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Cascades for Dynamical Games*

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Abstract

We study in this paper the hierarchical regulation of a control system with two controls. State and controls are subjected to (viability) constraints, and the controls have to be chosen in order to maintain the viability of the state. We characterize the (set–valued) subregulation maps governing the evolution of viable controls, and establish a metaregulation law for the evolution of viable state–control solutions. Moreover, among the viable evolutions, we select solutions regulated in a hierarchical manner : the regulation of the first control depends only on the present state and the present controls, whereas the regulation of the second control also takes into account the velocity of the first control. The main example is given by the heavy hierarchical solutions : both controls are constant as long as possible, but when the viability of the solution is at stake, the second control will vary first whereas the first control remains constant, and then, if this is no longer sufficient, also the first control starts moving.

We present our results in the framework of dynamical games, where these concepts can be used to model a hierarchical game, i.e., a game where one player has a stronger inertial power.

Key words: regulation of control systems with two controls under viability constraints, dynamical games, viability theory, differential inclusions

AMS Subject Classifications: 93C10, 93A13, 90D65

0 Introduction

We study in this paper the hierarchical regulation of a control system with two or more controls.

It has been motivated by a demo–economic model introduced by N. Bonneuil [4]. In this model, the state represents the per capita capital and the controls the per capita consumption and the crude birth rate. The controls

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are supposed to evolve with bounded velocity. Furthermore, capital, per capita consumption and crude birth rate are subjected to positivity (viability) constraints, and the controls have to be chosen in order to maintain the viability of the state.

In order to obtain numerical results, the crude birth rate has been fixed, and the viability of the control system has been studied only with respect to the per capita consumption [4]. Of particular interest are solutions obeying the *inertia principle*: the control maintaining the viability of the state evolves with minimal velocity. Such solutions are called *heavy solutions* (see [2, ch. 7]) for a rigorous definition).

Actually, both crude birth rate and per capita consumption are submitted to a *hierarchical inertia principle*, the crude birth rate being "heavier" than the per capita consumption. Namely, both controls evolve with minimal velocity, but when the viability of the solution is at stake, the per capita consumption will change first, and then, if this is no longer sufficient, also the crude birth rate will evolve. This motivates the terminology "cascades".

The controls are thus regulated in a hierarchical manner: in order to construct viable state solutions, the evolution of the velocity of the crude birth rate depends only on the present state and the present controls, whereas the regulation of the per capita consumption has also to take into account the present velocity of the crude birth rate.

These concepts can also be used in a dynamical game framework in order to model a *hierarchical dynamical game*. In this paper, we will present our results in the dynamical game terminology. We understand under the name "dynamical game" any problem related to dynamical systems controlled by two or more players, which is a more general notion than "differential game". Since Isaacs [9], the terminology "differential game" is usually reserved to control problems with two control vectors, each of them under the control of one of the two players. The purpose of a differential game is for one player, to reach a target or to maximize a cost, the purpose of the second player being opposite (see for example [8], [10]).

In this paper, the purpose of the dynamical game is to maintain for a given initial condition at least one solution of the dynamical system in a closed subset, which is a viability problem. The rule of the game is no longer minimaximization, but a hierarchical inertia principle. This means that one player has a stronger inertial power: he/she changes his/her control last, forcing the other player to move his/her control first.

Namely, we consider the following two–person game:

- The *dynamics of the game* are given by

$$\left\{ \begin{array}{llll} i. & x'(t) & = & f(x(t), u(t), v(t)) \\ ii. & u(t) & \in & U(x(t)) \\ iii. & v(t) & \in & V(x(t), u(t)), \end{array} \right. \tag{1}$$

where $u(\cdot)$, $v(\cdot)$ are regarded as strategies used by the players to govern the evolution of the state $x(\cdot)$ of the game.

- The *purpose of the game* is to maintain the state of the system in a given set $K = Dom(U)$, i. e., to find "playable" solutions of the game (1). (See [2, ch. 14] for a presentation of playability.)
- The players act on the velocities of the strategies, which are regarded as *decisions*.
- The players play a *hierarchical game*: in order to construct playable strategies, the first player's decision depends only on the present state and the present strategies, whereas the second player has also to consider the first player's decision.
- The main example for such a hierarchical game is given by the game, where the strategies obey a *hierarchical inertia principle*: both strategies evolve with minimal velocity, but when the viability of the solution is at stake, the inertia of the second strategy will be abandoned first, and then, if this is not longer sufficient, both strategies will move.

In this case, the players will try to minimize at each instant the velocity of their control. Therefore, we shall bound a priori these velocities by constants c_1 and c_2:

$$\left\{ \begin{array}{lcl} u'(t) & \in & c_1 B_1 \\ v'(t) & \in & c_2 B_2, \end{array} \right. \tag{2}$$

where B_1 and B_2 are the respective closed unit balls of the strategy spaces. We will hence act on the state–strategy space, which allows not only to find playable state–solutions, but also regular strategies. The idea for choosing decision rules such that the decision of the second player depends not only on the present state and the present strategies, but also on the first players decision, i. e.,

$$\left\{ \begin{array}{lcl} u'(t) & \in & S_1(x(t), u(t), v(t)) \\ v'(t) & \in & S_2(x(t), u(t), v(t); u'(t)), \end{array} \right. \tag{3}$$

is to introduce the concept of "cascades", which allows a hierarchical selection of a given set–valued map with images in a product space in an appropriate manner.

Naturally, these concepts can be extended to n–player dynamical games.

1 The Viability Theorem

We recall the Viability Theorem for differential inclusions and we adapt it in the framework of dynamical games.

Let X be a finite dimensional vector space. Recall that the domain of a set–valued map $F : X \rightsquigarrow X$ is defined by

$$Dom(F) = \{x \in X;\ F(x) \neq \emptyset\}.$$

Theorem 1.1 (Viability Theorem) *[2, th. 3.3.5,th. 4.1.2] Let $F : X \rightsquigarrow X$ be a nontrivial, upper semicontinuous set–valued map with compact convex images and linear growth, and let $K \subset Dom(F)$ be a closed set. The following properties are equivalent:*

i. For any $x_0 \in K$ there exists a viable solution on $[0, \infty[$ to the differential inclusion

$$\left\{ \begin{array}{lcl} x'(t) & \in & F(x(t)) \text{ for almost all } t \geq 0 \\ x(0) & = & x_0, \end{array} \right. \tag{4}$$

i. e., a solution of system (4) remaining in K for all $t \geq 0$.

ii. The set K is a viability domain*, i. e., it satisfies the following tangential condition*

$$F(x) \cap T_K(x) \neq \emptyset \quad \text{for all } x \in K.$$

When K is not a viability domain, there exists a largest closed viability domain contained in K, called the viability kernel $Viab_F(K)$ *of K.*

The viability kernel is the set of all initial conditions such that at least one solution starting from them is viable in K ([2, Th. 4. 1. 2.]). The linear growth of F ensures the existence of viable solutions defined on the interval $[0, \infty[$. The contingent cone to K at x is the set

$$T_K(x) \quad = \quad \{v \in X;\ \liminf_{\epsilon \longrightarrow 0+} \frac{d_K(x + \epsilon v)}{\epsilon} = 0\}, \tag{5}$$

where $d_K(y)$ denotes the distance of y to K, defined by

$$d_K(y) \quad := \quad \inf_{z \in K} \|y - z\|. \tag{6}$$

The viability kernel is an efficient concept used in many situations (see for example [5], [13]). Recently, algorithms have been designed to find it (see [14], [15]).

Let us now consider the two–person game

$$\left\{ \begin{array}{rlcl} i. & x'(t) & = & f(x(t), u(t), v(t)) \\ ii. & u(t) & \in & U(x(t)) \\ iii. & v(t) & \in & V(x(t), u(t)), \end{array} \right. \tag{7}$$

where Z_1 and Z_2 are finite dimensional vector spaces, $U : X \rightsquigarrow Z_1$ and $V : X \times Z_1 \rightsquigarrow Z_2$ are set–valued maps and $f : Graph(U \times V) \longrightarrow X$ is a single–valued map. Here, $U \times V$ denotes the composition $x \mapsto \{(u,v) \in Z_1 \times Z_2;\ u \in U(x) \text{ and } v \in V(x,u)\}$. To obtain state–solutions $x(\cdot)$ regulated by "smooth" strategies $u(\cdot)$ and $v(\cdot)$, we introduce two nonnegative functions φ_1, φ_2 on $X \times Z_1 \times Z_2$ bounding the growth of the strategies. We get the following associated two–person game

$$\left\{ \begin{array}{rrcl} i. & x'(t) & = & f(x(t),u(t),v(t)) \\ ii. & u'(t) & \in & \varphi_1(x(t),u(t),v(t))B_1 \\ iii. & v'(t) & \in & \varphi_2(x(t),u(t),v(t))B_2, \end{array} \right. \tag{8}$$

where B_1 and B_2 denote the closed unit balls of Z_1 and Z_2 respectively. Solutions of (8) are called *φ–smooth solutions.* We regard state–dependent constraints on the strategies as constraints on the state–strategy triples:

$$(x(t),u(t),v(t)) \in K := Graph(U \times V).$$

We posit the following assumptions

$$\left\{ \begin{array}{rl} \text{i.} & Graph(U) \text{ and } Graph(V) \text{ are closed.} \\ \text{ii.} & f \text{ is continuous and has linear growth.} \\ \text{iii.} & \varphi_1 \text{ and } \varphi_2 \text{ are continuous and have linear growth.} \end{array} \right. \tag{9}$$

We deduce from the Viability Theorem 1.1 applied to the two–person game (8) on $K = Graph(U \times V)$ (see also [2, th. 7.2.5]) the following

Theorem 1.2 (Subregulation and Metaregulation Map) *Let us assume that the two–person game (7) and the functions φ_1, φ_2 satisfy conditions (9). Let $R : X \rightsquigarrow Z_1 \times Z_2$ be a closed set–valued map contained in $U \times V$. Then the following two conditions are equivalent:*

i. For all initial state–strategy condition $(x_0,u_0,v_0) \in Graph(R)$, there exists a φ–smooth state–strategy solution $(x(\cdot),u(\cdot),v(\cdot))$ on $[0,\infty[$ to the two–person game (7) starting at (x_0,u_0,v_0) and viable in $Graph(R)$.

ii. R is a solution to the partial differential inclusion

$$0 \in DR(x,u,v)(f(x,u,v)) - \varphi_1(x,u,v)B_1 \times \varphi_2(x,u,v)B_2$$

for all $(x,u,v) \in Graph(R)$ satisfying the constraints

$$\forall x \in Dom(U \times V), \quad R(x) \subset (U \times V)(x).$$

*In this case, the map R is called a φ–*subregulation map *of $U \times V$. The* metaregulation law *regulating the evolution of the state–strategy solutions viable in $Graph(R)$ takes the form of the system of differential inclusions*

$$\begin{cases} i. & x'(t) = f(x(t), u(t), v(t)) \\ ii. & (u'(t), v'(t)) \in G_R(x(t), u(t), v(t)), \end{cases} \tag{10}$$

where the set–valued map G_R is defined by

$$G_R(x, u, v) := DR(x, u, v)(f(x, u, v)) \cap \varphi_1(x, u, v)B_1 \times \varphi_2(x, u, v)B_2$$

and called the metaregulation map *associated with R.*
Furthermore, there exists a largest φ–subregulation map R^φ contained in $U \times V$.

The graph of the largest φ–subregulation map R^φ is the viability kernel of the graph of $U \times V$ for the differential inclusion (8). It is very difficult to characterize it analytically, but it can be determined numerically by using the viability kernel algorithm (see [6], [7], [14], [15]). This is the reason why we use below not only the largest subregulation map R^φ, but any subregulation map R which can be characterized in an easier way (in the case of inequality constraints one can use Maderner's results [11], [12].)

2 Cascades of Two Controls

We shall first define cascades of a general set–valued map G, and then, in particular, consider cascades of the metaregulation map G_R of a two–person game. Our aim is to show the existence of state–strategy solutions of the two–person game (7) for all initial conditions in the graph of a given subregulation map, where the evolution of the strategies is governed by a cascade of the metaregulation map.

Let X, Z_1, Z_2 be finite dimensional spaces and $G : X \rightsquigarrow Z_1 \times Z_2$ be a set–valued map. The map G induces the projection $G_1 := \pi_1 \circ G$ of G on Z_1, i. e.,

$$G_1 : X \rightsquigarrow Z_1;\ x \mapsto \{u \in Z_1;\ \exists v \in Z_2 \text{ such that } (u, v) \in G(x)\}$$

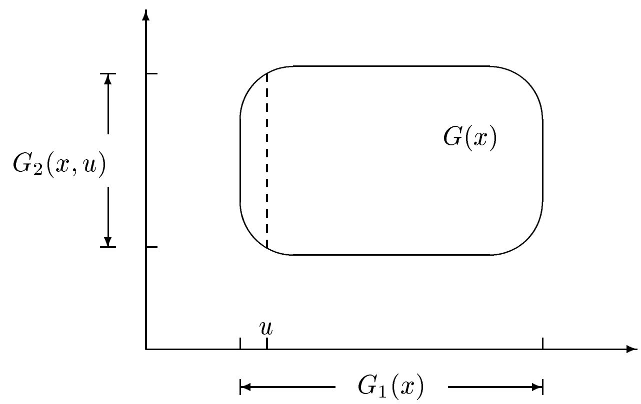
and further

$$G_2 : X \times Z_1 \rightsquigarrow Z_2;\ (x, u) \mapsto \{v \in Z_2;\ (u, v) \in G(x)\}.$$

We observe that $Dom(G_1) = Dom(G)$, $Graph(G_1) = \pi_{X \times Z_1}(Graph(G)) = Dom(G_2)$, and that $Graph(G_2) = Graph(G)$.

Figure 1 shows an example of the decomposition of a set–valued map with values in $\mathbb{R}^2$. For fixed $x \in X$, the set $G_1(x)$ is the projection of the

Figure 1: The sets $G(x)$, $G_1(x)$ and $G_2(x,u)$.



set $G(x)$ on the u–line, and for fixed $u \in I\!R$, the set $G_2(x,u)$ is the projection of the intersection of the vertical line through u and the set $G(x)$ on the v–line.

Definition 2.1 *Two set–valued maps*

$$S_1 : X \leadsto Z_1 \quad \textit{and} \quad S_2 : X \times Z_1 \leadsto Z_2$$

form a cascade of G *if* $Dom(S_1) = Dom(G_1)$, $Dom(S_2) = Dom(G_2)$, $Graph(S_1)$ *and* $Graph(S_2)$ *are closed, and*

$$\begin{cases} S_{G_1}(x) & := & S_1(x) \cap G_1(x) \neq \emptyset & \forall x \in Dom(S_1) \\ S_{G_2}(x,u) & := & S_2(x,u) \cap G_2(x,u) \neq \emptyset & \forall (x,u) \in Dom(S_2). \end{cases}$$

A cascade (S_1, S_2) of G induces a map $S : Dom(G) \leadsto Z_1 \times Z_2$ by the following relation:

$$(u,v) \in S(x) \iff u \in S_1(x) \text{ and } v \in S_2(x,u). \tag{11}$$

The map S has closed graph, and satisfies $S(x) \cap G(x) \neq \emptyset$ for all $x \in Dom(G)$, so that it defines a selection procedure of the map G in the sense of [2, ch. 6]. To apply the Viability Theorem 1.1, we need S to have convex values. An easy proof implies the statement below:

Lemma 2.1 *Let us assume that G_1 has convex values. The following two conditions are equivalent:*

i. S has convex values.

ii. S_1 and S_2 satisfy the convexity conditions

$$\left\{ \begin{array}{ll} (a) & tu_1 + (1-t)u_2 \in S_1(x) \\ (b) & tS_2(x,u_1) + (1-t)S_2(x,u_2) \subset S_2(x, tu_1 + (1-t)u_2) \end{array} \right. \tag{12}$$

for all $(x,u_1), (x,u_2) \in Dom(S_2) \cap Graph(S_1)$ and $t \in [0,1]$.

Naturally, the condition (12) is satisfied if S_1 and S_2 have convex values and if the intersection map $S_{G_1} = S_1 \cap G_1$ is single–valued, or in particular, if S_1 and S_2 are both single–valued.

We turn now to the dynamical game (7)

$$\left\{ \begin{array}{rlcl} i. & x'(t) & = & f(x(t), u(t), v(t)) \\ ii. & u(t) & \in & U(x(t)) \\ iii. & v(t) & \in & V(x(t), u(t)) \end{array} \right.$$

and fix two nonnegative functions φ_1, φ_2 and a φ–subregulation map $R : X \leadsto Z_1 \times Z_2$ of $U \times V$.

Theorem 2.1 (Cascade) *Let us assume that the conditions (9) hold true. Let (S_1, S_2) be a cascade of the metaregulation map G_R. If S_1 and S_2 satisfy the convexity condition (12), then for all initial conditions $(x_0, u_0, v_0) \in Graph(R)$ there exists a solution $(x(\cdot), u(\cdot), v(\cdot))$ of the hierarchical game*

$$\left\{ \begin{array}{rl} i. & x'(t) = f(x(t), u(t), v(t)) \\ ii. & u'(t) \in S_{G_{R,1}}(x(t), u(t), v(t)) \\ iii. & v'(t) \in S_{G_{R,2}}(x(t), u(t), v(t); u'(t)) \end{array} \right. \tag{13}$$

defined on $[0, \infty[$ with $x(0) = x_0$, $u(0) = u_0$, $v(0) = v_0$, and viable in $Graph(R)$.

We noted $S_{G_{R,1}} = (S_{G_R})_1$ of definition 2.1 in the theorem above.

Proof: We consider the set–valued map F defined by

$$F(x,u,v) := \{f(x,u,v)\} \times (S(x,u,v) \cap (\varphi_1(x,u,v)B_1 \times \varphi_2(x,u,v)B_2))$$

for all $(x,u,v) \in Graph(R)$, where S is the set–valued map defined by (11), and the following differential inclusion

$$\left\{ \begin{array}{l} (x'(t), u'(t), v'(t)) \in F(x(t), u(t), v(t)), \\ \text{where } (x(t), u(t), v(t)) \in Graph(R) \text{ for all } t \geq 0. \end{array} \right. \tag{14}$$

Since S has closed graph and convex values, and

$$(S(x,u,v)\cap(\varphi_1(x,u,v)B_1\times\varphi_2(x,u,v)B_2))\supset(S(x,u,v)\cap G_R(x,u,v)\neq\emptyset$$

for all $(x,u,v)\in Graph(R)$, F is upper semicontinuous with nonempty convex compact images and with linear growth. To apply the Viability Theorem 1.1, we have to verify that $Graph(R)$ is a viability domain of F, i. e.,

$$T_{Graph(R)}(x,u,v)\cap F(x,u,v)\neq\emptyset \quad \forall(x,u,v)\in Graph(R).$$

But this is implied by the fact that the sets $G_{R,1}(x,u,v)\cap S_1(x,u,v)$ and $G_{R,2}(x,u,v;u')\cap S_2(x,u,v;u')$ are always nonempty. Therefore, for all initial state $(x_0,u_0,v_0)\in Graph(R)$ there exists a solution $(x(\cdot),u(\cdot),v(\cdot))$ to (14) viable in $Graph(R)$. But this is also a solution of (13), because it satisfies

$$\begin{array}{rcl}(x'(t),u'(t),v'(t)) & \in & T_{Graph(R)}(x(t),u(t),v(t)) \\ & = & Graph(DR(x(t),u(t),v(t))\end{array}$$

almost everywhere, and hence

$$\begin{array}{rcl}(u'(t),v'(t)) & \in & DR(x(t),u(t),v(t))(f(x(t),u(t),v(t)) \\ & & \cap\varphi_1(x(t),u(t),v(t))B_1\times\varphi_2(x(t),u(t),v(t))B_2 \\ & = & G_R(x(t),u(t),v(t))\end{array}$$

almost everywhere. □

Corollary 2.1 *If the same assumptions as in the previous theorem hold true, and if the intersections $S_{G_{R,1}}=\{s_{G_{R,1}}\}$ and $S_{G_{R,2}}=\{s_{G_{R,2}}\}$ are single–valued, then for all initial conditions $(x_0,u_0,v_0)\in Graph(R)$ there exists a solution $(x(\cdot),u(\cdot),v(\cdot))$ of the hierarchical closed loop game*

$$\left\{\begin{array}{rl} i. & x'(t)=f(x(t),u(t),v(t)) \\ ii. & u'(t)=s_{G_{R,1}}(x(t),u(t),v(t)) \\ iii. & v'(t)=s_{G_{R,2}}(x(t),u(t),v(t);u'(t))\end{array}\right. \tag{15}$$

defined on $[0,\infty[$ with $x(0)=x_0$, $u(0)=u_0$, $v(0)=v_0$, and viable in $Graph(R)$.

3 Some Properties of the Metaregulation Map

Let $G: X\rightsquigarrow Z_1\times Z_2$ be a set–valued map and X, Z_1 and Z_2 be finite dimensional vector spaces. We shall investigate whether convexity and closedness of the images and lower semicontinuity are inherited by G_1 and G_2. These results will be needed in the next section to establish the existence of hierarchical heavy solutions.

Proposition 3.1 *If G has closed convex values, then G_1 has convex and G_2 has closed convex values. If in addition $G(x)$ is compact for all $x \in Dom(G)$, then G_1 has also closed values.*

Proof: The first statement is obvious. To verify the second, we fix $x \in Dom(G_1)$ and a sequence $(u_n)_n$ in $G_1(x)$ converging to $u \in Z_1$. For all $n \in \mathbb{N}$ there exists $v_n \in Z_2$ such that $(u_n, v_n) \in G(x)$. Since $G(x)$ is compact, there exists a subsequence converging to some $(\tilde{u}, v) \in G(x)$. But u_n converges to u, which yields $u = \tilde{u} \in G_1(x)$. $\square$

The lower semicontinuity of G implies directly the lower semicontinuity of G_1, but it is not sufficient for the lower semicontinuity of G_2.

Proposition 3.2 *If G is lower semicontinuous, then G_1 is lower semicontinuous.*

Proof: We fix $x \in Dom(G_1)$, $u \in G_1(x)$ and a sequence $(x_n)_n$ in $Dom(G_1)$ converging to x. Let $v \in Z_2$ such that $(u, v) \in G(x)$. Since G is lower semicontinuous, there exists $(u_n, v_n) \in G(x_n)$ for all $n \in \mathbb{N}$ converging to (u, v). In particular, $G_1(x) \ni u_n \longrightarrow u$. $\square$

Proposition 3.3 *Let us assume that G is lower semicontinuous and that for all $x \in Dom(G)$ and all $\epsilon > 0$ there exists $\delta > 0$ such that*

$$\left\{ \begin{array}{ll} & \|x - \tilde{x}\| < \delta, \ \ u_1, u_2 \in G_1(\tilde{x}) \textit{ with } \|u_1 - u_2\| < \delta \\ \Longrightarrow & \sup_{v_1 \in G_2(\tilde{x}, u_1)} \inf_{v_2 \in G_2(\tilde{x}, u_2)} \|v_1 - v_2\| < \epsilon. \end{array} \right. \tag{16}$$

Then G_2 is lower semicontinuous.

Proof: Fix $(x, u) \in Dom(G_2)$, $v \in G_2(x, u)$ and a sequence $(x_n, u_n)_n$ in $Dom(G_2)$ converging to (x, u). We have to construct a sequence $(v_n)_n$ converging to v with $v_n \in G_2(x_n, u_n)$ for all $n \in \mathbb{N}$. Because G is lower semicontinuous and $(x_n)_n$ converges to x, there exists $(\tilde{u}_n, \tilde{v}_n) \in G(x_n)$ for all $n \in \mathbb{N}$ such that $(\tilde{u}_n, \tilde{v}_n) \longrightarrow (u, v)$ if $n \longrightarrow \infty$. By assumption (16), for all $k \in \mathbb{N}$ there exists $\delta_k > 0$ (without restriction $\delta_k \downarrow 0$) such that

$$\begin{array}{ll} & \|x - \tilde{x}\| < \delta_k, \ \ u_1, u_2 \in G_1(\tilde{x}) \ \text{ with } \|u_1 - u_2\| < \delta_k \\ \Longrightarrow & \sup_{v_1 \in G_2(\tilde{x}, u_1)} \inf_{v_2 \in G_2(\tilde{x}, u_2)} \|v_1 - v_2\| < \frac{1}{k}. \end{array}$$

By induction we find an index sequence $n_1 < n_2 < \ldots < n_k < \ldots$ such that

$$\|x - x_n\| \le \delta_k, \ \ \|u - u_n\| \le \frac{\delta_k}{2}, \ \ \|u - \tilde{u}_n\| \le \frac{\delta_k}{2} \qquad \forall n \ge n_k.$$

Therefore for all $k \in \mathbb{N}$ and for all $n \geq n_k$ we have

$$\sup_{v_1 \in G_2(x_n, \tilde{u}_n)} \inf_{v_2 \in G_2(x_n, u_n)} \|v_1 - v_2\| < \frac{1}{k}.$$

With $k(n) := \max\{k \in \mathbb{N};\ n \geq n_k\}$ $(k(n) \uparrow \infty)$, we have in particular that for all $n \in \mathbb{N}$

$$\exists v_n \in G_2(x_n, u_n) \text{ such that } \|\tilde{v}_n - v_n\| < \frac{1}{k(n)}.$$

Letting $n \longrightarrow \infty$ yields $v_n \longrightarrow v$, which is the desired conclusion. □

If G has compact values, condition (16) is satisfied if the family of set–valued maps $G_2(x, \cdot) : G_1(x) \rightsquigarrow Z_2$ $(x \in X)$ is locally uniformly continuous relative to the Hausdorff semimetric $\mathbb{d}$ defined on the family of compact subsets of Z_2 by

$$K, L \subset Z \text{ compact} \quad \Longrightarrow \quad \mathbb{d}(K, L) := \sup_{v \in K} \inf_{w \in L} \|v - w\|.$$

Proposition 3.4 *If G is upper semicontinuous with compact values, then condition (16) is necessary for the lower semicontinuity of G_2.*

Proof: We suppose (16) is not satisfied. Then there exists $x \in Dom(G)$ and $\epsilon > 0$ such that for all $n \in \mathbb{N}$ there exist $x_n \in B(x, \frac{1}{n})$, $u_n, \tilde{u}_n \in G_1(x_n)$, $\|u_n - \tilde{u}_n\| < \frac{1}{n}$ such that

$$\sup_{v_1 \in G_2(x_n, \tilde{u}_n)} \inf_{v_2 \in G_2(x_n, u_n)} \|v_1 - v_2\| \geq \epsilon > \frac{\epsilon}{2}. \tag{17}$$

For all $n \in \mathbb{N}$ we can choose $\tilde{v}_n \in G_2(x_n, \tilde{u}_n)$ such that

$$\inf_{v_2 \in G_2(x_n, u_n)} \|\tilde{v}_n - v_2\| > \frac{\epsilon}{2}. \tag{18}$$

Since $x_n \longrightarrow x$, and G is upper semicontinuous with compact values, almost all $G(x_n)$ are contained in the compact set $\overline{B(G(x), 1)}$. The sequence $(\tilde{u}_n, \tilde{v}_n)_n$ contains hence a convergent subsequence. So we can assume that $(\tilde{u}_n, \tilde{v}_n)_n$ converges already to $(u, v) \in Z_1 \times Z_2$. Since $Graph(G)$ is closed and

$$Graph(G) \ni (x_n, \tilde{u}_n, \tilde{v}_n) \longrightarrow (x, u, v),$$

we obtain $(u, v) \in G(x)$. Condition (18) implies that $\|\tilde{v}_n - v_n\| > \frac{\epsilon}{2}$ for all $v_n \in G_2(x_n, u_n)$; in particular, there does not exist any sequence of $v_n \in G_2(x_n, u_n)$ converging to $v \in G_2(x, u)$. Hence G_2 cannot be lower semicontinuous. □

4 Hierarchical Inertia Principle

The simplest way to construct explicit dynamical closed loops in cascade of the dynamical game (7) associated with a given subregulation map R is to select the elements of $G_{R,1}$ and $G_{R,2}$ each with minimal norm. Such a state–strategy solution $(x(\cdot), u(\cdot), v(\cdot))$ will satisfy a hierarchical inertia principle: when the viability is at stake, the inertia of the second strategy will be relaxed first, and then, if this is no longer sufficient, both strategies will evolve. To realize the hierarchical inertia principle, we need the following definition.

Definition 4.1 *Let $G : X \rightsquigarrow Z_1 \times Z_2$ be a set–valued map, and X, Z_1 and Z_2 be finite dimensional vector spaces. We denote by $\mathrm{iner}_{G_1}(x)$ and $\mathrm{iner}_{G_2}(x,u)$ the elements of $G_1(x)$ (resp. $G_2(x,u)$) with minimal norm:*

$$\mathrm{iner}_{G_1}(x) = \{u \in G_1(x);\ \|u\| = \min_{\tilde{u} \in G_1(x)} \|\tilde{u}\|\} \quad \forall x \in Dom(G_1)$$

$$\mathrm{iner}_{G_2}(x,u) = \{v \in G_2(x,u);\ \|v\| = \min_{\tilde{v} \in G_2(x,u)} \|\tilde{v}\|\} \quad \forall (x,u) \in Dom(G_2).$$

If the set–valued maps G_1 and G_2 have convex compact values, the maps iner_{G_1} and iner_{G_2} are single–valued on $Dom(G_1)$ and $Dom(G_2)$ respectively. In this case, we denote by iner_{G_1} and iner_{G_2} also the corresponding functions. They arise from the intersection of G_1 and G_2 respectively with the set–valued maps iner_1 and iner_2 defined by

$$\begin{array}{lcll} \mathrm{iner}_1(x) & = & \{u \in Z_1;\ \|u\| \leq \inf_{\tilde{u} \in G_1(x)} \|\tilde{u}\|\} & \forall x \in Dom(G_1) \\ \mathrm{iner}_2(x,u) & = & \{v \in Z_2;\ \|v\| \leq \inf_{\tilde{v} \in G_2(x,u)} \|\tilde{v}\|\} & \forall (x,u) \in Dom(G_2). \end{array}$$

Figure 2 shows again a two–dimensional example. For fixed $x \in X$, the point $\mathrm{iner}_{G_1}(x) =: u$ is the point of the projection $G_1(x)$ of the set $G(x)$ on the u–line which has minimal absolute value. The point $\mathrm{iner}_{G_2}(x,u) =: v$ is the point of the projection $G_2(x,u)$ of the intersection of the vertical line through u and the set $G(x)$ on the v–line which has minimal absolute value.

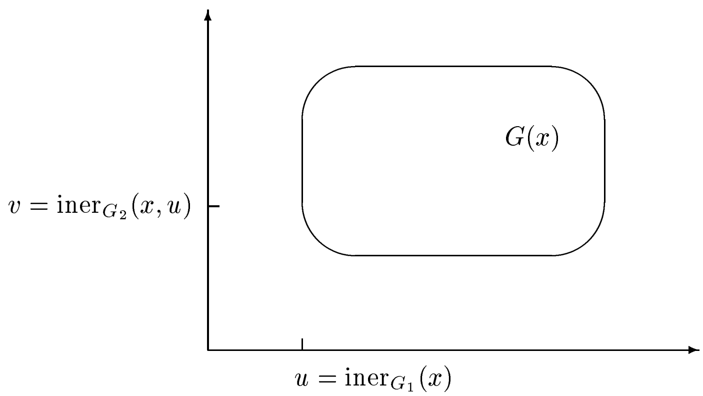
We have to verify that the pair $(\mathrm{iner}_1, \mathrm{iner}_2)$ defines a cascade of G, i. e., that their graphs are closed. This can be treated in a more general framework by an optimization procedure.

Proposition 4.1 *Let us assume that the set–valued map G has convex compact values and that the induced maps G_1 and G_2 are lower semicontinuous. Let $V_1 : Dom(G_1) \times Z_1 \longrightarrow \mathbb{R}$ and $V_2 : Dom(G_2) \times Z_2 \longrightarrow \mathbb{R}$ be continuous. Then the pair (S_1, S_2) defined by*

$$\begin{array}{lcl} S_1(x) & := & \{u \in Z_1;\ V_1(x,u) \leq \inf_{\tilde{u} \in G_1(x)} V_1(x,\tilde{u})\} \\ S_2(x,u) & := & \{v \in Z_2;\ V_2(x,u,v) \leq \inf_{\tilde{v} \in G_2(x,u)} V_2(x,u,\tilde{v})\} \end{array}$$

is a cascade of G.

Figure 2: The points $\mathrm{iner}_{G_1}(x) = u$ and $\mathrm{iner}_{G_2}(x,u) = v$.



Proof: We observe first that the set $S_1(x) \cap G_1(x)$ is always nonempty, because V_1 is continuous and $G_1(x)$ compact for all $x \in Dom(G_1)$. Since G_1 is lower semicontinuous and V_1 continuous, the Maximum Theorem [1, th. 1.4.16] implies that the function ϑ defined by

$$\vartheta(x,u) = V_1(x,u) + \sup_{\tilde{u} \in G_1(x)} (-V_1(x,\tilde{u}))$$

is lower semicontinuous as well. This yields the closedness of $Graph(S_1)$, because

$$Graph(S_1) = \{(x,u) \in X \times Z_1;\ \vartheta(x,u) \leq 0\}.$$

The same reasoning applies to G_2. □

Taking $V_1(x,u) = \|u\|$ and $V_2(x,u,v) = \|v\|$, we obtain the following corollary.

Corollary 4.1 *If G has convex compact images and if G_1 and G_2 are lower semicontinuous, then* $(\mathrm{iner}_1, \mathrm{iner}_2)$ *defines a cascade of G.*

For dynamical games, we can now combine the preceding results with the Cascade Theorem 2.1 to obtain the following

Corollary 4.2 *Let us assume that the conditions (9) hold true, that the metaregulation map G_R has convex compact values, and that the associated* *maps $G_{R,1}$ and $G_{R,2}$ are lower semicontinuous. Then, for all initial condition $(x_0, u_0, v_0) \in Graph(R)$, there exists a solution $(x(\cdot), u(\cdot), v(\cdot))$ of the hierarchical closed loop game*

$$\begin{cases} i. & x'(t) &=& f(x(t), u(t), v(t)) \\ ii. & u'(t) &=& \text{iner}_{G_{R,1}}(x(t), u(t), v(t)) \\ iii. & v'(t) &=& \text{iner}_{G_{R,2}}(x(t), u(t), v(t); u'(t)) \end{cases}$$

starting at (x_0, u_0, v_0), defined on $[0, \infty[$ and viable in $Graph(R)$.

Proof: The convexity condition (2.1) in Theorem 2.1 is satisfied because iner_1 and iner_2 have convex values and because $\text{iner}_{G_{R,1}}$ is single–valued. □

Corollary 4.3 *Let $V_1 : Graph(G_{R,1}) \to \mathbb{R}$ and $V_2 : Graph(G_{R,2}) \to \mathbb{R}$ be continuous and convex, and let the intersection map $S_{G_{R,1}} = S_1 \cap G_{R,1}$ be single–valued. If the same assumptions as in the previous corollary hold true, then for all initial condition $(x_0, u_0, v_0) \in Graph(R)$ there exists a solution $(x(\cdot), u(\cdot), v(\cdot))$ of the dynamical game (7) such that $u'(\cdot)$ and $v'(\cdot)$ minimize $V_1(x(\cdot), u(\cdot), v(\cdot);\ \cdot\ )$ and $V_2(x(\cdot), u(\cdot), v(\cdot); u'(\cdot),\ \cdot\ )$ respectively at each instant.*

Proof: V_1 and V_2 convex imply that S_1 and S_2 have convex values, and because $S_{G_{R,1}} = S_1 \cap G_{R,1}$ is single–valued, the convexity condition (2.1) is satisfied. □

The following simple example demonstrates the hierarchical inertia principle.

Example Consider the following dynamical game. The dynamics of the game are given by

$$\begin{cases} i. & x'(t) &=& 1 \\ ii. & u'(t) &\in& [-c, c] \\ iii. & v'(t) &\in& [-c, c], \end{cases} \tag{19}$$

where c is a positive constant. Both players' strategies have to satisfy the state–dependent constraints

$$u(t) + v(t) \le b_1 - \lambda_1 x(t) \quad \text{and} \quad u(t) + v(t) \le b_2 - \lambda_2 x(t)$$

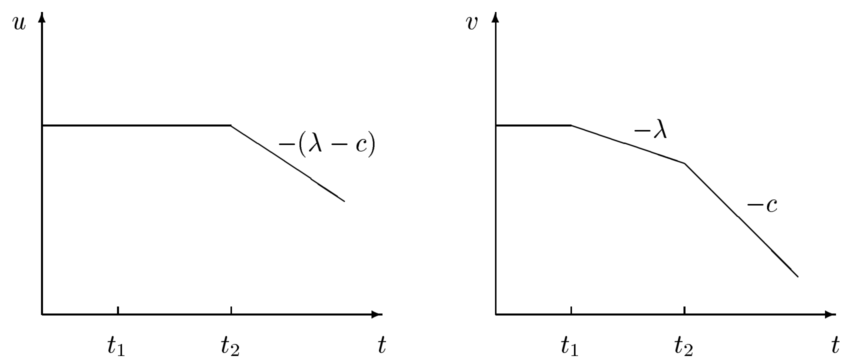
for all $t \ge 0$. Here, $b_1, b_2, \lambda_1, \lambda_2 \in \mathbb{R}$ are such that

$$0 < \lambda_1 < c < \lambda_2 < 2c.$$

It is easy to show that the graph of the set–valued map

$$U \times V : \mathbb{R}^+ \rightsquigarrow \mathbb{R} \times \mathbb{R},$$
$$x \mapsto \{(u, v) \in \mathbb{R}^2;\ u + v \le b_1 - \lambda_1 x \text{ and } u + v \le b_2 - \lambda_2 x\}$$

Figure 3:



is a viability domain for the differential inclusion (19). Hence, $R := U \times V$ is a regulation map for the game (19). Calculating the metaregulation map G_R belonging to R and selecting corresponding to the hierarchical inertia principle, we find the metaregulation of the heavy hierarchical strategies:

$$\begin{aligned} \text{iner}_{G_{R,1}}(x,u,v) &= \begin{cases} c - \lambda_2 & \text{if } u + v = b_2 - \lambda_2 x \\ 0 & \text{otherwise} \end{cases} \\ \text{iner}_{G_{R,2}}(x,u,v;\text{iner}_{G_{R,1}}(x,u,v)) &= \begin{cases} -\lambda_1 & \text{if } u + v = b_1 - \lambda_1 x \\ -c & \text{if } u + v = b_2 - \lambda_2 x \\ 0 & \text{otherwise .} \end{cases} \end{aligned}$$

A solution of the hierarchical dynamical game

$$\begin{cases} i. & x'(t) = 1, & x(0) = x_0 \\ ii. & u'(t) = \text{iner}_{G_{R,1}}(x(t),u(t),v(t)), & u(0) = u_0 \\ iii. & v'(t) = \text{iner}_{G_{R,2}}(x(t),u(t),v(t);u'(t)), & v(0) = v_0, \end{cases}$$

for $x_0 = 0$ and appropriate $u_0, v_0 \in I\!R$, is given by $x(t) = 1$ and by the functions $u(\cdot)$ and $v(\cdot)$ whose graphs are given in figure 3.

5 Hierarchical Viability Niches

In this section, we study evolutions obeying the hierarchical inertia principle. More precisely, we want to explain when both strategies are constant forever, when only the second strategy has to move whereas the first strategy remains constant, and when both strategies has to vary. For this purpose, we consider the dynamical game (8), and fix the largest subregulation map $R = R^{\varphi}$ associated with given φ_1, φ_2 according to Theorem 1.2. We introduce the following notation:

$$\begin{array}{rcl} C_{R,1}(u) & = & \{(x,v) \in X \times Z_2;\ 0 \in G_{R,1}(x,u,v)\} \\ C_R(u,v) & = & \{x \in X;\ (0,0) \in G_R(x,u,v)\} \\ & = & \{x \in X;\ (0,0) \in DR(x,u,v)(f(x,u,v))\} \end{array}$$

for all $u \in Z_1$ and $v \in Z_2$. The set–valued maps $C_{R,1}$ and C_R have closed graph provided that G_R has closed graph.

Besides the dynamical game (8), we consider the restricted game

$$\left\{ \begin{array}{rrcl} i. & x'(t) & = & f(x(t),u(t),v(t)) \\ ii. & u'(t) & = & 0 \\ iii. & v'(t) & \in & \varphi_2(x(t),u(t),v(t))B_2, \end{array} \right. \tag{20}$$

where $u(t) \in U(x(t))$ and $v(t) \in V(x(t),u(t))$ for all $t \geq 0$. Here, the first strategy a priori has to be constant. According to Theorem 1.2, we fix a subregulation map R^{01} associated with the functions $(0,\varphi_2)$, and define the set–valued map N_1 by

$$\forall u \in Z_1, \qquad N_1(u) = \{(x,v) \in X \times Z_2;\ (u,v) \in R^{01}(x)\} \subset C_{R,1}(u).$$

Finally, we restrict also the evolution of the second strategy considering

$$\left\{ \begin{array}{rrcl} i. & x'(t) & = & f(x(t),u(t),v(t)) \\ ii. & u'(t) & = & 0 \\ iii. & v'(t) & = & 0, \end{array} \right. \tag{21}$$

where $u(t) \in U(x(t))$ and $v(t) \in V(x(t),u(t))$ for all $t \geq 0$, and where both strategies are constant. We fix a subregulation map R^0 for the system above, and we get the set–valued map N as the inverse of R^0:

$$\forall (u,v) \in Z_1 \times Z_2, \qquad N(u,v) = \{x \in X;\ (u,v) \in R^0(x)\} \subset C_R(u,v).$$

We call $N_1(u)$ *partial viability niche* of u, and $N(u,v)$ *viability niche* of (u,v).

Proposition 5.1

i. The partial viability niche $N_1(u)$ of a strategy u is the viability kernel of $Graph(V)$ for the differential inclusion

$$\left\{ \begin{array}{rrcl} i. & x'(t) & = & f(x(t),u,v(t)) \\ ii. & v'(t) & \in & \varphi_2(x(t),u,v(t))B_2 \end{array} \right. \tag{22}$$

parameterized by the constant strategy u.

ii. *The viability niche $N(u,v)$ of (u,v) is the viability kernel of $(U \times V)^{-1}(u,v)$ for the equation $x'(t) = f(x(t),u,v)$ parameterized by the constant strategies u, v.*

Proof: Fix $u \in Z_1$. We have to show that $N_1(u) = Viab(Graph(V))$. For each $(x_0, v_0) \in N_1(u)$ there exists a viable solution $(x(\cdot), u(\cdot), v(\cdot))$ to the differential inclusion (20) starting at (x_0, u, v_0) and viable in $Graph(R^{01}) \subset Graph(U \times V)$. Hence $(x(\cdot), u(\cdot))$ is a solution to (22) satisfying $(x(t), v(t)) \in Graph(V)$ for all $t \geq 0$. Conversely, for all $(x_0, v_0) \in Viab(Graph(V))$ there exists a solution to (22) starting at (x_0, v_0) and viable in $Graph(V)$. Hence $(x(\cdot), u, v(\cdot))$ is a solution to (20) viable in $Graph(U \times V)$ and therefore $(u, v_0) \in R^{01}(x_0)$.

The proof of the second part is similar. $\Box$

We observe the following cascade–like behaviour of a heavy hierarchical solution. If for some $t_1 \geq 0$ the state–solution $x(\cdot)$ enters the subset $C_R(u,v)$ then both strategies $u(\cdot), v(\cdot)$ remain equal to u and v respectively as long as $x(t) \in C_R(u,v)$. Furthermore, we obtain the following alternative:

i. If $x(t_1)$ is contained in $N(u,v)$, then $(u(t), v(t))$ remains equal to (u,v) for all $t \geq t_1$, and $x(t)$ remains in the viability niche $N(u,v)$ for all $t \geq t_1$.

ii. If $x(t_1) \notin N(u,v)$, then $x(t)$ must eventually leave $C_R(u,v)$ in finite time $t_2 \geq t_1$ [2, prop. 4.1.4]. After that, $u(t)$ remains equal to u as long as $(x(t), v(t)) \in C_{R,1}(u)$. A second alternative emerges:

   (a) If $(x(t_2), v(t_2))$ is contained in $N_1(u)$, then $u(t)$ remains equal to u for all $t \geq t_2$, and $(x(t), v(t))$ remains in the partial niche $N_1(u)$ for all $t \geq t_2$.

   (b) If $(x(t_2), v(t_2)) \notin N_1(u)$, then $(x(t), v(t))$ must eventually leave $C_{R,1}(u)$ in finite time and $u(t)$ must start moving as well.

6 Implicitly–Defined Cascades

In this section, we give another example for a class of cascades. In fact, a cascade of a set–valued map $G : X \rightsquigarrow Z_1 \times Z_2$ can be defined selecting first a strategy $u \in G_1(x)$ satisfying the (implicit) inclusion $0 \in E_1(x,u)$ and then a strategy $v \in G_2(x,u)$ solution to $0 \in E_2(x,u,v)$, where $E_1 : X \times Z_1 \rightsquigarrow Z_1$ and $E_2 : X \times Z_1 \times Z_2 \rightsquigarrow Z_2$ are given set–valued maps.

Proposition 6.1 *Assume that G has compact convex values, that E_1 and E_2 satisfy $Dom(E_1) \supset Graph(G_1)$ and $Dom(E_2) \supset Graph(G_2)$, and that* *they are both upper semicontinuous with compact convex values. Assume further that*

$$\begin{array}{lclcl} E_1(x,u) & \cap & T_{G_1(x)}(u) & \neq \emptyset & \forall (x,u) \in Graph(G_1) \\ E_2(x,u,v) & \cap & T_{G_2(x,u)}(v) & \neq \emptyset & \forall (x,u,v) \in Graph(G_2). \end{array} \tag{23}$$

Then the pair (S_1, S_2) defined by

$$\begin{array}{lcll} S_1(x) & = & \{u \in Z_1;\ 0 \in E_1(x,u)\} & \forall x \in Dom(G_1) \\ S_2(x,u) & = & \{v \in Z_2;\ 0 \in E_2(x,u,v)\} & \forall (x,u) \in Dom(G_2) \end{array} \tag{24}$$

is a cascade of G.

Proof: $Graph(S_1) = E_1^{-1}(0)$ is closed since E_1 is upper semicontinuous with compact values. Condition (23) means that for all $x \in Dom(G_1)$, $G_1(x)$ is a viability domain for the map $E_1(x,\cdot)$. Since $G_1(x)$ is convex and compact for all $x \in Dom(G_1)$, the Equilibrium Theorem [1, th. 3. 2. 1] yields that $G_1(x)$ contains an equilibrium of $E_1(x,\cdot)$, i. e., there exists $u \in G_1(x)$ such that $0 \in E_1(x,u)$. Hence $S_1(x) \cap G_1(x) \neq \emptyset$ for all $x \in Dom(S_1)$.

The same reasoning applies to S_2 and G_2. □

We consider now the dynamical game (7) assuming it satisfying the conditions (9). To apply the Cascade Theorem 2.1, we need S_1 and S_2 to have the convexity properties (12). We obtain by an easy proof the following result:

Proposition 6.2 *If E_1 and E_2 have convex values, then S_1 has convex values and $S_2(x,\cdot)$ has convex graph for all $x \in Dom(G_1)$.*

The theorem below is an immediate consequence of the Cascade Theorem 2.1 and Proposition 6.1.

Theorem 6.1 *Let R be a regulation map for the dynamical game (7). Assume that the metaregulation map G_R has convex compact values, that*

$$\begin{array}{l} E_1 : X \times Z_1 \times Z_2 \times Z_1 \rightsquigarrow Z_1 \\ E_2 : X \times Z_1 \times Z_2 \times Z_1 \times Z_2 \rightsquigarrow Z_2 \end{array}$$

satisfy $Dom(E_1) \supset Graph(G_{R,1})$, $Dom(E_2) \supset Graph(G_{R,2})$, that they are upper semicontinuous with compact convex values, and that the intersection map $S_{G_{R,1}} = S_1 \cap G_{R,1}$ is single–valued. Assume further that

$$\begin{array}{rcll} E_1(x,u,v;u') \cap T_{G_{R,1}(x,u,v)}(u') & \neq & \emptyset & \forall (x,u,v) \in Dom(G_{R,1}) \\ & & & \forall u' \in G_{R,1}(x,u,v) \\ E_2(x,u,v;u',v') \cap T_{G_{R,2}(x,u,v;u')}(v') & \neq & \emptyset & \forall (x,u,v;u') \in Dom(G_{R,2}) \\ & & & \forall v' \in G_{R,2}(x,u,v;u'). \end{array}$$

Then for all $(x_0, u_0, v_0) \in Graph(R)$ *there exists a state–strategy solution* $(x(\cdot), u(\cdot), v(\cdot))$ *to the game (7) satisfying* $(u(t), v(t)) \in Graph(R)$ *for all* $t \geq 0$ *and*

$$\begin{array}{l} 0 \in E_1(x(t), u(t), v(t); u'(t)) \\ 0 \in E_2(x(t), u(t), v(t); u'(t), v'(t)) \end{array}$$

for all $t \geq 0$.

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