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Versions of Sontag's Input to State Stability Condition and Output Feedback Global Stabilization*

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Abstract

A new approach on the output feedback global stabilization problem for triangular nonlinear systems is presented. Our methodology extends some ideas from our recent works (1993) and is quite different and less technical than these proposed by earlier works where similar results are obtained in the presence of the "input to state stability condition". The main sufficient conditions we propose are versions of this condition.

Key words: input to state stability, feedback, global, stabilization

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1 Introduction

We deal with the output feedback global stabilizability problem for single-input systems of the form

$$\begin{aligned} \dot{x} &= f(x, y_1) \\ \dot{y}_i &= g_i(x, y_1, ..., y_i) + y_{i+1}\ , \quad 1 \le i \le m \\ y_{m+1} &\doteqdot u \end{aligned} \tag{1.1}$$

$$x \in \Re^n \quad ; \quad y \doteqdot (y_1, ..., y_m)' \in \Re^m$$

where u and y are the input and the output of (1.1), respectively, $'$ stands for transpose, and we assume that the mappings f and g_i are continuous (C°) vanishing at zero.

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In the present paper we extend previous results on the same problem for interconnected nonlinear systems (see [1-18] and references therein). Our approach is different from this employed in relative existing works (see for instance [2,3,9,12]) and is based on some ideas from [17,18] as well as on versions of the *"input to state stability condition" (I.S.S.C.)* concerning the stability behavior of the subsystem

$$\dot{x} = f(x, y) \ \text{ with } y \text{ as input} \tag{1.2}$$

In particular, the first version (Condition 2.1) is of Lyapunov type and as it has been recently proved by Lin, Sontag and Wang is *equivalent* to I.S.S.C. This condition implies that for the case $m = 1$ the corresponding $(n+1)$-dimensional system (1.1) is *globally asymptotically stabilizable* at the origin *(G.A.S.)* by *output* feedback which is *continuous* on $\Re^{n+1}$ (Theorem 3.1). The second version (Condition 2.2) is weaker than the I.S.S.C. and guarantees that (1.1) is G.A.S. by means of an output feedback which is *smooth* on $\Re^n$, provided that f and g_i are C^1 and the matrix $(\partial f/\partial x)(0,0)$ is Hurwitz (Theorem 3.2 and Corollary 3.4). Finally, a pair of stability conditions much more weaker than I.S.S.C. as well as Conditions 2.2 are presented in section 4 (Conditions 4.1 and 4.2). These conditions are applicable to global stabilization of (1.1) provided that the mappings g_i are *bounded* (Corollary 4.4). It should be noted that versions of the previous results have been originally presented in the recent works [3] of Jiang, Praly and Teel and [9] of Praly and Jiang. In particular, in [3] among other things it is shown that (1.1) is G.A.S. by smooth output feedback provided that $(\partial f/\partial x)(0,0)$ is Hurwitz and the subsystem (1.2) satisfies the I.S.S.C. Comparing with [3,9] our approach is based to weaker assumptions and is less technical. We also remark that our results can directly be extended for systems of the form $\dot{x} = f(x, y_1)$; $\dot{y}_i = g_i(x, y_1, ..., y_i) + y_{i+1} h_i(x, y_1, ..., y_i)$, $1 \leq i \leq m$, $y_{m+1} := u$, where h_i are everywhere strictly positive C° mappings.

2 Versions of the "Input to State Stability Condition"

It will be useful to recall first the definition of the I.S.S.C. We say that (1.2) satisfies the I.S.S.C. if it is *complete*, (namely, for every initial state x_0 and for any essentially bounded measurable control y there exists a solution $x(t, x_0; y)$ of (1.2) starting from x_0 at time $t = 0$, which is defined for every $t \geq 0$) and there exist a function $\beta : \Re^+ \to \Re^+$ of class K , (namely β is continuous, strictly increasing vanishing at zero) and a C° function $\alpha : \Re^+ \times \Re^+ \to \Re^+$ such that for each fixed t the function $\alpha(\cdot, t)$ is of class K, for each fixed s the function $\alpha(s, \cdot)$ decreases to zero at infinity and further for any x_0 , (essentially) bounded input y and for (almost) all t the following holds:

$$|x(t, x_0; y)| \leq \alpha(|x_0|, t) + \beta(\|y_t\|), \tag{2.1}$$

where $y_T(t)$ equals $y(t)$ for $0 \le t \le T$ and is zero otherwise, and $\|\cdot\|, |\cdot|$ are the L^∞ and the usual Euclidean norm, respectively. (Note: Completeness of (1.2) imposed by the previous definition does not consist an extra assumption; in particular, it is very easy to see that (2.1) implies completeness of (1.2)).

The previous condition has been used by several authors in order to face the global stabilizability problem (see for instance [2,3,9,11]). We now propose two versions of the I.S.S.C.

Condition 2.1 *There exist a positive definite, uniformly unbounded (p.d.u.u.) C^1 function $V : \Re^n \to \Re^+$ and a function $\gamma : \Re^+ \to \Re^+$ of class K_∞ (namely, $\gamma \in K$ and $\gamma(s) \to +\infty$ as $s \to +\infty$) such that*

$$DV(x)f(x,y) < 0, \ \ \forall |y| \le \gamma(|x|), \ x \neq 0 \tag{2.2}$$

(DV denotes the derivative of V).

It is important to note that Condition 2.1 is equivalent to the *"Lyapunov description"* of the I.S.S.C. (see equation (36) in [11] or condition (3) and Lemma 1 of the recent work [2] of Freeman; Kokotovic). To be more specific, if (2.2) holds then following the same arguments with those given in [10, Section VI] we can determine a p.d.u.u. C^1 map $\hat{V} : \Re^n \to \Re^+$ and a function $\zeta : \Re^+ \to \Re^+$ of class K_∞ such that

$$D\hat{V}(x)f(x,y) \le -\zeta(|x|), \ \ \forall |y| \le \gamma(|x|).$$

The latter, according to the proof of Theorem 1 in [10], implies I.S.S.C. Conversely, as we have mentioned in the introduction, it has been recently proved by Sontag and Wang in [14] that I.S.S.C. implies Condition 2.1 provided that f is at least C^1. This is a consequence of a general converse Lyapunov theorem proved by Lin, Sontag and Wang in [7] concerning the set stability of control systems with inputs taking values on a compact set.

Since I.S.S.C. is stronger than *"bounded input bounded state" (B.I.B.S.)* and also *"converging input converging state" (C.I.C.S.)* properties (see [12] for definitions) it follows that Condition 2.1 implies B.I.B.S., C.I.C.S. and *completeness* of (1.2).

The following condition is weaker than I.S.S.C.

Condition 2.2 *There exists a function $\gamma : \Re^+ \to \Re^+$ of class K_∞ such that*

(a) *for every initial state x_0 and input y such that $x(t, x_0; y)$ exists for all $t \ge 0$ and satisfies*

$$|y(t)| \le \gamma(|x(t, x_0; y)|) \tag{2.3}$$

it follows that

$$x(t,x_0;y) \to 0 \text{ as } t \to +\infty \tag{2.4a}$$

and since γ is C° vanishing at zero

$$y(t) \to 0, \quad \text{as } t \to +\infty \ ; \tag{2.4b}$$

(b) *for every initial state x_0 , input y and time T such that (2.3) is satisfied on $[0,T)$ it follows that $\lim_{t\to T}|x(t,x_0;y)| < +\infty$ and, since γ is of class K_∞ , $\lim_{t\to T}|y(t)| < +\infty$.*

Condition 2.2 says essentially the following: for each vector (x_0, y_0) belonging to the region $M := \{(x,y) \in \Re^{n+1} : |y| \le \gamma(|x|)\}$ and for every input y the trajectory $(x(t,x_0;y), y(t))$ either leaves $M\backslash\{0\}$ after some finite time, or stays in M for all t, approaching zero as $t \to +\infty$.

Example Consider the system $\dot{x}_1 = g(x_1), \dot{x}_2 = -x_2+y$, $(x_1,x_2) \in \Re^k \times \Re$ and assume that $0 \in \Re^k$ is a *global attractor* with respect to $\dot{x}_1 = g(x_1)$. Then we can easily justify that this system satisfies Condition 2.2 with $\gamma(s) = \frac{1}{2}s^{1/2}$, although it may fails to be input to state stable or even B.I.B.S. Note that I.S.S.C. is satisfied under the extra assumption that $0 \in \Re^k$ is globally asymptotically stable with respect to $\dot{x}_1 = g(x_1)$.

The following Propositions 2.3-2.6 summarize some interesting properties of Condition 2.2. In particular, in Propositions 2.3 and 2.4 some rather strict but useful Lyapunovlike descriptions of Condition 2.2 are provided and in Proposition 2.5 it is shown that I.S.S.C. implies Condition 2.2. This in conjunction with the previous example guarantees that Condition 2.2 is weaker than I.S.S.C.

Proposition 2.3

(i) Suppose that there exist a pair of positive definite C^1 functions V_1, V_2 : $\Re^n \to \Re^+, V_2$ being uniformly unbounded, functions $\zeta \in K$ and $\gamma \in K_\infty$, and $\theta : \Re^+ \to \Re^+$ being everywhere strictly positive and continuous with

$$\int_0^{+\infty} \frac{dr}{\theta(r)} = +\infty \tag{2.5}$$

and a positive constant R such that

$$DV_1(x)f(x,y) \ < \ -\zeta(|x|), \ \forall\, |y| \le \gamma(|x|), \ x \ne 0 \tag{2.6a}$$

$$DV_2(x)f(x,y) \ \le \ \theta(V_2(x)), \ \forall\, |y| \le \gamma(|x|), \ |x| \ge R \tag{2.6b}$$

Then Condition 2.2 is satisfied with the same characteristic function γ;

(ii) it turns out that Condition 2.1 implies Condition 2.2 (with the same characteristic function γ).

Proof (Outline): We show that (2.6b) plus (2.5) implies property (b) of Condition 2.2. (Its proof consists a slight generalization of a well known result due to Wintner [19]). Suppose on the contrary that there exist an x_0 , input y and time T such that the solution $x(t, x_0; y)$ of (1.2) exists and satisfies (2.3) on $[0, T)$, whereas $\lim |x(t, x_0; y)| \to \infty$ as $t \to T$. Then there would exist a pair of sequences $\{t_v\}$ and $\{\rho_v\}, (v = 1, 2, \ldots)$ with $T > t_v > \rho_v$, $t_v \to T$ such that $|x(t, x_0; y)| \geq |x(\rho_v, x_0; y)| = R$ for $t \in [t_v, \rho_v]$ and $|x(t_v, x_0; y)| \to +\infty$ as $t_v \to T$. Then by (2.5),(2.6) and the fact that V_2 is p.d.u.u. on $\Re^n$ we get

$$+\infty = \lim_{t_v \to T} \int_{V_2(x(\rho_v, x_0; y))}^{V_2(x(t_v, x_0; y))} \frac{dr}{\theta(r)} < T$$

a contradiction, hence property (b) of Condition 2.2 is satisfied. The latter in conjunction with (2.6a) implies property (a) of Condition 2.2 (The proof of this statement follows by using standard Lyapunov based arguments based on property (b) of Condition 2.2 and the fact that $\zeta \in K$). Finally, note that if Condition 2.1 is satisfied then obviously (2.6a) and (2.6b) hold with $V_1 = V_2 = V$ being uniformly unbounded, $\theta = 1$, $R = 0$ and the same γ.

A much more strong Lyapunov like description of Condition 2.2 is given by the following proposition.

Proposition 2.4 *Assume that there exist a pair of positive definite C^1 functions $V_1, V_2 : \Re^n \to \Re^+$, V_2 being uniformly unbounded on $\Re^n$, a function $\gamma \in K_\infty$ and a constant $R \geq 0$ such that*

$$DV_1(x) f(x, y) < 0, \ \forall \ |y| \leq \gamma(|x|), \ x \neq 0 \tag{2.7a}$$

$$DV_2(x) f(x, y) \leq 0, \ \forall |y| \leq \gamma(|x|), \ |x| \geq R. \tag{2.7b}$$

Then the system (1.2) satisfies Condition 2.2 as well as B.I.B.S. and C.I.C.S. properties. Obviously, Condition 2.1 implies both (2.7a) and (2.7b) with $V_1 = V_2 = V$, $R = 0$ and the same γ.

Proof: We only show that (2.7b) implies B.I.B.S. The rest part of the proof follows easily and it is left to the reader. First, note that (2.7b) is a special case of (2.6b) hence implies property (b) of Condition 2.2. Consider now any initial x_0 and bounded input y. We show that the solution $x(\cdot, x_0; y)$ is bounded. We distinguish three cases. The first is $|y(t)| \leq \gamma(|x(t, x_0; y)|)$ for all t after some finite time. Then the desired conclusion follows by taking into account (2.7b) and the fact that V_2 is p.d.u.u.

on $\Re^n$. Suppose next that there exist sequences $\{t_v\}$, $\{\rho_v\}$; $\{s_v\}$ with $0 < t_v \leq \rho_v \leq s_v < t_{v+1}$ and $t_v \to +\infty$ such that $|y(t)| \leq \gamma(|x(t, x_0; y)|)$ for $t \in [t_v, s_v]$, $|y(t)| \geq \gamma(|x(t, x_0; y)|)$ for $t \in [s_v, t_{v+1}]$ and $|x(\rho_v, x_0; y)| = \max\{|x(t, x_0; y)|,\ t \in [t_v, s_v]\}$ and assume on the contrary that

$$\overline{\lim}|x(\rho_v, x_0; y)| = +\infty.$$

By (2.7b) and the fact that V_2 is p.d.u.u. on $\Re^n$ we get

$$\overline{\lim} V_2(x(t_v, x_0; y)) \geq \overline{\lim} V_2(x(\rho_v, x_0; y)) = +\infty$$

and so $\overline{\lim}|x(t_v, x_0; y)| \to +\infty$. On the other hand $|y(t_v)| = \gamma(|x(t_v, x_0; y)|)$ and since $\gamma \in K_\infty$ it follows that $\overline{\lim}|y(t_v)| = +\infty$, a contradiction. The third case is $|y(t)| \geq \gamma(|x(t, x_0; y)|)$ for all t after some finite time. Then the desired conclusion follows directly from the boundedness of the input y. Hence B.I.B.S. property is satisfied. Similarly we can establish that (2.7a) implies C.I.C.S.

Proposition 2.5 *If the system (1.2) satisfies the I.S.S.C. then it also satisfies Condition 2.2 with γ being any real function of class K_∞ such that*

$$(\beta \circ \gamma)(s) < \lambda s,\ \forall\ s > 0 \tag{2.8}$$

where β is the function defined in (2.1) and λ being a positive constant with $\lambda < 1$.

Proof: The proof can be followed by using Proposition 2.4 and the equivalence of Condition 2.1 with I.S.S.C., which as we have already mentioned was established in [14]. For reasons of completeness a simple proof, which follows directly from the definition of the I.S.S.C., is presented here. Consider any function γ of class K_∞ which satisfies (2.1) and let an initial state x_0 , input y and time $T \leq +\infty$ such that $x(t, x_0; y)$ exists and (2.3) holds on $[0, T)$. We claim that $\overline{\lim}|x(t, x_0, y)| < +\infty$ as $t \to T$. Indeed, by (2.1) it follows

$$|x(t, x_0; y)| \leq \alpha(|x_0|, t) + (\beta \circ \gamma)(\|x(\cdot, x_0; y)_t\|) \tag{2.9}$$

for all $t \in [0, T)$. Suppose on the contrary that $\overline{\lim}|x(t, x_0; y)| = +\infty$ as $t \to T$. Then there would exist a sequence $\{t_v\}$ with $t \in [0, T)$ and $t_v \to T$ such that $|x(t, x_0; y)| \leq |x(t_v, x_0; y)_{t_v}|$ for all $t \leq t_v$. This implies $\|x(\cdot, x_0; y)_{t_v}\| = |x(t_v, x_0; y)|$ and so by (2.8) and (2.9) we get $(1 - \lambda)\overline{\lim}|x(t_v, x_0; y)| \leq \lim \alpha(|x_0|, t_v) < +\infty$ as $t_v \to T$, a contradiction. Therefore property (b) of Condition 2.2 is satisfied. We now establish property (a) of Condition 2.2, namely we show that (2.4) is fulfilled for any x_0 and input y satisfying (2.3). Suppose on the contrary that $\overline{\lim}|x(t, x_0; y)| = \rho > 0$ as $t \to +\infty$. The analysis given above implies that $\rho < +\infty$. It turns out that for every $\epsilon > 0$ there exists a $T = T(\epsilon) > 0$ such that

$$|x(t, x_0; y)| < \rho + \epsilon \text{ for all } t \geq T. \tag{2.10}$$

Let $\hat{x}_0 = x(T, x_0; y)$ and $\hat{y}(t) = y(t+T)$. By (2.1), (2.8) and (2.10) it follows that

$$\begin{aligned} |x(t,\hat{x}_0;\hat{y})| &\leq \alpha(|\hat{x}_0|,t) + \beta(\|\hat{y}_t\|) \\ &\leq \alpha(|\hat{x}_0|,t) + (\beta\circ\gamma)(\|x(\cdot,\hat{x}_0;\hat{y}_t)\|) \\ &\leq \alpha(|\hat{x}_0|,t) + (\beta\circ\gamma)(\rho+\epsilon), \ \ \forall\, t\geq 0. \end{aligned}$$

therefore $\rho = \overline{\lim}|x(t,x_0;y)| < \lim \alpha(|\hat{x}_0|,t) + (\beta\circ\gamma)(\rho+\epsilon) = (\beta\circ\gamma)(\rho+\epsilon)$. Since the latter holds for every $\epsilon > 0$ we conclude that $\rho \leq (\beta\circ\gamma)(\rho)$, $\rho \neq 0$ which contradicts (2.8), hence property (a) of Condition 2.2 is fulfilled.

We conclude this section by the following proposition that we shall need in the next section. (Analogous result is provided in [3] in the presence of the I.S.S.C.).

Proposition 2.6 *Suppose that f is C^1 near zero and there exists a C^1 Lyapunov function Φ of the origin with respect to $\dot{x} = f(x,0)$ such that*

$$\inf\{D\Phi(x)f(x,0)/|x||D\Phi(x)|,\ x\neq 0 \ \ \textit{near zero}\} < 0. \tag{2.11}$$

Then, if (1.2) satisfies Condition 2.2, there exist a smooth *function $\hat{\gamma}$: $\Re^+ \to \Re^+$ of class K_∞ and positive constants c and σ such that $\hat{\gamma}(s) = cs$ for $s \in [0,\sigma]$, $\hat{\gamma}(s) \leq \gamma(s)$ otherwise and further Condition 2.2 is satisfied with $\hat{\gamma}$ instead of γ. In particular, (2.11) holds if we assume that $(\partial f/\partial x)(0,0)$ is Hurwitz.*

Proof: Since f is C^1 is written $f(x,y) = f(x,0) + (\partial f/\partial y)(x,\hat{y})y$ for x, y and appropriate $\hat{y}$ near zero, hence by (2.11) we can find a constant $c > 0$ such that $D\Phi(x)f(x,y) < 0$ for all $|y| \leq c|x|$, x near zero. This implies the existence of a neighborhood N of $0 \in \Re^n$ such that the solution $x(t,x_0;y)$ is defined for all $t \geq 0$ and is contained in N approaching zero as $t \to +\infty$ for every $x_0 \in N$ and input y with $|y(t)| \leq c|x(t,x_0;y)|$. The desired conclusion follows then by combining the previous discussion and properties (a) and (b) of Condition 2.2.

3 Output Feedback Stabilizability

Our first result is the following theorem.

Theorem 3.1 *If Condition 2.1 is satisfied, then the system (1.1) with m=1 is G.A.S. by means of a continuous output feedback $u = \phi(y)$ with $\phi(0) = 0$ which is smooth on $\Re^n\backslash\{0\}$.*

Proof: In order to simplify the notation we denote $y = y_1$; $g = g_1$. Consider a pair of C° functions $a, b : \Re^+ \to \Re$ of class K_∞ such that according to the previous notations the following holds:

$$|g(x,y)| \leq a(|x|) + b(|y|), \ \ \forall\ (x,y) \in \Re^{n+1} \tag{3.1}$$

(see [9] for a proof of the previous inequality). We define $\rho_1(s) = \gamma(s)$ and $\rho_2(s) = (1/2)\gamma(s)$, γ being the function introduced in Condition 2.1. Without any loss of generality we may assume that a, b are C^∞ on $\Re\backslash\{0\}$ and γ is C^∞ on $\Re$. The latter is a direct consequence of (2.2) and Lemma 3.1 in [13]. We establish that there exists a C° map $y \to \theta(y)$ which is odd, C^∞ on $\Re\backslash\{0\}$, its restriction $\theta : \Re^+ \to \Re^+$ is of class K_∞ and such that the function

$$\phi(y) \doteqdot \begin{cases} -(1+E)y - b(y) - \theta(y) & , \quad y \geq 0 \\ -(1+E)y + b(-y) - \theta(y) & , \quad y < 0, \end{cases} \tag{3.2}$$

E being an arbitrary positive constant and b as defined in (3.1), is an output feedback stabilizer. Particularly, θ can be chosen in such a way that

$$\begin{aligned} 2a(|x|) \quad + \quad & \max\left\{|D\rho_1(|x|)f(x,y)|,\ \rho_2(|x|) \leq y \leq \frac{3}{2}\rho_1(|x|)\right\} \\ & < \frac{1}{4}\theta(\rho_2(|x|)), \forall x \neq 0. \end{aligned} \tag{3.3}$$

The existence of θ satisfying the previous properties follows from the fact that ρ_2 is of class K_∞ and the mappings $a(|x|)$, $\rho_1(|x|)$, $\rho_2(|x|)$, $D\rho_1(|x|)$ and $f(x,y)$ are continuous vanishing at zero.

In order to establish that the map ϕ as defined by (3.2) is the desirable output feedback we first prove that the following properties are satisfied:

(i) For every initial state $(x_0, y_0) \in \Re^{n+1}$ we have

$$\frac{d}{dt}(y^2(t)) = y(t)(g(x(t),y(t)) + \phi(y(t))) \leq -|y(t)|^2 \tag{3.4}$$

for each $t \geq 0$ such that $|y(t)| > \rho_2(|x(t)|)$ and for which the corresponding (not necessarily unique) solution

$$(x(t), y(t)) = (x(t, x_0, y_0),\ y(t, x_0, y_0))$$

of the closed - loop system

$$\dot{x} = f(x,y), \quad \dot{y} = g(x,y) + \phi(y) \tag{3.5}$$

exists;

(ii) the region

$$M \doteqdot \{(x,y) \in \Re^{n+1} : |y| \leq \rho_1(|x|)\} \tag{3.6}$$

is positively invariant with respect to (3.5).

We first prove property (i). We substitute the term

$$\phi(y)y = |y|\phi(|y|) = -(1+E)y^2 - |y|b(|y|) - |y|\theta(|y|) \tag{3.7}$$

in (3.4) and take into account (3.1). It suffices then to show that

$$|y|(a(|x|) + b(|y|)) - (1+E)y^2 - |y|b(|y|) - |y|\theta(|y|) < -|y|^2,$$

or $|y|(a(|x|) - \theta(|y|)) < 0$ for $|y| > \rho_2(|x|)$ which follows from (3.3). Next we establish property (ii). Let $(x_0, y_0) \in M$. Since $0 \in \Re^{n+1}$ is an equilibrium for (3.5) and belongs to M, we may assume that $(x_0, y_0) \neq 0$. We show that the corresponding solution of (3.5) lies in M for every $t \geq 0$ for which it exists. Equivalently

$$\begin{aligned} y^2(t) &\leq \rho_1^2(|x(t)|) \text{ or } y_0^2 + 2\int_0^t y(s)\left(g(x(s), y(s)) + \phi(y(s))\right) ds \\ &\leq \rho_1^2(|x_0|) + 2\int_0^t \rho_1(|x(s)|) D\rho_1(|x(s)|) \frac{x'(s)}{|x(s)|} f(x(s), y(s)) ds \end{aligned}$$

for every $t \geq 0$. Since $|y_0| \leq \rho_1(|x_0|)$ it suffices to prove that

$$y(g(x,y) + \phi(y)) < \rho_1(|x|) D\rho_1(|x|) \frac{x'}{|x|} f(x,y)$$

for $\rho_2(|x|) \leq y \leq \frac{3}{2}\rho_1(|x|)$ or by (3.1) and (3.7) that

$$|y|\left(a(|x|) + b(|y|)\right) - (1+E)y^2 - |y|b(|y|) - |y|\theta(|y|)$$

$$\leq \rho_1(|x|) D\rho_1(|x|) \frac{x'}{|x|} f(x,y)$$

which follows from (3.3) and the fact that $\frac{1}{2}\rho_1(|x|) \leq y \leq \frac{3}{2}\rho_1(|x|)$.

From property (i) it follows that for any initial state (x_0, y_0) the y-component of the solution $(x(t), y(t))$ of (3.5) satisfies the inequality

$$|y(t)| \leq |y_0| e^{-\frac{1}{2}t} \tag{3.8a}$$

as long as the solution exists and

$$\rho_1(|x(t)|) < |y(t)| \tag{3.8b}$$

By (3.8) and the positively invariance of M we distinguish two cases. The first is $|y(t)| \leq \rho_1(|x(t)|)$ after some finite time. The second is $|y(t)| > \rho_1(|x(t)|)$ for all $t \geq 0$ (for which the solution exists). This in conjunction with (3.8) implies that $y(t) \to 0$ and so $\rho_1(|x(t)|) \to 0$ provided that the solution exists for all $t \geq 0$. Since ρ_1 is C° vanishing at zero it follows that $(x(t), y(t)) \to 0$ as $t \to +\infty$. We can use the previous arguments in order to show that (3.5) is complete. Indeed, by (3.8) the solution $(x(t), y(t))$ exists for all $t \geq 0$ such that $(x(t), y(t)) \notin M$. Finally, we invoke Proposition 2.3 which asserts that Condition 2.1 implies Condition 2.2 with the same characteristic function $\gamma(= \rho_1)$. It follows that the solution of (3.5) also exists for all $t \geq 0$ with $(x(t), y(t)) \in M$ and thus by the positively invariance of M for every $t \geq 0$.

We are now in a position to prove that $0 \in \Re^{m+1}$ is globally asymptotically stable with respect to (3.5). Using the positively invariance of M and Condition 2.2 it follows that $x(t, x_0, y_0) \to 0$ as $t \to +\infty$ and consequently $y(t) \to 0$ as $t \to +\infty$ for every initial $(x_0, y_0) \in M$. The previous discussion, in conjunction with the fact that because of (3.8) each trajectory with initial state outside M is tending to zero as $t \to +\infty$ or enters M after some finite time, guarantees that $(x(t), y(t)) \to 0$ as $t \to +\infty$ for all $(x_0, y_0) \in \Re^{n+1}$. Next, we show that, $0 \in \Re^{n+1}$ is stable with respect to (3.5). By (2.2) and the fact that ρ_1 is C° vanishing at zero, we can find for each $\epsilon > 0$ a $\delta > 0$ with $\delta + \rho_1(\delta) \leq \epsilon/2$ such that

$$|(x(t, x_0, y_0), y(t, x_0, y_0))| \leq |x(t, x_0, y_0)| + \rho_1(|x(t, x_0, y_0)|) < \frac{\epsilon}{2} \tag{3.9}$$

for all $t \geq 0$ and $(x_0, y_0) \in M$ with $|x_0| < \delta$. Finally, from (3.8), (3.9), the positively invariance of M and since ρ_1 is strictly increasing it follows that for every $(x_0, y_0) \notin M$ with $|x_0| < \delta$ and $|y_0| < \rho_1(\delta)$ we have $\rho_1(|x(t, x_0, y_0)|) < |y(t, x_0, y_0)| < \rho_1(\delta)$ as long as the solution remains outside M, which in conjunction with (3.9) implies that

$$|(x(t, x_0, y_0), y(t, x_0, y_0))| < \rho_1(\delta) + \delta + \frac{\epsilon}{2} < \epsilon \tag{3.10}$$

for all $t \geq 0$. By (3.9) and (3.10) it follows that for each $\epsilon > 0$ we have $|(x(t, x_0, y_0), y(t, x_0, y_0))| < \epsilon$ for all $t \geq 0$ and (x_0, y_0) belonging to the region $\{(x, y) \in \Re^{n+1} : |x| < \delta,\ |y| < \rho_1(\delta)\}$. We conclude that $0 \in \Re^{n+1}$ is globally asymptotically stable with respect to (3.5).

Next we generalize Corollary 2.2 and Proposition 4.3 in [3] by proving that if the dynamics f and g_i are C^1 , $(\partial f/\partial x)(0, 0)$ is Hurwitz and Condition 2.2 is satisfied, then the system (1.1) is G.A.S. by a smooth output feedback. First, we examine the case $m = 1$. As in the proof of Theorem 3.1 we use the notations $y = y_1$ and $g = g_1$.

Theorem 3.2 *Consider the system (1.1) with $m = 1$ and suppose that the mappings f and $g = g_1$ are C^1 , the matrix $(\partial f/\partial x)(0, 0)$ is Hurwitz and either Condition 2.2 or one of its stronger versions ((2.5) plus (2.6), (2.7), Condition 2.1, I.S.S.C.) is satisfied. Then (1.1) with $m = 1$ is G.A.S. by means of a C^∞ output feedback $u = \phi(y)$; $\phi(0) = 0$, which is linear near* *zero and such that the matrix*

$$\begin{pmatrix} \frac{\partial f}{\partial x}(0,0), & \frac{\partial f}{\partial y}(0,0) \\ \frac{\partial g}{\partial x}(0,0), & \frac{\partial g}{\partial y}(0,0) + \frac{\partial \phi}{\partial y}(0) \end{pmatrix} \tag{3.11}$$

is Hurwitz and furthermore the system

$$\dot{x} = f(x,y),\ \dot{y} = g(x,y) + \phi(y) + w \tag{3.12}$$

with w as input satisfies Condition 2.2.

Proof: Notice first that, since g is C^1 near zero, there exists a pair of smooth functions a and b of class K_∞ such that (3.1) holds and further

$$\text{both } a \text{ and } b \text{ are linear near zero.} \tag{3.13}$$

Next we recall Proposition 2.6 which asserts that since $(\partial f/\partial x)(0,0)$ is Hurwitz, we can determine a C^∞ function $\hat{\gamma} : \Re^+ \to \Re^+$ of class K_∞ which is linear near zero such that Condition 2.2 is satisfied with respect to (1.2) with $\hat{\gamma}$ instead of γ. We define $\rho_1(s) = \hat{\gamma}(s)$; $\rho_2(s) = (1/2)\rho_1(s)$. Since f, ρ_1 and ρ_2 are C^1 vanishing at zero and because of (3.13), there exists a constant $R > 0$ such that

$$\max\{|D\rho_1(|x|)f(x,y)|,\ \rho_2(|x|) \le y \le \frac{3}{2}\rho_1(|x|)\} \le R|x|, \tag{3.14}$$

for x near zero. From (3.13) and (3.14) it follows that there exists an odd C^∞ function θ which is linear near zero such that the map $\theta : \Re^+ \to \Re^+$ is of class K_∞ and (3.3) is satisfied. This in conjunction with (3.13) implies that the output feedback $\phi : \Re \to \Re$ as defined in (3.2) is smooth and linear near zero. Moreover, since $(\partial f/\partial x)(0,0)$ is Hurwitz, we can select the constant E in (3.2) sufficiently large such that the matrix (3.11) is also Hurwitz. Using (3.3) and following exactly the same arguments with those given in the proof of Theorem 3.1 we can establish that (3.8a) is fulfilled as long as (3.8b) and

$$|w(t)| \le \frac{1}{2}a(|x(t)|) + \frac{1}{4}\theta(|y(t)|) \tag{3.15}$$

hold and for each $t \ge 0$ for which the solution

$$(x(t), y(t)) = (x(t, x_0, y_0; w),\ y(t, x_0, y_0; w))$$

of (3.12) exists. Furthermore (3.3) implies that the set M as defined by (3.6) is positively invariant with respect to (3.12), with inputs w satisfying (3.15); namely, for each $(x_0, y_0) \in M$ it holds $|y(t, x_0, y_0; w)| \le \rho_1(|x(t, x_0, y_0; w)|)$

as long as (3.15) is satisfied. Let , $: \Re^+ \to \Re^+$ be a function of class K_∞ such that

$$, (|(x,y)|) \leq \frac{1}{2}a(|x|) + \frac{1}{4}\theta(|y|),\ \forall\,(x,y) \in \Re^{n+1} \tag{3.16}$$

(The existence of , is guaranteed by the fact that both a and $\theta : \Re^+ \to \Re^+$ are of class K_∞). Using the previous properties, as well as property (a) of Condition 2.2 imposed for the subsystem (1.2), we can establish as in the proof of Theorem 3.1 that for each initial state $(x_0, y_0) \in \Re^{n+1}$ the trajectory $(x(t), y(t))$ of (3.12) is tending to zero as $t \to +\infty$ for every input w such that

$$|w(t)| < ,\ (|(x(t), y(t))|) \tag{3.17}$$

and providing that the solution exists for all $t \geq 0$. It follows that the system (3.12) satisfies property (a) of Condition 2.2 with , instead of γ. Moreover, as in the proof of Theorem 3.1 we can show that (3.5) (namely, the system (3.12) with zero input) is complete and so zero $0 \in \Re^{n+1}$ is a global attractor with respect to (3.5). This in conjunction with the fact that the matrix (3.11) is Hurwitz implies that $0 \in \Re^{n+1}$ is globally asymptotically stable with respect to (3.6).

Finally, we establish that the system (3.12) also satisfies property (b) of Condition 2.2 with , instead of γ. Consider any vector $(x_0, y_0) \in \Re^{n+1}$, time T and input w satisfying (3.17) for all $t \in [0, T)$ and assume that the corresponding solution of (3.12) exists on $[0, T)$. We show that

$$\lim_{t \to T} |(x(t), y(t); w(t))| < +\infty. \tag{3.18}$$

We distinguish two cases. The first is $\rho_1(|x(t)|) < |y(t)|$ for t near T which in conjunction with (3.8) and (3.15)-(3.17) implies that $y(t)$ and therefore $x(t)$ and $w(t)$ are bounded for t near T. The other case is $(x_0, y_0) \in M$ and $|y(t)| \leq \rho_1(|x(t)|)$ for all $t \in [0, T)$. Then by using property (b) of Condition 2.2, which has assumed for the system (1.2), it follows that the solution of (3.12) as well as the corresponding input w satisfying (3.17) are both bounded on $[0, T]$. The previous discussion asserts that (3.18) is satisfied and so the the proof is completed.

Remark 3.3 If the assumption "$(\partial f/\partial x)(0,0)$ is Hurwitz" in the statement of Theorem 3.2 is dropped and if we further assume that all the Dini derivatives of γ exist at zero, then similar to the proof of this theorem it can be shown that there exists a continuous output feedback $u = \phi(y)$ such that (3.12) also satisfies Condition 2.2 and the origin will be a *global attractor* with respect to (3.5). However in that case ϕ will be in general C^∞ on $\Re^n \backslash \{0\}$ and zero may fails to be stable with respect to (3.5).

We are now in a position to prove our result concerning the output stabilizability problem for the general case (1.1) with output $y = (y_1, y_2, ..., y_m)'$.

Corollary 3.4 *Consider the general $(n+m)$-dimensional case (1.1) and assume that ϕ and g_i are C^1 . Then under the same hypothesis of Theorem 3.2 the system (1.1) is G.A.S. by means of a smooth output feedback which is linear near zero.*

Proof: The proof of the general case is based on Theorem 3.2 and follows by using standard induction arguments like those given in [3, Proposition 4.3] and in other relative works (see for instance [6,8,13]). For reasons of simplicity we consider the case $m = 2$. The general case follows similarly by induction. First, we invoke Theorem 3.2 in order to establish that there exists a C^∞ map $y_2 = \phi_1(y_1)$ with $\phi_1(0) = 0$ which is linear near zero and such that if we define

$$X \doteqdot (x', y_1)',\ F(X,Y) \doteqdot (f'(x_1, y_1), g_1(x, y_1) + \phi_1(y_1) + Y)',$$

the origin $0 \in \Re^{n+1}$ is globally asymptotically stable with respect to

$$\dot{X} = F(X,Y) \tag{3.19}$$

with $Y = 0$, $(\partial F/\partial X)(0,0)$ is Hurwitz and the system (3.19) with Y as input satisfies Condition 2.2. Then we apply the transformation $x = x, y_1 = y_1;\ Y = y_2 - \phi_1(y_1)$.In the new coordinates the original system (3.19) takes the form

$$\dot{X} = F(X,Y);\ \dot{Y} = G(X,Y) + u \tag{3.20a}$$

where

$$\begin{aligned} G(X,Y) &\doteqdot -D\phi_1(y_1)(g_1(x,y_1) + \phi_1(y_1) + Y) \\ &+ \quad g_2(x, y_1, \phi_1(y_1) + Y) \end{aligned} \tag{3.20b}$$

The desired conclusion follows then by using the properties of (3.19) and applying once again Theorem 3.2 for (3.20).

4 Weaker Stability Conditions

It is worth remarking that the assumption $\gamma \in K_\infty$ that has been imposed in both Conditions 2.1 and 2.2 can be relaxed in some cases. For instance, let us assume that there exists a pair of increasing *bounded* functions a, b : $\Re^+ \to \Re^+$ satisfying (3.1). Then the results of Theorems 3.1 and 3.2 concerning the case (1.1) with $m = 1$ are also valid if instead of Conditions 2.1 or 2.2 the system (1.2) satisfies one of the following weaker versions.

Condition 4.1 *The system (1.2) is complete and there exist a p.d.u.u. C^1 function $V : \Re^n \to \Re^+$ and a* bounded *function $\gamma \in K$ such that (2.2) holds.*

Condition 4.2 *The system (1.2) is complete and there exists a bounded function $\gamma \in K$ such that properties (a) and (b) of Condition 2.2 are satisfied.*

Indeed, in that case we can determine an increasing function $\rho_1 : \Re^+ \to \Re^+$ and positive constants ξ and σ such that $\rho_1(s) \leq \gamma(s)$ for all $s \leq 0$ and $\rho_1(s) = \sigma$ for every $\sigma \leq \xi$. Furthermore we can assume that without any loss of generality both a and b satisfying (3.1) are constants on $[\xi, +\infty)$. The previous assumption guarantees the existence a *bounded* C° map θ which is constant for $|y| > \xi$, its restriction $\theta : \Re^+ \to \Re^+$ is increasing and satisfies all the required properties including (3.3). The desired feedback will given again by (3.2) whereas its derivative will be constant for $|y| > \xi$. Moreover in this case the resulting system (3.12) will be complete. This follows from completeness assumption for (1.2) and the fact that θ is bounded. Indeed, boundedness of θ together with (3.2) implies that for every pair of mappings $x(t)$ and $w(t)$, w being essentially bounded, the one-dimensional system $\dot{y} = g(x,y) + \phi(y) + w$ is complete. The latter in conjunction with the completeness of (1.2) implies completeness of (3.12).

Note at this point that both Conditions 4.1 and 4.2 are weaker than I.S.S.C. as well as Condition 2.2 and as in Section 2 we can easily establish that Condition 4.1 implies Condition 4.2.

We can also derive an extension of Corollary 3.4 in the case $\gamma \in K$ and under the extra assumption that each g_i is bounded. However, a more careful analysis, than this given in Theorem 3.2 and Corollary 3.4, is required. We first need the following lemma whose proof consists of a slight modification of the procedure used in the proof of Theorem 3.2.

Lemma 4.3 *Consider the system (1.1) with $m = 1$ and suppose that f and g_1 are smooth and g_1 is decomposed as $g_1(x,y) = G_1(x_1,x_2,y) + G_2(x_2,y)$ with $x = (x_1,x_2) \in \Re^{n_1} \times \Re^{n_2}$, $n_1 + n_2 = n$ for the case $n_2 \geq 1$ and $g_1(x,y) = G_1(x,y) + G_2(y)$ otherwise, where G_1 and G_2 are smooth vanishing at zero and G_1 is bounded. Furthermore assume that $(\partial f/\partial x)(0,0)$ is Hurwitz and the system (1.2) satisfies Condition 4.2 (or its stronger version Condition 4.1). Then there exists a linear map $\phi(y)$ such that the system (1.1) with $m = 1$ is G.A.S. by means of the smooth feedback $u = \phi(y) - G_2(x_2,y)$. Furthermore the derivative of the closed-loop dynamics $(f\prime, G_1 + \phi)\prime$ at the origin will be Hurwitz and the system $\dot{x} = f(x,y)$, $\dot{y} = G_1(x,y) + \phi(y) + w$ with w as input satisfies Condition 4.2.*

Proof (Outline): Let P be the positive definite solution of the matrix equation $P\frac{\partial f}{\partial x}(0,0) + \left(\frac{\partial f}{\partial x}(0,0)\right)' P = -I$ and define $\Phi(x,y) := x'Px\ y^2$, $u(y) := \phi(y) - G_2(x_2,y)$, $\phi(y) := -Ey, E > 0$. Then it can be easily justified that there exists a positive E_1 such that for all $E \geq E_1$ the derivative of Φ

along the trajectories of the linearization at zero of the resulting closed-loop system

$$\dot{x} = f(x, y_1), \dot{y} = G_1(x, y) + \phi(y) + w, \phi(y) = Ey \tag{4.1}$$

is strictly negative and for sufficiently small $c > 0$, being independent of E, the compact set

$$N := (x, y) \in \Re^{n+1} : \Phi(x, y) \leq c \tag{4.2}$$

is positively invariant with respect to (4.1). As in Theorem 3.2, consider the mappings $\rho_1(s)$ and $\rho_2(s)$ and without any loss of generality assume that $\rho_2(s) = \frac{1}{2}\rho_1(s) = \frac{1}{2}\alpha, \forall s \geq \xi$ for arbitrary small positive constants ξ and α. Pick ξ such that the sphere of radius 2ξ centered at $0 \in \Re^{n+1}$ is contained in N. Let E be a positive constant satisfying

$$E > \max\{E_1, 2\alpha^1 \sup G_1(x, y)\} + 1$$

whose existence is guaranteed by the boundedness of G_1. By a slight modification of the approach in Theorem 3.2 we can find a bounded function $, \in K$ in such a way that

(i) for every initial y_0 with $|y_0| > \alpha/4$ property (3.8a) holds for all inputs ω satisfying (3.17) with respect to (4.1) with E as defined by (4.3);

(ii) the set $N \cup M$, with M, N as defined by (3.6) and (4.2), respectively is positively invariant with respect to (4.1).

The previous properties imply as in the proof of Theorem 3.2 the desired conclusion.

Corollary 4.4 *Suppose that f is C^1, the subsystem (1.2) satisfies the same hypotheses of Lemma 4.3 and the mappings g_i , $i = 1, ..., m$ are C^1 and bounded. Then the system (1.1) is G.A.S. by means of a linear feedback depending only on $y_1, ...y_m$.*

Proof: For reasons of simplicity we consider the case $m = 2$. For the case $m = 1$ we define $G_1 = g_1$, $G_2 = 0$ and apply the result of Lemma 4.3. Let $\phi_1(y_1)$ be the corresponding linear feedback stabilizer. We proceed as in the proof of Corollary 3.4 following the same notations in order to obtain the system (3.20). Note that G is decomposed as

$$G(X, Y) \doteqdot G_1(X, Y) + G_2(y_1, Y)$$

where

$$G_1(X, Y) \doteqdot -D\phi_1(y_1)(g_1(x, y_1) + g_2(x, y_1, \phi_1(y) + Y)\,;$$

$$G_2(y_1, Y) = -D\phi_1(y_1)(Y + \phi_1(y_1)).$$

The desired conclusion then follows from Lemma 4.3, the properties of (3.19) and the fact that g_1 and g_2 are bounded. Particularly, there exists a linear map $\phi(Y)$ as in the statement of Lemma 4.3 such that the desired feedback stabilizer for (3.20) has the form $\phi_2(y_1, Y) = \phi(Y) - G_2(y_1, Y)$, namely it depends only on $Y = y_2 - \phi_1(y_1)$ and y_1 (the last variable of X). It turns out by the definition of G_2 that ϕ_2 is linear and depends only on y_1 and y_2.

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