

and that $\varphi_r$ is a viscosity solution of

$$H(x, \varphi_r(x), \nabla\varphi_r(x)) = 0.$$

The following properties of $\varphi_a$ and $\varphi_r$ provide motivation for the hypotheses to follow.

**Proposition 1.3.** *For any $t_0 < t_1 < t_2$,*

$$\sup_{w \in L_2^{n_w}([t_1,t_2])} \{\varphi_a(x(t_2)) - \varphi_a(x_0) - \int_{t_1}^{t_2} (\gamma^2 w(t)^T w(t) - z(t)^T z(t)) dt\} = 0. \tag{1.10}$$

*If the system is reachable, then*

$$\sup_{w \in L_2^{n_w}([t_0,t_1])} \{\varphi_r(x_0) - \varphi_r(x(t_0)) + \int_{t_0}^{t_1} (\gamma^2 w(t)^T w(t) - z(t)^T z(t)) dt\} = 0. \tag{1.11}$$

*Here our convention is*

$$x(t) = x(t, t_1; x_0, w), \;\; z(t) = z(t, t_1; x_0, w). \tag{1.12}$$

**Proof:** We prove only (1.10); equation (1.11) follows analogously. To condense notation, set

$$q(x, w) = \gamma^2 w^T w - [C(x) + D(x)w]^T [C(x) + D(x)w].$$

Then by definition

$$\begin{aligned} -\varphi_a(x_0) &\le \int_{t_1}^{t_f} q(x(t), w(t)) dt \\ &= \int_{t_1}^{t_2} q(x(t), w(t)) dt + \int_{t_2}^{t_f} q(x(t), w(t)) dt \end{aligned} \tag{1.13}$$

for all $w \in L_2^{n_w}([t_1, t_f])$ where $t_0 < t_1 < t_f$ and where $x(t)$ and $z(t)$ are as in (1.12). Taking the infimum over $w \in L_2^{n_w}(t_2, t_f)$ in the second term on the right hand side in (1.13) gives

$$-\varphi_a(x_0) \leq \int_{t_1}^{t_2} q(x(t), w(t))dt - \varphi_a(x(t_2)).$$

This gives (1.10) with $\leq$ in place of $=$.

To get the reverse inequality, let $\epsilon > 0$. Then by definition there is a $w_\epsilon \in L_2^{n_w}(t_1, t_f)$ so that

$$\begin{aligned}\varphi_1(x_0) + \epsilon &> \int_{t_1}^{t_f} q(x_\epsilon(t), w_\epsilon(t))dt \\ &= \int_{t_1}^{t_2} q(x_\epsilon(t), w_\epsilon(t))dt + \int_{t_2}^{t_f} q(x_\epsilon(t), w_\epsilon(t))dt \\ &\geq \int_{t_1}^{t_2} q(x_\epsilon(t), w_\epsilon(t))dt - \varphi_a(x_\epsilon(t_2)).\end{aligned}$$

From this the reverse inequality in (1.10) follows easily.

In order to prove that $\varphi_a$ is a viscosity solution of $-H(x, \nabla\varphi_a(x)) = 0$ we need a slight strengthening of (1.10), namely:

(H1) *Given $x_0 \in \mathbf{R^n}$ and $t_1 < t_2$ with $t_2 - t_1$ sufficiently small there is a bounded set $B_{x_0} \subset \mathbf{R^{n_w}}$ such that*

$$\sup_{\substack{w \in L_2^{n_w}([t_1, t_2]) \\ w(t) \in B_{x_0}\ \forall t}} \{\varphi_a(x(t_2)) - \varphi_a(x_0) - \int_{t_1}^{t_2} (\gamma^2 w(t)^T w(t) - z(t)^T z(t))dt\} = 0$$

*where $x(t)$ and $z(t)$ are given by (1.12).*

Similarly, to prove that $\varphi_r$ is a viscosity solution of $H(x, \nabla\varphi(x)) = 0$ we shall need the following strengthening of (1.11):

(H2) *Given $x_0 \in \mathbf{R^n}$ and $t_0 < t_1$ with $t_1 - t_0$ sufficiently small, there is a bounded set $B_{x_0} \subset \mathbf{R^{n_w}}$ so that*

$$\sup_{\substack{w \in L_2^{n_w}([t_0, t_1]) \\ w(t) \in B_{x_0}\ \forall t}} \{\varphi_r(x_0) - \varphi_r(x(t_0)) - \int_{t_0}^{t_1} (\gamma^2 w(t)^T w(t) - z(t)^T z(t))dt\} = 0$$

*where $x(t)$ and $z(t)$ are given by (1.12).*

We are now ready to state the following result.

**Theorem 1.4.** *Assume that the system $\Sigma$ in (1.1) has finite gain at most $\gamma$ and is uniformly controllable, so $\varphi_a$ and $\varphi_r$ are both well-defined and continuous storage functions for $\Sigma$. Then:*

*(i) $\varphi_a$ is a viscosity solution of $-H(x, \nabla\varphi(x)) = 0$ if (H1) is satisfied.*

*(ii) $\varphi_r(x)$ is a viscosity solution of $H(x, \nabla\varphi(x)) = 0$ if (H2) is satisfied.*

**Proof:** We prove assertion (i) only. Assertion (ii) follows in the same way by running the system in backwards time.

We have already observed that $\varphi_a$ is a storage function for the system $\Sigma$ (1.1). Hence, by Theorem 1.2 we know that $\varphi_a$ is a viscosity supersolution of $-H(x, \nabla\varphi(x)) = 0$. It remains to show that $\varphi_a(x)$ is a viscosity subsolution of $-H(x, \nabla\varphi(x)) = 0$ if hypothesis (H1) holds.

As a matter of notation throughout the proof, set

$$q(x,w) = \gamma^2 w^T w - [C(x) + D(x)w]^T [C(x) + D(x)w].$$

Suppose that $\varphi_a(x)$ is not a viscosity subsolution of $-H = 0$. Then there is an $x_0 \in \mathbf{R^n}$, a $p \in D^+\varphi(x_0)$ and an $\epsilon > 0$ so that

$$p^T[A(x_0) + B(x_0)w] - q(x_0, w) \le -\epsilon < 0 \tag{1.14}$$

for all $w \in \mathbf{R^{n_w}}$. By an equivalent characterization of the super gradient (see e.g. [3] or [4]) there is smooth function $\psi : \mathbf{R^N} \to \mathbf{R}$ so that (i) $\psi(x_0) = \varphi_a(x_0)$, (ii) $\nabla\psi(x_0) = p$ and (iii) $\varphi_a(y) - \psi(y) \le 0$ for all $y \in \mathbf{R^N}$. Then (1.14) becomes

$$\nabla\psi(x_0)^T[A(x_0) + B(x_0)w] - q(x_0, w) \le -\epsilon < 0 \tag{1.15}$$

for all $w \in R^{n_w}$. By assumption the expression on the left side of (1.15) is uniformly continuous. Hence there is a $\delta > 0$ so that

$$\nabla\psi(x(t))^T[A(x(t)) + B(x(t))] - q(x(t), w(t)) \le -\frac{\epsilon}{2} \tag{1.16}$$

for $t_1 \le t \le t_1 + \delta$ for any $w \in L_2^{n_w}([t_1, t_1 + \delta])$ as long as $w(t) \in B_{x_0}$, where $B_{x_0}$ is as in (H1). Integrating (1.16) from $t_1$ to $t_1 + \delta$ gives us

$$\psi(x(t_1 + \delta)) - \psi(x_0) - \int_{t_1}^{t_1+\delta} q(x(t), w(t))dt \le -\frac{\epsilon}{2}\delta. \tag{1.17}$$

for all $w \in L_2^{n_w}([t_1, t_1 + \delta])$ with $w(t) \in B_{x_0}$ for all $t$.

On the other hand, if we take $\delta$ sufficiently small, the hypothesis (H1) combined with the known relations

$$\psi(x_0) = \varphi_a(x_0), \ \ \psi(x) \ge \varphi_a(x) \text{ for all } x$$

gives us

$$\sup_{\substack{w \in L_2^{n_w}([t_1, t_1+\delta]) \\ w \in B_{x_0} \, \forall t}} \{\psi(x(t_1)) - \psi(x_0) - \int_{t_1}^{t_1+\delta} q(x(t), w(t)) dt\} \geq 0,$$

a direct contradiction to (1.17). It follows that $\varphi_a(x)$ is a viscosity solution of $-H(x, \nabla\varphi(x)) = 0$ as claimed.

**Remark 1.2.** The proof of Theorem 1.4 is a modification of the proof of Theorem 4.1 in [8]. There it is proved that upper and lower value functions for a differential game are viscosity solutions of associated (time-varying) upper and lower (respectively) Hamilton-Jacobi-Isaacs equations. Also it is assumed that both players use controls in a bounded set (the analogue of our hypotheses (H1) and (H2)). We also remark that the smooth case of Theorem 1.4 is obtained in [16] but with a somewhat different hypothesis.

## 2 The $H_\infty$ Control State Feedback Problem

In this section we consider the nonlinear $H_\infty$-control state feedback problem. For simplicity we consider only plants which are affine in the input variables. Thus we assume that we are given a plant $P : (w, u) \to z$ described by state space equations

$$\begin{aligned} \dot{x} &= A(x) + B_1(x)w + B_2(x)u \\ z &= C_1(x) + D_{12}(x)u. \end{aligned} \tag{2.1}$$

Here $w$ is a reference or disturbance signal with values in $\mathbf{R}^{\mathbf{n_w}}$, $u$ is the control signal with values in $\mathbf{R}^{\mathbf{n_u}}$, $z$ is the error signal with values in $\mathbf{R}^{\mathbf{n_z}}$ and $x$ is the state vector assumed to have values in $\mathbf{R}^{\mathbf{n}}$. We assume that $A, B_1, B_2, C_1, D_{12}$ are continuously differentiable matrix functions on $\mathbf{R}^{\mathbf{n}}$ of appropriate sizes such that $A(0) = 0, C_1(0) = 0$ (so 0 in $\mathbf{R}^{\mathbf{n}}$ is an equilibrium point when $w(t)$ and $u(t)$ are taken equal to 0). In order that certain explicit formulas make sense it is convenient to assume that $x \to e_1(x) := D_{12}(x)^T D_{12}(x)$ is uniformly positive definite and uniformly bounded on $\mathbf{R}^{\mathbf{n}}$, but the general analysis goes through without this assumption. The problem of interest here (the nonlinear $L_2$-gain problem with state feedback) is to design a state feedback $u = c(x)$ (where we take $c(0) = 0$) so that the resulting closed loop system $P_c : w \to z$ given by

$$\begin{aligned} \dot{x} &= A(x) + B_2(x)c(x) + B_1(x)w \\ z &= C_1(x) + D_{12}(x)c(x) \end{aligned} \tag{2.2}$$

is *well-posed* (i.e. there exists a unique solution $x(t)$ of the first of equations (2.2) for all $t > 0$ for any initial value $x(0)$) and *has $L_2$-gain at most* $\gamma$ (for some prespecified tolerance level $\gamma$). This is the nonlinear $H_\infty$-control state feedback problem, apart from the stability side constraint with which we do not deal in this paper (see the Introduction).

To analyze the $L_2$-gain property, we apply the analysis of the previous section and seek a storage function $\varphi : \mathbf{R^n} \to \mathbf{R}^+$ with $\varphi(0) = 0$ which satisfies the energy balance inequality (EB) for the closed loop system.

We therefore formulate the $\varphi$-dissipative state feedback problem

$(\varphi - DISSFBK)$ : *given a system as in* (2.1), *find a state feedback law* $u = c(x)$ *and a continuous nonnegative real-valued storage function* $\varphi$ : $\mathbf{R^n} \to \mathbf{R}^+$ *so that the energy balance inequality*

$$\varphi(x(t_2)) - \varphi(x(t_1)) \leq \int_{t_1}^{t_2} \{\gamma^2 \|w(t)\|^2 - \|z(t)\|^2\} dt$$

*holds on all paths* $(x(t), w(t), z(t))$ *of the closed loop system* (2.2).

Note that if $u = c_*(x)$ is a solution of the $L_2$-gain state feedback problem as posed above, then the results of [12] imply the existence of a nonnegative, possibly extended real valued, not necessarily smooth storage function $\varphi$ satisfying (EB) which has finite values on reachable states. Conversely, if $(u = c_*(x), \varphi)$ is a solution of $(\varphi - DISSFBK)$ such that the associated closed loop system ((2.1) with $u = c_*(x)$ is well-posed, then $u = c_*(x)$ provides a solution of the $L_2$-gain state feedback problem. Previous works on the nonlinear $H_\infty$-control state feedback problem [1, 21, 22] assume that there exist a smooth solution of (EB). We point out also that in [21] it is shown that there exists a solution of (EB) which is smooth at least in a neighborhood of the origin if the linearized problem is in the strictly suboptimal case. In any case, the $(\varphi - DISSFBK)$ problem (with $\varphi$ only assumed to be semicontinuous) appears to be a reasonable next step towards consideration of the general $H_\infty$ problem.

A consequence of Theorem 1.2 is that, under certain circumstances, the integral form of the $(\varphi - DISSFBK)$ can be translated to an infinitesimal form; the following result summarizes the situation.

**Theorem 2.1.** *If the pair* $(c_*, \varphi)$ *(where* $\varphi : \mathbf{R^n} \to \mathbf{R}^+$ *is continuous with* $\varphi(0) = 0$ *and* $c_* : \mathbf{R^n} \to \mathbf{R^{n_u}}$ *is continuously differentiable) solves the* $(\varphi - DISSFBK)$ *problem, then* $\varphi$ *is a viscosity supersolution of*

$$-H_{c_*(x)}(x, \nabla\varphi(x)) = 0 \tag{2.3}$$

*where*

$$H_c(x,p) = \max_w \{p^T[A(x) + B_1(x)w + B_2(x)c] + [C_1(x) + D_{12}(x)c]^T[C_1(x) + D_{12}(x)c] - w^T w\},$$

*or equivalently,*

$$\begin{aligned} H_c(x,p) =& p^T[A(x) + B_2(x)c + \frac{1}{4}B_1(x)B_1(x)^T p] \\ &+ [C_1(x) + D_{12}(x)c]^T[C_1(x) + D_{12}(x)c]. \end{aligned} \tag{2.4}$$

*In particular*

$$\min_c \max_{p\in D^-\varphi(x)} H_c(x,p) \leq 0. \tag{2.5}$$

*Conversely, if the continuous function* $\varphi : \mathbf{R^n} \to \mathbf{R}^+$ *and* $C^1$ *function* $c_* : \mathbf{R^n} \to \mathbf{R^{n_u}}$ *are such that*

$$\max_{p\in D^-\varphi(x)} H_{c_*(x)}(x,p) \leq 0,$$

*then* $(u = c_*(x), \varphi)$ *is a solution of the* $(\varphi - DISSFBK)$ *problem.*

**Proof:** Simply apply Theorem 1.2 to the closed loop system associated with the $L^2$-gain state-feedback problem.

**Remark 2.1.** In the case where $\varphi$ is smooth, ($D^-\varphi(x) = \{\nabla\varphi(x)\}$ is a singleton), (2.5) can be solved explicitly for $c_*(x)$—

$$c_*(x) = -e_1(x)^{-1}[\frac{1}{2}B_2(x)^T\nabla\varphi(x) + D_{12}(x)^T C_1(x)]$$

where we assume that $e_1(x) = D_{12}(x)^T D_{12}(x)$ is uniformly invertible—from which (2.5) assumes the form

$$\begin{aligned} H_{c_*(x)}(x,\nabla\varphi(x)) = & \nabla\varphi(x)^T[A(x) - B_2(x)e_1(x)^{-1}D_{12}(x)^T C_1(x)] \\ &+ \frac{1}{4}\nabla\varphi(x)^T[B_1(x)B_1(x)^T - B_2(x)e_1(x)^{-1}B_2(x)^T]\nabla\varphi(x) \\ &+ C_1(x)^T[I - D_{12}(x)e_1(x)^{-1}D_{12}(x)^T]C_1(x) \leq 0. \end{aligned}$$

This agrees with the solution of the $H_\infty$-control state feedback problem for the nonlinear case as presented elsewhere (see [22]) and specializes to the inequality version of the $X$-Riccati equation in [7] in the linear case (with $\frac{1}{2}\nabla\varphi(x) = Xx$).

**Remark 2.2.** Note that $H_c(x,p)$ is quadratic in $p$ with positive semidefinite Hessian. Hence the maximum over $p \in D^-\varphi(x)$ necessarily is attained at a boundary point of $D^-\varphi(x)$.

**Remark 2.3.** Note that if (2.5) holds for some continuous $\varphi$, then a natural candidate for $c_*$ so that $(c_*, \varphi)$ solves (2.3) is any function $x \to c_*(x)$ such that

$$\max_{p\in D^-\varphi(x)} H_{c_*(x)}(x,p) = \min_c \max_{p\in D^-\varphi(x)} H_c(x,p). \tag{2.6}$$

Unfortunately, there may be no solution $c_*$ of (2.6) which is smooth.

## 3 The $H_\infty$-Control Measurement Feedback Problem

In this section we consider a plant $P$ as in Section 2 but augmented by an additional output signal $y$ with values in $\mathbf{R^{n_y}}$ which serves as a measurement signal. We assume that the plant $P : (w, u) \to (z, y)$ is described by state space equations of the form

$$\begin{aligned} \dot{x} &= A(x) + B_1(x)w + B_2(x)u; \\ z &= C_1(x) + D_{12}(x)u; \\ y &= C_2(x) + D_{21}(x)w. \end{aligned} \tag{3.1}$$

In addition to the assumptions on $A(x), B_1(x), B_2(x), C_1(x), D_{12}(x)$ as in Section 2, we assume that $C_2$ and $D_{21}$ are continuously differentiable matrix functions of appropriate sizes with $C_2(0) = 0$. The $L_2$-gain measurement feedback problem which we consider here is to construct a dynamic compensator $K : y \to u$ given in terms of a state space realization of the form

$$\begin{aligned} \dot{\zeta} &= a(\zeta) + b(\zeta)y \\ u &= c(\zeta) \end{aligned} \tag{3.2}$$

(where $\zeta$ takes values in the compensator state space $\mathbf{R^{n_o}}$, $a$, $b$ and $c$ are smooth (continuously differentiable) functions with $a(0) = 0, c(0) = 0$) so that the resulting closed loop system (3.1) and (3.2) is *well-posed* and *has $L^2$-gain at most* $\gamma$ for some prespecified tolerance level $\gamma$. By the discussion in Section 1, the $L_2$-gain condition is satisfied once we find an energy (or storage) function $\varphi = \varphi(x, \zeta) : \mathbf{R^n} \times \mathbf{R^{n_o}} \to \mathbf{R}^+$ such that the energy balance inequality

$$\begin{aligned} &\varphi(x(t_2), \zeta(t_2)) - \varphi(x(t_1), \zeta(t_1)) \\ &\leq \int_{t_1}^{t_2} \{\gamma^2 \|w(t)\|^2 - \|z(t)\|^2\} dt \end{aligned} \tag{$EB$}$$

is satisfied over all trajectories $(x(t), \zeta(t), w(t), z(t))$ of the closed loop system. In general there is no a priori reason why such a $\varphi$ need be smooth. Here we focus on what we call the $\varphi$-dissipative measurement feedback problem:

$(\varphi - DISMFBK)$: *given a plant $P$ as in (3.1), find a compensator $K = (a, b, c)$ as in (3.2) and a continuous $\varphi : \mathbf{R^n} \times \mathbf{R^{n_o}} \to \mathbf{R}^+$ so that $(EB)$ is satisfied for the closed loop system (3.1) - (3.2).*

We remark that in [2] the smooth version of this problem (where $\varphi$ was required to be smooth) was called simply the $\varphi$-dissipative feedback ($\varphi - DISFBK$) problem.

Application of Theorem 1.2 to the ($\varphi - DISMFBK$) problem leads immediately to the following infinitesimal version of the dissipation inequality (EB).

**Theorem 3.1.** *Suppose the collection $(a_*, b_*, c_*, \varphi)$ is a solution of the $(\varphi - DISMFBK)$ problem, where $\varphi : \mathbf{R^n} \to \mathbf{R}^+$ is continuous and where $a_*, b_*, c_*$ are all of class $C^1$. Then $\varphi$ is a viscosity supersolution of*

$$-H_{a_*(\zeta), b_*(\zeta), c_*(\zeta)}(x, \nabla_x \varphi(x,\zeta), \nabla_\zeta \varphi(x,\zeta)) = 0 \tag{3.4}$$

*where*

$$\begin{aligned} H_{a,b,c}(x, p_z, p_\zeta) :=& \\ & \max_w \{ p_x^T [A(x) + B_1(x)w + B_2(x)c] \\ & + p_\zeta^T [a + bC_2(x) + bD_{21}(x)w] \\ & + [C_1(x) + D_{12}(x)c]^T [C_1(x) + D_{12}(x)c] - w^T w \} \end{aligned}$$

*or equivalently,*

$$\begin{aligned} H_{a,b,c}(x, p_z, p_\zeta) = & \, p_x^T [A(x) + B_2(x)c + \frac{1}{4} B_1(x) B_1(x)^T p_x] \\ & + p_\zeta^T [a + bC_2(x) + \frac{1}{4} b e_2(x) b^T p_\zeta + \frac{1}{2} b D_{21}(x) B_1(x)^T p_x] \\ & + [C_1(x) + D_{12}(x)c]^T [C_1(x) + D_{12}(x)c] \end{aligned} \tag{3.5}$$

*where $e_2(x) = D_{21}(x) D_{21}(x)^T$. In particular,*

$$\max_\zeta \min_{a,b,c} \max_x \max_{(p_x, p_\zeta) \in D^- \varphi(x,\zeta)} H_{a,b,c}(x, p_x, p_\zeta) \le 0. \tag{3.6}$$

*Conversely, if the continuous function $\varphi : \mathbf{R^n} \times \mathbf{R^{n_0}} \to \mathbf{R}^+$ with $\varphi(0,0) = 0$ and the $C^1$ functions $a_*(\zeta), b_*(\zeta), c_*(\zeta)$ are such that*

$$\max_{(p_x, p_\zeta) \in D^+ \varphi(x,\zeta)} H_{a_*(\zeta), b_*(\zeta), c_*(\zeta)}(x, p_x, p_\zeta) \le 0 \tag{3.7}$$

*for all $(x, \zeta)$, then $(a_*, b_*, c_*, \varphi)$ is a solution of the $(\varphi - DISMFBK)$ problem.*

In the rest of this section we work with the *infinitesimal* version of the $(\varphi - DISMFBK)$ problem, namely: *find a nonnegative continuous function $\varphi$ on $\mathbf{R^n} \times \mathbf{R^{n_0}}$ with $\varphi(0,0) = 0$ such that (3.6) holds, where $H$ is given by (3.5).*

As a first reduction, we present the following necessary condition for the existence of a solution to the infinitesimal $(\varphi - DISMFBK)$ problem.

**Proposition 3.2.** *A necessary condition for the function $\varphi$ to be a solution to the infinitesimal $(\varphi - DISMFBK)$ problem is*

$$\inf_{a,b,c} \max_{(p_x,p_\zeta)\in D^-\varphi(x,\zeta)} H_{a,b,c}(x,p_x,p_\zeta) \leq 0 \tag{3.8}$$

*for all $x$ and $\zeta$.*

**Proof:** By definition a solution $\varphi$ of the infinitesimal $(\varphi - DISMFBK)$ problem satisfies (3.6). For each fixed $\zeta$, interchange of the order of *max* and *min* gives the inequality

$$\begin{aligned}&\sup_x \inf_{a,b,c} \max_{(p_x,p_\zeta)} H_{a,b,c}(x,p_x,p_\zeta)\\ &\leq \min_{a,b,c} \max_x \max_{(p_x,p_\zeta)} H_{a,b,c}(x,p_x,p_\zeta)\end{aligned}$$

from which (3.8) follows.

As we shall see (as was done in [2]), the key step in analyzing this condition further is the introduction of the subset $Z_\varphi$ of $\mathbf{R^n} \times \mathbf{R^{n_o}}$ defined by

$$Z_\varphi = \{(x,\zeta) : (p_x, 0) \in D^-\varphi(x,\zeta) \text{ for some } p_x \in \mathbf{R^n}\} \tag{3.9}$$

Since $D^-\varphi(x,\zeta)$ is a closed and convex subset of $\mathbf{R^n}\times\mathbf{R^{n_o}}$ , the complement $Z^c_\varphi$ of $Z_\varphi$ in $\mathbf{R^n} \times \mathbf{R^{n_o}}$ has the characterization

$$\begin{aligned}Z^c_\varphi = \{(x,\zeta) :\ &\text{there exists a nonzero vector } a \in \mathbf{R^{n_o}}\\ &\text{so that } a^T p_\zeta \leq -\delta < 0 \text{ for some } \delta > 0 \text{ for all } p_\zeta \in D^-_\zeta\varphi(x,\zeta)\}\end{aligned} \tag{3.10}$$

We then have the following result.

**Proposition 3.3.** *Assume that $D^-\varphi(x,\zeta)$ is a bounded set whenever $(x,\zeta) \notin Z_\varphi$. If $(x,\zeta) \notin Z_\varphi$ then*

$$\inf_{a,b,c} \max_{(p_x,p_\zeta)\in D^-\varphi(x,\zeta)} H_{a,b,c}(x,p_x,p_\zeta) = -\infty. \tag{3.11}$$

*Hence a necessary condition for the infinitesimal $(\varphi - DISMFBK)$ problem to have a solution is that*

$$\sup_{(x,\zeta)\in Z_\varphi} \inf_{a,b,c} \max_{(p_x,p_\zeta)\in D^-\varphi(x,\zeta)} H_{a,b,c}(x,p_x,p_\zeta) \leq 0. \tag{3.12}$$

**Remark 3.1.** As in the state feedback case (see Remark 2.2 after Theorem 2.1) the maximum over $(p_x,p_\zeta) \in D^-\varphi(x,\zeta)$ in (3.12) necessarily occurs on the boundary of $D^-\varphi(x,\zeta)$ since $H_{a,b,c}(x,p_x,p_\zeta)$ is quadratic in $(p_x,p_\zeta)$ with positive semidefinite Hessian.

**Remark 3.2.** The hypothesis that $D^-\varphi(x,\xi)$ is bounded is imposed for convenience in the proof. The result is probably true more generally. In any case in many circumstances viscosity solutions $\varphi$ of Hamilton-Jacobi equations are Lipschitz, in which case this boundedness hypothesis is automatically satisfied (see [10]).

**Proof:** We work with the necessary condition (3.8) given by Proposition 3.2. First note that

$$\inf_a \max_{(p_x,p_\zeta)\in D^-\varphi(x,\zeta)} p_\zeta^T a = \begin{cases} -\infty & \text{if } (x,\zeta)\notin Z_\varphi \\ 0 & \text{if } (x,\zeta)\in Z_\varphi. \end{cases} \tag{3.13}$$

Indeed, if $(x,\zeta)\notin Z_\varphi$, then by (3.10) there is always a choice of $a$ so that

$$\max_{(p_x,p_\zeta)\in D^-\varphi(x,\zeta)} p_\zeta^T a < 0.$$

By rescaling $a$ we can arrange the quantity $\max_{p\in D^-\varphi(x,\zeta)} p_\zeta^T a$ to be as small as we like. This proves (3.13) in case $(x,\zeta)\notin Z_\varphi$. If $(x,\zeta)\in Z_\varphi$ then for any $a$ the choice $p_\zeta=0$ achieves $p_\zeta^T a=0$ and therefore

$$\max_{(p_x,p_\zeta)\in D^-\varphi(x,\zeta)} p_\zeta^T a \geq 0.$$

But the choice $a=0$ always achieves

$$\max_{(p_x,p_\zeta)\in D^-\varphi(x,\zeta)} p_\zeta^T a|_{a=0} = 0$$

Thus (3.13) follows in case $(x,\zeta)\in Z_\varphi$ as well.

Next we argue that

$$\inf_{a,b,c} \max_{(p_x,p_\zeta)\in D^-\varphi(x,\zeta)} h_{a,b}(x,p_x,p_\zeta) = \begin{cases} -\infty & \text{if } (x,\zeta)\notin Z_\varphi \\ 0 & \text{if } (x,\zeta)\in Z_\varphi. \end{cases} \tag{3.14}$$

where

$$\begin{aligned} &h_{a,b}(x,p_x,p_\zeta) = \\ &p_\zeta^T[a+bC_2(x)+\frac{1}{4}be_2(x)b^T p_\zeta+\frac{1}{2}bD_{21}(x)B_1(x)^T p_x]. \end{aligned} \tag{3.15}$$

Indeed, if $(x,\zeta)\notin Z_\varphi$ again by using the assumption that $D^-\varphi(x,\zeta)$ is bounded and (3.10), we can choose $a$ and then rescale $a$ (with $b,c$ fixed) to make the quantity

$$\max_{(p_x,p_\zeta)\in D^-\varphi(x,\zeta)} h_{a,b}(x,p_x,p_\zeta)$$

as small as we like, and (3.14) follows in case $(x,\zeta) \notin Z_\varphi$. The case $(x,\zeta) \in Z_\varphi$ in (3.14) follows in the same way as for (3.13).

Finally note from (3.5) and (3.15) that

$$\begin{aligned} H_{a,b,c}(x,p_x,p_\zeta) = p_x^T[A(x) + B_2(x)c + \frac{1}{4}B_1(x)B_1(x)^T p_x] \\ + h_{a,b}(x,p_x,p_\zeta) \\ + [C_1(x) + D_{12}(x)c]^T[C_1(x) + D_{12}(x)c]. \end{aligned} \tag{3.16}$$

In (3.16), if $(x,\zeta) \notin Z_\varphi$ then by appropriate choice of $a$ while holding $b$ and $c$ fixed, we can make the second term of (3.16) approach $-\infty$ uniformly with respect to $(p_x,p_\zeta) \in D_\varphi^-(x,\zeta)$ while the first term is uniformly bounded with respect to $(p_x,p_\zeta)$ (by the assumed boundedness of $D^-\varphi(x,\zeta)$) and the last term is fixed. The assertion of Proposition 3.2 now follows.

We next obtain the analogues of the $X$- and $Y$-Riccati equations appearing in the linear theory (see [7]) as necessary conditions for solutions of the infinitesimal $(\varphi - DISMFBK)$ problem to exist.

**Theorem 3.4.** *Assume that $\varphi(x,\xi)$ is a continuous function such that $D^-\varphi(x,\xi)$ is a bounded set whenever $(x,\xi) \notin Z_\varphi$ (where $Z_\varphi$ is given by (3.9)). Then necessary conditions for $\varphi$ to be a solution to the infinitesimal ($\varphi$ - DISMFBK) problem are:*
*(i)*

$$\min_c \max_{(p_x,0)\in D^-\varphi(x,\zeta)} H_c(x,p_x) \le 0 \tag{3.17}$$

*for all $(x,\zeta) \in Z_\varphi$, where $H_c(x,p)$ is the Hamiltonian for the $(\varphi - DISSFBK)$ problem given by (2.4) and $Z_\varphi$ is given by (3.9)*
*(ii)*

$$\inf_{\widetilde{b}} \max_{p_x\in D_1^-\varphi(x,0)} K_{\widetilde{b}}(x,p_x) \le 0 \tag{3.18}$$

*for all $x$ in $\mathbf{R^n}$, where $K_{\widetilde{b}}(x,p_x)$ is given by*

$$\begin{aligned} K_{\widetilde{b}}(x,p_x) = p_x^T[A(x) + \frac{1}{4}B_1(x)B_1(x)^T p_x] \\ + \widetilde{b}[C_2(x) + \frac{1}{2}D_{21}(x)B_1(x)^T p_x] + \frac{1}{4}\widetilde{b}e_2(x)\widetilde{b}^T + C_1(x)^T C_1(x). \end{aligned}$$

**Remark 3.3.** In the smooth case where $D^-\varphi(x,\zeta) = \{\nabla\varphi(x,\zeta)\}$, as in Remark 2.1 (3.17) takes the explicit form

$$\begin{aligned} &\nabla_x\varphi(x,\zeta)^T[A(x) - B_2(x)e_1(x)^{-1}D_{12}(x)^T C_1(x)] \\ &+ \frac{1}{4}\nabla_x\varphi(x,\zeta)^T[B_1(x)B_1(x)^T - B_2(x)e_1(x)B_2(x)^T]\nabla_x\varphi(x,\zeta) \\ &+ C_1(x)^T[I - D_{12}(x)e_1(x)^{-1}D_{12}(x)^T]C_1(x) \le 0 \end{aligned}$$

for all $(x,\zeta) \in Z_\varphi$. If we assume that $Z_\varphi$ has the form $Z_\varphi = \{(x,\zeta(x)) : x \in \mathbf{R^n}\}$ of a graph space over the first coordinate space and specialize to the linear case, we pick up the inequality version of the $X$-Riccati equation from [7] (with $\frac{1}{2}\nabla_x\varphi(x,\zeta(x)) = Xx$).

As for condition (3.18), in the case where $\varphi$ is smooth, we have

$$\begin{aligned} K_{\widetilde{b}}(x,\nabla_x\varphi(x,0)) = \nabla_x\varphi(x,0)^T[A(x) + \frac{1}{4}B_1(x)B_1(x)^T\nabla_x\varphi(x,0)] \\ + \widetilde{b}[C_2(x) + \frac{1}{2}D_{21}(x)B_1(x)^T\nabla_x\varphi(x,0)] + \frac{1}{4}\widetilde{b}e_2(x)\widetilde{b}^T. \end{aligned}$$

As this expression is quadratic in $\widetilde{b}$, the minimum over $\widetilde{b}$ can be calculated explicitly if we assume that $e_2(x)$ is invertible. In this case, the critical value of $\widetilde{b}$ is given by

$$\widetilde{b}_* = -2[C_2(x)^T + \frac{1}{2}\nabla_x\varphi(x,0)^TB_1(x)D_{21}(x)^T]$$

and condition (3.18) becomes

$$\begin{aligned} \min_{\widetilde{b}}\ K_{\widetilde{b}}(x,\nabla_x\varphi(x,0)) =& K_{\widetilde{b}_*}(x,\nabla_x\varphi(x,0)) \\ =& \nabla_x\varphi(x,0)^T[A(x) - B_1(x)D_{21}(x)^Te_2(x)^{-1}C_2(x)] \\ &+ C_1(x)^TC_1(x) - C_2(x)^Te_2(x)^{-1}C_2(x) \\ &+ \frac{1}{4}\nabla_x^T\varphi(x,0)^T[B_1(x)B_1(x)^T \\ &- B_1(x)D_{21}(x)^Te_2(x)^{-1}D_{21}(x)B_1(x)^T]\nabla_x\varphi(x,0) \\ \leq& 0. \end{aligned}$$

In the linear case, this condition specializes to the inequality version of the Riccati equation for $Y^{-1}$ in the solution presented in [7] (with $\frac{1}{2}\nabla_x\varphi(x,0) = Y^{-1}x$).

**Proof of Theorem 3.4:** To prove (i), note that

$$H_{a,b,c}(x,p_x,0) = H_c(x,p_x)$$

where $H_{a,b,c}(x,p_x,p_\zeta)$ is the Hamiltonian for the $(\varphi - DISMFBK)$ problem given by (3.5) while $H_c(x,p)$ in general is the Hamiltonian for the $(\varphi - DISSFBK)$ problem given by (2.4). Moreover we have the trivial inequality

$$\inf_{a,b,c}\ \max_{(p_x,0)\in D^-\varphi(x,\zeta)} H_{a,b,c}(x,p_x,0) \leq \inf_{a,b,c}\ \max_{(p_x,p_\zeta)\in D^-\varphi(x,\zeta)} H_{a,b,c}(x,p_x,p_\zeta)$$

whenever $(x,\zeta) \in Z_\varphi$. ¿From the necessity of condition (3.8), we see that (3.17) is necessary for the existence of a solution of the infinitesimal $(\varphi - DISMFBK)$ problem as asserted. To prove (ii), simply restrict $(x,\zeta)$ to $(x,0)$ in (3.8).

**Remark 3.4.** As an example, consider the case where the storage function $\varphi$ is piecewise smooth, i.e., the discontinuities of its first order derivatives all occur on a hypersurface and have simple jumps. As the simplest such scenario, write $x \in \mathbf{R^n}$ as $x = (x_1, \cdots, x_n)$ and suppose that $\varphi$ is continuously differentiable in $(x_2, \cdots, x_n, \zeta)$ for each fixed $x_1$ and that, for each fixed $(x_2, \cdots, x_n, \zeta)$, the derivative of $\varphi$ with respect to $x_1$ is continuous except for possibly a simple jump discontinuity at the point $x_{10} = x_{10}(x_2, \cdots, x_n, \zeta)$. Then for each $(x,\zeta) \in Z_\varphi$, the set $\{p_x : (p_x, 0) \in D^-\varphi(x,\zeta)\}$ is a singleton if $(x,\zeta)$ does not have the form $(x,\zeta) = (x_{10}(x_2, \cdots, x_n, \zeta), x_2, \cdots, x_n, \zeta)$ and has the form of an interval

$$\{(1-s)p_{x-}(x,\zeta) + sp_{x+}(x,\zeta)\}$$

otherwise, where $p_{x-}(x,\zeta)$ and $p_{x+}(x,\zeta)$ are the one-sided gradients of $\varphi$ computed from either side of the hypersurface $x_1 = x_{10}(x_2, \cdots, x_n, \zeta)$. Since (as in Remark 2.2) the maximum in (3.17) necessarily is attained on the boundary of the set $D^-\varphi(x,\zeta)$, (3.17) assumes the form

$$\min_c \max \{H_c(x, p_{x-}(x,\zeta)), H_c(x, p_{x+}(x,\zeta))\}.$$

The minimizing $c$ can be computed explicitly as

$$c_*(x,\zeta) = -e_2(x)^{-1}[B_2(x)^T X(x,\zeta) + D_{12}(x)C_1(x)]$$

where

$$\begin{aligned} X(x,\zeta) =& \frac{1}{2}p_{x+}(x,\zeta) \text{ if } H_{c_{*+}}(x, p_{x+}(x,\zeta)) \geq H_{c_{*-}}(x, p_{x-}(x,\zeta)) \\ =& \frac{1}{2}p_{x-}(x,\zeta), \text{ otherwise} \end{aligned}$$

where

$$\begin{aligned} c_{*+}(x,\zeta) =& - e_2(x)^{-1}[\frac{1}{2}B_2(x)^T p_{x+}(x,\zeta) + D_{12}(x)C_1(x)] \\ c_{*-}(x,\zeta) =& - e_2(x)^{-1}[\frac{1}{2}B_2(x)^T p_{x-}(x,\zeta) + D_{12}(x)C_1(x)]. \end{aligned}$$

If $Z_\varphi$ can be expressed as a graph space $Z_\varphi = \{(x(\zeta), \zeta) : \zeta \in R^{n_0}\}$ over the second coordinate space, then a natural candidate (however possibly not smooth) for the compensator output map then is $c_{K*}(\zeta) = c_*(x(\zeta), \zeta)$.

**Remark 3.5.** An interesting open question concerns the Separation Principle proved in [2]. It says that in the smooth case, if (3.4) holds with equality and certain other nondegeneracy conditions hold, then the compensator dynamics $a_K(\zeta)$ necessarily has the form $a_K(\zeta) = a_{K*}(\zeta)$ where

$$\begin{aligned} a_{K*}(\zeta) &= A(\zeta) + B_1(\zeta)B_1(\zeta)^T X(\zeta) + B_2(\zeta)c_{K*}(\zeta) \\ &\quad - b_K(\zeta)[C_2(\zeta) + D_{21}(\zeta)B_1(\zeta)^T X(\zeta)] \end{aligned}$$

where the input map $b_K(\zeta)$ for the compensator still remains to be found and $c_{K*}(\zeta)$ is as in Remark 3.4. Extensions of these results to the non-smooth case requires further understanding of the calculus of generalized gradients.

## References


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