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Disturbance Decoupling Via Differential Forms*

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Abstract

We formulate and study the question of finding a static feedback which will cause an affine control system (with some restrictions on the number of inputs) to leave a given codistribution invariant. Specifically, our contribution is threefold: i) we show that such a study has numerous benefits even when the codistribution is nonsingular. For instance, one can recover both the standard results and results as manifestations of the same phenomenon; ii) for analytic data and singular codistributions we relate this question to the work of [[18]]; and iii) in the process of studying the same question for partially smooth data, we develop a method for solving degenerate, overdetermined systems of first-order partial differential equations which is reminiscent of the method of successive integration.

Key words: one forms, integrability conditions, blow ups

AMS Subject Classifications: 93C10, 93C15, 93C25

1 Introduction

A problem that has attracted quite some research in control theory is the *controlled invariance* problem. Its standard formulation runs as follows (see [16, 21] and the references therein): given a nonlinear control system

$$\dot{x} = f(x) + \sum_{i=1} g_i(x)u_i \tag{1.1}$$

and an involutive distribution (or vector field system) Δ, find a feedback control law of the form $u = \alpha(x) + \beta(x)v$, where v is the new control, so that the closed loop system will leave Δ invariant. A necessary condition (if the feedback is to be regular) for its solvability is that the so-called *weak*

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(f, g)-invariance(or local controlled invariance) condition, $[F, \Delta] \subseteq \Delta + G$ hold. Here F is in any of the one of the vector fields $f, g_i, i = 1, \ldots, m$ and G is the distribution spanned by the $g_i, i = 1, \ldots, m$. It is also known to be sufficient under the following regularity conditions: a) Δ has constant rank in a neighborhood U of the operating point, and b) the distribution $(G + \Delta)\backslash\Delta$ has constant rank on U of the operating point. Furthermore, the feedback $\alpha(x)$ and $\beta(x)$ may be found locally (i.e., on some open subset $V \subseteq U$) by solving the system of equations:

$$X_i \alpha_j(x) = c_j^i(x), i = 1, \ldots, d, j = 1, \ldots, m$$

and (on denoting by $\beta_l(x)$ the lth column of $\beta(x)$)

$$X_i \beta_l(x) = C_i(x)\beta_l(x), i = 1, \ldots, d$$

where $X_1, \ldots, X_d$ form a basis for the distribution Δ. The $c_i^j(x)$(resp.$C_i(x)$) are smooth functions (resp. matrices of smooth functions) on U concocted out of the data (see [22]).

A natural problem that arises is the study of the same question with Δ replaced by a codistribution Ω. (Throughout this paper we will use the term codistribution to mean the span of a set of one-forms, since this terminology is by now established in the control literature.) Our aim is to show that an answer to this question has umpteen practical benefits as well. To begin with, it is known that in most applications of the controlled invariance the point of departure is actually a codistribution which is spanned by exact differentials of smooth functions obtained directly from the data of the design problem. The vector field system, Δ, of the paragraph above is merely the annihilator of this codistribution. Passage to the annihilator is indeed one of the main reasons for imposing the regularity conditions mentioned in the first paragraph. Furthermore, passing to the annihilator imposes the additional burden of pairing down; i.e, one has to find basis for the annihilating distribution. This involves solving systems of analytic equations. We shall see in the balance of the paper that there are innumerable other advantages of analysing this problem via differential forms. One such prominent advantage is the fact that the analysis of singularities in the controlled invariance problem is rendered very elegant and rather complete, so much so that our earlier attempt at analysing this issue via distributions (see [23]) seems innocuous in comparison. We will assume that Ω has a basis of exact differentials of functions $h_1, \ldots, h_p$ which are real analytic on U—a connected open set of interest in R^n which contains the reference point (assumed to be the origin in R^n), and on which all the data are defined. For details on when one may take Ω to be spanned by exact differentials see, for instance, [18, 19]. By abuse of language, we will allude to the $h_i, i = 1, \ldots, p$ as the outputs. As we shall show below, it is more useful to consider these outputs as being more intrinsic to the problem than Ω itself (cf. **P2** below). There are two reasons for taking the h_i to be analytic. First, in the application to the disturbance decoupling problem it is almost necessary to take the actual outputs of the plant to be analytic (primarily because one cannot guarantee the completeness of the closed loop vector fields). Since, at least some of the $h_i, i = 1, \ldots, p$ are the actual outputs (in the disturbance decoupling problem) this partially justifies taking Ω to be analytic. Another reason is that we will have to make certain density arguments and this necessitates analyticity. Finally, at a later stage in the paper we will need a canonical form for the h_i (either before or after blowing-up). In this paper we will restrict attention to the case that $p \geq m$. Since practically all the techniques in the $m > 1$ case are exactly the same as those for $m = 1$, we will, for the most part, deal only with the single-input case. The case for multiple inputs will be illustrated via an example which will clearly reveal the intervening calculations. The case where $m > p$ is physically the easier situation. We speculate that this case could also be subsumed by our methods if we employ generalised inverses.

The first question that one has to address is that of finding the correct analog of the controlled invariance problem. Below we will list three versions. There are two motivating factors in arriving at these versions. First, any generalization must reduce to the standard formulation under the usual regularity conditions, and second, must lead to a resolution of the disturbance decoupling problem (which, in this author's opinion, is the archtypal application of the controlled invariance problem). Denoting by F, either of the closed-loop vector fields, (i.e, either $f + \sum_i g_i \alpha_i$ or $(g\beta)_l, l = 1, \ldots, m$), these versions are: *find a feedback control law* $u(t) = \alpha(x(t)) + \beta(x(t))v(t)$, *so that:*

P1 $dL_F h_i(x) = \sum_{l=1}^{p} a_{li}(x) dh_l i = 1, \ldots, p$ for some smooth (resp. analytic) functions $a_{li}(x)$.

P2 a) $dL_F h_i \wedge dh_1 \wedge \ldots \wedge dh_p = 0, i = 1, \ldots, p$; b) $dL_F^k h_i \wedge dh_1 \wedge \ldots \wedge dh_p = 0, i = 1, \ldots, p, k = 0, 1, \ldots$; and c) the $L_F^k h_i$ are constant on the connected components of the level sets of the h_i for all $k = 0, 1, \ldots$ and all $i = 1, \ldots, p$.

P3 The $L_F h_i$ are each some functions, $A_{i,F}(h_1, \ldots, h_p)$ of the outputs. The regularity of the $A_{i,F}$ as a function of its arguments is for the moment unspecified. It will be clarified below.

Remark 1.1 We have been deliberately vague about the domains of validity of the statements in **P1**, **P2** and **P3**. Ideally, we would like these to hold on all of U. In the discussion which will immediately follow this remark, we will clarify when this is likely to be true. For instance, if F and the $h_i, i = 1, \ldots, p$ are all globally real analytic (i.e., we have succeeded in finding global real analytic feedbacks) we cannot still guarantee that the conditions which locally ensure the existence of C^ω germs $A_{i,F}$ to satisfy **P3** extend to a yield the global real-analyticity of these $A_{i,F}$. On the other hand, the reason for which we are demanding that **P3** be true will not require the global real-analyticity of these $A_{i,F}$. In fact, we could get by with weaker smoothness assumptions (as will be clarified below), in which case the chances of the global existence of the $A_{i,F}$'s increase dramatically. The point of the discussion below would be to argue that part a) of **P2** yields the best possible conclusions and at the same time leads to a tractable theory of solvability. So, the discussion below will clarify, to the extent possible, the situations where the global solvability of part a) of **P2** will lead to the global solvability of the other formulations. In seeking global solvability of these other formulations, we will be willing to sacrifice the regularity of the various objects involved, so long as they are adequate for the resolution of the associated control problems.

1.1 Making the case for part a) of P2

Now **P1** clearly is the dual of the standard formulations of the controlled invariance [at least when Δ can be written as $ker\ (dh_i), i = 1, \dots, p$] . It also implies parts a) and b) of **P2**. That it implies part a) is trivial. For part b) we just observe that if **P1** is valid, then by taking Lie derivatives with respect to the vector field F of the inclusion in **P1**, we get iteratively:

$$dL_F^k h_i(x) = \sum_{l=1}^{p} a_{li}^k(x) dh_l i = 1, \dots, p, k = 1, 2, \dots$$

which, of course, implies part b) of **P2**.

Now, the reverse implication (viz., **P2 a** implies **P1**) does not always hold. This is precisely the content of the so-called division theorems (see [7, 18, 19, 26]). They assert, under certain conditions, that the equality $dA \wedge dh_1 \wedge \dots \wedge dh_p = 0$, for germs of functions (smooth, real analytic, etc.) A and $h_i, i = 1, \dots, p$, implies that $dA(x) = \sum_{i=1}^{p} a_i(x) dh_i(x)$, for germs of (smooth, real analytic, etc.) functions, $a_i(x)$. These "certain conditions" are on the ideal generated by the $h_i, i = 1, \dots, p$ in the ring of (germs) of C^∞ (resp C^ω) functions. In [19], a global division theorem is also proved for the C^∞ case; i.e., if A and the $h_i, i = 1, \dots, p$ are globally defined C^∞ functions which satisfy the technical hypotheses for a local division theorem, then the functions $a_i(x)$ may be taken to be globally defined. Since this argument makes use of partitions of unity, it cannot be expected to hold in the C^ω case, in general.

Assuming that the division theorems hold for the $h_i, i = 1, \dots, p$, then part c) of **P2** follows from part b) of **P2**. This in turn implies that F preserves, for small t, the level sets of the $h_i, i = 1, \dots, p$ provided the Lie series for $h_i(\phi_F(t,x_0))$ converges. It is this convergence of the Lie series that leads to the resolution of the disturbance decoupling problem. Now if all the h_i are real analytic, then indeed F preserves the level sets of the h_i for all t for which the quantity $h_i(\phi_F(t,x_0))$ is well defined. If the division theorems are not valid globally, then these assertions are only valid for those initial conditions x_0 in the open set V where these division theorems hold. This point is pertinent since, as we stated before, it is unlikely that they hold globally in the C^ω case. On the other hand, real analyticity is typically required for any Lie series argument to be of any validity. In this regard, we note that if both the $h_i, i = 1, \ldots, p$ and F are real analytic on their common domains, then so are the $h_i(\phi_F(t,x_0))$, (see [27]). This issue makes the solvability of **P3** (globally) that much more pertinent.

Indeed, assuming that **P3** is globally solvable, let us show that F preserves the level sets of the h_i for all t for which the quantity $h_i(\phi_F(t,x_0))$ is defined (regardless of x_0). To see this, consider the initial value problem for the system of ordinary differential equations

$$\dot{y}_i = A_{i,F}(y_1, \ldots, y_p), i = 1, \ldots, p$$

with initial conditions $y_i(0) = h_i(x_0), i = 1, \ldots, p$.

It is clear that $h_i(\phi_F(t,x_0))$ satisfies this system. Assuming uniqueness of solutions yields the desired property. Thus, the only regularity property that we need impose on the $A_{i,F}$, as functions of the $h_i, i = 1, \ldots, p$ is that they be regular enough to guarantee uniqueness of the associated IVP.

Now, if part c) of **P2** is valid, then the $A_{i,F}, i = 1, \ldots, p$ certainly exist as set theoretical objects. The question of when they are C^∞ is the subject of the so-called composite differentiable functions theorems, [4, 11, 18, 19].

Let us briefly explain the contents of [4, 11] These papers give sufficient conditions for the global existence of C^∞ $A_{i,F}$, assuming (amongst other things) the real analyticity of the $h_i, i = 1, \ldots, p$. Unfortunately, the sufficient conditions they require are not immediately related to an easily verifiable condition like part a) of **P2**. Since we will illustrate how to use a condition like part a) of **P2** to arrive at the sufficient conditions of [11] in the penultimate section of this paper, we will not dwell on it here.

Remark 1.2

- If Ω is nonsingular on U, then **P1**, **P2** and **P3** are at least locally equivalent, even if the h_i are only C^∞ on U.

- In our opinion, **P3** (for sufficiently regular $A_{i,F}$) is at the basis of the appearance of the techniques of the disturbance decoupling problem in many synthesis problems which are not really related to it.

It seems reasonable, therefore, to consider these three versions to be equivalent (at least locally) under a fairly wide set of circumstances. However, as we saw, it is part a) of **P2** which is instrumental in leading to **P3**. It is also cast in the form of an equality (as opposed to **P1**—the dual of all the standard formulation of the controlled invariance problem). *On the basis of the above discussion, we will, therefore, concentrate our efforts on solving on part a) of* **P2** *in the remainder of this paper.*

The balance of the paper is organised as follows. In the next section we will detail certain results on differential forms which we will need for the analysis of the problem at hand. This section also contains a theorem of Malgrange ([18]) which is crucial for the results in this paper. In the third section we will obtain our analog of the *weak* (f, g)-invariance criterion. As the reader will notice, our analog *is an equality condition and not an inclusion condition.* In the next section we will illustrate, via the $m = 2$ case, how several inputs may be handled. Furthermore, in this section we recover, as examples, the standard results and also the theorem of [8] when $m = 1$.

The remainder of the paper concentrates on solving the controlled invariance problem for systems with partially smooth data when p is just 1. We will take f to be C^∞ and h, the sole output, to be C^ω. This is as weak as one can relax the smoothness assumptions when $p = 1$ if the results are to be of practical relevance. For this reason it is important to note that after a blow-up, any C^ω function is a monomial. There is, however, another reason for our considering only C^∞ f. We wish to obtain a result of the "Formal + Hyperbolic Implies Smooth" genre for certain degenerate and overdetermined systems of partial differential equations (which arise in connection with the controlled invariance problem). Monomials provide outputs which are annihilated by linear hyperbolic (with parameters) vector fields, thereby providing a relatively simple model for such questions and at the same time continuing to be of some practical relevance. During the course of obtaining our result we also develop a procedure for solving certain first order overdetermined systems of partial differential equations which degenerate, which is reminiscent of the method of successive integration. Mathematically speaking, this is the novel feature of this paper. We hope this method will be of independent interest. Our technique is partly inspired by Nelson's spectacular proof of the Sternberg linearization theorem (see [20]). The contents of this section should really be thought of as a method of obtaining C^∞ solutions to part a) of **P2**, once a formal solution is known (with the understanding that the additional hypotheses needed to make the method work will vary from case to case).

2 Preliminaries on Differential Forms

We assume that the reader is conversant with the basic facts about the Lie derivative, the interior product and the exterior derivative (see [1]). Let α be a k-form and β be a l-form. We will allow these forms to have, in local coordinate charts, coefficients that are meromorphic functions. More precisely, each of their coefficients can be a quotient of C^∞ or real-analytic functions. We will say two such p-forms are equal if their coefficients agree at every point that the two are well defined. Under this convention all the standard results relating the exterior, interior and Lie derivatives to wedge products continue to be valid. We assume that the reader is conversant with these basic facts (see [1]. We will only record the following not so well-known, though easily deduced, consequence of Cartan's magic formula ($L_X = d \circ i_X + i_X \circ d$):

Proposition 2.1 *[1] Let μ be a p-form(with meromorphic coefficients) and X, Y two smooth vector fields. Then we have:*

$$L_X(i_Y\mu) - L_Y(i_X\mu) - i_{[X,Y]}\mu = d(i_X(i_Y\mu)) - i_X(i_Y(d\mu)).$$

Next we turn to the result of Malgrange, alluded to in the introduction.

Theorem 2.1 *([18]) Let $h_1, \ldots, h_p$ be germs of p holomorphic functions at the origin in C^n. Let $S(dh_1, \ldots, dh_p)$ be the germ of the analytic set where the rank of the codistribution spanned, over the ring of germs of holomorphic functions, by the $dh_1, \ldots, dh_p$ is less than p. Assume that the codimension of $S(dh_1, \ldots, dh_p)$ is at least 3. Let θ be the germ of a holomorphic 1-form which satisfies the condition*

$$d\theta \wedge dh_1 \wedge \ldots dh_p = 0.$$

Then there exists a germ of a holomorphic function α at so that

$$(\theta - d\alpha) \wedge dh_1 \wedge \ldots \wedge dh_p = 0.$$

It is useful to make the following definition:

Definition 2.1 *The real analytic functions $h_i, i = 1, \ldots, p$, defined on some connected open subset U containing the origin in R^n, are* **good outputs** *if their complexifications, $h_i^C, i = 1, \ldots, p$, have the property that the set $S(\Omega)$, consisting of points where the holomorphic p-form $dh_1^C \wedge \ldots \wedge dh_p^C$ vanishes, has codimension at least three in U^C, the complexification of the set U.*

Several remarks are in order regarding Theorem (2.1):

a) Malgrange's result is a local existence theorem in the holomorphic category. However, the usual practice of complexification enables us to deduce the existence of real valued solutions when the data is real analytic provided the $h_i, i = 1, \ldots, p$ are good outputs. The texts [15, 17] contain readable expositions of complexification. For explicit statements in the smooth category on a related problem (see also (c) below) we refer the reader to [19]. We will not state these results here, since it is not our purpose to obtain results of the utmost generality on **P2**. We only wish to illustrate, via applying Theorem (2.1), the utility of these ideas to nonlinear control problems. We also direct the attention of the reader to the material in [2] which could be construed as solving the controlled invariance problem for outputs which are multivalued (in the sense of complex variables). These authors also relate our problem to the 16th problem of Hilbert.

b) It is important to notice that the form θ is required to be holomorphic. This may, in our applications, be a little excessive. It is this author's belief that a result analogous to Theorem (2.1) would hold if only some components of this form were holomorphic. To be more precise, we conjecture that these results would still hold even if only $\theta \wedge dh_1 \wedge \ldots dh_p$ is the germ of a holomorphic form. The main step towards proving such a theorem would be to extend Saito's generalisation of the De-Rham division lemma to partially meromorphic forms, since his generalisation is a crucial ingredient in the proof of Theorem (2.1), see [26]. We can, however, prove such a result if the domain over which the data is prescribed has certain desirable attributes (see Proposition (3.1) later in the paper).

c) Malgrange and Moussu ([18, 19]) study a similar and related problem. They consider the generalization of the Frobenius' theorem to the case when the forms in question have singularities. More precisely, consider a collection of the germs at 0 of p holomorphic 1-forms $\omega_i, i = 1, \ldots, p$ which also satisfy the integrability condition $d\omega_l \wedge \omega_1 \wedge \ldots \omega_p = 0, l = 1, \ldots, p$. Then, under the hypothesis that the codimension of the set where the codistribution spanned by the $\omega_i, i = 1, \ldots, p$ has rank less than p is at least 3, Malgrange shows that there exist germs of holomorphic functions $f_i, g_{ik}, i = 1, \ldots, p; k = 1, \ldots, p$ so that $\omega_i = \sum_k g_{ik} df_k i = 1, \ldots, p$. Furthermore $det(g_{ik}(0)) \neq 0$ and $df_i(0) = 0$ for all $i = 1, \ldots, p$. Moussu also proves similar results in the smooth category. The connection between this result and our problem is the following. Let θ be the germ of a holomorphic 1-form satisfying $d\theta \wedge dh_1 \wedge \ldots dh_p = 0$. Now on the space coordinatized by the dependent and independent variables (x, α) (with the exterior derivative on this space denoted by d_e) consider the collection of germs of one form $\{\theta - d\alpha, dh_1, \ldots, dh_p\}$. Note that $d_e(\theta - d\alpha) = d\theta$. Therefore the integrability conditions $d_e\omega_l \wedge \omega_1 \wedge \ldots \omega_p = 0$ follow from the condition $d\theta \wedge dh_1 \wedge \ldots dh_p = 0$. Clearly the codimension of the singular set of these 1-forms is the same as the codimension of the singular set of the collection $dh_1, \ldots, dh_p$ as a set in the space coordinatized by the x's alone. Thus if this codimension is at least 3, then we can find first integrals for this system—in particular for $\theta - d\alpha$. Since the "α" coefficient of this form is non-zero, we can apply the implicit function theorem to its first integral to conclude the existence of a solution to $(\theta - dU) \wedge dh_1 \wedge \ldots \wedge dh_p = 0$. For a related but different notion of integrability we refer the reader to the paper [10].

We next discuss two theorems which relate to the contents of the final three sections of the paper. (It is presumed that the reader is familiar with the concept of flat functions and Borel's lemma—see, for instance, [12]). The first is a result of Glaesser [11] on composed functions.

Theorem 2.2 *[11] Consider a p-tuple of real analytic functions $h_1, \ldots, h_p$, defined on an open neighborhood, U, of the origin in R^n, such that:*

1. *The codistribution spanned by their differentials has maximal rank p on an open dense subset of U.*
2. *The map $h = (h_1, \ldots, h_p)$ from U to R^p is semi-proper; i.e., it satisfies: a) its image is closed; and b) for every compact subset K in its range there is a compact subset, L, of U so that $K = h(L)$.*

Let $A \in C^\infty(U)$ be such that it satisfies the "appartenance biponctuelle" condition; i.e., for every set, S, consisting of two points in U there exists a C^∞ function F_S on U such that the function $A - F_S \circ h$ is a flat function on S. Then, there exists a C^∞ function F satisfying $A = F \circ h$ on all of U.

Remark 2.1

- We can dispense with first hypothesis, namely the density of the set of regular points, if U is connected and if $dh_1 \wedge \ldots \wedge dh_p$ is non-zero at at least one point of U - **something which we will assume from this point onwards**. This follows from the the real analyticity of the map h.
- All proper maps are semi-proper. So are monomials $\Pi_{i=1}^n x_i^{p_i}, p_i \geq 0$, even if some of the p_i are zero.

Remark 2.2 As is pointed out in the "remarque" at the end of Section 2 of [11] one can replace the "appartenance biponctuelle" condition for the set $\{x, y\}$ by the verification of the existence of functions F_x and F_y, smooth in neighborhoods of $h(x)$ and $h(y)$ respectively, so that $A - F_x \circ h$ (resp $A - F_y \circ h$) is a flat function on $\{x\}$ (resp. $\{y\}$), if $h(x) \neq h(y)$. One may not be able to do this if $h(x) = h(y)$, because one cannot guarantee that $F_x = F_y$ (See, in this regard the discussion on Page 143, and also section 4 of [3]). In our situation, however, this is not a problem because we will use a condition of the type $dA \wedge dh = 0$ to obtain these functions F_x, F_y. We will do this by first setting up an explicit formal power series solution and then invoking Borel's lemma. If $h_x = h_y$, it will be obvious that the formal powers series (this will be a formal power series in the indeterminate h) which solves $dA \wedge dh = 0$ about x will be identical to the one about the point y.

We end this section with the following version of Hironaka's resolution of singularities theorem (see [5]).

Theorem 2.3 *([5], Corollary 4.9) Let M be a real analytic manifold. Let h be a real analytic function on M, which is not identically zero on any component of M. Then there is a real analytic manifold N and a proper, surjective real analytic map $\pi : N \to M$ so that: a) $h \circ \pi$ is locally normal crossings on N; i.e., about each point $p \in N$, there is a coordinate system $(U_p, x_1, \ldots, x_n)$ so that on U_p, $h \circ \pi$ is of the form $t(x)\Pi_{i=1}^{n} x_i^{n_i}$, where the $n_i, i = 1, \ldots, n$ are all non-negative integers and $t(x)$ is real-analytic and non-vanishing on all of U_p; and b) π is a real-analytic diffeomorphism on an open dense subset of N.*

In [28] a simultaneous desingularization theorem is proved for a finite collection of real-analytic functions.

Remark 2.3 The version of Theorem (2.3) we will use will have M replaced by an open neighborhood of the origin in R^n. Furthermore we, will take the origin to be a critical point of the function h, since otherwise one does not need to blow up to get a locally normal crossings model. Also we may take π to be a real analytic diffeomorphism on an open dense subset whose complement contains $\pi^{-1}(0)$. For a further simplification in the statement see Theorem (7.1) later.

Remark 2.4 The factor $t(x)$ in Th (2.3) can be taken to be 1, as was pointed out to us by Prof. H. Sussmann. As ought to be clear, we may assume that $t(x) > 0$ for our purposes. If it is not, we will work with $-dh$. Indeed, if we keep in mind that the purpose of using Th (2.3) is to pull dh back via π^* and then work with the total differential of a simpler C^ω function, we see that:

$$\pi^*(-dh) == -d(h \circ \pi) = d(-h \circ \pi).$$

So, now assuming $n_1 > 0$ without loss of generality, we make a change of coordinates according to:

$$\psi_1 = t^{\frac{1}{n_1}}(x)x_1, \psi_k = x_k, k = 2, \ldots, n.$$

This is a real analytic change of coordinates, which renders $t(x) = 1$ in the new set of coordinates.

We mention, in passing, that resolution of singularities was suggested as a tool for problems in geometric control in [6]. The approach taken there, however, differs from that in this paper significantly.

3 The Analog of the Weak (f, g)-Invariance Condition

In this section, we will obtain a necessary condition for the solvability of **P2**. We will assume that (after a preliminary feedback has been applied) $dL_g h_i \wedge dh_1 \wedge \ldots \wedge dh_p = 0, i = 1, \ldots, p$. We will not carry out the construction of such a feedback here, in order to keep the length of this paper within reasonable limits. We just remark that construction of β is usually easier and typically goes through under weaker regularity hypotheses than those for α (see [25]).

Consider, therefore, the problem of finding $\alpha(x)$ so that $dL_F h_i \wedge dh_1 \wedge \ldots \wedge dh_p = 0, i = 1, \ldots, p$, where $F(x) = f(x) + g(x)\alpha(x)$. The last equation reads as:

$$(dL_f h_i + \alpha dL_g h_i + L_g h_i d\alpha) \wedge dh_1 \wedge \ldots \wedge dh_p = 0, i = 1, \ldots, p,$$

which yields:

$$(dL_f h_i + L_g h_i d\alpha) \wedge dh_1 \wedge \ldots \wedge dh_p = 0 i = 1, \ldots, p. \tag{3.1}$$

If we formally divide by $L_g h_i$ we have the following problem to solve:

P: Find a smooth function $\alpha(x)$ in a neighborhood of the operating point (assumed to be the origin) so that :

$$(-\frac{dL_f h_i}{L_g h_i} - d\alpha) \wedge dh_1 \wedge dh_2 \ldots \wedge dh_p. = 0 i = 1, \ldots, p \tag{3.2}$$

For **P** to make sense we have to make some smoothness assumptions on the forms $\theta_i = -\frac{dL_f h_i}{L_g h_i}, i = 1, \ldots, p$. There are either of two conditions which we will impose from this point onwards:

S1 At least one of the forms $\theta_i, i = 1, \ldots, p$ are smooth in a neighborhood U of the operating point.

S2 The forms $\theta_i \wedge dh_i \wedge \ldots \wedge dh_p, i = 1, \ldots, p$ are smooth on a neighborhood U of the operating point.

Remark 3.1 There is a disparity of sorts between **S1** and **S2** in that we require only one of the forms in the former to satisfy the prescribed condition whereas in the latter all the forms have to satisfy the imposed condition. As the reader will notice, later we only require **S2** for the results to go through. **S1** will serve only the purposes of a comparison, to be made in the sequel, with the standard formulation. On the other hand, in the light of (3.4) below, we could even afford to demand that **S2** be only satisfied *a priori* by one of the forms, θ_i, since it will then be satisfied by the rest as well.

We seek α on a neighborhood $V \subseteq U$. Note that as long as α exists on this subset V, **P2** will have a solution on V. Indeed, the only points where the $\theta_i, i = 1, \ldots, p$ could conceivably fail to be smooth are the points where $L_g h_i = 0, i = 1, \ldots, p$. But if α exists at these points, then the equation in **P2** holds there. Indeed, under either **S1** or **S2** we must have $dL_f h_i \wedge dh_1 \ldots \wedge dh_p = 0, i = 1, \ldots, p$ at points where $L_g h_i = 0, i = 1, \ldots, p$.

If we take the exterior derivative of both sides of (3.2), we arrive at the following necessary condition for the solvability of the above problem:

$$d\theta_i \wedge dh_1 \ldots \wedge dh_p = 0, i = 1, \ldots, p. \tag{3.3}$$

However a more careful analysis shows that there is a stronger condition which implies (3.3), and which is also necessary for the solvability of the problem P. Indeed, if there exists an $\alpha(x)$ which solves P, then we necessarily have $(\theta_1 - \theta_i) \wedge dh_1 \ldots \wedge dh_p = 0$ for $i = 1, \ldots, p$. If now in addition (3.3) is satisfied *a priori* only for $i = 1$, then exterior differentiation of the preceding equality shows that (3.3) holds for the remaining indices as well. *Furthermore, it is easy to see that if this condition holds and* P *is solvable for just $i = 1$, then its solution α solves the remaining equations in* P *as well.* We, therefore, take these two conditions as our analogue of the weak (f, g)-invariance condition, viz:

$$\begin{aligned} d\theta_1 \wedge dh_1 \ldots \wedge dh_p &= 0 \\ (\theta_1 - \theta_i) \wedge dh_1 \ldots \wedge dh_p &= 0, i = 1, \ldots, p. \end{aligned} \tag{3.4}$$

Remark 3.2

a) If one writes out every component of the equality expressed by P, then an overdetermined system of partial differential equations for the unknown α is obtained. The important thing to observe about this system is that it is obtained *without having to pair down* from Ω (cf. section 1). Once Ω is given, these equations are extremely simple to describe.

b) The necessary conditions (3.4) are equalities and *not inclusion conditions.*

c) When $p = 1$ (3.4) always holds. The first equality holds due to the assumption $dL_g h \wedge dh = 0$, and the second for trivial reasons.

Next we will relate P to an equivalent problem which uses the set of vector fields which annihilate each of the $dh_i, i = 1, \ldots, p$. We assume such a distribution exists for the moment. If such a distribution exists only locally, then the results below are valid only on its domain of existence. If it does not exist at all, then this circle of ideas is irrelevant. So, let $X_i, i \in I$ (where I is some indexing set) be the collection of all vector fields which satisfy $L_{X_i}h_j = i_{X_i}dh_j = 0, i \in I, j = 1, \ldots, p$. If Δ denotes this collection, then it is certainly involutive. *We further assume that Δ has maximal rank p on a dense subset of U.* Denote by c_i the smooth functions defined as $i_{X_i}\theta_1$ (the smoothness follows from **S2** and the formula for the interior product of a vector field and a wedge product). *We claim that the solution, if one exists, to the overdetermined system of partial differential equations*:

$$X_i\alpha = c_i, i \in I \tag{3.5}$$

is the desired feedback.

Indeed, a solution to (3.5) satisfies

$$i_{X_i}(\theta_1 - d(\alpha)) = 0, i \in I.$$

This follows from Cartan's magic formula. Now the $h_l, l = 1, \ldots, p$ being analytic, Ω has maximal rank equal to p on a dense set in U. Therefore, the distribution spanned by the X_i has maximal rank equal to $n - p$ on a dense subset. This implies that $\theta - d\alpha$ is in the span of the $dh_i, i = 1, \ldots, p$ (possibly with meromorphic coefficients) on this dense subset. Hence its wedge product with $dh_1 \wedge \ldots \wedge dh_p$ equals 0 on the same dense set. But the wedge product being 0 on a dense subset implies its being 0 on all of U since it is, after all, a system of equalities amongst everywhere defined continuous functions.

Remark 3.3 Observe that the second condition in (3.4) also yields the equality:

$$i_{X_i}\theta_j = i_{X_i}\theta_k, i \in I, j, k = 1, \ldots, p.$$

This follows from taking the interior product of both sides of the second equation in (3.4) with respect to the $X_i, i \in I$. In the context of (3.5) this means that there is no ambiguity in choosing h_1 in writing down the system of partial differential equations (3.5). Any other choice of output would yield exactly the same set of equations.

Next, we will show that the first equation of (3.4), provides the integrability conditions for the overdetermined system of partial differential equations (3.5). For purposes of brevity let us denote, by σ, the p-form $dh_1 \wedge \ldots \wedge dh_p$. Let us apply Proposition (2.1) to the $p+1$-form $\theta_1 \wedge \sigma$. Doing so yields, as a first step:

$$(L_{X_k}c_l - L_{X_l}c_k - i_{[X_k, X_l]}\theta_1)\sigma = d[(i_{X_k}i_{X_l}\theta_1) \wedge \sigma) - i_{X_k}i_{X_l}(d\theta_1 \wedge \sigma).$$

In arriving at the left hand side we have made use of Leibnitz formulae for the interior product and the fact that all the $X_i, i \in I$ annihilate the $dh_j, j = 1, \ldots, p$. The second term on the right hand side of the equation above vanishes on account of (3.4). The first term also equals 0, since $i_{X_l} i_{X_k}(\zeta) = 0$ for any 1-form, ζ. Thus we get:

$$L_{X_k} c_l - L_{X_l} c_k - i_{[X_k, X_l]}\theta_1 = 0, \; k, l \in I$$

on the dense subset where $\sigma \neq 0$, and hence everywhere. These are indeed the integrability conditions for the system (3.5).

We now have all the tools we need to prove a slightly improved version of Theorem (2.1).

Proposition 3.1 *Let $h_1, \ldots, h_p$ be p real analytic functions defined on a connected open subset U of R^n whose complexification, U^C, is a Stein manifold in C^n. Let $S(dh_1, \ldots, dh_p)$ be the analytic set where the rank of the codistribution spanned, over the ring of germs of holomorphic functions, by the complexifications of the $dh_1, \ldots, dh_p$ is less than p. Assume that the codimension of $S(dh_1, \ldots, dh_p)$ in U^C is at least 3. Let θ be a real analytic 1-form which satisfies the condition*

$$d\theta \wedge dh_1 \wedge \ldots dh_p = 0.$$

Then there exists a real analytic function α defined on all of U so that

$$(\theta - d\alpha) \wedge dh_1 \wedge \ldots \wedge dh_p = 0.$$

Proof: The linearity of the question allows us to complexify. Let us denote by O the set $U^C \backslash S(\Omega)$ and cover it by open subsets $O_i, i \in I$, over which the systems (3.5) corresponding to the data at hand are solvable (with solutions denoted by α_i). Now since the $h_l, l = 1, \ldots, p$ and their complexifications $h_l^C, l = 1, \ldots, p$ are globally defined, we have the following equality on $O_{ij} = O_i \cap O_j$:

$$d(\alpha_i - \alpha_j) \wedge dh_1^C \wedge \ldots \wedge dh_p^C = 0$$

for each pair of indices $i, j \in I$ for which O_{ij} is non-empty. By the non-singularity of the complexification of Ω on O, we conclude that for such indices we have

$$\alpha_i - \alpha_j = t_{ij}(h_1^C, \ldots, h_p^C)$$

for some function t_{ij} which is holomorphic on O_{ij}. Since on triple overlaps

$$\alpha_i - \alpha_j + \alpha_j - \alpha_k + \alpha_k - \alpha_i = 0,$$

the identity $t_{ij} + t_{jk} + t_{ki} = 0$ follows. Now on the complements of analytic subsets of Stein spaces which have codimension at least 3, Cousin's Problem A is solvable ([14], p. 138). We deduce that there exist functions t_i which are holomorphic on the O_i for $i \in I$ such that $t_{ij} = t_i - t_j$.

Furthermore, an examination of the proof on page 132 of [14] reveals that the t_i are functions of $h_1^C, \ldots, h_p^C$. Indeed, the functions t_i are constructed in page 132 of [14] out of the Taylor series expansions of the functions t_{ij}. Since the t_{ij} are dependent only on $h_i, i = 1, \ldots, p$, and on O the map $h^C = (h_1^C, \ldots, h_p^C)$ has full rank, it is easy to see that the functions t_i are also functions only of h.

Now, defining $\hat{\alpha_i}$ as $\alpha_i - t_i$, we see that the $\hat{\alpha_i}$ patch up to yield a function which is holomorphic on all of O and also satisfies $(\theta^C - d\alpha) \wedge dh_1^C \wedge \ldots \wedge dh_p^C = 0$ on the subset O. Since the complement of O is an at least codimension 3 subset in U^C, the holomorphic function constructed above can be extended holomorphically to all of U^C to yield a function, again denoted by α, which by density satisfies the equation $(\theta^C - d\alpha) \wedge dh_1^C \wedge \ldots \wedge dh_p^C = 0$ on U^C. This last extension result may be found in standard several complex variables texts (e.g., [14]). This completes the proof of the proposition.

We make the following remarks regarding Proposition (3.1):

- R^n itself satisfies the requirement on U.
- The proof above contains a global result which depends on two crucial assumptions : a) the analyticity of the data at hand; b) the fact that the $h_i, i = 1, \ldots, p$ are globally defined. Whilst a) could perhaps be circumvented under special circumstances for C^∞ data (e.g., the extension past the singularities could fail without additional hypotheses), b) cannot. It is here that the global results of [9] are beyond the scope of our techniques. On the other hand, it is reasonable to assume the global existence of the $h_i, i = 1, \ldots, p$ in applications.
- We still believe that this version is valid for germs of the $h_i, i = 1, \ldots, p$; i.e., we need not assume that U^C is Stein—the method of the proof used here will, of course, not work in this case.
- In [18] Malgrange proves the solvability (in the holomorphic category) even if $CodS(\Omega) = 2$ provided θ satisfies **S1**, and **P** is *known to be formally solvable.* We cannot improve this result to the case that only $\theta \wedge dh^C$ is holomorphic (at least via the techniques of Proposition (3.1)), since we do not know if Cousin's Problem A is solvable on domains which are the complements of codimension 2 subsets of Stein manifolds.

The global problem considered in Proposition (3.1) is called a Cousin A problem in the several complex variables literature (see, for instance, [14]). The paper [[2]] also proves a "local implies global" result by solving a Cousin A problem. However, the hypotheses and the conclusions are different from ours.

Example A - The Nonsingular Case: We first treat the results from the standard formulation of the controlled invariance problem. Recall the regularity assumptions from the introductory section regarding the distributions Δ and G. Since $m = 1$, the second regularity condition translates to $G \cap \Delta = 0$; i.e, $L_g h_i \neq 0, i = 1, \ldots, p$. Let Ω be the annihilator of Δ. By the Frobenius theorem we may assume that there exists a local coordinate system in which $h_i(x) = x_i, i = 1, \ldots, p$. Since $L_g h_i \neq 0, i = 1, \ldots, p$ the corresponding $\theta_i, i = 1, \ldots, p$ satisfy the assumption **S1**. *The assumptions $G \cap \Delta = 0$ is thus the strongest requirement in order that* **S1** *be valid.* We assume that (3.4) holds. Clearly, then **P** is solvable. If the reader wishes he may appeal to (3.5) to see this.

Example B: In this example we will show that one can recover the results of ([8]) on the solvability of the controlled invariance problem in the presence of singularities in the inputs. Of course, [[8]] obtains results even when $m \neq 1$. The most general result of this type can be found in [13]. More precisely, the situation considered is that of a smooth nonlinear control system, and an involutive, *nonsingular* distribution Δ which the authors seek to render controlled invariant locally via feedback. They make the following assumption:

$$[f, X_i] = V_i + c^i(x)g(x), i = 1, \ldots, n - p. \tag{3.6}$$

Here p is the codimension of Δ and the $V_i, i = 1, \ldots, n-p$ are vector fields which take values in Δ, and most importantly *the $c_i(x), i = 1, \ldots, n - p$ are C^∞ functions on U.* They do *not* make the standard assumption that $G \cap \Delta = 0$. Under the assumption (3.6), they demonstrate the existence of a smooth feedback $\alpha(x)$ which causes the closed loop vector field $f(x) + \alpha(x)g(x)$ to leave Δ invariant. We will now recover their result using the formalism of this section.

To that end, let Ω be the annihilator of Δ. Since Δ is nonsingular and involutive, we may assume that there exists a system of coordinates under which the $h_i = x_i, i = 1, \ldots, p$. In the same coordinates the coordinate vector fields $\frac{\partial}{\partial x_j}, j = p+1, \ldots, n$ constitute, visibly, a basis for Δ. Denote by $f_i, g_i, i = 1, \ldots, n$ the components of f and g respectively in these coordinates. We then have:

$$\theta_i = \sum_{j=1}^{n} \frac{\frac{\partial f_i}{\partial x_j}}{g_i} dx_j i = 1, \ldots, p.$$

Using the formula $[X, Y] = (DX).Y - (DY).X$ for the Lie bracket of two vector fields X and Y to express (3.6) in coordinates shows that $i_{X_k}\theta_l = c^k(x),\ k = p+1, \ldots, n,\ l = 1, \ldots, p$. *Thus we see that (3.6) yields, in addition to the integrability conditions for (3.5), the fact that the $\theta_i, i = 1, \ldots, p$ satisfy* **S2**. Thus the system of partial differential equations, (3.5), corresponding to this situation can indeed be solved. This yields the desired conclusion.

Before proceeding to the case of a singular Ω, we wish to illustrate by means of two examples that it is possible that both **S1** holds and fails, and yet in both situations **P** is solvable.

Example C: Let $h(x) = x_1x_2x_3,\, x \in R^3$. Let the control vector field be $g(x) = \frac{1}{3}(x_1, x_2, x_3)^t$ and the drift vector field, $f(x)$ equal $(x_1^2x_2x_3^2, 0, 0)^t$. Then $L_gh = h$, so that $dL_gh \wedge dh = 0$. However, $dL_fh \wedge dh \neq 0$. In fact, $L_fh = h^2x_3$, so that $dL_fh = 2hx_3dh + h^2dx_3$. Therefore, $\theta_1 = -(x_3dh + hdx_3)$, which equals $-d(hx_3)$. Hence, **S1** is valid. Trivially, (3.4) holds, and a solution to **P** is $\alpha(x) = -hx_3$.

Now consider, the same data except for the drift, $f(x)$, which is now given by the vector field $(x_1^2x_2, x_2^2, x_3)^t$. Now, the differential form θ_1 equals $-[(4x_2 + x_1^{-1})dx_1 + (4x_1 + x_2^{-1})dx_2 + (\frac{2(x_1x_2+1}{x_3})dx_3]$. Clearly, **S1** does not hold. However, $\theta_1 \wedge dh$ is smooth as can be easily seen. Hence, **S2** is valid. In addition, (3.4) holds and a solution, $\alpha(x)$, to **P** is $-2x_1x_2$.

We now analyse the singular case under the assumption of real analyticity of all data.

The Analytic Case: We now allow Ω to become singular. Let us make the following assumptions about the data of the problem:

1. The 1-forms $\theta_i \wedge dh_1 \wedge \ldots \wedge dh_p, i = 1, \ldots, p$ are real-analytic on a connected open subset U of R^n.

2. The $h_i, 1 = 1, \ldots, p$ are all good outputs on a connected open subset U whose complexification is a Stein manifold.

3. Equation (3.4) is satisfied by the data of the problem.

We then have:

Proposition 3.2 *Under the above assumptions there exists a real analytic feedback $\alpha(x)$ on U under which the closed loop system corresponding to the control system (1.1) leaves Ω controlled invariant.*

Proof: The result is a consequence of Proposition (3.1) and complexification. Of course, if U's complexification is not a Stein manifold, then we can prove the local existence of such a feedback if the $\theta_i, i = 1, \ldots, p$ are themselves real analytic.

4 The Case of Multiple Inputs

We will briefly illustrate how all of the foregoing analysis can be extended to the case of multiple inputs if $p \geq m$. We will display the intervening calculations for the case $p = 3,\ m = 2$.

We form $m \times C_m^p$ one forms in the following fashion. We first take h_1 and h_2 together and go through the steps in the single-input case. More precisely, we define a pair of one forms:

$$(\theta_1^{12}, \theta_2^{12})^{tr} = A^{-1}(x)(-dL_f h_1, -dL_f h_2)^{tr} \tag{4.1}$$

where, the matrix $A(x)$ has for its ijth element the quantity $L_{g_j} h_i, i, j = 1, 2$.

By considering h_2, h_3 together we form, likewise, a second pair of one forms denoted $(\theta_1^{23}, \theta_2^{23})^{tr}$. Similarly we form a pair of one forms $(\theta_1^{31}, \theta_2^{31})^{tr}$. Of course, all these forms will typically be meromorphic. We can demand that they either satisfy **S1** or **S2**. The α_i can then be found by solving, say,

$$(\theta_i^{12} - d\alpha_i) \wedge dh_1 \wedge \ldots \wedge dh_p = 0, i = 1, 2. \tag{4.2}$$

The version of the equations (3.4) now become:

$$\begin{aligned} d\theta_l^{12} \wedge dh_1 \ldots \wedge dh_p &= 0 \\ (\theta_l^{12} - \theta_l^{rs}) \wedge dh_1 \ldots \wedge dh_p &= 0, l = 1, 2 \end{aligned} \tag{4.3}$$

for each index rs from the set $\{(23), (31)\}$.

We may now apply the reasoning of the previous sections to the above system of equations - see [25] for the exact technical hypotheses needed.

5 Outline for Partially Smooth Data

In this section we will outline the structure of the proof of the solvability of **P** (when $p = 1$) when f and g are only C^∞, so that $\theta = \theta_1 \wedge dh_1$ is just C^∞. We take $h = h_1$ to be a good output defined on an open subset U of R^n with a Stein complexification U^C. The solution to **P** will then be carried out in the following steps:

Step 1: Since θ satisfies **S2** and h is a good output there exists a formal solution to **P** about each point of U. After choosing a operating point, call this formal solution γ. By Remark (2.4) h is locally normal crossings. Consider the effect of the map π^* on the equation $(\theta - d\gamma) \wedge dh = 0$. Denoting, by abuse of notation, $\pi^*(\theta)$, also by θ we see that we have a formal solution to the equation $(\theta - d\alpha) \wedge d(\Pi_{i=1}^n x_i^{p_i}) = 0$ about each point of N. For use in the last section, we denote by $\bar{\theta}$, the object $\theta - d(\bar{\gamma})$, where $\bar{\gamma}$ is the smooth function whose Taylor jet is the series γ. Thus, $\bar{\theta} \wedge dh$ has all its components flat at the origin.

Step 2: In the next section we will construct a C^∞ solution to the equation $(\theta - d\alpha) \wedge d(\Pi_{i=1}^n x_i^{p_i}) = 0$. We need only consider the case where the functions c^i in the system (3.5) corresponding to this data all vanish to infinite order at the origin. However, we will also briefly examine the formal solvability of these equations because we need a slightly stronger property than the vanishing to infinite order at the origin of the c^i.

Remark 5.1 We leave it to the reader to verify the fact that we may take the p_i all to be relatively prime. Clearly if **Step 2** can be executed in this case, then so can it in the general case.

As a matter of fact, we will need the p_i to be relatively prime only for the next step. More precisely, we will need it to be able to conclude the "appartenance biponctuelle" condition of Theorem (2.2) from a condition like $dA \wedge dh = 0$, where A is a C^∞ function obtained from the C^∞ functions of **Step 2**.

Step 3: The smooth solution obtained in the step above will have to be blown down. We will carry this out in the last section.

6 Successive Integration and the Monomial Case

Let $h = \Pi_{i=1}^n x_i^{p_i}$. Assume that the first r of the p_i's are not zero and the remaining are. One has to then solve the following system of equations:

$$X_l \alpha = p_l x_1 \frac{\partial \alpha}{\partial x_1} - p_1 x_l \frac{\partial \alpha}{\partial x_l} = c^l = i_{X_l}\theta, l = 2, \ldots, r \tag{6.1}$$

along with

$$\frac{\partial \alpha}{\partial x_l} = c^l = i_{X_l}\theta, l = r+1, \ldots, n. \tag{6.2}$$

This follows either from writing down directly the equations (3.2), or by using the equations (3.5) and noticing that the X_l form a basis for the vector fields which annihilate dH. We first claim that we may suppose that $r = n$. This is a special case of the following theorem(see [24]):

Theorem 6.1 *Suppose that (6.1) has a C^∞ solution when one sets $(x_{r+1}, \ldots, x_n) = (0, \ldots, 0)$. Then the entire system of equations (6.1) and (6.2) also has a C^∞ solution. Furthermore, if the c^i vanish to infinite order at the origin, and the solution to the corresponding (6.1) with $(x_{r+1}, \ldots, x_n)$ $= (0, \ldots, 0)$ also is flat at the origin, then so does the solution to the complete system (6.1) and (6.2).*

Therefore, from this point onwards, we shall concentrate our energies on (6.1) with $r = n$.

In the subsections which follow, we will construct the solution to the system (3.5), corresponding to H, as the sum of three smooth functions $U(x)$, $V(x)$ and $W(x)$.

6.1 Construction of U(x)

We *define $U(x)$ to be the C^∞ function whose Taylor jet equals the formal power series solution to (6.1) (with $r = n$).* The formal solvability, of course, follows from the hypotheses that θ satisfies the condition **S1** and Theorem (2.1). The existence of $U(x)$ then follows from Borel's lemma.

Let $m = (m_1, \ldots, m_n)$ be a multi-index of non-negative integers. Associate to the vector field $X_i, i = 1, \ldots, n$ and every multi-index m the quantity $P^i(m), i = 1, \ldots, n$ defined to be the sum $p_1 m_1 - p_i m_i$. It is easily seen that if $A(x_1, \ldots, x_n) = \sum_m A_m x^m$ is any formal power series then

$$X_i A(x) = \sum_m A_m P^i(m) x^m. \tag{6.3}$$

The reader should use this fact and the formal solvability of (6.1) to set up a tangible formal power series solution. We will not carry out the details here for reasons of brevity. Doing so, however, is very useful to understand the motivation behind the steps of the remaining two subsections. Notice that the formal power series solution obtained this way will differ from any other formal solution only by a formal first integral of all the $X_i, i = 2, \ldots, n$. One can use (6.3) and the last observation to ascertain:

Remark 6.1 Let us denote by d^i the functions $c^i - X_i U, i = 2, \ldots, n$. Then it is easy to see from the structure of the formal power series solution that the d^i vanish to infinite order with respect to $(x_1, x_i), i = 2, \ldots, n$ at $(x_1, x_i) = (0, 0)$. This will be of importance in the subsequent subsection.

6.2 Construction of the function $V(x)$

For later use we observe that the flows of the vector fields $X_i, i = 2, \ldots, n$ denoted $\phi_i(t, x), i = 1, \ldots, n$, consist of diffeomorphisms (denoted ϕ^i_t) defined for all t.

We first form the auxiliary system of equations

$$X_i v = d^i, i = 2, \ldots, n \tag{6.4}$$

for the unknown function v.

Let us first observe that the integrability conditions for the system (6.4), namely $X_k d^l = X_l d^k, k, l = 2, \ldots, n$, follow from those for the system (6.1). The function $V(x)$ will now be concocted out of the sum of $n-1$ integrals $I_i, i = 2, \ldots, n$.

Let $I_2 = -\int_0^\infty d^2(\phi^2_t(x))dt$. I_2 is well defined if $x_1 = 0$. Indeed, since d^2 vanishes at the origin, we have the estimate:

$$\mid d^2(x) \mid \leq K \mid x \mid \tag{6.5}$$

and since $x_1 = 0$ this, in turn, implies that for $x_1 = 0$

$$| I_2 | \leq KL \int_0^{\infty} e^{-p_1 t} | x | dt \tag{6.6}$$

where L is another constant. The latter estimate shows that I_2 is well defined if $x_1 = 0$. Furthermore, since d^2 vanishes to infinite order with respect to (x_1, x_2) whenever $(x_1, x_2) = (0, 0)$ we have estimates similar to (6.5) for derivatives of all possible orders of d^2. These yield estimates similar to (6.6) and justify differentiation under the integral sign, and thereby, shows that I_2 is actually a C^{∞} function.

We will next show that I_2 actually satisfies $X_2 I_2 = d^2$ on the hyperplane $\{x_1 = 0\}$. We have $X_2 I_2 = -\int_0^{\infty} X_2 d^2(\phi_t^2(x)dt$. (Notice that we have already justified differentiation under the integral sign.) We will now use the following facts (see [1]):

Proposition 6.1

1. *For any diffeomorphism ϕ, any smooth vector field Y and any smooth function y we have:*

$$\phi^* L_X y(x) = L_{(\phi^*)X}(\phi^* y(x)). \tag{6.7}$$

2. *Let X, Y be two smooth vector fields with respective(local) flows ϕ_t and ψ_t. Then $[X, Y] = 0$ iff a) $\phi_t^* Y = Y$ and b) $\psi_t^* X = X$.*

3. *Let X be a vector field and ϕ_t be its (local) flow. Let y be a smooth function. Then:*

$$\frac{d}{dt}(\phi_t^* y) = \phi_t^* L_X y(x). \tag{6.8}$$

Applying (6.7) to each of the diffeomorphisms ϕ_t^2, along with Proposition (6.1), we get:

$$X_2 I_2 = -\int_0^{\infty} (\phi_t^2)^* (L_{X_2} d^2) dt.$$

Upon using the formula (6.8), we now get

$$X_2 I_2 = -\int_0^{\infty} \frac{d}{dt} (\phi_t^2)^* d^2(x) dt.$$

This finally yields $X_2 I_2 = d^2(x) - Lim_{t \to \infty} (\phi_t^2)^* d^2(x)$. Since $x_1 = 0$, this limit tends to $d^2(0, 0, x_3, \ldots, x_n)$, which equals 0, since d^2 vanishes whenever $(x_1, x_2) = (0, 0)$. *Thus I_2 indeed satisfies the equation $X_2 v = d^2$ on $\{x_1 = 0\}$.*

Next, let us compute $X_3 I_2$. We get $X_3 I_2 = -\int_0^\infty X_3 d^2(\phi_t^2(x)dt$. The integrability condition $X_3 d^2 = X_2 d^3$, therefore yields:

$$X_3 I_2 = -\int_0^\infty X_2 d^3(\phi_t^2(x)dt.$$

Keeping in mind the fact that $[X_2, X_3] = 0$, we get upon using Proposition (6.1):

$$X_3 I_2 = d^3(x) - Lim_{t\to\infty}(\phi_t^2)^* d^3(x).$$

Once again, since $x_1 = 0$, the limit in the last equation equals $d^3(0, 0, x_3, \ldots, x_n)$. *Therefore we have to amend I_2 by another function so that on $\{x_1 = 0\}$ we have $X_3 I_2 = d^3(x)$.*

To that end, we define I_3 as $-\int_0^\infty d^3(0, 0, e^{-p_1 t}x_3, x_4, \ldots, x_n)dt$. Once again, since d^3 vanishes to infinite order with respect to (x_1, x_3) when $(x_1, x_3) = (0, 0)$, estimates similar to (6.5) and (6.6) show that the integral I_3 is a C^∞ function of all its variables. Clearly, the arguments above also show that $X_3 I_3 = d^3(0, 0, x_3, \ldots, x_n)$. Therefore, $X_3(I_2 + I_3) = d^3(x)$ on $\{x_1 = 0\}$. *Furthermore, since the definition of I_3 does not involve x_1 or x_2, we have $X_2 I_3 = 0$, and hence $X_2(I_2 + I_3) = d^2(x)$ on $\{x_1 = 0\}$.*

Let us now evaluate $X_4(I_2 + I_3)$. We get, using the commutativity of X_4 with X_2 and X_3:

$$\begin{aligned} X_4(I_2 + I_3) &= d^4(x) - d^4(0, 0, x_3, x_4, \ldots, x_n) + d^4(0, 0, \ldots, x_n) \\ &\quad - d^4(0, 0, 0, x_4, \ldots, x_n) = d^4(x) - d^4(0, 0, 0, x_4, \ldots, x_n). \end{aligned}$$

If we define I_4 as $-\int_0^\infty d^4(0, 0, 0, e^{-p_1 t}x_4, \ldots, x_n)dt$, we see, using the same yoga once again, that $X_4 I_4 = d^4(0, 0, 0, x_4, \ldots, x_n)$. Thus, $X_4(I_2 + I_3 + I_4) = d^4(x)$ on $\{x_1 = 0\}$ and, since the definition of I_4 does not involve any of the variables, the x_1, x_2, x_3, I_4 is a C^∞ first integral of both X_2 and X_3. Therefore, $X_i(I_2 + I_3 + I_4) = d^i(x), i = 2, 3, 4$ on $\{x_1 = 0\}$. Iterating this argument we arrive at the conclusion that I_0 defined as $I_2 + I_3 + \ldots + I_n$, where $I_n = -\int_0^\infty d^n(0, 0, \ldots, 0, e^{-p_1 t}x_n)dt$ satisfies $X_i v = d^i, i = 2, \ldots, n$ on $\{x_1 = 0\}$.

Now by a generalization of Borel's lemma (Lemma 2.5, p. 98 of [12]) there exists a C^∞ function $V(x)$ dependent on the variables $(x_1, \ldots, x_n)$ such that:

$$\frac{\partial^{|m|}}{\partial x_1^{m_1} \ldots \partial x_n^{m_n}} V(x) = I_m(x)$$

for all multi-indices $m = (m_1, \ldots, m_n)$ on the set $\{x_1 = 0\}$. Here, $\mid m \mid = \sum_{i=1}^n m_i$, and

$$I_m = \frac{\partial^{|m|-m_1}}{\partial x_2^{m_2} \ldots \partial x_n^{m_n}} I_0.$$

If we now define $E^i(x)$ to be $d^i(x) - X_i V(x)$ for $i = 2, \ldots, n$, then it is clear that the functions $E^i(x), i = 2, \ldots, n$ vanish to infinite order on $\{x_1 = 0\}$.

6.3 Construction of the function $W(x)$

We first form another system of partial differential equations, associated to the problem:

$$X_i w = E^i(x), i = 2, \ldots, n. \tag{6.9}$$

The integrability conditions for the system (6.9) are $X_k e^l = X_l e^k, k, l = 2, \ldots, n$, and their validity is an immediate consequence of the integrability conditions for (6.4). We will now construct a solution $W(x)$ to (6.9) as a certain integral I. I is defined to be $\int_{-\infty}^{0} E^2(\phi_t^2(x))dt$. Deferring the issues of the convergence of this integral and the validity of differentiation under the integral sign for the moment, let us verify that I formally solves $X_2 v = E^2(x)$. We get:

$$X_2 I = \int_{-\infty}^{0} X_2 E^2(\phi_t^2(x))dt. \tag{6.10}$$

In exact analogy with the techniques of the previous subsection we get:

$$X_2 I = E^2(x) - E^2(Lim_{t \to -\infty} \phi_t^2(x)).$$

However, as $t \to -\infty$ the flow of X_2 approaches the hyperplane $\{x_1 = 0\}$. Thus the limit in the previous equation is actually 0(since E^2 vanishes on that hyperplane). Thus I formally satisfies $X_2 w = E^2(x)$. Furthermore, I also formally satisfies the remaining equations in the system (6.9). Indeed,

$$X_i I = \int_{-\infty}^{0} X_i E^2(\phi_t^2(x))dt, i = 2, \ldots, n$$

and this equals $\int_{-\infty}^{0} X_2 E^i(\phi_t^2(x))dt, i = 2, \ldots, n$ and this, of course, equals $E^i(x) - E^i(Lim_{t \to -\infty} \phi_t^2(x)), i = 2, \ldots, n$. Therefore, as a consequence of all the $E^i, i = 2, \ldots, n$ vanishing on $\{x_1 = 0\}$, I satisfies $X_i w = E^i(x), i = 2, \ldots, n$.

Finally, the convergence and smoothness of the integral I follows from the facts: a) on the hyperplane $\{x_1 = 0\}$ $E^i, i = 2, \ldots, d$ vanish to infinite order so that we have an estimate $\mid E^i(x) \mid \leq C_i(dist(x, \{x_1 = 0\}))^n, i = 2, \ldots, d$ for all positive integers n and for some positive constants $C_i, i = 2, \ldots, d$ and b) the estimate $dist(\phi_2(t, x), \{x_1 = 0\}) \leq e^{-Nt} dist(x, \{x_1 = 0\})$ for some positive constant N. Here, of course, $dist(,)$ stands for the (squared) distance of the first entry from the set in the second entry. Once again, these estimates provide the desired conclusion (for instance by using the same arguments to show that, say, I_2 was smooth).

Remark 6.2 The results of this section should be viewed as a "Formal + Hyperbolic Implies C^∞" result for systems of the type (3.5). Of course, the $X_i, i = 2, \ldots, d$ are not quite hyperbolic, so in some ways we have obtained a stronger result. However, we ought to remember that the commutativity of the fields $X_i, i = 2, \ldots, d$ played a crucial role.

7 Blowing Down the Solution

Since it is only the origin and its vicinity that we are really interested in, we can use b) of Th (2.3) to arrive at the following statement:

Theorem 7.1 *Let h be a real-analytic function on a neighborhood U of the origin in R^n. Then there is an open subset $V \subseteq U$ and a proper real-analytic map $\pi : N \to V$, where N is a real-analytic manifold, which is a diffeomorphism outside of the set $H^{-1}(0)$ and, in addition, has the property that h is locally a monomial. In other words, about each point p in N there are local coordinates $(x_1, \ldots, x_n)$ in which the map H is of the form $\Pi_{i=1}^n x_i^{p_i}$ for certain non-negative integers $p_i, i = 1, \ldots, n$.*

Now **Step 3** will be executed in the following steps:

- Each point $p \in \pi^{-1}(0)$ has a neighborhood U_p in N in which H is a monomial. We take a finite subcovering (using the fact that $\pi^{-1}(0)$ is a compact set—owing to the properness of the map π), say $U_1, \ldots, U_N$, and cover a neighborhood of $\pi^{-1}(0)$. Without loss of generality, we may take these $U_i, i = 1, \ldots, N$ to be connected open sets (otherwise we work with their connected components). In each of these, the system $(\pi^*\theta - d\alpha) \wedge dH = 0$ admits a solution, $\alpha_i, i = 1, \ldots, N$, which is furthermore flat on the subset $U_i \cap \pi^{-1}(0)$. This, of course, follows from the previous section.

- Thus whenever i and $j (i, j = 1, \ldots, N)$ satisfy $U_i \cap U_j \neq \Phi$, we obviously have $d(\alpha_i - \alpha_j) \wedge dH = 0$. Now H, being a monomial, is semiproper. Furthermore, since both π and h are real-analytic, so is H. Now the relation $d(\alpha_i - \alpha_j) \wedge dH = 0$ implies that about each point $p \in U_i \cap U_j$, $(\alpha_i - \alpha_j)$ is a formal power series in the variable H. This can be seen, keeping in mind Remark (5.1), by setting up a formal power series solution to the system of partial differential equations $X_l(\alpha_i - \alpha_j) = 0, l = 1, \ldots, K$, where the $X_l, l = 1, \ldots, K$ are vector fields which annihilate $d(h \circ \pi)$ as in the previous section. By Borel's lemma, therefore, there exists a smooth function H_p near each point $p \in U_i \cap U_j$ satisfying: $\alpha_i - \alpha_j - H_p \circ H$ is defined in a neighborhood of the point p and is flat at the point p. Now, given any two point set $\{x, y\}$ in $U_i \cap U_j$ we appeal to Remark (2.2) to construct a smooth function B so that $(\alpha_i - \alpha_j - B \circ H)$ is defined on all of $U_i \cap U_j$ and is flat on the set $\{x, y\}$. Thus the function also satisfies the "appartenance biponctuelle" criterion of Th (2.2). Therefore, we can conclude the existence of a function h_{ij} which is smooth on $U_i \cap U_j$ which satisfies $\alpha_i - \alpha_j = h_{ij} \circ H$ on $U_i \cap U_j$. Furthermore, it should be clear that the h_{ij} are all *flat on the subset* $U_i \cap U_j \cap \pi^{-1}(0)$.

- It is obvious that these locally defined functions h_{ij} satisfy the identities $h_{ij} + h_{jk} + h_{ki} = 0$ (on all triple intersections). We would like to conclude the existence of functions, h_i, which are smooth in U_i and also flat on the subset $U_i \cap \pi^{-1}(0)$ so that $h_{ij} = h_i - h_j$ on the subset $U_i \cap U_j$. We would also like the functions h_i to be functions of only H. Why this extra property is needed will become clear presently. Normally, the existence of such h_i (without the above additional property) would follow from the solvability of a Cousin A problem—which, fact in turn, follows from the fact the sheaf of C^∞ functions is actually a fine sheaf (see, for instance, Ch 5 of [29]). Let us directly illustrate this so that it becomes clear what is needed in addition.

  We set $X = \cup_{i=1}^{N} U_i$. We then chose a smooth partition of unity $s_l(x), l = 1, \ldots, N$ subordinate to the $U_l, l = 1, \ldots, N$. We now define the function h_i with domain $U_i, i = 1, \ldots, N$ by

$$h_i(x) = \sum_l s_l(x) h_{il}(x).$$

  Clearly the $h_i(x), i = 1, \ldots, N$ are C^∞ and flat on $U_i \cap \pi^{-1}(0)$. Now we write, for every $x \in U_a \cap U_b$ (for any two fixed indices a, b), the equations $h_{ab} = h_{ac} - h_{bc}, c = 1, \ldots, N$. We next multiply both sides of the last equality by the $s_c, c = 1, \ldots, N$ and then sum over c. This yields the conclusion.

  The problem with the above construction is that the functions s_l of the partition of unity cannot be written as functions of H. To circumvent this problem, we denote by V_i the complement of the set $U_i \cap \pi^{-1}(0)$ in U_i. We may assume, without any loss of generality, that all these open sets V_i are connected (otherwise we carry out the step below with their connected components). Clearly, H is a submersion on the $V_i, i = 1, \ldots, N$. Therefore, the image of the open sets V_i under H, denoted W_i are open sets in R. Clearly, the functions $h_{ij}, i, j = 1, \ldots, N$ on X give rise to well defined functions $r_{ij}, i, j = 1, \ldots, i_a$ which are smooth on the the open sets $W_i \cap W_j$. They also satisfy the identities $r_{ab} + r_{bc} + r_{ca} = 0$ on the corresponding triple overlaps. Thus we can find r_a, smooth on $W_a, a = 1, \ldots, N$ which satisfy $r_a - r_b = r_{ab}$ on $W_a \cap W_b$.

Finally, we define $h_i : U_i \rightarrow R, i = 1, \ldots, N$ by letting i) h_i equal $r_i \circ H$ on V_i; ii) and by letting it and all its derivatives be identically 0 on $U_i \cap \pi^{-1}(0)$. Clearly the h_i, so defined, is C^∞ on U and flat on $U_i \cap \pi^{-1}(0)$. The $h_i, i = 1, \ldots, N$ thus constructed have all the desired properties.

- Now, let us define a function α on X, according to $\alpha = \alpha_i - h_i$ on U_i. Clearly, these local pieces patch together on the overlaps $U_i \cap U_j$ to yield a smooth function on the neighborhood X which is flat on the subset $\pi^{-1}(0)$. Furthermore, the function α satisfies: a) it is constant on $\pi^{-1}(0)$, *so that it can be blown-down*, and b) $(\pi^*\theta - d\alpha) \wedge dH = 0$ on X. Therefore, the function $\kappa = \alpha \circ \pi^{-1}$ is well-defined, smooth in a neighborhood of $0 \in V$, is flat at 0 and satisfies $(\bar{\theta} - d\kappa) \wedge dh = 0$. Hence, $\kappa + \bar{\gamma}$ is the solution to $(\theta - d\alpha) \wedge dh = 0$ (where $\bar{\gamma}$ is as in Section 5).

This completes the blowing down process.

Remark 7.1 We see, by bearing in mind Remark (6.2), that we can prove the (local) smooth solvability of $\mathbf{P}$ even if $CodS(dh^C) = 2$, provided θ is C^∞ *and we know, beforehand, the formal solvability* of $\mathbf{P}$.

8 Conclusion

In this paper we introduced the study of controlled invariance for single—input systems, with differential forms as the starting point. In addition to the natural "functorial" appeal of this problem, several practical benefits are briefly recapitulated here:

1. The partial differential equations for the feedback can be written down from the data at hand, obviating thereby, the need to pair down to the annihilating distribution.

2. The necessary condition for the solvability of this problem is an equality condition and not an inclusion condition. Likewise, several other conditions regarding symmetry aspects (to mention just one) can be expressed as equality conditions. These are not discussed in the present paper.

3. It is possible to analyse regularity conditions in the codistributions and the control vector fields simultaneously. It also makes it possible to visualize what one might mean by a singularity in the controlled invariance problem. We also note in passing that the methods of this paper can be extended to cover the case where the disturbance signal is also available for measurement and thereby to some aspects of the

model matching problem. It is also possible, via the same techniques, to analyze the noninteracting control problem in the presence of singularities in the decoupling matrix (details may be found in [25]).

There is at least one problem which deserves further research, viz., generalizing these results to the case when $m > p$. Finally, it would be interesting to obtain global results in the presence of singularities. The formulation of the controlled invariance problem in this paper has the advantage that it renders possible the formulation of global questions in terms of cohomological data. Whilst this does not simplify global questions substantially, it has the psychological value of being similar to questions of the same genre in other disciplines. It would be interesting to examine the relation of the formalism here, with regard to global results, to the results of the excellent paper [9]. Loosely speaking, the motivation behind the suggested problem is that if a singularity does not cause problems locally, then it is not likely to do so globally either.

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