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Transcendence in Simultaneous Stabilization*

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Abstract

We show that the simultaneous stabilization question: *When are three linear systems stabilizable by the same controller?* cannot be solved by a semialgebraic set description nor be answered by computational machines. Contrary to the case of two systems, the underlying infinite-dimensional space of controllers cannot be bypassed. Simultaneous stabilization of three systems is truly transcendental.

Key words: linear systems, stabilization, simultaneous stabilization, interpolation, semialgebraic, decidability, computability

AMS Subject Classifications: 93D15, 68Q05, 03D35, 30C80

1 Introduction

Only few open problems in systems and control theory can be stated in a single sentence that is understandable by the whole control research community. One such problem, known as the simultaneous stabilization problem for three or more systems, is studied in this paper. The one-sentence question that has until now defied all attempts of solution is: *When are three (or more) linear systems stabilizable by the same controller?*

In this paper we derive explicit necessary and sufficient conditions for the stabilizability of one particular family of first order systems. From this analysis we draw general conclusions on the structure of the set of systems that are simultaneously stabilizable. Our conclusions are twofold. First, the set of triplets of systems that are simultaneously stabilizable is not semialgebraic; second, simultaneous stabilizability of more than two systems is not decidable by standard computation machines.

Before detailing our contributions we start with a description of the problem and with a survey of existing results.

We consider systems that are linear, time-invariant, single-input, single-output and that are given by their (real rational) transfer functions. A

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rational function will be called stable if it has no poles in the closed unit disc. Consideration of other stability regions would not modify the conclusions obtained in this paper; we choose the above definition of stability for convenience only. A controller c (internally) stabilizes a system p if the four closed loop transfer functions $cp/(1+cp)$, $p/(1+cp)$, $c/(1+cp)$, $1/(1+cp)$ are stable. The k systems p_i $(i = 1, \cdots, k)$ are simultaneously stabilizable (or, for short, stabilizable) if there exists a controller c that stabilizes all systems p_i. The simultaneous stabilization problem is one of finding necessary and sufficient conditions for systems to be simultaneously stabilizable.

Simultaneous stabilization was first studied more than a decade ago [28, 23] and has since then received considerable attention [6, 7, 8, 9, 14, 15, 16, 17, 26, 31]. It is shown in [23] and [28] that the case of two systems can be translated, with the help of the Youla-Kucera parametrization, into the problem of stabilizing a single related system by a stable controller. The latter question is elegantly solved by Youla et al. in [33]: a system is stabilizable by a stable controller if and only if it has an even number of poles between each pair of real unstable zeros. This condition is known as the parity interlacing condition. By successively using the Youla-Kucera parametrization and checking the parity interlacing condition we thus obtain a general tractable test for the case of two systems.

For three systems it is shown by Ghosh [14, 15] that, similarly to the two systems situation, the problem can generically be translated into the problem of unstable pole assignment of a related system by a controller that is both stable and minimum phase. A necessary condition for this problem to be solvable, the 3-interlacing condition, is given by Blondel in [5, 7]. This condition can be seen as a counterpart of the parity interlacing condition for three systems. Unfortunately, the 3-interlacing condition (and the k-interlacing conditions for k $(k \geq 3)$ systems) was shown not to be sufficient for simultaneous stabilizability (see [9, Theorem 5.3]).

Many other results have been obtained for simultaneous stabilization. We mention here: the analysis of simultaneous stabilization for multivariable systems [28], results on genericity of simultaneous stabilization [29, 25, 17], simultaneous stabilization in a state-space framework [21, 27], sufficient conditions for simultaneous stabilization [6, 13, 20, 31], or simultaneous stabilization using time-varying control [19]. Examples of applications and motivations for using simultaneous stabilization control design are given in Ackermann [1].

Despite all the above mentioned progress, the genuine stabilization problem for three or more systems remains essentially unsolved. Even in very simplified situations, where for example we consider first order systems, no satisfactory necessary and sufficient conditions are known. Simultaneous stabilization constitutes an outstanding open problem in linear system theory.

In this paper we consider a particular class of systems for which we have obtained complete stabilization conditions. Throughout the paper we analyse the three first order systems

$$\frac{z}{1+\beta_1 z}, \qquad \frac{z}{1+\beta_2 z}, \qquad \frac{z}{1+\beta_3 z}$$

where β_i are real numbers. If $\beta_i = \beta_j$ for some $i \neq j$, then the three systems degenerate into two systems that are easily seen simultaneously stabilizable. If $|\beta_i| < 1$ then the systems are stable, hence stabilizable. Finally, using the main result of [8], one can show that, for $\beta_1 = 0$ and $\beta_2 = -\beta_3 = \beta$, the systems are simultaneously stabilizable if and only if $|\beta| < \Gamma^4(\frac{1}{4})/(4\pi^2) = 4.377\cdots$. In this paper we are interested by general necessary and sufficient conditions in terms of the parameters β_i.

In Section 2 we show that the stabilizability of $z/(1+\beta_i z)$ depends upon the existence of a real rational function q that does not assume the values $0, 1$ and ∞ in the closed unit disc, and that takes the values $q(0) = (\beta_2 - \beta_3)/(\beta_2 - \beta_1)$ and $q'(0) = (\beta_2 - \beta_3)(\beta_1 - \beta_3)/(\beta_2 - \beta_1)$. We then use a theorem from geometric function theory, Landau's theorem, to derive necessary and sufficient conditions for the existence of such a rational function. The condition obtained involves the coefficients β_i and the so-called elliptic modular function. It is a necessary and sufficient condition for the stabilizability of the three systems.

In Section 3 we use the condition obtained in Section 2 to show that the set of coefficients β_i for which the systems are simultaneously stabilizable is not a semialgebraic set. These results extend an earlier result of Blondel and Gevers [8] and also answers a question raised by Ghosh in several of his papers (e.g., in [15] and [16]). Semialgebraic sets are particular subsets of $\mathbb{R}^n$ that can be used to describe a wide variety of decision problems (see [3] for examples from control theory). As an illustration of this, we show that if the number of systems is limited to two, or if the controller is constrained to have an order that is less than a given constant, then the set of simultaneously stabilizable systems is semialgebraic. In some sense, the non semialgebraicity of the set of coefficients β_i for which the systems $z/(1+\beta_i z)$ are simultaneously stabilizable expresses that, in this case, the infinite dimensional space of the controller cannot be bypassed; the problem is truly transcendental. The argument of the proof crucially relies on transcendence properties of the elliptic modular function appearing in Landau's theorem.

In the last section we interpret the results of Section 3 in terms of computability. We prove that the problem of determining whether our three systems are simultaneously stabilizable cannot be decided by certain computational machines. The first machine that we consider was recently introduced by Blum, Shub and Smale in a seminal paper [11] and has since then received considerable attention from the theoretical computer science community. Roughly speaking, the Blum-Shub-Smale machine (nicknamed BSS machine) is a real number counterpart of the Turing machine. We show that simultaneous stabilization cannot be decided by such a machine.

BSS machines are not allowed to compute roots of polynomials. In a last contribution we introduce a machine that extends the range of BSS decidable problems by including, among infinitely many other operations, the computation of roots of polynomials as possible operation. In a final theorem we show that simultaneous stabilization of the systems $z/(1+\beta_i z)$ remains undecidable in this extended computational framework.

The results obtained in this paper are for triplets of systems belonging to the set $\{z/(1+\beta z) \mid \beta \in \mathbb{R}\}$. It is clear that our conclusions also hold for larger subsets of $\mathbb{R}(z)$ and for any number of systems greater or equal to three. More precisely, assume that $k \geq 3$, P is a subset of $\mathbb{R}(z)$, and P contains $\{z/(1+\beta z) \mid \beta \in \mathbb{R}\}$. Then the set of k-uples of systems in P that are simultaneously stabilizable does not form a semialgebraic set, and simultaneous stabilizability of k systems in P is not decidable by the machines introduced in the fourth section.

We use the following notation: $\mathbb{C}$ and $\mathbb{R}$ are the sets of complex and real numbers, $\Re(z)$ and $\Im(z)$ are the real and imaginary parts of z. $\mathbb{R}(z)$ is the set of real rational functions. $D(R) = \{z \in \mathbb{C} \mid |z| < R\}$ is the open disc with center 0 and radius R. $D = D(1)$ is the open unit disc, and $\overline{D}$ is its closure. $\Pi^+ = \{z \in \mathbb{C} \mid \Im(z) > 0\}$ is the open upper half plane.

2 Three Special Systems

In this section we derive explicit necessary and sufficient conditions for the three systems

$$\frac{z}{1+\beta_1 z}, \quad \frac{z}{1+\beta_2 z}, \quad \frac{z}{1+\beta_3 z},$$

to be simultaneously stabilizable. As will be clear from the proof of Theorem 2, the stabilizability of $z/(1+\beta_i z)$ depends upon the existence of a real rational function q that does not assume the values $0, 1$ and ∞ in the closed unit disc $\overline{D}$, and that takes the values $q(0) = (\beta_2 - \beta_3)/(\beta_2 - \beta_1)$ and $q'(0) = (\beta_2 - \beta_3)(\beta_1 - \beta_3)/(\beta_2 - \beta_1)$. A criteria for the existence of such a real rational function is given almost explicitly in a theorem from complex analysis, Landau's theorem. The main difference is that Landau's theorem is stated for analytic rather than real rational functions. In the proof of Theorem 1 we use an approximation argument to obtain a necessary and sufficient condition for the existence of a rational function q that satisfies the requested interpolation conditions. The criterion, when used on the three systems, gives a stabilization condition in the form of an inequality involving the coefficients β_i.

The proof of Landau's theorem relies on properties of a well-known analytic function, the *elliptic modular function*. There are several related functions associated with this name. The one that we are considering here is a conformal mapping λ of the upper half plane Π^+ onto $\mathbb{C} \setminus \{0,1\}$. For the construction and properties of λ, see Segal [24, pp. 68-76] and Rudin [22, sec. 16.17-16.20]. Local inverses of λ will be denoted by ν. From the discussion in Rudin it is easy to see that λ has a local inverse $\nu_* : \mathbb{C} \setminus [0,\infty) \to \mathbb{C}$ that is such that

$$\nu_*(x) = i|\nu_*(x)| \quad \text{and} \quad \nu_*'(x) = -i|\nu_*'(x)| \tag{1}$$

for all $x < 0$.

Using the properties of the elliptic modular function we now prove an extended version of Landau's theorem.

Landau's theorem: *Suppose that $f : D(R) \to \mathbb{C} \setminus \{0,1\}$ is analytic. Let $a_0 = f(0)$, $a_1 = f'(0)$ and let ν be a local inverse of λ in a neighbourhood of a_0. Then*

$$R|a_1| \leq \frac{2\Im(\nu(a_0))}{|\nu'(a_0)|}. \tag{2}$$

Equality holds in (2) if and only if $f = \lambda \circ \psi$, where

$$\psi(z) = \frac{\overline{\nu(a_0)}\alpha z + R\nu(a_0)}{\alpha z + R}, \tag{3}$$

and α is a complex number of modulus 1.

Conversely, suppose that a_0 and a_1 are complex numbers such that $a_0 \neq 0, 1$ and inequality (2) holds. Then there exist an analytic function $f : D(R) \to \mathbb{C} \setminus \{0,1\}$ such that $f(0) = a_0$ and $f'(0) = a_1$. If a_0 and a_1 are real numbers, then f can be chosen so that $f(\overline{z}) = \overline{f(z)}$.

Proof: For the first part, see Segal [24, pp. 76-77].

The condition for equality can be derived from Segal's proof by noting that equality holds in Schwarz' lemma only for functions $F : D \to D$ of the type $F(z) = \alpha z$, where $|\alpha| = 1$.

For the converse part, the case $a_1 = 0$ is trivial, so we may without loss of generality assume that $a_1 \neq 0$ and that equality holds in (2). Taking $f = \lambda \circ \psi$, we have $f(0) = a_0$ and $|f'(0)| = |a_1|$. By adjusting α we can obtain $f'(0) = a_1$.

When a_0 and a_1 are real, we claim that α can be selected so that $f = \lambda \circ \psi$ maps the interval $(-R, R)$ into the reals. We prove this in three cases:

Case 1: $a_0 < 0$. Choose $\alpha = 1$. We can assume that $\nu = \nu_*$. By equations (1), we have $\Re(\nu_*(a_0)) = 0$, so that

$$\psi(z) = \nu_*(a_0)\frac{R-z}{R+z}.$$

It is easy to see that ψ maps $(-R, R)$ into the positive imaginary axis. Since λ maps the positive imaginary axis into the negative real axis, our function $f = \lambda \circ \psi$ maps $(-R, R)$ into the reals.

Case 2: $0 < a_0 < 1$. We can take $\alpha = 2\nu(a_0) - 1$, where ν is a local inverse of λ that maps $(0, 1)$ into the half-circle $C = \{z \in \Pi^+ \mid |z - \frac{1}{2}| = \frac{1}{2}\}$. Then ψ maps $(-R, R)$ into C, and λ maps C into $(0, 1)$.
Case 3: $a_0 > 1$. Let $\alpha = 1$ and assume that ν maps the interval $(1, +\infty)$ into the half-line $L = \{z \in \Pi^+ \mid \Re(z) = 1\}$. Then ψ maps $(-R, R)$ into L, and λ maps L into $(1, +\infty)$.

The details of case 2 and 3 are similar to case 1.

Thus, for a suitable choice of α, $f = \lambda \circ \psi$ maps the interval $(-R, R)$ into the reals. But then $f'(0)$ and a_1 are real and satisfy $|f'(0)| = |a_1|$, so by reverting the sign of α if needed, we obtain $f'(0) = a_1$. Finally, it is clear that $f(\overline{z}) = \overline{f(z)}$.

For the mapping properties of λ used here, see Rudin [22, the proof of Theorem 16.20]. ■

We now prove a version of Landau's theorem for real rational functions.

Theorem 1: *Suppose that a_0 and a_1 are real numbers such that $a_0 \neq 0, 1$. Then there exists a real rational function q such that $q(0) = a_0$, $q'(0) = a_1$ and $q(z) \neq 0, 1, \infty$ for all z in the closed unit disc $\overline{D}$ if and only if*

$$|a_1| < \frac{2\Im(\nu(a_0))}{|\nu'(a_0)|}, \tag{4}$$

where ν is a local inverse of the elliptic modular function λ. The right hand side in (4) does not depend on the particular choice of ν.

Proof: *"Only if" part:* Assume that q is such a function. By continuity, there exists an $R > 1$ such that $q(z) \neq 0, 1, \infty$ for all z in the open disc $D(R)$. Landau's theorem now gives us the inequality (2), from which the strict inequality (4) follows since $R > 1$.

"If" part: Assume that (4) holds. In the trivial case $a_1 = 0$ we may choose q to be constant. Otherwise, define

$$R = \frac{2\Im(\nu(a_0))}{|a_1||\nu'(a_0)|}. \tag{5}$$

The strict inequality (4) shows that $R > 1$. By Landau's theorem and equality (5), there exists an analytic function $f : D(R) \to \mathbb{C} \setminus \{0,1\}$ such that $f(0) = a_0, f'(0) = a_1$ and $f(\overline{z}) = \overline{f(z)}$. We now approximate f with a real polynomial q which does not assume the values 0 and 1 in $\overline{D}$ and satisfies $q(0) = a_0$ and $q'(0) = a_1$. This will complete the "if" part. For this purpose, let

$$\eta = \min\{\inf_{z\in\overline{D}} |f(z)|, \inf_{z\in\overline{D}} |f(z) - 1|\}.$$

Since $f(z) \neq 0, 1$ for all $z \in D(R)$, and since $\overline{D}$ is compact, $\eta > 0$. The function defined by $h(z) \triangleq (f(z) - a_0 - a_1 z)/z^2$ is analytic and such that $h(\overline{z}) = \overline{h(z)}$ in $D(R)$. By Runge's theorem (see Rudin [22, Theorem 13.7]) there is a polynomial p such that

$$|h(z) - p(z)| < \eta \qquad (\forall z \in \overline{D}). \tag{6}$$

Define a real polynomial by $p_1(z) \triangleq (p(z) + \overline{p(\overline{z})})/2$. Since $h(\overline{z}) = \overline{h(z)}$, it is easy to see that p_1 satisfies (6) also. Now the real polynomial $q(z) \triangleq a_0 + a_1 z + z^2 p_1(z)$ satisfies

$$|f(z) - q(z)| = |z^2 h(z) - z^2 p_1(z)| < \eta \qquad (\forall z \in \overline{D}).$$

In conjunction with the definition of η, this shows that $q(z) \neq 0, 1$ for all $z \in \overline{D}$. ■

We now have all that is needed to show:

Theorem 2: *Let β_i $(i = 1, 2, 3)$ be distinct real numbers. The systems*

$$\frac{z}{1+\beta_1 z}, \quad \frac{z}{1+\beta_2 z}, \quad \frac{z}{1+\beta_3 z}, \tag{7}$$

are simultaneously stabilizable if and and only if

$$|a_1| < \frac{2\Im(\nu(a_0))}{|\nu'(a_0)|}, \tag{8}$$

where a_0 and a_1 are defined by

$$\begin{aligned} a_0 &= \frac{\beta_2 - \beta_3}{\beta_2 - \beta_1}, \\ a_1 &= \frac{(\beta_2 - \beta_3)(\beta_1 - \beta_3)}{\beta_2 - \beta_1}, \end{aligned} \tag{9}$$

and ν is a local inverse of the elliptic modular function λ.

Proof: We adopt the factorization approach given in Vidyasagar [25]. A controller n/d (where n and d are coprime real polynomials) in closed loop with a system n_i/d_i (where n_i and d_i are coprime real polynomials) leads to a stable closed-loop configuration if and only if $n_i n + d_i d$ is a stable polynomial, i.e., has no zeros in the closed unit disc $\overline{D}$. The controller $n/d = -\beta_3/1$ stabilizes the third system $z/(1+\beta_3 z)$. Hence, by the Youla-Kucera parametrization (see Vidyasagar [25]), a factorization of all the controllers n/d that stabilize this system is given by

$$n(z) = -\beta_3 + r(z)(1+\beta_3 z), \qquad d(z) = 1 - r(z)z, \tag{10}$$

where r is an arbitrary real rational function with no poles in $\overline{D}$. This controller n/d also stabilizes the first and second systems if and only if

$$zn(z) + (1+\beta_i z)d(z) \neq 0 \qquad \text{for all } z \in \overline{D},\ i = 1, 2. \tag{11}$$

Putting (10) into (11) and simplifying, we get

$$1 + (\beta_i - \beta_3)z + (\beta_3 - \beta_i)z^2 r(z) \neq 0 \qquad \text{for all } z \in \overline{D},\ i = 1, 2. \tag{12}$$

After division by $\beta_i - \beta_3$ and introduction of

$$\begin{aligned} a_0 &= \frac{\frac{1}{\beta_1-\beta_3}}{\frac{1}{\beta_1-\beta_3} - \frac{1}{\beta_2-\beta_3}} = \frac{\beta_2-\beta_3}{\beta_2-\beta_1}, \\ a_1 &= \frac{1}{\frac{1}{\beta_1-\beta_3} - \frac{1}{\beta_2-\beta_3}} = \frac{(\beta_2-\beta_3)(\beta_1-\beta_3)}{\beta_2-\beta_1}, \end{aligned}$$

the condition (12) can be written as

$$a_0 + a_1 z - a_1 z^2 r(z) \neq 0, 1 \qquad \text{for all } z \in \overline{D}. \tag{13}$$

We have thus shown that the systems (7) are simultaneously stabilizable if and only if there exists a real rational function r with no poles in $\overline{D}$ such that (13) holds. Denoting the left-hand side of (13) by $q(z)$ it is easy to see that the existence of a rational function r that has the required properties is equivalent to the existence of a real rational function q that is such that $q(0) = a_0$, $q'(0) = a_1$ and $q(z) \neq 0, 1, \infty$ for all z in $\overline{D}$. By Theorem 1, this is equivalent to the strict inequality (8). ■

3 Semialgebraic Sets

Let S be the set of all triplets $(\beta_1, \beta_2, \beta_3) \in \mathbb{R}^3$ for which the systems $z/(1+\beta_i z)$ are simultaneously stabilizable. In this section we use the explicit description of S given in Theorem 2 to show that S is not a semialgebraic set.

As already mentioned in the introduction, this result answers a question addressed by Ghosh [15, Question 2.2 p. 1094] and [16, Conclusion and future developments p. 495], and extends significantly a recent result obtained by Blondel and Gevers [8]. The extension is to be taken in the following sense. In [8] the authors proved that the set of coefficients of triplets of systems that are simultaneously stabilizable can not be described by unions and intersections of solutions of multivariable *rational* polynomial equalities and inequalities. We extend this result in two respects. First, we prove the same result for the class of *real* polynomials rather than rational polynomials. In a certain sense this first extension shows that one may add one type of operation among those permitted in [8], namely, rational operations (addition, multiplication, subtraction or division) with given real constants. A second difference with the result in [8] is that we show that the complement of S is *not a countable* union of semialgebraic sets. This statement is much stronger than the one on semialgebraicity because it involves a countable number of sets. It is this result that is needed in the next section for proving decidability properties.

We start with a definition of semialgebraic sets and with some examples from systems and control theory that motivate their use. General references for semialgebraic sets are [4] and [12].

Definition: *A set $X \subseteq \mathbb{R}^n$ is semialgebraic if it is a finite union of sets of the type*

$$\{x \in \mathbb{R}^n \quad | \quad P_1(x) = 0, ..., P_k(x) = 0, \\ P_{k+1}(x) > 0, ..., P_m(x) > 0\},$$

where $P_i(x) = P_i(x_1, ..., x_n)$ $(i = 1, 2, ..., m)$ are real polynomials in n variables.

It is easy to see that, together with the equalities $P_i(x) = 0$ and inequalities $P_j(x) > 0$, we may add conditions involving any of $\geq, <, \leq, \neq$. All such sets can be reduced to sets of the form above.

Unions and intersections of semialgebraic sets are semialgebraic. Also, the complement of a semialgebraic set is semialgebraic.

In the next section we will also need the concept of a semialgebraic function.

Definition: *Let X be a semialgebraic subset of $\mathbb{R}^n$. A function $f : X \to \mathbb{R}^k$ is called semialgebraic if its graph $\{(x, y) \in X \times \mathbb{R}^k \mid f(x) = y\}$ is a semialgebraic subset of $\mathbb{R}^{n+k}$.*

The next five examples illustrate the notion of semialgebraic sets.

Example 1: Polynomial stability

Let H_n be the set of coefficients $(a_0, a_1, ..., a_n) \in \mathbb{R}^{n+1}$ for which the polynomial $a_0 + a_1 s + ... + a_n s^n$ is Hurwitz stable, i.e., has no zeros in the closed right half plane. By the Routh-Hurwitz criterion, H_n can be described by a logical expression involving polynomial equalities and inequalities in the coefficients $a_0, ..., a_n$. For instance, when $n = 3$ we have

$$
\begin{aligned}
H_3 = \{&(a_0, a_1, a_2, a_3) \in \mathbb{R}^4 \mid \\
&(a_3 > 0 \text{ and } a_2 > 0 \text{ and } a_1 > 0 \text{ and } a_0 > 0 \text{ and } a_1 a_2 - a_0 a_3 > 0) \\
&\text{or } (a_3 < 0 \text{ and } a_2 < 0 \text{ and } a_1 < 0 \text{ and } a_0 < 0 \text{ and } a_1 a_2 - a_0 a_3 < 0) \\
&\text{or } (a_3 = 0 \text{ and } a_2 > 0 \text{ and } a_1 > 0 \text{ and } a_0 > 0) \\
&\text{or } (a_3 = 0 \text{ and } a_2 < 0 \text{ and } a_1 < 0 \text{ and } a_0 < 0) \\
&\text{or } (a_3 = 0 \text{ and } a_2 = 0 \text{ and } a_1 a_0 > 0) \\
&\text{or } (a_3 = 0 \text{ and } a_2 = 0 \text{ and } a_1 = 0 \text{ and } a_0 \neq 0)\},
\end{aligned}
$$

so that H_3 is semialgebraic. Similarly, H_n is easily seen to be semialgebraic for all n. More generally, any logical expression involving equalities and inequalities between multivariable real polynomials (such as the expression describing H_3) describes a semialgebraic set.

Example 2: The space of systems

Let Σ_n be the set of vectors $(a_0, ..., a_n, b_0, ..., b_n) \in \mathbb{R}^{2n+2}$ for which

1) The polynomial $b_0 + b_1 z + ... + b_n z^n$ has highest order coefficient equal to 1.
2) The polynomials $a_0 + a_1 z + ... + a_n z^n$ and $b_0 + b_1 z + ... + b_n z^n$ are coprime.

We use the symbol $\neg$ to denote logical negation. The set Σ_n may be described by

$$
\begin{aligned}
\Sigma_n \quad = \quad &\{(a_0, ..., a_n, b_0, ..., b_n) \in \mathbb{R}^{2n+2} \mid \\
&\forall z \; \neg (a_0 + a_1 z + ... + a_n z^n = 0 \text{ and } b_0 + b_1 z + ... + b_n z^n = 0) \\
&\text{ and} \\
&((b_n = 1) \\
&\text{or } ((b_n = 0) \text{ and } (b_{n-1} = 1)) \\
&\text{or } (...) \\
&\text{or } ((b_n = 0) \text{ and } ... \text{ and } (b_1 = 0) \text{ and } (b_0 = 1)))\}.
\end{aligned}
$$

By using the Tarski-Seidenberg theorem (see [4]) the universal quantifier $\forall$ and the corresponding variable, variable z may be eliminated from this expression, leaving a semialgebraic condition on the coefficients $a_0, ..., a_n$, $b_0, ..., b_n$. Thus the set Σ_n is a semialgebraic subset of $\mathbb{R}^{2n+2}$.

To every vector $(a_0, ..., a_n, b_0, ..., b_n)$ in Σ_n there corresponds in a one-to-one fashion a system

$$p(z) = \frac{a_0 + a_1 z + ... + a_n z^n}{b_0 + b_1 z + ... + b_n z^n}$$

of order at most n. In the sequel we often identify a system with its coefficient vector in Σ_n, and we speak of Σ_n as the space of systems of order at most n. The cartesian product space Σ_n^k is the space of all k-tuples of systems of order at most n. A set of k-tuples of systems of order at most n is called semialgebraic if its corresponding subset of Σ_n^k is semialgebraic. This notion of semialgebraicity is used in the next three examples.

Example 3: Simultaneous stabilization of two systems

Consider two systems p_1 and p_2 of order at most n. Using the Youla-Kucera parametrization, one can construct a system p that is such that p_1 and p_2 are simultaneously stabilizable if and only if p is stabilizable by a stable controller. For instance, when p_1 and p_2 have no common real unstable poles then we can take $p = p_2 - p_1$ (see for example [7]). A system p is stabilizable by a stable controller if and only if between any two zeros of p in the closed interval $[-1, 1]$ there is an even number of poles of p (counted with multiplicity); see Youla et al. [33]. This interlacement condition can be translated into a semialgebraic condition on the coefficients of p; see Anderson [2]. This implies that the set of pairs of systems that are simultaneously stabilizable is semialgebraic.

Example 4: Simultaneous stabilization with a controller of a priori bounded order

Consider the set $S_{n,N}^k$ of all k-tuples of systems $(p_1, ..., p_k)$ that are of order at most n and that are are simultaneously stabilizable by a controller *of order at most* N. If we use a parametrization of the set of controllers of order at most N, and the Routh-Hurwitz criteria on the k closed loop polynomials obtained with this parametrization, it is clear that the set $S_{n,N}^k$ can be described by a finite number of equalities and inequalities between real polynomials in the coefficients of the systems and in the coefficients of the controller connected with logical operators and the existential quantifier ($\exists$).

By the Tarski-Seidenberg theorem, the quantifiers $\exists$ and the corresponding variable (in this case, the coefficients of the controller) can be eliminated from this description, leaving a semialgebraic condition on the coefficients of the systems. Thus the set $S_{n,N}^k$ of k-tuples of systems that are of order at most equal to n and that are simultaneously stabilizable by a controller of order at most N is semialgebraic. For a more comprehensive treatment of this example, see Ghosh [16].

Example 5: Functions with rational components

This more technical example is introduced here because it is needed in the proof of Theorem 3. Let n_i, d_i $(i = 1, 2, ..., k)$ be real polynomials of n

variables $(x_1, ..., x_n) = x$. A function

$$f(x) = (\frac{n_1(x)}{d_1(x)}, ..., \frac{n_k(x)}{d_k(x)})$$

is defined on the semialgebraic subset X of $\mathbb{R}^n$ where all the denominators $d_i(x)$ are nonzero. The graph of f is

$$\{(x, y) \in X \times \mathbb{R}^k \; ; y_i = n_i(x)/d_i(x) \text{ for } i = 1, 2, ...k\},$$

which is a semialgebraic set. Hence f is a semialgebraic function.

We now come to our main theorem, which shows that the result of Example 3 does not extend to the case of three systems, nor does Example 4 extend to the case where there is no a priori bound on the order of the controller.

First we need a lemma that shows that the bound on a_1 in Theorem 2 is given by a non-algebraic function of a_0. Recall that an analytic function f is termed *algebraic* if there exists a nonzero polynomial P such that $P(z, f(z)) \equiv 0$.

Lemma: *The analytic function* $F : \mathbb{C} \setminus [0, \infty) \to \mathbb{C}$ *defined by*

$$F(z) = -2\frac{\nu_*(z)}{\nu_*'(z)}$$

is not algebraic. In addition to this we have

$$F(x) = \frac{2\Im(\nu_*(x))}{|\nu_*'(x)|}$$

for all $x < 0$.

Proof: Assume, to get a contradiction, that P is a nonzero polynomial such that

$$P(z, F(z)) = 0 \qquad \text{for all } z \in \mathbb{C} \setminus [0, \infty).$$

As proved in Theorem 16.20(a) of Rudin [22], the modular function λ has the property that $\lambda(z + 2n) = \lambda(z)$ for all integers n and $z \in \Pi^+$. This implies that $\nu_n(z) \triangleq 2n + \nu_*(z)$ is a local inverse of λ for every integer n. Since ν_n and ν_* are both local inverses of the analytic function λ, they must be analytic continuations of each other (see Theorem 10.7.2 in Hille [18]). Hence $F_n(z) \triangleq -2\frac{\nu_n(z)}{\nu'(z)}$ is an analytic continuation of F, and likewise $P(z, F_n(z))$ is an analytic continuation of $P(z, F(z))$. But by assumption $P(z, F(z)) \equiv 0$, and so $P(z, F_n(z)) \equiv 0$. Now fix $z_0 \in \mathbb{C} \setminus [0, \infty)$.

Since $F_n(z_0) = -2\frac{2n+\nu_*(z_0)}{\nu_*'(z_0)}$ has infinitely many values as n ranges over the integers, this shows that the polynomial $P_{z_0}(w) \triangleq P(z_0, w)$ has infinitely many zeros. Hence $P_{z_0} = 0$, so $P = 0$. This contradiction shows that the assumption that F is algebraic was false.

The statement about the values of $F(x)$ for $x < 0$ follows immediately from the fact that, for all $x < 0$, we have

$$\nu_*(x) = i|\nu_*(x)| \text{ and } \nu_*'(x) = -i|\nu_*'(x)|.$$

■

Theorem 3: The set S of triplets $(\beta_1, \beta_2, \beta_3) \in \mathbb{R}^3$ for which the systems $z/(1 + \beta_i z)$ are simultaneously stabilizable is not semialgebraic. Furthermore, the set S is a countable union of semialgebraic sets but its complement in $\mathbb{R}^3$ is not.

Proof: If S was semialgebraic so would be its complement S^c. Hence, the first assertion follows from the second one.

We first prove the easy part of the second assertion. Namely, we prove that S is a countable union of semialgebraic sets.

For that purpose, define S_n by

$$S_n = \{(\beta_1, \beta_2, \beta_3) \in \mathbb{R}^3 \mid$$

$$\frac{z}{1+\beta_i z} \text{ are simultaneously stabilizable by a controller of order } n\}.$$

Then $S = \bigcap_{n=0}^{\infty} S_n$. By Example 4, the sets S_n are semialgebraic and thus the first part is proved.

We now prove the second part of the assertion. Assume, to get a contradiction, that the complement S^c is a countable union of semialgebraic sets. Since the set $\{(\beta_1, \beta_2, \beta_3) \in \mathbb{R}^3 \mid \beta_1 < \beta_2 < \beta_3\}$ is semialgebraic, the set

$$B \triangleq S^c \cap \{(\beta_1, \beta_2, \beta_3) \in \mathbb{R}^3 \mid \beta_1 < \beta_2 < \beta_3\}$$

is a countable union of semialgebraic sets. Theorem 2 implies that

$$B = \{(\beta_1, \beta_2, \beta_3) \in \mathbb{R}^3 \mid \beta_1 < \beta_2 < \beta_3 \text{ and } |a_1| \geq \frac{2\Im(\nu(a_0))}{|\nu'(a_0)|}\},$$

where a_0 and a_1 are defined by

$$\begin{aligned} a_0 &= \frac{\beta_2 - \beta_3}{\beta_2 - \beta_1}, \\ a_1 &= \frac{(\beta_2 - \beta_3)(\beta_1 - \beta_3)}{\beta_2 - \beta_1}. \end{aligned}$$

The function which maps $(\beta_1, \beta_2, \beta_3)$ to (a_0, a_1) maps B onto

$$A \stackrel{\triangle}{=} \{(a_0, a_1) \in \mathbb{R}^2 \mid a_0 < 0 \text{ and } a_1 \geq \frac{2\Im(\nu(a_0))}{|\nu'(a_0)|}\}.$$

By Example 6, this function is semialgebraic. By using the Tarski-Seidenberg theorem, it is easy to prove that a semialgebraic function maps semialgebraic sets onto semialgebraic sets. Thus A is a countable union of semialgebraic sets, so we can write $A = \bigcup_{n=1}^{\infty} A_n$, where

$$\begin{aligned} A_n = \{(x,y) \in \mathbb{R}^2 \quad | \quad & P_{n,1}(x,y) = 0, ..., P_{n,k_n}(x,y) = 0, \\ & P_{n,k_n+1}(x,y) > 0, ..., P_{n,m_n}(x,y) > 0\}, \end{aligned}$$

$0 \leq k_n < \infty$, $k_n \leq m_n < \infty$ and $P_{n,i}(x,y)$ are nonzero real polynomials. With the help of the non-algebraic function F in the lemma we can write

$$A = \{(x,y) \in \mathbb{R}^2 \mid x < 0 \text{ and } y \geq F(x)\}.$$

Now fix $x_0 < 0$. Then $(x_0, F(x_0)) \in A$, so $(x_0, F(x_0)) \in A_{n_0}$ for some n_0. If $k_{n_0} = 0$, then A_{n_0} would be open, so $(x_0, F(x_0))$ would be an interior point of A. But $(x_0, y) \notin A$ if $0 < y < F(x_0)$. Hence $k_{n_0} > 0$, so $P_{n_0,1}(x_0, F(x_0)) = 0$.

Therefore the sets $Z_n \stackrel{\triangle}{=} \{x < 0 \mid P_{n,1}(x, F(x)) = 0\} \quad (n = 1, 2, ...)$ have union $(-\infty, 0)$. One of these sets, say Z_{n_1}, must be uncountable, and must thus have a limit point in $(-\infty, 0)$. This means that the set of zeros of the analytic function $P_{n_1,1}(z, F(z))$ has a limit point in its domain of definition, so it must be identically zero. Since $P_{n_1,1}$ is nonzero, this shows that F is an algebraic function, a contradiction. ■

From Theorem 3 we deduce a corollary that gives a general property of simultaneously stabilizable systems.

Corollary: *Let Σ_n^k be the space of all k-tuples of systems of order at most n and let S_n^k be the subset of Σ_n^k that correspond to k-tuples of systems that are simultaneously stabilizable. If $n \geq 1$ and $k \geq 3$ then S_n^k is not semialgebraic.* ■

4 Computability

In this section we interpret the result of the previous section in terms of computability. We show that the question of deciding whether the three systems $z/(1 + \beta_i z)$ $(i = 1, 2, 3)$ are simultaneously stabilizable cannot be decided by certain *computational machines.*

The classical types of machines considered in computer science use the binary digit as the fundamental unit of information. Recently, a new type of machine which uses exact real numbers as information unit, have been introduced by Blum, Shub and Smale [11]. These machines (BSS machines) are allowed to perform exact rational operations (additions, multiplications, subtractions and divisions) on real numbers and are regarded as counterparts of Turing machines for real numbers. In their original context, BSS machines are defined over ordered rings. The ring of integers $\mathbb{Z}$ and the field of real numbers $\mathbb{R}$ are only two examples of ordered rings. The case of integers leads, roughly speaking, to Turing machines whereas the machine that we are considering here is the BSS machine over the reals. We refer the reader to the original paper [11] for a precise description of BSS machines.

Proposition 2 of [11] shows that if a set $E \subseteq \mathbb{R}^d$ is decidable by a BSS machine (i.e., for every input $x \in \mathbb{R}^d$, the machine will answer in a finite time whether or not $x \in E$), then both E and its complement $\mathbb{R}^d \setminus E$ are countable unions of semialgebraic sets. We have shown in Theorem 3 that the complement S^c of the set of triplets $(\beta_1, \beta_2, \beta_3) \in \mathbb{R}^3$ for which the systems $z/(1 + \beta_i z)$ are simultaneously stabilizable is not a countable union of semialgebraic set. Hence S does not belong to the class of sets that are decidable by a BSS machine and the stabilizability of $z/(1 + \beta_i z)$ is not decidable by a BSS machine.

One of the drawbacks of BSS machines for our purposes is that they do not include the computation of roots of polynomial as a possible elementary operation. Extraction of roots is considered in systems theory as a standard operation that can possibly be included in any decision algorithm. In what follows we introduce a machine that is more powerful than that of Blum, Shub and Smale and that allows the computation of roots. We will see that, even with this machine, the question of simultaneous stabilizability of the three systems $z/(1 + \beta_i z)$ is not decidable.

Definition: *A machine consists of a possibly infinite set of nodes N. Associated to each node n there is:*

1) *A set X_n, the input space.*

2) *A function $t_n : X_n \to N$, the transition function.*

3) *For each node m in the range of t_n, a function $f_{n \to m} : t_n^{-1}(m) \to X_m$, the data transformation.*

One node n_s is singled out as the *start node*, and another one is the *end node* n_e. The machine works in discrete time steps $0, 1, 2, \cdots$. At each time k the machine is at a certain node n_k and has a certain value $x_k \in X_{n_k}$ of its stored data. The machine starts at the start node ($n_0 = n_s$) and its data is initialized with an input x_0 belonging to the input space X_{n_s} of the start node. At time k two things can happen: if the machine is at the end node ($n_k = n_e$), then it stops and outputs the value x_k. Otherwise, the machine goes to node $n_{k+1} = t_{n_k}(x_k)$ and transforms the data according to $x_{k+1} = f_{n_k \to n_{k+1}}(x_k)$.

We impose the following restrictions on our machines:
For each node n

a) The input space X_n is a semialgebraic subset of some euclidean space $\mathbb{R}^{d_n}$.

b) The transition function t_n has a finite range. This means that at each node there is only a finite number of nodes to which the machine may make a transition.

c) The inverse images $X_{n\to m} \triangleq t_n^{-1}(m)$ are semialgebraic subsets of $\mathbb{R}^{d_n}$. Note that the set $X_{n\to m}$ is the set of data in $\mathbb{R}^{d_n}$ that make the machine transit from node n to node m.

d) The data transformations $f_{n\to m}$ are semialgebraic functions.

Behind these abstract definitions lies a very natural idea of machine; basically one that uses an algorithm that involve only semialgebraic functions. Rational operations are examples of operations that lead to algebraic functions. A less trivial example is that of root extraction.

Theorem 4: *Define a function $f : \mathbb{C}^n \to \mathbb{C}^n$ by*

$$f(a_0, a_1, ..., a_{n-1}) = (z_1, z_2, ..., z_n),$$

where $z_1, z_2, ..., z_n$ are the roots of the polynomial equation

$$a_0 + a_1 z + ... + a_{n-1} z^{n-1} + z^n = 0$$

ordered increasingly according to the following order $\preceq$ of the complex numbers:

$$z \preceq w \text{ if and only if } \Re(z) < \Re(w) \text{ or } (\Re(z) = \Re(w) \text{ and } \Im(z) \leq \Im(w)).$$

Then f is semialgebraic when considered as a mapping from $\mathbb{R}^{2n}$ to $\mathbb{R}^{2n}$ in the obvious way.

Proof: The graph of f is

$$\{(a_0, ..., a_{n-1}, z_1, ..., z_n) \in \mathbb{C}^n \times \mathbb{C}^n \mid z_1 \preceq z_2 \preceq ... \preceq z_n \text{ and}$$
$$a_0 = \prod_{i=1}^{n} (-z_i)$$
$$...$$
$$a_{n-1} = -\sum_{i=1}^{n} (z_i)\},$$

which is semialgebraic considered as a subset of $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$. ∎

Decidable sets for our machine are defined in the following way:

Definition: *Let E and I be sets such that $E \subseteq I$. We say that E is decidable in I, if there is a machine such that:*

1) The input space of the start node is I. The input space of the end node is $\{0,1\}$.

2) For every input $x_0 \in E$, the machine eventually stops and outputs 1.

3) For every input $x_0 \in I \setminus E$, the machine eventually stops and outputs 0.

We can characterize the decidable sets as follows.

Theorem 5: *Let I be a semialgebraic set and let E be a subset of I. Then E is decidable in I if and only if both E and $I \setminus E$ are countable unions of semialgebraic sets.*

Proof:
"Only if" part: Assume that we have a machine that decides E; i.e., 1)-3) of the previous definition holds. Consider a particular sequence of nodes $(n_0, n_1, ..., n_k)$ that the machine may follow upon input x_0, where $n_0 = n_s$ and $n_k = n_e$. Let $X^1(n_0, ..., n_k)$ be the set of all inputs $x_0 \in X_{n_s}$ that makes the machine follow this sequence of nodes and then output 1. We have

$$\begin{aligned} X^1(n_0, ..., n_k) \quad = \quad & \{x_0 \in X_{n_s} \mid x_0 \in X_{n_0 \to n_1}, \\ & f_{n_0 \to n_1}(x_0) \in X_{n_1 \to n_2}, \\ & f_{n_1 \to n_2} \circ f_{n_0 \to n_1}(x_0) \in X_{n_2 \to n_3}, \\ & ..., \\ & f_{n_{k-2} \to n_{k-1}} \circ ... \circ f_{n_0 \to n_1}(x_0) \in X_{n_{k-1} \to n_k}, \\ & f_{n_{k-1} \to n_k} \circ ... \circ f_{n_0 \to n_1}(x_0) = 1\}. \end{aligned}$$

By the restrictions c) and d) the sets $X_{n \to m}$ and the functions $f_{n \to m}$ are semialgebraic. Using the Tarski-Seidenberg theorem it is easy to prove that the composition of two semialgebraic functions is a semialgebraic function, and that the inverse image of a semialgebraic set under a semialgebraic mapping is a semialgebraic set. Hence the set $X^1(n_0, ..., n_k)$ is semialgebraic. By assumption, E is the union of all $X^1(n_0, ..., n_k)$, where $(n_0, ..., n_k)$ ranges over all possible paths the machine may take. Restriction b) above implies that this set of possible paths is countable. Hence E is a countable union of semialgebraic sets. Similarly, $I \setminus E$ is a countable union of semialgebraic sets.

"If" part: Assume that $E = \bigcup_{i=1}^{\infty} E_i$ and $I \setminus E = \bigcup_{i=1}^{\infty} C_i$, where E_i and C_i are semialgebraic sets. The machine described by the following diagram decides E in I.

$$
\begin{array}{ccccccccc}
\text{output } 1 & & & & \text{output } 1 & & & & \\
{\scriptstyle \text{yes}} \uparrow & & & & {\scriptstyle \text{yes}} \uparrow & & & & \\
x \in E_1? & \xrightarrow{\text{no}} & x \in C_1? & \xrightarrow{\text{no}} & x \in E_2? & \xrightarrow{\text{no}} & x \in C_2? & \xrightarrow{\text{no}} & \ldots \\
 & & {\scriptstyle \text{yes}} \downarrow & & & & {\scriptstyle \text{yes}} \downarrow & & \\
 & & \text{output } 0 & & & & \text{output } 0 & &
\end{array}
$$

■

We now collect Theorem 3 and Theorem 5 into a single sentence:

Theorem 6: *The stabilizability of the three systems $z/(1+\beta_i z)$ $(i = 1, 2, 3)$ is not decidable by a BSS machine nor is it by a machine defined above.*

Proof: The statement for BSS machines follow from Theorem 3 and from the fact that sets that are decidable by BSS machines must have a complement that is a countable union of semialgebraic sets. The second statement follows, in the same way, from Theorem 3 and Theorem 5. ■

As a corollary of Theorem 6 we have:

Corollary: *Let P be a subset of $\mathbb{R}(z)$ such that $z/(1+\beta z) \in P$ for all $\beta \in \mathbb{R}$. The stabilizability of three systems in P is not decidable by a BSS machine nor by a machine defined above.* ■

5 Conclusion

We have shown that simultaneous stabilizability of three systems is not a "semialgebraic problem" and that it cannot be decided by our machines, which are allowed to evaluate semialgebraic functions.

Thus, every solution of the simultaneous stabilization problem for three or more systems must necessarily include some transcendental function. We have given one example of this, namely the systems $z/(1+\beta_i z)$ $(i = 1, 2, 3)$, whose stabilization condition can be expressed in terms of an inequality involving the elliptic modular function.

Can the general simultaneous stabilization problem be solved in terms of the elliptic modular function only? We believe not, but this remains an open problem.

Our result is a negative one, but in practice it has less significance, since there are often practical limitations on the order of the controllers that one can implement. By Example 4, the simultaneous stabilizability problem with a controller of a priori bounded order is solvable in terms of rational operations on the coefficients of the systems, although this computation may be very involved.

As a final comment let us note that simultaneous stabilization of the systems analysed in this paper cannot be rephrased as a stabilization problem of a single system by a controller that is both stable and minimum phase. It is yet unknown whether the latter problem is semialgebraic.

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