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Robust H^∞ Control of Uncertain Systems with Structured Uncertainty*

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Abstract

This paper considers a problem of absolute stabilization with a specified level of disturbance attenuation for a class of uncertain systems in which the uncertainty is structured and satisfies a certain integral quadratic constraint. The paper shows that an uncertain system is absolutely stabilizable with a specified level of disturbance attenuation if and only if there exists a solution to a corresponding H^∞ control problem. The paper also shows that if an uncertain system can be absolutely stabilized with a specified level of disturbance attenuation via nonlinear output feedback control, then it can be absolutely stabilized with a specified level of disturbance attenuation via linear output feedback control.

Key words: robust H^∞ control, absolute stabilizability, uncertain systems, integral quadratic constraint, structured uncertainty, nonlinear control

AMS Subject Classifications: 93B36

1 Introduction

An important idea to emerge in recent years is the connection between the Riccati equation approach to H^∞ control and the problem of stabilizing an uncertain system containing norm bounded uncertainty; e.g., see [1, 2, 3]. In these papers, the notion of stabilizability considered is that of quadratic stabilizability. This notion can be extended to consider the problem of robustly stabilizing a system and also giving a specified level of disturbance attenuation; e.g., see [4].

In contrast to the quadratic stabilizability approach mentioned above, an alternative approach involves the notion of absolute stabilizability for

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uncertain systems in which the uncertainty satisfies a certain integral quadratic constraint; e.g., see [5, 6, 7]. The advantage of this approach is that it allows for non-conservative results to be obtained for the case of uncertain systems with structured uncertainty. Furthermore, using this framework, it was shown in [5] that if an uncertain system with structured uncertainty can be absolutely stabilized using nonlinear output feedback control, then it can also be absolutely stabilized using linear output feedback control. The main contribution of this paper is to extend the results of [5] to consider a problem of absolute stabilization with a specified level of disturbance attenuation. Also, the results of this paper enable some of the strong assumptions made in [5] to be weakened.

As in [5], a key technical result used in this paper is an extension of the "S-procedure" result of [8]. An important feature of the nonlinear S-procedure result of this paper is that the assumptions required are significantly weaker than the assumptions required in the nonlinear S-procedure of [5]. It is this nonlinear S-procedure result which enables us to consider the problem of absolute stabilization with a specified level of disturbance attenuation for the case of uncertain systems with structured uncertainty. Furthermore, this result also allows us to consider the possible use of nonlinear controllers.

The remainder of the paper proceeds as follows. In Section 2, we define the class of uncertain systems under consideration. For this class of uncertain systems, we define our notion of absolute stabilizability with a specified level of disturbance attenuation. In Section 3, we establish our technical result concerning the extension of the "S-procedure" result of [8] to the case of nonlinear systems. Section 4 of the paper presents our main result which is a necessary and sufficient condition for absolute stabilizability with a specified level of disturbance attenuation. This condition is given in terms of the existence of solutions to a pair of parameter dependent algebraic Riccati equations of the game type.

2 Problem Statement

We consider an output feedback H^∞ control problem for an uncertain system of the following form:

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + B_1 w(t) + B_2 u(t) + \sum_{s=1}^{k} D_s \xi_s(t); \\
z(t) &= C_1 x(t) + D_{12} u(t); \\
\zeta_1(t) &= K_1 x(t) + G_1 u(t); \\
&\vdots
\end{aligned}$$

$$\begin{aligned}\zeta_k(t) &= K_k x(t) + G_k u(t);\\ y(t) &= C_2 x(t) + D_{21} w(t) \end{aligned} \tag{2.1}$$

where $x(t) \in \mathbf{R}^n$ is the *state*, $w(t) \in \mathbf{R}^p$ is the *disturbance input*, $u(t) \in \mathbf{R}^m$ is the *control input*, $z(t) \in \mathbf{R}^q$ is the *error output*, $\zeta_1(t) \in \mathbf{R}^{h_1}, \ldots, \zeta_k(t) \in \mathbf{R}^{h_k}$ are the *uncertainty outputs*, $\xi_1(t) \in \mathbf{R}^{r_1}, \ldots, \xi_k(t) \in \mathbf{R}^{r_k}$ are the *uncertainty inputs* and $y(t) \in \mathbf{R}^l$ is the *measured output*. The uncertainty in this system is described by a set of equations of the form

$$\begin{aligned}\xi_1(t) &= \phi_1(t, \zeta_1(\cdot)|_0^t)\\ \xi_2(t) &= \phi_2(t, \zeta_2(\cdot)|_0^t)\\ &\vdots\\ \xi_k(t) &= \phi_k(t, \zeta_k(\cdot)|_0^t)\end{aligned} \tag{2.2}$$

where the following Integral Quadratic Constraint is satisfied.

Definition 2.1 *(Integral Quadratic Constraint; see [9, 10, 11, 12, 13, 14, 15, 5, 6, 16, 7].) An uncertainty of the form (2.2) is an admissible uncertainty for the system (2.1) if the following conditions hold: Given any locally square integrable control input $u(\cdot)$ and locally square integrable disturbance input $w(\cdot)$, and any corresponding solution to the system (2.1), (2.2), let $(0, t_*)$ be the interval on which this solution exists. Then there exist constants $d_1 \geq 0, \ldots, d_k \geq 0$ and a sequence $\{t_i\}_{i=1}^\infty$ such that $t_i \to t_*$, $t_i \geq 0$ and*

$$\int_0^{t_i} \|\xi_s(t)\|^2 dt \leq \int_0^{t_i} \|\zeta_s(t)\|^2 dt + d_s \quad \forall i \quad \forall s = 1, \ldots, k. \tag{2.3}$$

Here $\|\cdot\|$ denotes the standard Euclidean norm and $\mathbf{L}_2[0,\infty)$ denotes the Hilbert space of square integrable vector valued functions defined on $[0,\infty)$. Note that t_i and $t_\star$ may be equal to infinity. The class of all such admissible uncertainties $\xi(\cdot) = [\xi_1(\cdot), \ldots, \xi_k(\cdot)]$ is denoted Ξ.

In references [14] and [15], a number of examples are given of physical systems in which the uncertainty naturally fits into the above framework. Also, note that the above uncertainty description allows for uncertainties in which the uncertainty input ξ_s depends dynamically on the uncertainty output ζ_s. In this case, the constant d_s may be interpreted as a measure of the size of the initial condition on the uncertainty dynamics.

Also, it is clear that the uncertain system (2.1), (2.3) allows for uncertainty satisfying a norm bound condition. In this case, the uncertain system would be described by the state equations

$$\dot{x}(t) = [A + \sum_{s=1}^k D_s \Delta_s(t) K_s] x(t) + [B_2 + \sum_{s=1}^k D_s \Delta_s(t) G_s] u(t) + B_1 w(t);$$

$$y(t) = C_2x(t) + D_{21}w(t); \qquad \sup \|\Delta_s(t)\| \leq 1 \tag{2.4}$$

where $\Delta_s(t)$ are the uncertainty matrices (see e.g. [2, 3]). Indeed, let $\xi_s(t) = \Delta_s(t)[K_sx(t) + G_su(t)]$. Then the uncertainties $\xi_s(\cdot)$ satisfy conditions (2.3) with $d_s = 0$ and with any t_i.

For the uncertain system (2.1), (2.3), we consider a problem of absolute stabilization with a specified level of disturbance attenuation. The class of controllers considered are nonlinear output feedback controllers of the form

$$\begin{aligned} \dot{x_c}(t) &= \Lambda(x_c(t), y(t)); \\ u(t) &= \lambda(x_c(t), y(t)), \end{aligned} \tag{2.5}$$

where $\Lambda(x_c, y)$ and $\lambda(x_c, y)$ are continuous vector functions. Note that the dimension of the controller state vector $x_c(t)$ in (2.5) may be arbitrary.

Definition 2.2 *The uncertain system (2.1), (2.3) is said to be* absolutely stabilizable with disturbance attenuation γ *(via nonlinear output feedback control) if there exists an output feedback controller (2.5) and constants* $c_1 > 0$ *and* $c_2 > 0$ *such that the following conditions hold:*

(i) For any initial condition $[x(0), x_c(0)]$*, any admissible uncertainty inputs* $\xi(\cdot)$ *and any disturbance input* $w(\cdot) \in \mathbf{L}_2[0, \infty)$*, then*

$$[x(\cdot), x_c(\cdot), u(\cdot), \xi_1(\cdot), \ldots, \xi_k(\cdot)] \in \mathbf{L}_2[0, \infty)$$

(hence, $t_* = \infty$ *) and*

$$\begin{aligned} &\|x(\cdot)\|_2^2 + \|x_c(\cdot)\|_2^2 + \|u(\cdot)\|_2^2 + \sum_{s=1}^{k} \|\xi_s(\cdot)\|_2^2 \\ &\leq c_1 [\|x(0)\|^2 + \|x_c(0)\|^2 + \|w(\cdot)\|_2^2 + \sum_{s=1}^{k} d_s]. \end{aligned} \tag{2.6}$$

(ii) The following H^∞ *norm bound condition is satisfied: If* $x(0) = 0$ *and* $x_c(0) = 0$*, then*

$$J \triangleq \sup_{w(\cdot) \in \mathbf{L}_2[0,\infty)} \sup_{\xi(\cdot) \in \Xi} \frac{\|z(\cdot)\|_2^2 - c_2 \sum_{s=1}^{k} d_s}{\|w(\cdot)\|_2^2} < \gamma^2. \tag{2.7}$$

Here, $\|q(\cdot)\|_2$ *denotes the* $\mathbf{L}_2[0, \infty)$ *norm of a function* $q(\cdot)$*. That is,* $\|q(\cdot)\|_2^2 \triangleq \int_0^\infty \|q(t)\|^2 dt$*.*

Observation 2.1 It follows from the above definition that if the uncertain system (2.1), (2.3) is absolutely stabilizable with disturbance attenuation γ, then the corresponding closed loop system (2.1), (2.3), (2.5) with $w(\cdot) \in \mathbf{L}_2[0,\infty)$, has the property that $x(t) \to 0$ as $t \to \infty$. Indeed, since $[x(\cdot), u(\cdot), \xi(\cdot), w(\cdot)] \in \mathbf{L}_2[0,\infty)$, we can conclude from (2.1) that $\dot{x}(\cdot) \in \mathbf{L}_2[0,\infty)$. However, using the fact that $x(\cdot) \in \mathbf{L}_2[0,\infty)$ and $\dot{x}(\cdot) \in \mathbf{L}_2[0,\infty)$, it now follows that $x(t) \to 0$ as $t \to \infty$.

3 S-Procedure for Nonlinear Systems

In this section, we present a result which extends the "S-Procedure" of [8]. The result is closely related to the result presented in [5]. However, the assumptions required for this result are considerably weaker than the assumptions required in [5]. The main result of this section applies to a nonlinear, time-invariant system of the form

$$\dot{h}(t) = \Pi(h(t), \psi(t)) \tag{3.1}$$

where $h(t) \in \mathbf{R}^N$ is the *state* and $\psi(t) \in \mathbf{R}^M$ is the *input*. Associated with the system (3.1) is the following set of functionals:

$$\begin{aligned}
f_0\left(h(\cdot), \psi(\cdot)\right) &= \int_0^\infty \nu_0(h(t), \psi(t))dt, \\
f_1\left(h(\cdot), \psi(\cdot)\right) &= \int_0^\infty \nu_1(h(t), \psi(t))dt, \\
&\vdots \\
f_k\left(h(\cdot), \psi(\cdot)\right) &= \int_0^\infty \nu_k(h(t), \psi(t))dt.
\end{aligned}$$

Assumptions The system (3.1) and associated set of functionals satisfy the following assumptions:

3.1 The functions $\Pi(\cdot,\cdot), \nu_0(\cdot,\cdot), \ldots, \nu_k(\cdot,\cdot)$ are continuous.

3.2 For all $\psi(\cdot) \in \mathbf{L}_2[0,\infty)$ and all initial conditions $h(0) \in \mathbf{R}^N$, the corresponding solution $h(\cdot)$ belongs to $\mathbf{L}_2[0,\infty)$ and the corresponding quantities $f_0\left(h(\cdot), \psi(\cdot)\right)$, $f_1\left(h(\cdot), \psi(\cdot)\right)$, …, $f_k\left(h(\cdot), \psi(\cdot)\right)$ are finite.

3.3 Given any $\varepsilon > 0$, there exists a constant $\delta > 0$ such that the following condition holds: For any input function $\psi_0(\cdot) \in \{\psi_0(\cdot) \in \mathbf{L}_2[0,\infty), \|\psi_0(\cdot)\|_2^2 \leq \delta\}$ and any $h_0 \in \{h_0 \in \mathbf{R}^N : \|h_0\| \leq \delta\}$, let $h_0(t)$ denotes the corresponding solution to (3.1) with initial condition $h_0(0) = h_0$. Then $|f_s\left(h_0(\cdot), \psi_0(\cdot)\right)| < \varepsilon$ for $s = 0, 1, \ldots, k$.

Note, Assumption 3.3 is a stability type assumption on the system (3.1).

Notation For the system (3.1) satisfying the above assumptions, we define $\Omega \subset \mathbf{L}_2[0,\infty)$ as follows: Ω is the set of $\{h(\cdot),\psi(\cdot)\}$ such that $\psi(\cdot) \in \mathbf{L}_2[0,\infty)$ and $h(\cdot)$ is the corresponding solution to (3.1) with initial condition $h(0)=0$.

Theorem 3.1 *Consider the system (3.1) and associated functionals and suppose the Assumptions 3.1 – 3.3 are satisfied. If* $f_0\left(h(\cdot),\psi(\cdot)\right) \geq 0$ *for all* $\{h(\cdot),\psi(\cdot)\} \in \Omega$ *such that* $f_1\left(h(\cdot),\psi(\cdot)\right) \geq 0, \ldots, f_k\left(h(\cdot),\psi(\cdot)\right) \geq 0$*, then there exist constants* $\tau_0 \geq 0, \tau_1 \geq 0, \ldots, \tau_k \geq 0$ *such that* $\sum_{s=0}^{k} \tau_s > 0$ *and*

$$\tau_0 f_0\left(h(\cdot),\psi(\cdot)\right) \geq \tau_1 f_1\left(h(\cdot),\psi(\cdot)\right) + \tau_2 f_2\left(h(\cdot),\psi(\cdot)\right) + \cdots + \tau_k f_k\left(h(\cdot),\psi(\cdot)\right) \tag{3.2}$$

for all $\{h(\cdot),\psi(\cdot)\} \in \Omega$.

In order to prove this theorem, we will use the following convex analysis result, the proof of which was given in [5]. However, we first introduce some notation.

Notation Given $S \subset \mathbf{R}^n$ and $T \subset \mathbf{R}^n$, then $S+T := \{x+y : x \in S, y \in T\}$. Also, $\mathrm{cl}(S)$ denotes the closure of the set S.

Lemma 3.1 *(See [5] for proof) Consider a set* $M \subset \mathbf{R}^{k+1}$ *with the property that* $a+b \in \mathrm{cl}(M)$ *for all* $a,b \in M$*. If* $x_0 \geq 0$ *for all vectors* $\begin{bmatrix} x_0 & \cdots & x_k \end{bmatrix}' \in M$ *such that* $x_1 \geq 0, \ldots x_k \geq 0$*, then there exist constants* $\tau_0 \geq 0, \ldots, \tau_k \geq 0$ *such that* $\sum_{s=0}^{k} \tau_s > 0$ *and* $\tau_0 x_0 \geq \tau_1 x_1 + \tau_2 x_2 + \cdots + \tau_k x_k$ *for all* $\begin{bmatrix} x_0 & \cdots & x_k \end{bmatrix}' \in M$.

Proof of Theorem 3.1: In the order to prove this theorem we establish the following claim.

Claim: Given any $\epsilon_0 > 0$ and any input $\psi_0(\cdot) \in \mathbf{L}_2[0,\infty)$, then there exists a constant $\delta_0 > 0$ such that the following condition holds: for any $h_0 \in \{h_0 \in \mathbf{R}^N : \|h_0\| \leq \delta_0\}$, let $h_1(t)$ denotes the corresponding solution to (3.1) with initial condition $h_1(0)=0$ and let $h_2(t)$ denotes the corresponding solution to (3.1) with initial condition $h_2(0)=h_0$. Then $|f_s\left(h_1(\cdot),\psi_0(\cdot)\right) - f_s\left(h_2(\cdot),\psi_0(\cdot)\right)| < \epsilon_0$ for $s=0,1,\ldots,k$.

Indeed, let $\epsilon_0 > 0$ be some constant and let δ be the constant from Assumption 3.3 corresponding to $\varepsilon = \frac{\epsilon_0}{4}$. According to Assumption 3.2, $[h_1(\cdot),\psi_0(\cdot)] \in \mathbf{L}_2[0,\infty)$ and, therefore, there exists a $T > 0$ such that $\|h_1(T)\| \leq \frac{\delta}{2}$ and $\int_T^\infty \|\psi_0(t)\|^2 dt \leq \delta$. Assumption 3.1 implies that there exists a constant $\delta_0 > 0$ such that for all $\|h_0\| < \delta_0$, the solution $h_2(\cdot)$ of the system (3.1) with input $\psi_0(\cdot)$ and initial condition $h_2(0)=h_0$ satisfies condition

$$\int_0^T |\nu_s(h_1(t),\psi_0(t)) - \nu_s(h_2(t),\psi_0(t))| dt < \frac{\epsilon_0}{2} \tag{3.3}$$

for all s and

$$\|h_2(T) - h_1(T)\| < \frac{\delta}{2}. \tag{3.4}$$

Since $\|h_1(T)\| \leq \frac{\delta}{2}$, we have from (3.4) that $\|h_2(T)\| \leq \delta$. Furthermore, Assumption 3.3 and time invariance of the system (3.1) imply that

$$\int_T^\infty |\nu_s(h_1(t),\psi_0(t)) - \nu_s(h_2(t),\psi_0(t))|dt \leq$$
$$\int_T^\infty (|\nu_s(h_1(t),\psi_0(t))| + |\nu_s(h_2(t),\psi_0(t))|)dt \leq \frac{\epsilon_0}{4} + \frac{\epsilon_0}{4} = \frac{\epsilon_0}{2}.$$

From this and the inequality (3.3) the claim follows immediately.

Now suppose $f_0\left(h(\cdot),\psi(\cdot)\right) \geq 0$ for all $\{h(\cdot),\psi(\cdot)\} \in \Omega$ such that

$$f_1\left(h(\cdot),\psi(\cdot)\right) \geq 0, \ldots, f_k\left(h(\cdot),\psi(\cdot)\right) \geq 0$$

and let

$$M := \left\{ \begin{bmatrix} f_0\left(h(\cdot),\psi(\cdot)\right) & \ldots & f_k\left(h(\cdot),\psi(\cdot)\right) \end{bmatrix}' \in \mathbf{R}^{k+1} : \{h(\cdot),\psi(\cdot)\} \in \Omega \right\}.$$

It follows from the assumption on the set Ω that $x_0 \geq 0$ for all vectors $\begin{bmatrix} x_0 & \cdots & x_k \end{bmatrix}' \in M$ such that $x_1 \geq 0, \ldots, x_k \geq 0$. Let $\{h_a(\cdot),\psi_a(\cdot)\} \in \Omega$ and $\{h_b(\cdot),\psi_b(\cdot)\} \in \Omega$ be given. Since $h_a(\cdot) \in \mathbf{L}_2[0,\infty)$, then there exists a sequence $\{T_i\}_{i=1}^\infty$ such that $T_i > 0$ for all i, $T_i \to \infty$ and $h_a(T_i) \to 0$ as $i \to \infty$. Now consider the corresponding sequence $\{h_i(\cdot),\psi_i(\cdot)\}_{i=1}^\infty \subset \Omega$, where

$$\psi_i(t) = \begin{cases} \psi_a(t) & t \in [0,T_i); \\ \psi_b(t-T_i) & t \geq T_i. \end{cases}$$

We will establish that $f_s(h_i(\cdot),\psi_i(\cdot)) \to f_s(h_a(\cdot),\psi_a(\cdot)) + f_s(h_b(\cdot),\psi_b(\cdot))$ as $i \to \infty$ for $s = 0,1,\ldots,k$. Indeed, let $s \in \{0,1,\ldots,k\}$ be given and fix i. Now suppose $\tilde{h}_b^i(\cdot)$ is the solution to (3.1) with input $\psi(\cdot) = \psi_b(\cdot)$ and initial condition $\tilde{h}_b^i(0) = h_a(T_i)$. It follows from the time invariance of the system (3.1) that $h_i(t) \equiv \tilde{h}_b^i(t-T_i)$. Hence,

$$\begin{aligned} f_s(h_i(\cdot),\psi_i(\cdot)) &= \int_0^\infty \nu_s(h_i(t),\psi_i(t))dt \\ &= \int_0^{T_i} \nu_s(h_a(t),\psi_a(t))dt + \int_{T_i}^\infty \nu_s(h_i(t),\psi_b(t-T_i))dt \\ &= \int_0^{T_i} \nu_s(h_a(t),\psi_a(t))dt + f_s(\tilde{h}_b^i(\cdot),\psi_b(\cdot)). \end{aligned}$$

Using the fact that $h_a(T_i) \to 0$, the above claim implies $f_s(\tilde{h}_b^i(t),\psi_b(t)) \to f_s(h_b(\cdot),\psi_b(\cdot))$ as $i \to \infty$. Also, $\int_0^{T_i} \nu_s(h_a(t),\psi_a(t))dt \to f_s(h_a(\cdot),\psi_a(\cdot))$. Hence,

$$f_s(h_i(\cdot),\psi_i(\cdot)) \to f_s(h_a(\cdot),\psi_a(\cdot)) + f_s(h_b(\cdot),\psi_b(\cdot)).$$

From the above, it follows that the set M has the property that $a+b \in \mathrm{cl}(M)$ for all $a, b \in M$. Hence, Lemma 3.1 implies that there exist constants $\tau_0 \geq 0, \ldots, \tau_1 \geq 0$ such that $\sum_{s=0}^{k} \tau_s > 0$ and $\tau_0 x_0 \geq \tau_1 x_1 + \cdots + \tau_k x_k$ for all $\left[\begin{array}{ccc} x_0 & \cdots & x_k \end{array}\right]' \in M$. That is, condition (3.2) is satisfied. □

4 The Main Results

In this section, we present the main result of this paper which establishes a necessary and sufficient condition for the uncertain system (2.1), (2.3) to be absolutely stabilizable with a specified level of disturbance attenuation. This condition is given in terms of the existence of solutions to a pair of parameter dependent algebraic Riccati equations. The Riccati equations under consideration are defined as follows:
Let $\tau_1 > 0, \ldots, \tau_k > 0$ be given constants and consider the algebraic Riccati equations

$$\begin{array}{l}(A - B_2 E_1^{-1} \hat{D}_{12}' \hat{C}_1)' X + X(A - B_2 E_1^{-1} \hat{D}_{12}' \hat{C}_1) \\ \quad + \quad X(\hat{B}_1 \hat{B}_1' - B_2 E_1^{-1} B'_2) X + \hat{C}_1' (I - \hat{D}_{12} E_1^{-1} \hat{D}_{12}') \hat{C}_1 = 0; \end{array} \quad (4.1)$$

$$\begin{array}{l}(A - \hat{B}_1 \hat{D}_{21}' E_2^{-1} C_2) Y + Y(A - \hat{B}_1 \hat{D}_{21}' E_2^{-1} C_2)' \\ \quad + \quad Y(\hat{C}_1' \hat{C}_1 - C'_2 E_2^{-1} C_2) Y + \hat{B}_1 (I - \hat{D}_{12}' E_2^{-1} \hat{D}_{21}) \hat{B}_1' = 0 \end{array} \quad (4.2)$$

where

$$\begin{array}{rcl} \hat{C}_1 & = & \left[\begin{array}{c} C_1 \\ \sqrt{\tau_1} K_1 \\ \vdots \\ \sqrt{\tau_k} K_k \end{array}\right]; \hat{D}_{12} = \left[\begin{array}{c} D_{12} \\ \sqrt{\tau_1} G_1 \\ \vdots \\ \sqrt{\tau_k} G_k \end{array}\right]; \\ & & \hat{D}_{21} = \left[\begin{array}{cccc} \gamma^{-1} D_{21} & 0 & \ldots & 0 \end{array}\right]; \\ E_1 & = & \hat{D}_{12}' \hat{D}_{12}; E_2 = \hat{D}_{21} \hat{D}_{21}'; \\ \hat{B}_1 & = & \left[\begin{array}{cccc} \gamma^{-1} B_1 & \sqrt{\tau_1}^{-1} D_1 & \ldots & \sqrt{\tau_k}^{-1} D_k \end{array}\right]. \end{array} \quad (4.3)$$

Assumptions The uncertain system (2.1), (2.3) will be required to satisfy the following additional assumptions:

4.1 The pair (A, C_1) is observable.

4.2 $E_1 > 0$.

4.3 The pair (A, B_1) is controllable.

4.4 $E_2 > 0$.

Theorem 4.1 *Consider the uncertain system (2.1), (2.3) and suppose that Assumptions 4.1-4.4 are satisfied. Then the following statements are equivalent:*

- *(i) The uncertain system (2.1),(2.3) is absolutely stabilizable with disturbance attenuation γ via the nonlinear output feedback control (2.5).*
- *(ii) There exist constants $\tau_1 > 0, \ldots, \tau_k > 0$ such that the Riccati equations (4.1) and (4.2) have solutions $X > 0$ and $Y > 0$ and such that the spectral radius of their product satisfies $\rho(XY) < 1$.*

If condition (ii) holds, then the uncertain system (2.1), (2.3) is absolutely stabilizable with disturbance attenuation γ via a linear controller of the form

$$\begin{aligned} \dot{x}_c(t) &= A_c x_c(t) + B_c y(t), \\ u(t) &= C_c x_c(t) \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} A_c &= A + B_2 C_c - B_c C_2 + (\hat{B}_1 - B_c \hat{D}_{21})\hat{B}_1' X \\ B_c &= (I - YX)^{-1}(Y C'_2 + \hat{B}_1 \hat{D}_{21}')E_2^{-1} \\ C_c &= -E_1^{-1}(B'_2 X + \hat{D}_{12}' \hat{C}_1). \end{aligned} \tag{4.5}$$

Proof: $(i) \Rightarrow (ii)$ Consider the set Ω of vector functions

$$\lambda(\cdot) = [x(\cdot), x_c(\cdot), \xi(\cdot), w(\cdot)]$$

in $\mathbf{L}_2[0,\infty)$ connected by (2.1), (2.5) and the initial condition $[x(0), x_c(0)] = 0$. Condition (2.7) implies that there exists a constant $\delta_1 > 0$ such that $J < \gamma^2 - 2\delta_1$. Let $\delta_2 = \min[(2c_1 k)^{-1}, \delta_1(2c_2 k(c_1+1))^{-1}, \delta_1(2(c_1+1))^{-1}]$, where c_1 and c_2 are the constants from Definition 2.2. Consider the functionals $f_0, f_1, \ldots, f_k$ from Ω to $\mathbf{R}$ where

$$\begin{aligned} f_0(\lambda(\cdot)) &= -(\|z(\cdot)\|_2^2 - \gamma^2\|w(\cdot)\|_2^2 + \delta_2\|\lambda(\cdot)\|^2); \\ f_1(\lambda(\cdot)) &= \|\zeta_1(\cdot)\|_2^2 - \|\xi_1(\cdot)\|_2^2 + \delta_2\|\lambda(\cdot)\|^2; \\ &\vdots \\ f_k(\lambda(\cdot)) &= \|\zeta_k(\cdot)\|_2^2 - \|\xi_k(\cdot)\|_2^2 + \delta_2\|\lambda(\cdot)\|^2. \end{aligned} \tag{4.6}$$

Here $\|\lambda(\cdot)\|^2 = \|x(\cdot)\|_2^2 + \|x_c(\cdot)\|_2^2 + \|\xi(\cdot)\|_2^2 + \|w(\cdot)\|_2^2$. We will prove that $f_0(\lambda(\cdot)) \geq 0$ for all $\lambda(\cdot) \in \Omega$ such that $f_s(\lambda(\cdot)) \geq 0$ for $s = 1, \ldots, k$. Indeed, if $f_s(\lambda(\cdot)) \geq 0$ for $s = 1, \ldots, k$, then the vector function $\lambda(\cdot)$ satisfies the constraints (2.3) with $d_s = \delta_2\|\lambda(\cdot)\|_2^2$ and $t_i = \infty$. Hence, condition (i) of Definition 2.2 implies that $\|\lambda(\cdot)\|^2 \leq (c_1+1)\|w(\cdot)\|_2^2 + c_1 k\delta_2\|\lambda(\cdot)\|^2$. Since $\delta_2 \leq (2c_1k)^{-1}$, then

$$\|\lambda(\cdot)\|^2 \leq 2(c_1+1)\|w(\cdot)\|_2^2. \tag{4.7}$$

Condition (ii) of Definition 2.2 implies that $-(\|z(\cdot)\|_2^2 - (\gamma^2-2\delta_1)\|w(\cdot)\|_2^2) + c_2k\delta_2\|\lambda(\cdot)\|^2 \geq 0$ for all $\lambda(\cdot) \in \Omega$ such that $f_s(\lambda(\cdot)) \geq 0$ for $s = 1,\ldots,k$. Since $\delta_2 \leq \delta_1(2c_2k(c_1+1))^{-1}$ and $\delta_2 \leq \delta_1(2(c_1+1))^{-1}$, inequality (4.7) implies that $\delta_1\|w(\cdot)\|_2^2 \geq c_2k\delta_2\|\lambda(\cdot)\|^2$ and $\delta_1\|w(\cdot)\|_2^2 \geq \delta_2\|\lambda(\cdot)\|^2$. Therefore,

$$\begin{aligned}
f_0(\lambda(\cdot)) &= -(\|z(\cdot)\|_2^2 - \gamma^2\|w(\cdot)\|_2^2 + \delta_2\|\lambda(\cdot)\|^2) \\
&\geq -\left(\begin{array}{c} \|z(\cdot)\|_2^2 - \gamma^2\|w(\cdot)\|_2^2 + \delta_2\|\lambda(\cdot)\|^2 + \delta_1\|w(\cdot)\|_2^2 \\ -c_2k\delta_2\|\lambda(\cdot)\|^2 + \delta_1\|w(\cdot)\|_2^2 - \delta_2\|\lambda(\cdot)\|^2 \end{array}\right) \\
&= -(\|z(\cdot)\|_2^2 - (\gamma^2-2\delta_1)\|w(\cdot)\|_2^2) + c_2k\delta_2\|\lambda(\cdot)\|^2 \\
&\geq 0.
\end{aligned}$$

Now the closed loop system (2.1), (2.5) can be rewritten as the system (3.1) where $h(\cdot) = [x(\cdot), x_c(\cdot)]$ and $\psi(\cdot) = [\xi_1(\cdot),\ldots,\xi_k(\cdot), w(\cdot)]$. Using the continuity of the coefficients in the controller (2.5)), it follows immediately that the closed loop system satisfies Assumption 3.1. Furthermore, given any $\psi(\cdot) = [\xi_1(\cdot),\ldots,\xi_k(\cdot), w(\cdot)] \in \mathbf{L}_2[0,\infty)$, it follows that the uncertainty inputs $[\xi_1(\cdot),\ldots,\xi_k(\cdot)]$ satisfy the constraints (2.3) with $d_s = \|\xi_s(\cdot)\|_2^2$ and $t_i = \infty$. Hence, condition (i) of Definition 2.2 implies that Assumption 3.2 is satisfied. Also, given any $\psi(\cdot) = [\xi_1(\cdot),\ldots,\xi_k(\cdot), w(\cdot)] \in \mathbf{L}_2[0,\infty)$ such that $\|\psi(\cdot)\|_2^2 \leq \delta$, then the constraints (2.3) are satisfied with $d_s \leq \delta$. Thus, condition (i) of Definition 2.2 also implies that Assumption 3.3 is satisfied.

We can now apply Theorem 3.1 to the above closed loop system. Using this theorem, it follows that there exist constants $\tau_0 \geq 0, \tau_1 \geq 0, \ldots, \tau_k \geq 0$ such that $\sum_{s=0}^k \tau_s > 0$ and the inequality (3.2) is satisfied for all $\lambda(\cdot) \in \Omega$.

Now we prove that $\tau_s > 0$ for all $s = 0,1,\ldots,k$. Condition (3.2) for the functionals (4.6) implies that

$$\begin{aligned}
\tau_0(\|z(\cdot)\|_2^2 - \gamma^2\|w(\cdot)\|_2^2) + \sum_{s=1}^k \tau_s(\|\zeta_s(\cdot)\|_2^2 - \|\xi_s(\cdot)\|_2^2) \\
\leq -\delta_0(\sum_{s=1}^k \|\xi_s(\cdot)\|_2^2 + \|w(\cdot)\|_2^2)
\end{aligned} \tag{4.8}$$

where $\delta_0 = \delta_2\sum_{s=0}^k \tau_s > 0$. If $\tau_j = 0$ for some $j = 1,\ldots,k$ then we can take $w(\cdot) \equiv 0, \xi_s(\cdot) \equiv 0$ for all $s \neq j$ and $\xi_j(\cdot) \not\equiv 0$. Then, the inequality (4.8) is not satisfied, because the left side of (4.8) is non-negative and the right side of (4.8) is negative. (Analogously, if $\tau_0 = 0$, we can take $w(\cdot) \neq 0$

and $\xi(\cdot) \equiv 0$.) Therefore, $\tau_s > 0$ for $s = 0, 1, \ldots, k$. In this case, we can take in (4.8) $\tau_0 = 1$.

Consider the following linear system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + \hat{B}_1 \hat{w}(t) + B_2 u(t); \\ \hat{z}(t) &= \hat{C}_1 x(t) + \hat{D}_{12} u(t); \\ y(t) &= C_2 x(t) + \hat{D}_{21} \hat{w}(t) \end{aligned} \tag{4.9}$$

where

$$\begin{aligned} \hat{w}(\cdot) &= [\gamma w(\cdot), \sqrt{\tau_1}\xi_1(\cdot), \ldots, \sqrt{\tau_k}\xi_k(\cdot)], \\ \hat{z}(\cdot) &= [z(\cdot), \sqrt{\tau_1}\zeta_1(\cdot), \ldots, \sqrt{\tau_k}\zeta_k(\cdot)] \end{aligned}$$

and the matrix coefficients $\hat{B}_1, \hat{C}_1, \hat{D}_{12}$ and $\hat{D}_{21}$ are defined by (4.3). In this system, $\hat{w}$ is the disturbance input and $\hat{z}$ is the controlled output.

The inequality (4.8) with $\tau_0 = 1$ may be rewritten as

$$\|\hat{z}(\cdot)\|_2^2 - \|\hat{w}(\cdot)\|_2^2 \leq -\delta_0 \|\hat{w}(\cdot)\|_2^2. \tag{4.10}$$

That is,

$$\hat{J} \triangleq \sup_{\hat{w}(\cdot) \in \mathbf{L}_2[0,\infty), x(0)=0, x_c(0)=0} \frac{\|\hat{z}(\cdot)\|_2^2}{\|\hat{w}(\cdot)\|_2^2} < 1. \tag{4.11}$$

Therefore, the controller (2.5) with the initial condition $x_c(0) = 0$ solves a standard output feedback H^∞ control problem for the system (4.9). Hence, condition (ii) of the theorem follows directly using Theorems 5.5 and 5.6 of [17]. This completes the proof of this part of the theorem.

$(ii) \Rightarrow (i)$ It is a standard result from H^∞ control theory that if condition (ii) holds then the linear controller (4.4) solves a standard output feedback H^∞ control problem defined by the system system (4.9) and H^∞ cost bound (4.11); e.g., see [17, 18]. Furthermore, condition (4.11) implies that there exists a constant $\delta_0 > 0$ such that the inequality (4.10) is satisfied for all the solutions of the closed loop system (4.9), (4.4) with $\hat{w}(\cdot) \in \mathbf{L}_2[0, \infty)$ and the initial condition $[x(0), x_c(0)] = 0$.

Now the closed loop uncertain system defined by (2.1) and (4.4) may be rewritten as

$$\dot{h}(t) = Ph(t) + Q\hat{w}(t), \tag{4.12}$$

where

$$h = \begin{bmatrix} x \\ x_c \end{bmatrix}, P = \begin{bmatrix} A & C_c \\ B_c C_2 & A_c \end{bmatrix}, Q = \begin{bmatrix} \hat{B}_1 \\ \hat{D}_{12} \end{bmatrix}.$$

Since the controller (4.4) solves the H^∞ control problem described above, the matrix P is stable. Furthermore, condition (2.3) implies that any disturbance input $w(\cdot) \in \mathbf{L}_2[0, \infty)$ and admissible uncertainty inputs $\xi_1(\cdot), \ldots, \xi_k(\cdot)$ satisfy the following integral quadratic constraint:

There exists a constant $d \geq 0$ and a sequence $\{t_i\}_{i=1}^{\infty}$ such that $t_i \to t_*$ (where $[0, t_*)$ is the interval of existence for the corresponding solution) and

$$\int_0^{t_i} \|\hat{w}(t)\|^2 dt \leq \int_0^{t_i} \|Th(t)\|^2 dt + d \ \forall i \tag{4.13}$$

where

$$T = \left[\begin{array}{cc} \hat{C}_1 & \hat{D}_{12} C_c \end{array} \right].$$

Moreover, the constant d is given by

$$d = \gamma^2 \|w(\cdot)\|_2^2 + \sum_{s=1}^{k} \tau_s d_s.$$

However, since $\hat{z} = Th$, condition (4.10) and stability of the matrix P imply that there exists a constant $\delta > 0$ such that

$$\int_0^{\infty} (\|Th(t)\|^2 - \|\hat{w}(t)\|^2) dt \leq -\delta \int_0^{\infty} (\|h(t)\|^2 + \|\hat{w}(t)\|^2) dt \tag{4.14}$$

for all $[h(\cdot), \hat{w}(\cdot)] \in \mathbf{L}_2[0, \infty)$ connected by (4.12) with $h(0) = 0$. Using Theorem 1 of [11], condition (4.14) and the stability of the matrix P imply absolute stability of the uncertain system (4.12), (4.13). Thus, there exists a constant $c_0 > 0$ such that for any initial condition $[x(0), x_c(0)]$ and any uncertainty $\hat{w}(\cdot)$ described by (4.13), then $[x(\cdot), x_c(\cdot), \hat{w}(\cdot)] \in \mathbf{L}_2[0, \infty)$ and

$$\int_0^{\infty} (\|x(t)\|^2 + \|x_c(t)\|^2 + \|\hat{w}(t)\|^2) dt \leq c_0 [\|x(0)\|^2 + \|x_c(0)\|^2 + d]. \tag{4.15}$$

Since $u(t) = C_c x_c(t)$ in controller (4.4),

$$\hat{w}(\cdot) = [\gamma w(\cdot), \sqrt{\tau_1} \xi_1(\cdot), \ldots, \sqrt{\tau_k} \xi_k(\cdot)]$$

and $d = \gamma^2 \|w(\cdot)\|_2^2 + \sum_{s=1}^{k} \tau_s d_s$, condition (i) of Definition 2.2 follows from the inequality (4.15).

We now establish condition (ii) of Definition 2.2. We have established that condition (4.10) is satisfied for all the solutions of the closed loop system (2.1), (4.4) with zero initial condition. This may be rewritten as condition (4.8) with $\tau_0 = 1$. Furthermore, all solutions to the closed loop system (2.1), (4.4) from $\mathbf{L}_2[0, \infty)$ satisfy condition (2.3) with $d_s = \max[0, -(\|z_s(\cdot)\|_2^2 - \|\xi_s(\cdot)\|_2^2)]$ and $t_i = \infty$. Therefore, it follows from (4.8) that condition (2.7) holds with $c_1 = \max[\tau_1, \ldots, \tau_k]$. This completes the proof of the theorem. □

The following corollary is an immediate consequence of the above theorem and the remarks following the definition of the uncertain system (2.1), (2.3).

Corollary 4.1 *The uncertain system with norm bounded uncertainties, (2.4) will be absolutely stabilizable with disturbance attenuation γ via the linear controller (4.4),(4.5) if condition (ii) of Theorem 4.1 is satisfied.*

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