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Zeros of Discrete-Time Spectral Factors, and the Internal Part of a Markovian Splitting Subspace*

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Abstract

The internal subspace of a Markovian realization is characterized using tools from geometric control theory. Zeros of spectral factors are studied and it is shown that the forward and backward spectral factors of a Markovian realization have the same zero structure. Moreover, it is shown that the number of zeros—including zeros at infinity and all zeros counted with multiplicity—of the spectral factor corresponding to a realization is equal to the dimension of the internal subspace. Finally, the forward and backward zero-dynamics operators are introduced, being stochastic counterparts of certain feedback matrices appearing in geometric control theory.

Key words: zeros, stochastic realization theory, geometric control theory

AMS Subject Classifications: 93E03, 94A12

1 Introduction

Given a stationary process y, obtained by passing white noise through a linear system with transfer function $W(z)$, what information is carried by the *zeros* of $W(z)$?

Moreover, supposing y has a Markovian realization with forward model

$$\begin{cases} x(t+1) &= Ax(t) + Bw(t) \\ y(t) &= Cx(t) + Dw(t), \end{cases}$$

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corresponding to the spectral factor $W(z) = C(zI - A)^{-1}B + D$, to which extent can we solve for x given y?

More specifically, if we let H_0 be the space generated by y defined as

$$H_0 = \overline{\text{span}}\,\{a'y(t) : a \in \mathbb{R}^m,\, t \in \mathbb{Z}\},$$

and X be the splitting subspace defined as

$$X = \{a'x(0) : a \in \mathbb{R}^n\},$$

can we then characterize the *internal* subspace $X \cap H_0$ of the realization in terms of the parameters (A, B, C, D)? In other words, which linear functionals of the form $a'x(0)$ can we estimate *exactly* given the output y?

In this paper we characterize $X \cap H_0$ using tools from *geometric control theory*. The internal subspace can be written as a direct sum $X \cap H_0 = X \cap H^- + X \cap H^+$, where H^- and H^+ are the *past* and *future* spaces of y respectively. We show that

$$X \cap H^- = \{a'x(0) : a \in \mathcal{V}_S^*(\mathcal{A}', \mathcal{C}', \mathcal{B}', \mathcal{D}'),$$

where $\mathcal{V}_S^*(\mathcal{A}', \mathcal{C}', \mathcal{B}', \mathcal{D}')$ is a certain subspace of the maximal output-nulling subspace of the control system defined by (A', C', B', D'). Moreover, let $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ be the matrices of a minimal *backward* model for X, then it holds that

$$X \cap H^+ = \{a'\bar{x}(0) : a \in \mathcal{V}_S^*(\bar{\mathcal{A}}', \bar{\mathcal{C}}', \bar{\mathcal{B}}', \bar{\mathcal{D}}'),$$

where $\mathcal{V}_S^*(\bar{\mathcal{A}}', \bar{\mathcal{C}}', \bar{\mathcal{B}}', \bar{\mathcal{D}}')$ is a certain subspace of the maximal output-nulling subspace of the control system defined by $(\bar{A}', \bar{C}', \bar{B}', \bar{D}')$.

We study zeros of spectral factors, and in particular we show that the dimension of $X \cap H_0$ is equal to the number of zeros of the spectral factor $W(z)$, where the zeros (finite and infinite) are counted with multiplicity. We also show that for a fixed Markovian realization of y, the forward and backward spectral factors have the same zero structure.

Moreover, we show that certain closed-loop system matrices appearing in geometric control theory have abstract counterparts in stochastic realization theory, called the zero-dynamics operators. In particular it turns out that the zeros of $W(z)$ are related to the eigenvalues of the zero-dynamics operators.

Similar results were obtained for continuous-time coercive stochastic systems by Lindquist, Michaletzky and Picci in [8]. However, the discrete-time problem of this paper has some interesting features not present in the continuous-time case. For example, it is necessary to consider both the backward and forward models of a Markovian realization to characterize $X \cap H_0$ completely.

Let us mention that zeros of discrete-time spectral factors were studied in the regular case, i.e., when the spectral density of y is positive definite on the unit circle and at infinity, by Michaletzky in [11].

In a very recent paper by Michaletzky and Ferrante [12], zeros of acausal spectral factors are studied (including zeros at infinity) and a state-space definition of zeros and (generalized) zero directions is given. We show that the notion of zero directions is consistent with geometric control theory in the sense that the set of generalized zero directions is in one-one correspondence with the set of generalized left eigenvectors of the induced mapping $(A+BF)|_{\mathcal{V}^*/\mathcal{R}^*}$, where $\mathcal{V}^*$ is the maximal output-nulling subspace, $\mathcal{R}^*$ is the maximal reachability subspace and F is a friend of $\mathcal{V}^*$.

The paper is organized as follows. In Section 2 we recall some facts from stochastic realization theory and geometric control theory. In Section 3 we discuss zeros of spectral factors and show that the forward and backward spectral factors have the same zero structure. In Section 4 we characterize $X \cap H_0$, and in Section 5 we introduce the zero-dynamics operators. In Section 6 we summarize the results of this paper.

2 Preliminaries

In this section we recall and derive some facts from geometric control theory and stochastic realization theory. We also introduce, and comment on, our assumption about the spectral density of y.

2.1 Some results from geometric control theory

In this paper we shall make extensive use of *geometric control theory* for minimal systems of the form

$$\begin{cases} x(t+1) &= Ax(t)+Bu(t) \\ y(t) &= Cx(t)+Du(t), \end{cases} \tag{2.1}$$

with k inputs and m outputs. Moreover, to avoid trivialities we shall assume that $\begin{bmatrix} B' & D' \end{bmatrix}'$ has full column rank.

The monograph by Wonham [16] is a standard reference on the subject, but it deals almost exclusively with systems for which $D=0$. Geometric control theory in the general case, i.e., when $D \neq 0$, was developed by Anderson in [2]. The paper by Aling and Schumacher [1] summarizes definitions and theorems for the general case. We here recall some definitions and results.

A subspace $\mathcal{V}$ is said to be *output-nulling* if there is a feedback matrix F such that

$$(A+BF)\mathcal{V} \subseteq \mathcal{V} \subseteq \ker(\mathcal{C}+\mathcal{D}\mathcal{F}).$$

Any such F is called a *friend* of $\mathcal{V}$. In terms of subspace inclusion there is a *maximal* output-nulling subspace denoted $\mathcal{V}^*(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D})$.

The concept of an output-nulling subspace has the following interpretation. Under the feedback law $u = Fx$ the system (2.1) becomes

$$\left\{ \begin{array}{rcl} x(t+1) &=& (A+BF)x(t) \\ y(t) &=& (C+DF)x(t). \end{array} \right.$$

If now $x(0) \in \mathcal{V}^*(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D})$, then $x(t) \in \mathcal{V}^*(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D})$ and $y(t) = 0$ for $t \geq 0$.

The map $(A + BF)|_{\mathcal{V}^*(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D})}$ will be of special interest; let $\mathcal{V}^*_s(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D})$ denote its stable eigenspace and let $\mathcal{V}^*_a(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D})$ denote its antistable eigenspace. If $(A+BF)|_{\mathcal{V}^*}$ has no eigenvalue on the unit circle, then

$$\mathcal{V}^* = \mathcal{V}^*_s + \mathcal{V}^*_a,$$

where the vector sum is direct.

The following lemma will be needed. A proof of the lemma can be found in [14].

Lemma 2.1 *If for a given input* $\{u(t) : t \geq 0\}$ *the corresponding output* $\{y(t) : t \geq 0\}$ *of the system* (2.1) *is zero, then the state trajectory* $\{x(t) : t \geq 0\}$ *is in* $\mathcal{V}^*(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D})$.

The set of states that can be reached from the origin while keeping the output zero is called the *maximal reachability subspace* $\mathcal{R}^*(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D})$ and is given as

$$\mathcal{R}^* = \langle \mathcal{A} + \mathcal{BF} |\ \mathrm{Im}\,(\mathcal{B}|_{\ker \mathcal{D}}) \cap \mathcal{V}^* \rangle,$$

where F is any friend of $\mathcal{V}^*(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D})$. The map induced in $\mathcal{V}^*/\mathcal{R}^*$ by $(A+BF)$, denoted $(A+BF)|_{\mathcal{V}^*/\mathcal{R}^*}$, will be of great interest in this paper. The map is independent of the particular choice of F.

The following lemma collects some results from geometric control theory that will be used in the following. A proof of the lemma can be found in [14].

Lemma 2.2 *Consider the system* (2.1), *assuming* $D \neq 0$. *Decompose the input space* $\mathcal{U}$ *and output space* $\mathcal{Y}$ *as the direct sums* $\mathcal{U} = \mathcal{U}_\infty + \ker \mathcal{D}$ *and* $\mathcal{Y} = \mathrm{Im}\ \mathcal{D} + \mathcal{Y}_\in$. *With respect to these decompositions the system* (2.1) *takes the form*

$$\left\{ \begin{array}{rcl} x(t+1) &=& Ax(t) + B_1u_1(t) + B_2u_2(t) \\ y_1(t) &=& C_1x(t) + D_1u_1(t) \\ y_2(t) &=& C_2x(t), \end{array} \right. \tag{2.2}$$

where D_1 *is invertible. Now, defining* $A_2 := A - B_1D_1^{-1}C_1$, $\mathcal{V}^*_\in := \mathcal{V}^*(\mathcal{A}_\in,\mathcal{B}_\in,\mathcal{C}_\in,\prime)$ *and* $\mathcal{R}^*_\in := \mathcal{R}^*(\mathcal{A}_\in,\mathcal{B}_\in,\mathcal{C}_\in,\prime)$ *it holds that* *(i)* $\mathcal{V}^*(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}) = \mathcal{V}^*_\in$

(ii) $\mathcal{R}^*(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}) = \mathcal{R}^*_\in$

(iii) *with respect to the decomposition* (2.2), *a friend of* $\mathcal{V}^*(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D})$ *can be taken to be of the form*

$$F := \begin{bmatrix} -D_1^{-1}C_1 \\ F_2 \end{bmatrix},$$

where F_2 *is a friend of* $\mathcal{V}^*_\in$.

(iv) $(A+BF)|_{\mathcal{V}*/\mathcal{R}^*} = (A_2+B_2F_2)|_{\mathcal{V}^*_\in/\mathcal{R}^*_\in}$.

We shall also deal with minimal systems evolving backwards in time of the form

$$\begin{cases} x(t) &= \bar{A}x(t+1) + \bar{B}u(t) \\ y(t) &= \bar{C}x(t+1) + \bar{D}u(t), \end{cases} \tag{2.3}$$

with k inputs and m outputs.

The transfer function of (2.3) is $\overline{W}(z) = z\bar{C}(I - z\bar{A})^{-1}\bar{B} + \bar{D}$. The results of geometric control theory are applicable for backward systems as well. Note that the appropriate feedback law should now be of the form $u(t) = \bar{F}x(t+1)$.

The following lemma is analogous to Lemma 2.1.

Lemma 2.3 *Consider the backward system* (2.3). *If for a given input* $\{u(t) : t \leq 0\}$ *the corresponding output* $\{y(t) : t \leq 0\}$ *of the system* (2.3) *is equal to zero, then the state trajectory* $\{x(t) : t \leq 1\}$ *is in* $\mathcal{V}^*(\bar{\mathcal{A}},\bar{\mathcal{B}},\bar{\mathcal{C}},\bar{\mathcal{D}})$.

2.2 Zeros

Research on zeros of multivariable systems has generated an extensive literature during the last decades, see e.g. the survey [15].

Intuitively, a zero is a complex number z_0 such that the transfer function $W(z)$ looses column rank at z_0, and consequently there is a nonzero input that produces zero output. The actual definition of a zero is a little bit more delicate.

In this section we recall some definitions and results on zeros from the literature. Moreover, we shall present a definition of a *zero direction* proposed by Michaletzky and Ferrante in [12] and show that the (generalized) zero directions are in one-one correspondence with the (generalized) left eigenvectors of $(A+BF)|_{\mathcal{V}^*/\mathcal{R}^*}$.

Finally, we shall give the appropriate definition of a zero of a backward system.

Definition 2.4 (Zeros) *Consider the minimal realization* (2.1) *and let*

$$W(z) := C(zI - A)^{-1}B + D.$$

(a) Let

$$P(z) := \begin{bmatrix} A - zI & B \\ C & D \end{bmatrix}.$$

A complex number z_0 is a zero *of the system* (2.1) *if*

$$\text{rank } P(z_0) < n + \min(k, m). \tag{2.4}$$

(b) We say that z_0 is a zero *of $W(z)$ if z_0 is a zero of* (2.1).

(c) A zero at infinity *of $W(z)$ is defined as a zero of*

$$T(z) := W(\frac{az + b}{cz + d})$$

at $z_0 = -d/c$, where $c \neq 0$.

The matrix $P(z)$ in Definition 2.4 is called the *Rosenbrock matrix* of the system (2.1). By the finite zero structure of the system (2.1) we shall refer to the nonunity invariant factors of $P(z)$. By the zero structure at infinity we mean the invariant factors corresponding to $z_0 = -d/c$ of the Rosenbrock matrix corresponding to a minimal realization of $T(z)$.

Remark 2.5 *Note that the zeros of the system defined by (A, B, C, D) are equal to the zeros of the dual system defined by (A', C', B', D'), having the transfer function $W(z)'$.*

The following well-known theorem can be found in [2]. A special case of theorem, namely for the case when $D = 0$, is given in [3].

Theorem 2.6 *Assume $\begin{bmatrix} C' & D' \end{bmatrix}'$ has full column rank. Then the nonunity invariant factors of*

$$(\, zI - (A + BF)\,)|_{\mathcal{V}^*/\mathcal{R}^*}$$

are the same as the nonunity invariant factors of $P(z)$.

In particular, the finite zeros of (2.1) *are precisely the eigenvalues of*

$$(A + BF)|_{\mathcal{V}^*/\mathcal{R}^*}.$$

Next we shall briefly discuss an aspect of the (generalized) zero directions of the system (2.1), as defined in the recent paper by Michaletzky and Ferrante [12]. More precisely, we shall show that the definition of zero directions is consistent with geometric control theory, in the sense that the (generalized) zero directions of the system (2.1) are in one-one correspondence with the (generalized) eigenvectors of the map $(A + BF)|_{\mathcal{V}^*/\mathcal{R}^*}$.

Definition 2.7 (Zero directions) *Consider the minimal realization* (2.1) *having at least as many inputs as outputs.*

(a) The row vector π *is a (left)* zero direction *corresponding to the zero* z_0 *if*

$$\begin{bmatrix} \pi & m \end{bmatrix} \begin{bmatrix} A - z_0 I & B \\ C & D \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix} \tag{2.5}$$

for some vector m.

(b) Consider a matrix triplet (Π, Λ, M) *such that*

(i) Π *has* d *linearly independent rows,*

(ii) Λ *is a* $d \times d$ *matrix having all eigenvalues equal to* z_0,

(iii) (Π, Λ, M) *is a solution of*

$$\begin{bmatrix} \Pi & M \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \Lambda\Pi & 0 \end{bmatrix}, \tag{2.6}$$

which is maximal in the sense that there is no other triplet $(\tilde{\Pi}, \tilde{\Lambda}, \tilde{M})$ *solving* (2.6) *such that* $\operatorname{rank} \tilde{\Pi} > \operatorname{rank} \Pi$ *and* $\tilde{\Lambda}$ *has all eigenvalues equal to* z_0.

Then the rows of Π *are called the* generalized zero directions *corresponding to the zero* z_0.

It easy to see that the matrix Λ *can without loss of generality be assumed to be in its Jordan form.*

The zero directions have a straight-forward interpretation in the stochastic case. Consider the stochastic system

$$\begin{cases} x(t+1) &= Ax(t) + Bw(t) \\ y(t) &= Cx(t) + Dw(t), \end{cases}$$

where w is a white noise, and suppose that

$$\begin{bmatrix} \pi & m \end{bmatrix} \begin{bmatrix} A - z_0 I & B \\ C & D \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}.$$

It now follows that

$$\pi x(t+1) = z_0 \pi x(t) - m y(t).$$

Hence, $\pi x(t)$ is a linear combination of the state variables that is driven by the process y. Similarly, we obtain

$$\Pi x(t+1) = \Lambda \Pi x(t) - M y(t)$$

for the generalized zero directions.

The following theorem is very much related to Theorem 2.6.

Theorem 2.8 *There is a one-one correspondence between the zeros and generalized zero directions of* (2.1) *on the one hand, and the eigenvalues and generalized left eigenvectors of* $(A+BF)|_{\mathcal{V}^*/\mathcal{R}^*}$ *on the other hand.*

Proof: The proof is based on ideas in the papers by Anderson [2, 3].

We first prove the theorem for the strictly proper case, i.e, when $D=0$. Write Im B as a direct sum Im $B = (\text{Im } B \cap \mathcal{V}^*) + \mathcal{M}$. If we decompose the state space $\mathbb{R}^n$ as a direct sum

$$\mathbb{R}^n = \mathcal{R}^* + \mathcal{V}^*/\mathcal{R}^* + \mathcal{W},$$

where the complement $\mathcal{W}$ has been chosen as an extension of $\mathcal{M}$, then in an adapted basis the matrices $(A+BF, B, C)$ takes the form

$$A+BF = \begin{bmatrix} A_{11}+B_1F_{11} & A_{12}+B_1F_{12} & A_{13}+B_1F_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{23}+B_3F_{23} \end{bmatrix}, \tag{2.7}$$

$$B = \begin{bmatrix} B_1 & 0 \\ 0 & 0 \\ 0 & B_3 \end{bmatrix} \text{ and } C = \begin{bmatrix} 0 & 0 & C_3 \end{bmatrix},$$

where F is a friend of $\mathcal{V}^*(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$. We shall now show that the zero directions correspond to the left eigenvectors of $(A+BF)_{\mathcal{V}^*/\mathcal{R}^*}$. To this end, let z_0 be a zero and consider the defining equation

$$\begin{bmatrix} \pi & m \end{bmatrix} \begin{bmatrix} A - z_0 I & B \\ C & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}. \tag{2.8}$$

Let F be a friend of $\mathcal{V}^*(\mathcal{A}, \mathcal{B}, \mathcal{C})$ and multiply (2.8) from the right with the invertible matrix

$$\begin{bmatrix} I & 0 \\ F & I \end{bmatrix},$$

which yields

$$\begin{bmatrix} \pi & m \end{bmatrix} \begin{bmatrix} A + BF - z_0 I & B \\ C & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}. \tag{2.9}$$

Next, insert the structured matrices of (2.7), partition π as $\pi = \begin{bmatrix} \pi_1 & \pi_2 & \pi_3 \end{bmatrix}$ and permute some columns to arrive at the equation

$$\begin{bmatrix} \pi_1 & \pi_2 & \pi_3 & m \end{bmatrix} \begin{bmatrix} A_{11}+B_1F_{11}-z_0I & B_1 & A_{12}+B_1F_{12} & A_{13}+B_1F_{13} & 0 \\ 0 & 0 & A_{22}-z_0I & A_{23} & 0 \\ 0 & 0 & 0 & A_{33}+B_3F_3-z_0I & B_3 \\ 0 & 0 & 0 & C_3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{2.10}$$

By construction, the pair (A_{11}, B_1) is reachable and it follows from the so-called Popov-Belevitch-Hautus lemma that the matrix

$$\begin{bmatrix}(A_{11}+B_1F_{11}-z_0I) & B_1\end{bmatrix}$$

has full row rank. Hence, $\pi_1 = 0$.

Moreover, the matrix

$$Q := \begin{bmatrix} A_{33}+B_3F_3-z_0I & B_3 \\ C_3 & 0 \end{bmatrix}$$

has full row rank, see [3, p. 590], and therefore $\begin{bmatrix}0 & \pi_2 & \pi_3 & m\end{bmatrix}$ solves (2.2) if and only if π_2 is a left eigenvector of A_{22} with eigenvalue z_0. Finally, π_3 and m are uniquely determined by

$$\pi_2 \begin{bmatrix}A_{23} & 0\end{bmatrix} + \begin{bmatrix}\pi_3 & m\end{bmatrix} Q = 0. \tag{2.11}$$

We now turn to the case of generalized zero directions; i.e., we shall show that the triplet (Π, M, Λ) is a maximal solution of (2.6), where Λ has all eigenvalues equal to z_0, if and only if the rows of Π are in one-one correspondence with the generalized left eigenvectors of $(A+BF)|_{\mathcal{V}^*/\mathcal{R}^*}$, corresponding to the eigenvalue z_0, and Λ is a block-diagonal matrix consisting of the Jordan blocks with eigenvalue z_0 of the Jordan form of $(A+BF)|_{\mathcal{V}^*/\mathcal{R}^*}$.

We first consider the 2-dimensional case when

$$\begin{bmatrix}\Pi & M\end{bmatrix} = \begin{bmatrix}\pi_{11} & \pi_{12} & \pi_{13} & m_1 \\ \pi_{21} & \pi_{22} & \pi_{23} & m_2\end{bmatrix} \text{ and } \Lambda = \begin{bmatrix} z_0 & 1 \\ 0 & z_0\end{bmatrix}.$$

Note that π_2 is a zero direction, which implies that $\pi_{21} = 0$. The vector π_1 is a generalized zero direction.

In the same way as we obtained (2.2) we get

$$\begin{bmatrix}\pi_{11} & \pi_{12} & \pi_{13} & m_1 \\ 0 & \pi_{22} & \pi_{23} & m_2\end{bmatrix} \begin{bmatrix} A_{11}+B_1F_{11} & B_1 & A_{12}+B_1F_{12} & A_{13}+B_1F_{13} & 0 \\ 0 & 0 & A_{22} & A_{23} & 0 \\ 0 & 0 & 0 & A_{33}+B_3F_3 & B_3 \\ 0 & 0 & 0 & C_3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} z_0\pi_{11} & 0 & z_0\pi_{12}+\pi_{22} & z_0\pi_{13}+\pi_{23} & 0 \\ 0 & 0 & z_0\pi_{22} & \pi_{23} & 0\end{bmatrix}. \tag{2.12}$$

It now follows from the reachability of (A_{11}, B_1) that $\pi_{11} = 0$. Moreover,

$$\pi_{12}(A_{22} - z_0 I) = \pi_{22};$$

i.e., π_{12} is a generalized left eigenvector of $(A+BF)|_{\mathcal{V}^*/\mathcal{R}^*}$. Finally, recalling that π_{23} is determined by (2.11), we find that π_{13} and m_1 are uniquely determined by

$$\begin{bmatrix}\pi_{13} & m_1\end{bmatrix} Q = \pi_{23} - \pi_{12}A_{23}.$$

The 2-dimensional case is now complete, and the general case follows by induction.

Finally, for the proper case, i.e., when $D \neq 0$, we invoke Lemma 2.2. Without loss of generality we can assume that the system has the form

$$\left\{ \begin{array}{rcl} x(t+1) & = & Ax(t) + B_1 u_1(t) + B_2 u_2(t) \\ y_1(t) & = & C_1 x(t) + D_1 u_1(t) \\ y_2(t) & = & C_2 x(t), \end{array} \right. \tag{2.13}$$

where D_1 is invertible.

By Lemma 2.2, there is a friend of $\mathcal{V}^*(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ having the form

$$F = \begin{bmatrix} -D_1^{-1} C_1 \\ F_2 \end{bmatrix}.$$

It is now straight-forward to show that (2.6) can be rearranged as

$$\begin{bmatrix} \Pi & M_2 & M_1 \end{bmatrix} \begin{bmatrix} A_2 + B_2 F_2 & B_2 & B_1 \\ C_2 & 0 & 0 \\ 0 & 0 & D_1 \end{bmatrix} = \begin{bmatrix} \Lambda\Pi & 0 & 0 \end{bmatrix},$$

where $A_2 := A - B_1 D_1^{-1} C_1$. Moreover, by Lemma 2.2

$$(A + BF)|_{\mathcal{V}^*/\mathcal{R}^*} = (A_2 + B_2 F_2)|_{\mathcal{V}^*_\in/\mathcal{R}^*_\in},$$

where $\mathcal{V}^*_\in := \mathcal{V}^*(\mathcal{A}_\in, \mathcal{B}_\in, \mathcal{C}_\in, \prime)$, $\mathcal{R}^*_\in := \mathcal{R}^*(\mathcal{A}_\in, \mathcal{B}_\in, \mathcal{C}_\in, \prime)$ and F_2 is a friend of $\mathcal{V}^*_\in$, and it follows from the analysis for the strictly proper case that (Π, M_2) is in one-one correspondence with the eigenstructure of $(A+BF)|_{\mathcal{V}^*/\mathcal{R}^*}$, corresponding to the eigenvalue z_0. Finally, M_1 is uniquely determined by the equation $\Pi B_1 + M_1 D_1 = 0$. □

The following theorem by Antsaklis [4, p. 48] will be important to our purposes.

Theorem 2.9 *Consider a system defined by* (A, B, C, D), *for which the transfer function* $W(z) = C(zI - A)^{-1}B + D$ *is of full rank. If* $m \geq k$, *i.e., the number of outputs is greater than or equal to the number of inputs, then* $\dim \mathcal{R}^*(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}) = \prime$.

Next we define zeros for the backward system (2.3) and the transfer function $\overline{W}(z) = z\bar{C}(I - z\bar{A})^{-1}\bar{B} + \bar{D}$.

Definition 2.10 *Consider the minimal realization* (2.3) *and let*

$$\overline{W}(z) := z\bar{C}(I - z\bar{A})^{-1}\bar{B} + \bar{D}.$$

(a) Let

$$\bar{P}(z) := \begin{bmatrix} z\bar{A} - I & \bar{B} \\ z\bar{C} & \bar{D} \end{bmatrix}.$$

A complex number z_0 is a zero *of the system* (2.3) *if*

$$\text{rank } \bar{P}(z_0) < n + \min(k, m). \tag{2.14}$$

(b) We say that z_0 is a zero *of $\overline{W}(z)$ if z_0 is a zero of* (2.3).

(c) A zero at infinity *of $\overline{W}(z)$ is defined as a zero of*

$$\overline{T}(z) := \overline{W}(\frac{az+b}{cz+d})$$

at $z_0 = -d/c$, where $c \neq 0$.

We shall call the matrix $\bar{P}(z)$ the *backward Rosenbrock matrix* of the system (2.3). By the finite zero structure of the backward system (2.3) we shall refer to the nonunity invariant factors of $\bar{P}(z)$. By the zero structure at infinity we mean the invariant factors corresponding to $z_0 = -d/c$ of the backward Rosenbrock matrix corresponding to a minimal realization of $\bar{T}(z)$.

2.3 Some results from stochastic realization theory

In this section we recall some facts from stochastic realization theory. The subject has a vast literature, see the papers by Lindquist, Picci and Pavon [9], [10], [13], and references therein.

One of the fundamental problems of stochastic realization theory is the following. Given an $m-$dimensional stationary process y defined on the integers having the rational spectral density $\Phi(z)$, find all minimal stochastic realizations of y of the form

$$\begin{cases} x(t+1) &= Ax(t) + Bw(t) \\ y(t) &= Cx(t) + Dw(t), \end{cases} \tag{2.15}$$

where A is a stability matrix, x is a wide-sense Markov process, and w is a normalized white noise. The realization (2.15) is a *forward model* in the sense that $w(t)$ is uncorrelated to $x(s)$ for $t \geq s$.

To each realization (A, B, C, D) there corresponds a stable spectral factor

$$W(z) = C(zI - A)^{-1}B + D$$

such that $W(z)W(1/z)' = \Phi(z)$. Note that in general the spectral factor is rectangular, having more inputs than outputs.

We emphasize that for a fixed process y there is in general a family of minimal stochastic realizations; with a particular choice of basis each of them can be taken to have the same A and C matrices. The family of realizations can be parameterized by the set $\mathcal{P}$ of solutions of the positive-real lemma equations associated with $\Phi(z)$, see e.g. [13].

Many aspects of stochastic realization theory, such as minimality, are best understood in terms of Hilbert space geometry. To each realization of y we can assign the $n-$dimensional space of random variables $X := \{a'x(0) : a \in \mathbb{R}^n\}$. This space is a subspace of an ambient space H of the model (2.15), defined as

$$H := \overline{\text{span}}\{a'w(t) : a \in \mathbb{R}^k,\, t \in \mathbb{Z}\},$$

where the closure is taken in the topology of the inner product $(\eta, \lambda) := E\,\eta\lambda$.

The ambient space H is naturally equipped with the unitary shift U induced by w, having the property that $Uw_i(t) = w_i(t+1)$.

Defining the *past* and *future output spaces* as

$$H^- := \overline{\text{span}}\{a'y(t) : a \in \mathbb{R}^m,\, t < 0\}$$

and

$$H^+ := \overline{\text{span}}\{a'y(t) : a \in \mathbb{R}^m,\, t \geq 0\}$$

respectively, it is well established in the literature that X is a *minimal Markovian splitting subspace* for H^- and H^+. In particular, H^- and H^+ are conditionally orthogonal given X.

Moreover, let H_0 be the vector sum of H^- and H^+, i.e.,

$$H_0 := H^- \vee H^+.$$

If $X \subset H_0$ we say that the realization is *internal*, which happens if and only if $W(z)$ is square [13, p. 169].

Actually, a coordinate free description of a Markovian realization of y is given by the triplet (H, U, X), since given this we can introduce a basis for X and derive a model for y of the form (2.15). However, we can equally well derive a model evolving backwards in time of the form

$$\begin{cases} \bar{x}(t) &= \bar{A}\bar{x}(t+1) + \bar{B}\bar{w}(t) \\ y(t) &= \bar{C}\bar{x}(t+1) + \bar{D}\bar{w}(t), \end{cases} \tag{2.16}$$

where $\bar{A}$ is a stability matrix and $\bar{w}$ is a normalized white noise. The transfer function

$$\overline{W}(z) = z\bar{C}(I - z\bar{A})^{-1}\bar{B} + \bar{D}$$

of (2.16) is called a backward spectral factor of $\Phi(z)$. The realization (2.16) is a *backward* model in the sense that $\bar{w}(t) \perp \bar{x}(s)$ for $s \geq t+1$.

Note that the noises w and $\bar{w}$ are different, but $x(0)$ and $\bar{x}(0)$ define the same splitting subspace X. Therefore, it is more appropriate to speak of (H, U, X) as the stochastic realization of y, and refer to (2.15) and (2.16) as the forward and backward model respectively, of (H, U, X).

If we agree to set $\bar{x}(t) := P^{-1}x(t)$, there are formulas in closed form relating (A, B, C, D) to $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ (see the book by Caines [5, p. 237]).

We here mention that a Markovian realization is minimal if and only if both (C, A) and $(\bar{C}, \bar{A})$ are observable pairs. More on this can be found in [10] and [13].

A Markovian splitting subspace can be represented as the intersection $X = S \cap \bar{S}$ of a pair $(S, \bar{S})$ of subspaces of the ambient space H which satisfy $S \supset H^-$ and $\bar{S} \supset H^+$, the invariance properties $U\bar{S} \subset \bar{S}$ and $U^*S \subset S$, and intersect perpendicularly in the sense that $H = S^\perp \oplus X \oplus \bar{S}^\perp$, where the complements are taken with respect to H. We shall write $X \sim (S, \bar{S})$ to refer to this representation.

Finally, we state as a lemma a result from [8, p. 17]. Although the paper [8] deals with the continuous-time case, the proof, which is completely geometric, shows that the lemma is valid in the discrete-time case as well.

Lemma 2.11 *Suppose the vector sum $H_0 = H^- + H^+$ is direct, and let X be a splitting subspace. Then*

$$X \cap H_0 = (X \cap H^-) + (X \cap H^+),$$

where the sum is direct.

2.4 Assumption on the spectral density

In this paper we assume that the spectral density $\Phi(z)$ is rational, full rank, and positive definite for $|z| = 1$, i.e., $\Phi(e^{i\omega})$ is coercive. This implies that the spaces H^- and H^+ are linearly independent [7], i.e., $H^- \cap H^+ = 0$, and that $H_0 = H^- + H^+$, where $+$ denotes *direct* vector sum. Furthermore, it follows that each element $\lambda \in H^-$ has a representation of the form

$$\lambda = \sum_{k=1}^{\infty} u(k)' y(-k),$$

where the sequence u is square-summable. Similarly, an element $\lambda \in H^+$ can be represented as

$$\lambda = \sum_{k=0}^{\infty} u(k)' y(k).$$

The fact that $\Phi(z)$ is positive definite on the unit circle also implies that no minimal spectral factor of $\Phi(z)$ has zeros on the unit circle.

The assumption that the spectral density be positive definite on the unit circle is common in stochastic systems theory. The continuous-time counterpart of this assumption, employed by Lindquist, Michaletzky and Picci in [8], is that the spectral density be coercive on the imaginary axis.

However, the discrete-time problem has some interesting features not present in the continuous-time case, even though we assume that the spectral density is coercive. In the continuous-time case the coercivity of the spectral density implies that no spectral factor has zeros at infinity; this is not true in the discrete-time case. As a consequence, the characterization of $X \cap H_0$ differs from that in continuous time.

Moreover, in continuous time the D-term of each realization satisfies

$$DD' = \Phi(\infty) > 0,$$

and thus we have full row rank. In the discrete-time case the D-term varies over the set of realizations, and cannot be be assumed to be of full rank for an arbitrary realization. As a consequence, different techniques are needed in the proofs of some of the theorems.

3 Zeros of Spectral Factors

In this paper we only consider minimal spectral factors. From Theorem 2.6 it follows that the finite zeros of the spectral factor $W(z)$ are precisely the eigenvalues of $(A + BF)|_{\mathcal{V}^*/\mathcal{R}^*}$, where F is a friend of $\mathcal{V}^*(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$.

Note that for a minimal spectral factor of a full rank spectral density, the number of inputs is greater than, or equal to, the number of outputs. Now, since $W(z)$ and $W(z)'$ have the same zeros, we can combine Theorem 2.6 and Theorem 2.9 to conclude that the *finite* zeros of $W(z)$ are precisely the eigenvalues of

$$(A' + C'F')|_{\mathcal{V}^*(\mathcal{A}', \mathcal{C}', \mathcal{B}', \mathcal{D}')},$$

where F' is a friend of $\mathcal{V}^*(\mathcal{A}', \mathcal{C}', \mathcal{B}', \mathcal{D}')$.

In this paper we shall sometimes suppress the dependence of the system matrices in $\mathcal{V}^*$ and $\mathcal{R}^*$; it should be clear from the context whether (A, B, C, D) or (A', C', B', D') is referred to. However, if backward quantities are involved, we shall indicate this with a bar.

The next theorem is a backward version of Theorem 2.6. Note, however, that now the *nonzero* zeros (including $z = \infty$) of $\overline{W}(z)$ correspond to the *inverses* of the eigenvalues of $(\bar{A} + \bar{B}\bar{F})|_{\bar{\mathcal{V}}^*/\bar{\mathcal{R}}^*}$, where $\bar{F}$ is a friend of $\bar{\mathcal{V}}^*(\bar{A}, \bar{B}, \bar{C}, \bar{D})$.

Theorem 3.1 *The nonzero zeros (including $z = \infty$) of the backward system* (2.3) *are precisely the inverses of the eigenvalues of* $(\bar{A} + \bar{B}\bar{F})|_{\bar{\mathcal{V}}^*/\bar{\mathcal{R}}^*}$, *where* $1/0 := \infty$.

Proof: If z_0 is nonzero and finite, the backward Rosenbrock matrix looses rank at z_0 if and only if

$$\begin{bmatrix} \bar{A} - (\frac{1}{z_0})I & \bar{B} \\ \bar{C} & \bar{D} \end{bmatrix}$$

looses rank. From Theorem 2.6 it now follows that this happens if and only if $1/z_0$ is an eigenvalue of $(\bar{A} + \bar{B}\bar{F})|_{\bar{\mathcal{V}}^*/\bar{\mathcal{R}}^*}$.

For $z_0 = \infty$ we proceed as follows. By theorem Theorem 2.6 the eigenstructure at the origin of $(\bar{A} + \bar{B}\bar{F})|_{\bar{\mathcal{V}}^*/\bar{\mathcal{R}}^*}$ is given by the matrix

$$\begin{bmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{bmatrix}. \tag{3.1}$$

To reveal the zero structure at infinity of $\overline{W}(z)$, we may consider the zero structure at $z = 1$ of the transfer function

$$\overline{T}(z) = \overline{W}(\frac{1}{z-1}).$$

Since A is a stability matrix, $\bar{A}_1 := \bar{A} + I$ is invertible, and by using the matrix-inversion lemma, $\overline{T}(z)$ can be written

$$\overline{T}(z) = -\bar{C}\bar{A}_1^{-1}[\frac{1}{z}I - \bar{A}_1^{-1}]^{-1}\bar{A}_1^{-1}\bar{B} + (\bar{D} - \bar{C}\bar{A}_1^{-1}\bar{B}),$$

and thus has the backward realization

$$(\bar{A}_1^{-1},\, \bar{A}_1^{-1}\bar{B},\, -\bar{C}\bar{A}_1^{-1},\, \bar{D} - \bar{C}\bar{A}_1^{-1}\bar{B}). \tag{3.2}$$

The backward Rosenbrock matrix of (3.2) is

$$\begin{bmatrix} z\bar{A}_1^{-1} - I & \bar{A}_1^{-1}\bar{B} \\ -z\bar{C}\bar{A}_1^{-1} & \bar{D} - \bar{C}\bar{A}_1^{-1}\bar{B} \end{bmatrix},$$

and the zero structure at $z = 1$ of (3.2) is given by the matrix

$$\begin{bmatrix} \bar{A}_1^{-1} - I & \bar{A}_1^{-1}\bar{B} \\ -\bar{C}\bar{A}_1^{-1} & \bar{D} - \bar{C}\bar{A}_1^{-1}\bar{B} \end{bmatrix}. \tag{3.3}$$

The matrix (3.1) can be written

$$\begin{bmatrix} \bar{A}_1 - I & \bar{B} \\ \bar{C} & \bar{D} \end{bmatrix}, \tag{3.4}$$

and since the matrices (3.4) and (3.3) are related as

$$\begin{bmatrix} \bar{A}_1 - I & \bar{B} \\ \bar{C} & \bar{D} \end{bmatrix} \begin{bmatrix} -\bar{A}_1^{-1} & -\bar{A}_1^{-1}\bar{B} \\ 0 & I \end{bmatrix} = \begin{bmatrix} \bar{A}_1^{-1} - I & \bar{A}_1^{-1}\bar{B} \\ -\bar{C}\bar{A}_1^{-1} & \bar{D} - \bar{C}\bar{A}_1^{-1}\bar{B} \end{bmatrix}$$

the conclusion follows. □

The next theorem states that for a fixed Markovian realization (H, U, X) of a process y the forward and backward models have the same zero structure; in particular, $W(z)$ and $\overline{W}(z)$ have the same zeros. For this theorem we do not assume that the spectral density is positive definite on the unit circle. The analogous result for the continuous-time case was shown by Green in [6], and Lindquist and Picci in [9].

Theorem 3.2 *Let y be a process with rational spectral density. If (H, U, X) is a Markovian realization of y with forward and backward spectral factors $W(z)$ and $\overline{W}(z)$ respectively, then $W(z)$ and $\overline{W}(z)$ have the same zero structure.*

Proof: We first show that $W(z)$ and $\overline{W}(z)$ have the same finite zero structure. This amounts to showing that

$$P(z) = \begin{bmatrix} A - zI & B \\ C & D \end{bmatrix} \text{ and } \bar{P}(z) = \begin{bmatrix} z\bar{A} - I & \bar{B} \\ z\bar{C} & \bar{D} \end{bmatrix}$$

are equivalent; i.e.,

$$P(z) = S(z)\,\bar{P}(z)\,T(z),$$

where S and T are some unimodular matrices. Without loss of generality we can choose basis in X such that $P = E\,x(0)x(0)' = I$. Then $x(0) = \bar{x}(0)$ and the parameters of the backward model can be expressed in terms of (A, B, C, D) as

$$\left\{ \begin{array}{rcl} \bar{A} &=& A' \\ \bar{B} &=& -(I + A')(I + A)^{-1}B \\ \bar{C} &=& CA' + DB' \\ \bar{D} &=& D - \bar{C}(I + A)^{-1}B - C(I + A)^{-1}B, \end{array} \right. \tag{3.5}$$

see e.g. Caines [5, p. 237]. Conversely, (A, B, C, D) can be obtained from $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ as

$$\left\{ \begin{array}{rcl} A &=& \bar{A}' \\ B &=& -(I + \bar{A}')(I + \bar{A})^{-1}\bar{B} \\ C &=& \bar{C}\bar{A}' + \bar{D}\bar{B}' \\ D &=& \bar{D} - C(I + \bar{A})^{-1}\bar{B} - \bar{C}(I + \bar{A})^{-1}\bar{B}. \end{array} \right. \tag{3.6}$$

Letting

$$A_1 := I + A$$

we can write

$$\bar{P}(z) = \begin{bmatrix} zA_1' - zI - I & -A_1'A_1^{-1}B \\ zCA_1' - zC + zDB' & D - DB'A_1^{-1}B - CA_1'A_1^{-1}B \end{bmatrix}.$$

Define

$$S(z) := \begin{bmatrix} -A_1(A_1')^{-1} & 0 \\ -DB'(A_1')^{-1} - C & I \end{bmatrix}$$

and

$$T(z) := \begin{bmatrix} I & 0 \\ -B'(A_1')^{-1} - zB'(A_1')^{-1} & I \end{bmatrix}.$$

It is now straight-forward to verify that

$$P(z) = S(z)\,\bar{P}(z)\,T(z).$$

In establishing this, it is useful to note that $(A_1 - BB')(A_1')^{-1} = A$, which is a consequence of the Lyapunov equation $I = AA' + BB'$.

In particular, we have shown that $W(z)$ and $\overline{W}(z)$ have the same zero structure at the origin, i.e.,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \sim \begin{bmatrix} -I & \bar{B} \\ 0 & \bar{D} \end{bmatrix}. \tag{3.7}$$

Next, we show that $W(z)$ and $\overline{W}(z)$ have the same zero structure at infinity. To this end, let $T(z) := W(-1 + 1/z)$. The zero structure at infinity of the forward model is now given by the zero structure at the origin of $T(z)$. Since

$$\begin{aligned} T(z) = W(-1 + \frac{1}{z}) &= C(\frac{1}{z}I - (A+I))^{-1}B + D \\ &= C(\frac{1}{z}I - A_1)^{-1}B + D \\ &= -CA_1(zI - A_1^{-1})^{-1}A_1^{-1}B + [D - CA_1^{-1}B], \end{aligned}$$

where the last equality follows from the matrix-inversion lemma, $T(z)$ can be realized as

$$(A_1^{-1},\, A_1^{-1}B,\, -CA_1^{-1},\, D - CA_1^{-1}B).$$

Hence, the zero structure at the origin of $T(z)$ is given by the matrix

$$\begin{bmatrix} A_1^{-1} & A_1^{-1}B \\ -CA_1^{-1} & D - CA_1^{-1}B \end{bmatrix},$$

which is equivalent to

$$\begin{bmatrix} -I & B \\ 0 & D \end{bmatrix} \tag{3.8}$$

since

$$\begin{bmatrix} A_1^{-1} & A_1^{-1}B \\ -CA_1^{-1} & D - CA_1^{-1}B \end{bmatrix} = \begin{bmatrix} A_1^{-1} & 0 \\ -CA_1^{-1} & I \end{bmatrix} \begin{bmatrix} -I & B \\ 0 & D \end{bmatrix} \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix}.$$

Moreover, from Theorem 3.1 it follows that the zero structure at infinity of $\overline{W}(z)$ is given by

$$\begin{bmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{bmatrix}. \tag{3.9}$$

Hence, we must finally show that the matrices (3.8) and (3.9) are equivalent, but this equivalence is just a "dual" version of (3.7), and can therefore be derived in a similar way. □

Combining Theorem 3.1 and Theorem 3.2 we obtain the following corollary.

Corollary 3.3 *The stable zeros of $W(z)$ are precisely the stable eigenvalues of*

$$(A + BF)|_{\mathcal{V}^*/\mathcal{R}^*},$$

where F is a friend of $\mathcal{V}^*(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$, and the antistable zeros of $W(z)$ are precisely the inverses of the stable eigenvalues of

$$(\bar{A} + \bar{B}\bar{F})|_{\bar{\mathcal{V}}^*/\bar{\mathcal{R}}^*},$$

where $\bar{F}$ is a friend of $\bar{\mathcal{V}}^*(\bar{A}, \bar{B}, \bar{C}, \bar{D})$.

Since $W(z)$ and $W(z)'$ have the same zeros, there is a dual version of the preceding corollary.

Corollary 3.4 *The stable zeros of $W(z)$ are precisely the stable eigenvalues of*

$$(A' + C'F')|_{\mathcal{V}^*},$$

where F' is a friend of $\mathcal{V}^*(\mathcal{A}', \mathcal{C}', \mathcal{B}', \mathcal{D}')$, and the antistable zeros of $W(z)$ are precisely the inverses of the stable eigenvalues of

$$(\bar{A}' + \bar{C}'\bar{F}')|_{\bar{\mathcal{V}}^*},$$

where $\bar{F}'$ is a friend of $\bar{\mathcal{V}}^*(\bar{A}', \bar{C}, \bar{B}', \bar{D}')$.

4 Characterization of $X \cap H_0$

In this section we characterize the internal part $X \cap H_0$ of a Markovian realization. In [8] it was shown that in the *continuous-time* case $X \cap H_0 = \{a'x(0) : a \in \mathcal{V}^*(\mathcal{A}', \mathcal{C}', \mathcal{B}', \mathcal{D}')\}$. In the discrete-time case this result does not hold in general, as is shown in the example below. However, one of the inclusions remains valid.

Note that a minimal spectral factor $W(z)$ of a full rank spectral density has at least as many inputs as outputs and that the converse holds for

$W(z)'$ and the dual quadruple (A', C', B', D'). From Theorem 2.9 it now follows that

$$\mathcal{R}^*(\mathcal{A}', \mathcal{C}', \mathcal{B}', \mathcal{D}') = \imath,$$

and therefore, by Theorem 2.6, the number of *finite* zeros of $W(z)$ (counted with multiplicity) is equal to the dimension of $\mathcal{V}^*(\mathcal{A}', \mathcal{C}', \mathcal{B}', \mathcal{D}')$.

Example 4.1 *Consider the scalar spectral density*

$$\Phi(z) = \frac{z}{(z-a)(1-az)},$$

where $|a| < 1$*. The two stable minimal scalar spectral factors are*

$$W_+(z) = \frac{1}{z-a} \text{ and } W_-(z) = \frac{z}{z-a} = \frac{a}{z-a} + 1.$$

Note that $W_+(z)$ *has no zero, whereas* $W_-(z)$ *has a zero at the origin, i.e., a stable zero.*

Letting (A, B_+, C, D_+) *and* (A, B_-, C, D_-) *be minimal realizations of* $W_+(z)$ *and* $W_-(z)$ *respectively, it follows from Theorem 2.6 and Theorem 2.9 that*

$$\mathcal{V}^*(\mathcal{A}', \mathcal{C}', \mathcal{B}'_+, \mathcal{D}'_+) = \imath,$$

whereas $\mathcal{V}^*(\mathcal{A}', \mathcal{C}', \mathcal{B}'_-, \mathcal{D}'_-)$ *is one-dimensional. Since the spectral factors are square, the realizations are internal and* $\dim(X_- \cap H_0) = \dim(X_+ \cap H_0) = 1$*, where* X_- *and* X_+ *are the minimal splitting subspaces corresponding to* $W_-(z)$ *and* $W_+(z)$ *respectively. Hence, the characterization of* $X \cap H_0$ *differs from that in the continuous-time case.*

Proposition 4.2 *Let* y *be a stationary process on* $\mathbb{Z}$ *and suppose the stochastic system*

$$\begin{cases} x(t+1) &= Ax(t) + Bw(t) \\ y(t) &= Cx(t) + Dw(t) \end{cases} \tag{4.1}$$

is a minimal realization of y*. If* $a \in \mathcal{V}^*_s(\mathcal{A}', \mathcal{C}', \mathcal{B}', \mathcal{D}')$ *then* $a'x(0) \in X \cap H^-$*, and if* $a \in \mathcal{V}^*_a(\mathcal{A}', \mathcal{C}', \mathcal{B}', \mathcal{D}')$*, then* $a'x(0) \in X \cap H^+$*.*

Proof: In the proof we shall need the (backward) dual control system

$$\begin{cases} z(t) &= A'z(t+1) + C'u(t) \\ v(t) &= B'z(t+1) + D'u(t). \end{cases} \tag{4.2}$$

Using the system equations (4.1) and (4.2), we get

$$z(t+1)'x(t+1) - z(t)'x(t) = v(t)'w(t) - u(t)'y(t). \tag{4.3}$$

To prove the first claim, let $a \in \mathcal{V}_s^*(\mathcal{A}',\mathcal{C}',\mathcal{B}',\mathcal{D}')$. Expanding (4.3) backwards gives

$$z(0)'x(0) - z(t)'x(t) = \sum_{k=t}^{-1} v(k)'w(k) - \sum_{k=t}^{-1} u(k)'y(k).$$

Letting $u(t) = F'z(t+1)$, where F' is a friend of $\mathcal{V}^*(\mathcal{A}',\mathcal{C}',\mathcal{B}',\mathcal{D}')$, gives

$$z(t) = (A' + C'F')z(t+1),$$

and if $z(0) = a$ then $z(t) \in \mathcal{V}_s^*(\mathcal{A}',\mathcal{C}',\mathcal{B}',\mathcal{D}')$ for $t \leq 0$ and

$$v(t) = (B' + D'F')z(t+1) = 0$$

for $t \leq -1$. By the stability of $(A' + C'F')|_{\mathcal{V}_s^*}$ it follows that $z(t) \to 0$ as $t \to -\infty$ and we get

$$z(0)'x(0) = -\sum_{k=-\infty}^{-1} u(k)'y(k) \in H^-.$$

Hence, $a'x(0) \in X \cap H^-$.

To prove the second claim, let $a \in \mathcal{V}_a^*(\mathcal{A}',\mathcal{C}',\mathcal{B}',\mathcal{D}')$. Expanding (4.3) forwards gives

$$z(t)'x(t) - z(0)'x(0) = \sum_{k=0}^{t-1} v(k)'w(k) - \sum_{k=0}^{t-1} u(k)'y(k).$$

Letting $u(t) = F'z(t+1)$, where F' is a friend of $\mathcal{V}^*(\mathcal{A}',\mathcal{C}',\mathcal{B}',\mathcal{D}')$, gives

$$z(t) = (A' + C'F')z(t+1).$$

Since $(A' + C'F')|_{\mathcal{V}_a^*}$ has no eigenvalue at the origin, it is invertible and we have the invariance

$$[(A' + C'F')|_{\mathcal{V}_a^*}]^{-1}\mathcal{V}_a^* \subseteq \mathcal{V}_a^*. \tag{4.4}$$

Hence, with the initial condition $z(0) = a$ the closed-loop system can be iterated forwards in time, and due to the invariance (4.4) it follows that

$$z(t) \in \mathcal{V}_a^*(\mathcal{A}',\mathcal{C}',\mathcal{B}',\mathcal{D}')$$

and $v(t) = 0$ for $t \geq 0$. By the stability of $[(A' + C'F')|_{\mathcal{V}_a^*}]^{-1}$, it follows that $z(t) \to 0$ as $t \to \infty$ and we get

$$z(0)'x(0) = \sum_{k=0}^{\infty} u(k)'y(k) \in H^+.$$

Hence, $a'x(0) \in X \cap H^+$. □

The following proposition gives the converse of the first claim of the preceding proposition. Recall, as was shown in Example 4.1, that the converse of the second claim is false. The proof of the proposition also explains why the dual quadruplet (A', C', B', D') enters the analysis.

Proposition 4.3 *Let y be a stationary process on $\mathbb{Z}$ and suppose the stochastic system*

$$\begin{cases} x(t+1) &= Ax(t) + Bw(t) \\ y(t) &= Cx(t) + Dw(t) \end{cases} \tag{4.5}$$

is a realization of y. Then $X \cap H^- = \{a'x(0) : a \in \mathcal{V}_s^*(\mathcal{A}', \mathcal{C}', \mathcal{B}', \mathcal{D}')\}$.

Proof: By Proposition 4.2 it follows that

$$a'x(0) \in X \cap H^-$$

if

$$a \in \mathcal{V}_s^*(\mathcal{A}', \mathcal{C}', \mathcal{B}', \mathcal{D}').$$

To show the converse, let $\eta = a'x(0) \in X \cap H^-$. Since $\eta \in X$, solving the first of the equations (4.5) gives

$$\eta = \sum_{k=-\infty}^{-1} a'A^{-k-1}Bw(k) = \sum_{k=-\infty}^{\infty} f(-k)'w(k), \tag{4.6}$$

where

$$f(k) := \begin{cases} B'(A')^{k-1}a \text{if } k \geq 1 \\ 0 \qquad\qquad \text{if } k \leq 0. \end{cases} \tag{4.7}$$

The spectral-domain equivalent of (4.6) is

$$\eta = \int_{-\pi}^{\pi} \hat{f}(e^{i\lambda})' d\hat{w}(\lambda). \tag{4.8}$$

Since $\eta \in H^-$ and $\Phi(e^{i\omega})$ is coercive, η has a time-domain representation

$$\eta = \sum_{k=-\infty}^{\infty} u(-k)'y(k),$$

where u is square summable and $u(k) = 0$ if $k \leq 0$, and a spectral-domain representation

$$\eta = \int_{-\pi}^{\pi} \hat{u}(e^{i\lambda})' d\hat{y}(\lambda) = \int_{-\pi}^{\pi} \hat{u}(e^{i\lambda})' W(e^{i\lambda}) d\hat{w}(\lambda), \tag{4.9}$$

where $W(z) = C(zI - A)^{-1}B + D$. By comparing (4.8) and (4.9), it follows from the uniqueness of spectral representations that

$$\hat{f}(e^{i\lambda}) = W(e^{i\lambda})'\hat{u}(e^{i\lambda}). \tag{4.10}$$

The equation (4.10) suggests the introduction of the (forward) dual control system

$$\begin{cases} z(t+1) &= A'z(t) + C'u(t) \\ v(t) &= B'z(t) + D'u(t) \end{cases} \tag{4.11}$$

having the transfer function $W(z)' = B'(zI - A')^{-1}C' + D'$. Transforming (4.10) into the time domain gives

$$f(t) = \sum_{k=-\infty}^{t-1} B'(A')^{t-k-1}C'u(k) + D'u(t). \tag{4.12}$$

Comparing (4.7) and (4.12) for $t \geq 1$, using the fact that $u(t) = 0$ for $t \leq 0$, gives

$$\sum_{k=1}^{t-1} B'(A')^{t-k-1}C'u(k) + D'u(t) = B'(A')^{t-1}a,$$

i.e.,

$$B'(A')^{t-1}(-a) + B'\sum_{k=1}^{t-1}(A')^{t-k-1}C'u(k) + D'u(t) = 0 \tag{4.13}$$

for $t \geq 1$. Equation (4.13) can be interpreted as that the input u is output-nulling for the dual system (4.11) initialized at $z(1) = -a$. By Lemma 2.1 it follows that $a \in \mathcal{V}^*(\mathcal{A}', \mathcal{C}', \mathcal{B}', \mathcal{D}')$.

It remains to show that $a \in \mathcal{V}^*_s(\mathcal{A}', \mathcal{C}', \mathcal{B}', \mathcal{D}')$. Since

$$\mathcal{V}^*(\mathcal{A}', \mathcal{C}', \mathcal{B}', \mathcal{D}') = \mathcal{V}^*_s(\mathcal{A}', \mathcal{C}', \mathcal{B}', \mathcal{D}') + \mathcal{V}^*_a(\mathcal{A}', \mathcal{C}', \mathcal{B}', \mathcal{D}'),$$

where the sum is direct, we can write

$$a = a_1 + a_2,$$

where $a_1 \in \mathcal{V}^*_s(\mathcal{A}', \mathcal{C}', \mathcal{B}', \mathcal{D}')$ and $a_2 \in \mathcal{V}^*_a(\mathcal{A}', \mathcal{C}', \mathcal{B}', \mathcal{D}')$. Consequently,

$$a'x(0) = a_1'x(0) + a_2'x(0), \tag{4.14}$$

and by Proposition 4.2 it follows that $a_1'x(0) \in X \cap H^-$ and $a_2'x(0) \in X \cap H^+$. Now rearrange (4.14) as

$$(a - a_1)'x(0) = a_2'x(0). \tag{4.15}$$

The left-hand side of (4.15) is in $X \cap H^-$ and the right-hand side is in $X \cap H^+$. Recall that by Lemma 2.11 the spaces $X \cap H^-$ and $X \cap H^+$ are linearly independent, and therefore

$$(a - a_1)' x(0) = 0.$$

Finally, since the components of $x(0)$ form a basis in X it follows that $a = a_1$ and $a \in \mathcal{V}_s^*(\mathcal{A}', \mathcal{C}', \mathcal{B}', \mathcal{D}')$. □

The next step is to characterize $X \cap H^+$. Since Proposition 4.3 states that $X \cap H^-$ and $\mathcal{V}_s^*(\mathcal{A}', \mathcal{C}', \mathcal{B}', \mathcal{D}')$ are isomorphic it is natural to utilize the *backward* model $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ of X and try to show that

$$X \cap H^+ = \{a' \bar{x}(0) : a \in \bar{\mathcal{V}}_s^*(\bar{A}', \bar{C}', \bar{B}', \bar{D}')\}.$$

Proposition 4.4 *Let y be a stationary process on $\mathbb{Z}$ and suppose that the backward stochastic system*

$$\left\{ \begin{array}{rcl} \bar{x}(t) & = & \bar{A}\bar{x}(t+1) + \bar{B}\bar{w}(t) \\ y(t) & = & \bar{C}\bar{x}(t+1) + \bar{D}\bar{w}(t) \end{array} \right. \tag{4.16}$$

is a realization of y. Then

$$X \cap H^+ = \{a' \bar{x}(0) : a \in \bar{\mathcal{V}}_s^*(\bar{A}', \bar{C}', \bar{B}', \bar{D}')\}.$$

Moreover, if $a \in \bar{\mathcal{V}}_a^*(\bar{A}', \bar{C}', \bar{B}', \bar{D}')$ then $a' x(0) \in X \cap H^-$.

The proof Proposition 4.4 is similar to the proofs of Proposition 4.2 and Proposition 4.3 and is therefore omitted.

Combining Proposition 4.2, Proposition 4.3 and Proposition 4.4 we obtain the following corollary.

Corollary 4.5 *Choose a basis in X such that $E\, x(0)x(0)' = I$ (which implies that $x(0) = \bar{x}(0)$). With this choice of basis it holds that*

$$\bar{\mathcal{V}}_a^*(\bar{A}', \bar{C}', \bar{B}', \bar{D}') \subseteq \mathcal{V}_s^*(\mathcal{A}', \mathcal{C}', \mathcal{B}', \mathcal{D}'),$$

and

$$\mathcal{V}_a^*(\mathcal{A}', \mathcal{C}', \mathcal{B}', \mathcal{D}') \subseteq \bar{\mathcal{V}}_s^*(\bar{\mathcal{A}}', \bar{\mathcal{C}}', \bar{\mathcal{B}}', \bar{\mathcal{D}}').$$

We summarize the results on the characterization of $X \cap H_0$ as a theorem.

Theorem 4.6 *Let y be stationary process with a rational coercive spectral density. Suppose (H, U, X) is a Markovian realization of y with forward model (A, B, C, D) and backward model $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$. Then*

$$X \cap H^- = \{a' x(0) : a \in \mathcal{V}_s^*(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})\}$$

and

$$X \cap H^+ = \{a'\bar{x}(0) : a \in \bar{\mathcal{V}}_s^*(\bar{A}, \bar{B}, \bar{C}, \bar{D})\}.$$

In particular, $\dim(X \cap H^-)$ *is equal to the number of stable zeros of* $W(z)$*, and* $\dim(X \cap H^+)$ *is equal to number of antistable zeros of* $W(z)$ *(including zeros at infinity).*

We illustrate the theorem in a continuation of Example 4.1.

Example 4.7 *Consider again the spectral density*

$$\Phi(z) = \frac{z}{(z-a)(1-az)},$$

where $|a| < 1$*, with the two minimal scalar spectral factors*

$$W_+(z) = \frac{1}{z-a} \text{ and } W_-(z) = \frac{z}{z-a} = \frac{a}{z-a} + 1.$$

A minimal realization of $W_+(z)$ *is*

$$\left\{ \begin{array}{rcl} x(t+1) & = & ax(t) + w(t) \\ y(t) & = & x(t). \end{array} \right. \tag{4.17}$$

Since $W_+(z)$ *has no zero it follows that* $\mathcal{V}^*(\dashv, \infty, \infty, \imath) = \imath$*, which also can be seen directly from* $\mathcal{V}^* \subseteq \ker \infty = \imath$*. This agrees with* $X_+ \cap H^- = 0$.

The backward model corresponding to (4.17) *is*

$$\left\{ \begin{array}{rcl} \bar{x}(t) & = & a\,\bar{x}(t+1) + (1-a^2)\,\bar{w}(t) \\ y(t) & = & a/(1-a^2)\,\bar{x}(t+1) + \bar{w}(t) \end{array} \right. \tag{4.18}$$

and the backward spectral factor is

$$\overline{W}_+(z) = \frac{1}{1-za}.$$

Since $\overline{W}_+(z)$ *is strictly proper, it has a zero at infinity, i.e., an antistable zero. From Theorem 2.9 and Theorem 3.1 it follows that* $\bar{\mathcal{V}}_s^*(a, a/(1-a^2), (1-a^2), 1)$ *is one-dimensional, which agrees with* $\dim(X_+ \cap H^+) = 1$.

A minimal realization of $W_-(z)$ *is*

$$\left\{ \begin{array}{rcl} x(t+1) & = & ax(t) + aw(t) \\ y(t) & = & x(t) + w(t). \end{array} \right. \tag{4.19}$$

Note that the processes x *and* w *in* (4.19) *are different from those in* (4.17)*. Since* $W_-(z)$ *has a zero at the origin it holds that* $\mathcal{V}_s^*(\dashv, \infty, \dashv, \infty)$ *is one-dimensional, which agrees with* $\dim(X \cap H^-) = 1$.

The backward model corresponding to (4.19) *is*

$$\begin{cases} \bar{x}(t) &= \quad a\,\bar{x}(t+1) + (1-a^2)/a\,\bar{w}(t) \\ y(t) &= a/(1-a^2)\,\bar{x}(t+1), \end{cases} \tag{4.20}$$

and the backward spectral factor is

$$\overline{W}_-(z) = \frac{z}{1-az}$$

with no antistable zero. Hence, $\bar{\mathcal{V}}_a^*(a,\, a/(1-a^2),\, (1-a^2)/a, 0) = 0$*, which agrees with* $\dim(X_- \cap H^+) = 0$*.*

Example 4.8 (AR(p) process) *Consider the AR(p) process*

$$y(t) = a_1 y(t-1) + a_2 y(t-2) + \ldots a_p y(t-p) + e(t),$$

where e is a normalized white noise.

One state space model is found by letting

$$x(t) := \begin{bmatrix} y(t-1), & y(t-2), & \ldots, & y(t-p) \end{bmatrix}',$$

$w(t) := e(t)$*, and defining the matrices*

$$A := \begin{bmatrix} a_1 & a_2 & \cdot & \cdot & a_{p-1} & a_p \\ 1 & 0 & \cdot & \cdot & 0 & 0 \\ 0 & 1 & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & 1 & 0 \end{bmatrix}, \; B := \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \tag{4.21}$$

$C := \begin{bmatrix} a_1, & a_2, & \ldots, & a_p \end{bmatrix}$ *and* $D := 1$*. The spectral factor corresponding to this realization is*

$$W(z) = \frac{z^p}{z^p - a_1 z^{p-1} - \ldots - a_p},$$

having a zero of multiplicity p *at the origin. Hence,* $W(z)$ *has* p *stable zeros and it follows from Theorem 4.6 that* $\dim(X \cap H^-) = p$*, as can be seen by inspection.*

Another state space model is found as

$$x(t) := \begin{bmatrix} y(t), & y(t-1), & \ldots, & y(t-p+1) \end{bmatrix}',$$

$w(t) := e(t+1)$*,* A *and* B *as in* (4.21)*,* $C := \begin{bmatrix} 1, & 0, & \ldots, & 0 \end{bmatrix}$ *and* $D := 0$*. The spectral factor corresponding to this realization is*

$$W(z) = \frac{z^{p-1}}{z^p - a_1 z^{p-1} - \ldots - a_p},$$

having a zero of multiplicity $p-1$ *at the origin. Hence,* $W(z)$ *has* $p-1$ *stable zeros and it follows from Theorem 4.6 that* $\dim(X \cap H^-) = p-1$ *and* $\dim(X \cap H^+) = 1$*, as can be seen by inspection.*

5 The Zero-Dynamics Operators

What is the stochastic interpretation of the mappings

$$(A' + C'F')|_{\mathcal{V}_S^*} \tag{5.1}$$

and

$$(\bar{A}' + \bar{C}'\bar{F}')|_{\bar{\mathcal{V}}_S^*}\,? \tag{5.2}$$

In this section we show that the mappings (5.1) and (5.2) have counterparts in stochastic realization theory, called the zero-dynamics operators. In particular, the eigenvalues of the mappings (5.1) and (5.2), i.e., the zeros of $W(z)$, are the eigenvalues of the zero-dynamics operators.

Recall that there are stochastic interpretations of the mappings A and $\bar{A}$, and thus also the poles of $W(z)$, see e.g. [10]. In fact, the mapping A' is a matrix representation of the compressed shift $U(X) : X \to X$ defined as $U(X) := E^X U|_X$. In particular, it follows that the set of poles of $W(z)$ is equal to the spectrum of $U(X)$. Moreover, $U^*(X) := E^X U^*|_X$ has the matrix representation $\bar{A}'$.

We now give a lemma, which is a discrete-time version of a lemma in [8], showing that $X \cap H^-$ and $X \cap H^+$ enjoy certain invariance properties, analogous to controlled invariance in geometric control theory.

Lemma 5.1 *Let $Y_t \triangleq \{a'y(t) : a \in \mathbb{R}^m\}$. If $\eta \in X \cap H^-$ then*

$$U\eta \in (X \cap H^-) + Y_0,$$

and if $\eta \in X \cap H^+$ then

$$U^*\eta \in (X \cap H^+) + Y_{-1}.$$

Proof: Let $\eta \in X \cap H^-$. Recall that $X = S \cap \bar{S}$ and $U\bar{S} \subseteq \bar{S}$. Thus, $\eta \in X$ implies that $\eta \in \bar{S}$ and $U\eta \in \bar{S}$. Moreover, $\eta \in H^-$ implies that $U\eta \in H^- + Y_0$. Hence, $U\eta \in \bar{S} \cap (H^- + Y_0)$.

We shall now show that

$$\bar{S} \cap (H^- + Y_0) = (\bar{S} \cap H^-) + (\bar{S} \cap Y_0) = (X \cap H^-) + Y_0. \tag{5.3}$$

We first show the second equality of (5.3). This follows from

$$X \cap H^- = \bar{S} \cap S \cap H^- = \bar{S} \cap H^-,$$

where $H^- \subseteq S$ has been used, and

$$\bar{S} \cap Y_0 = Y_0,$$

where $Y_0 \subseteq H^+ \subseteq \bar{S}$ has been used.

Next, we show the first equality of (5.3). Note that one of the inclusions is trivial. Now, let $\lambda \in \bar{S} \cap (H^- + Y_0)$. Since $\lambda \in H^- + Y_0$, we can write $\lambda = \alpha + \beta$ with $\alpha \in H^-$ and $\beta \in Y_0$. From $\lambda \in \bar{S}$ and $\beta \in Y_0 \subseteq H^+ \subseteq \bar{S}$ it follows that $\alpha \in \bar{S}$. Hence, $\alpha \in \bar{S} \cap H^-$ and $\beta \in \bar{S} \cap Y_0$. □

In view of the preceding lemma we can construct an operator acting on $X \cap H^-$, in a way resembling the construction of the compression of the shift. To this end, let

$$\pi_- : H_0 \to H^-$$

be the oblique projection onto H^- along H^+. Suppose

$$\eta = a'x(0) \in X \cap H^-,$$

i.e., $a \in \mathcal{V}_s(\mathcal{A}', \mathcal{C}', \mathcal{B}', \mathcal{D}')$. From Lemma 5.1 it follows that

$$U\eta = \eta_1 + b'y(0), \tag{5.4}$$

where $\eta_1 \in X \cap H^-$ and $b'y(0) \in Y_0$, and that the representation is unique. Applying π_- to (5.4) yields

$$\pi_- U\eta = \eta_1.$$

Definition 5.2 *The operator $V_f(X) : X \cap H^- \to X \cap H^-$ defined as*

$$V_f(X) := \pi_- U|_{X \cap H^-}$$

is called the forward zero-dynamics operator.

In a similar manner we can define an operator acting on $X \cap H^+$. Let

$$\pi_+ : H_0 \to H^+$$

be the oblique projection onto H^+ along H^-. Suppose

$$\eta = a'x(0) \in X \cap H^+,$$

i.e., $a \in \mathcal{V}_s(\bar{\mathcal{A}}', \bar{\mathcal{C}}', \bar{\mathcal{B}}', \bar{\mathcal{D}}')$. From Lemma 5.1 it follows that

$$U^*\eta = \eta_1 + b'y(-1), \tag{5.5}$$

where $\eta_1 \in X \cap H^+$ and $b'y(-1) \in Y_{-1}$, and that the representation is unique. Applying π_+ to (5.5) yields

$$\pi_+ U^* \eta = \eta_1.$$

Definition 5.3 *The operator $V_b(X) : X \cap H^+ \to X \cap H^+$ defined as*

$$V_b(X) := \pi_+ U^*|_{X \cap H^+}$$

is called the backward zero-dynamics operator.

The following theorem states that the mappings (5.1) and (5.2) are matrix representations of $V_f(X)$ and $V_b(X)$ respectively.

Theorem 5.4 *The forward and backward zero-dynamics operators have the matrix representations*

$$V_f(X) \simeq (A' + C'F')|_{\mathcal{V}_S^*(\mathcal{A}',\mathcal{C}',\mathcal{B}',\mathcal{D}')},$$

where F' is a friend of $\mathcal{V}_S^*(\mathcal{A}',\mathcal{C}',\mathcal{B}',\mathcal{D}')$, and

$$V_b(X) \simeq (\bar{A}' + \bar{C}'\bar{F}')|_{\bar{\mathcal{V}}_S^*(\bar{A}',\bar{C}',\bar{B}',\bar{D}')},$$

where $\bar{F}'$ is a friend of $\bar{\mathcal{V}}_S^*(\bar{A}',\bar{C}',\bar{B}',\bar{D}')$.

In particular, the stable zeros of $W(z)$ are precisely the eigenvalues of $V_f(X)$, and the antistable zeros of $W(z)$ are precisely the reciprocals of the eigenvalues of $V_b(X)$.

Proof: Suppose $\eta = a'x(0) \in X \cap H^-$, i.e., $a \in \mathcal{V}_S^*(\mathcal{A}',\mathcal{C}',\mathcal{B}',\mathcal{D}')$. Letting $\eta_1 = V_f(X)\eta$, we must show that

$$[(A' + C'F')a]'x(0) = \eta_1,$$

where F' is a friend of $\mathcal{V}^*(\mathcal{A}',\mathcal{C}',\mathcal{B}',\mathcal{D}')$. Since $\mathcal{V}_S^*(\mathcal{A}',\mathcal{C}',\mathcal{B}',\mathcal{D}')$ is invariant for $(A' + C'F')$ we have

$$[(A' + C'F')a]'x(0) \in X \cap H^-,$$

i.e.,

$$a'Ax(0) + a'FCx(0) = \eta_2, \tag{5.6}$$

for some $\eta_2 \in X \cap H^-$. Adding $a'Bw(0)$ to (5.6) yields

$$a'Ax(0) + a'Bw(0) + a'FCx(0) = \eta_2 + a'Bw(0). \tag{5.7}$$

Since $a \in \mathcal{V}_S^*(\mathcal{A}',\mathcal{C}',\mathcal{B}',\mathcal{D}')$, we have $(B' + D'F')a = 0$, i.e., $a'B = -a'FD$. Using this, (5.7) can be rearranged as

$$\begin{aligned} a'x(1) &= \eta_2 - a'FCx(0) - a'FDw(0) \\ &= \eta_2 - a'Fy(0). \end{aligned}$$

Now, since $a'Fy(0) \in Y_0$ and $\eta_2 \in X \cap H^-$ it follows by uniqueness that $\eta_1 = \eta_2$, and

$$V_f(X) \simeq (A' + C'F')|_{\mathcal{V}_S^*(\mathcal{A}',\mathcal{C}',\mathcal{B}',\mathcal{D}')}.$$

The proof of $V_b(X) \simeq (\bar{A}' + \bar{C}'\bar{F}')|_{\bar{\mathcal{V}}_S^*(\bar{A}',\bar{C}',\bar{B}',\bar{D}')}$ is completely analogous. □

6 Conclusions

We have characterized the internal part of a stochastic realization, i.e., the part of the stochastic state that can be determined from the output process. The characterization is stated in terms of geometric control theory, and the internal part is easy to compute when numerical values are given. In order to parameterize the internal part completely, it is necessary to consider both the forward and backward models of a fixed Markovian realization.

Moreover, we have shown that the forward and backward models of a fixed Markovian realization have the same zero structure.

The number of stable zeros of a (possibly nonsquare) spectral factor is equal to the dimension of the intersection between the splitting subspace and the past of the output process. The number of antistable zeros (including zeros at infinity) equals the dimension of the intersection between the splitting subspace and the future of the output process.

Finally, we have introduced the forward and backward zero-dynamics operators, having as eigenvalues the stable zeros and the inverses of the antistable zeros of the spectral factor. These operators are stochastic counterparts of certain feedback matrices appearing in geometric control theory.

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