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High Gain Limits of Trajectories and Attractors for a Boundary Controlled Viscous Burgers' Equation*

Christopher I. Byrnes† David S. Gilliam†
Victor I. Shubov†

Abstract

In this paper we consider a boundary control problem for a forced Burgers' equation on a finite interval. The controls enter as gain parameters in the boundary conditions as in [7, 6] and the forcing term is allowed to be time dependent and square integrable in the spatial variable for all time. The uncontrolled problem is obtained by equating the control parameters to zero while the zero dynamics is obtained by constraining the output to be zero. The main result of the paper is that for H^1-smooth initial data the trajectories of the closed loop system (positive gains) converge uniformly in space and time, to the trajectories of the zero dynamics system as the feedback gains are increased to infinity. This result is similar to the property of asymptotic phase for lumped nonlinear systems. For forcing terms which are independent of time, we also establish the existence of a compact local attractor for the nonlinear semigroup. Moreover, as a consequence of the uniform convergence of the trajectories, we show that the attracting sets converge to the attractor for the forced zero dynamics, which in this case always consists of a single point.

1 Introduction

One of the important feedback design methods of classical automatic control is root locus theory, based on the observation that in the frequency domain the closed loop poles of a system vary from the open loop poles to

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the open loop zeros as the gain is increased from zero to infinity. Successfully exploited for decades for finite dimensional systems, this fundamental method has been extended to the nonlinear finite dimensional case ([11]), where it is shown that as certain gain parameters are tuned, the closed-loop trajectories approach the trajectories of the zero dynamics. On the other hand, root locus methods have also recently been extended to the infinite dimensional case in [5] where a fairly complete analog of finite dimensional root locus theory is developed for a class of parabolic boundary control problems in which the inputs and outputs occur through certain boundary operators and a closed loop system is obtained by employing a proportional error feedback law, $u = -ky$. In this case, in [5] it is shown that the infinitely many closed loop poles vary from the open loop poles to the open loop zeros as the gain is increased from zero to plus or minus infinity, depending on the sign of the instantaneous gain. Defining the zero dynamics to be the system obtained by constraining the output y to zero, or equivalently, as the system obtained in the high gain limit, it is possible to enhance many of the classical results on stabilization of minimum phase systems (i.e., systems with exponentially stable zero dynamics). In particular, in [5] it is shown that the one parameter family of closed loop spatial operators A_k – the analog of $(A+BkC)$ in classical finite dimensional theory – form an analytic family, in the gain parameter k, of unbounded operators, in Hilbert state space, in the sense of norm resolvent convergence (cf. Kato [17]). This result together with a generalization the Trotter-Kato theorem provides a simple proof that the semigroups $S_k(t)$, with infinitesimal generators A_k, converge in the uniform operator topology to the semigroup generated by the zero dynamics, $S_\infty(t)$.

The main result of the present paper is to provide a nonlinear enhancement of the root locus results obtained for linear distributed parameter systems in [5] for a boundary controlled, viscous Burgers' equation. This work extends the results obtained in [10] for the unforced problem. In the unforced case all trajectories converge to zero for increasing time. This is in marked contrast with the present case in which the forced dynamics can have more complicated behavior as time increases. In fact we will show that for all positive values of the gain parameters and for a forcing term independent of time, there is always a compact local attractor. For the zero dynamics we can show (cf. [2, 14]) that this attractor consists of a single point. Our main result is that the closed-loop trajectories uniformly approach the trajectories of the zero dynamics as the gain parameters are tuned. And a corollary to our main convergence result is that for sufficiently small initial data and forcing term the attractors of the closed loop system converge to the attractor for the zero dynamics. Thus we establish, in this case, that boundary control provides a design method for stabilizing the system about the attractor of the zero dynamics. The most important result used in the proofs is that the linear part of the closed loop spatial Burgers' operator form a holomorphic family, with respect to the gain parameters, in the sense of norm resolvent convergence and that the nonlinear term is a Lipschitz operator with respect to the Hilbert scale norm generated by the square root of the linear part. This paper represents an extension of our earlier work [10] in which we proved convergence of trajectories for the unforced case. In that case we were able to obtain exponential stability along with the convergence of the trajectories while in this case we obtain the existence of local attractors and convergence of the local attractor to the single global asymptotically stable equilibrium for the zero dynamics.

As a final comment, we note that in this work we are particularly interested in the control of a viscous Burgers' equation as a continuation of our earlier work [7]-[10]. Nevertheless, it is possible to specify, in a variety of ways, classes of problems which contain the Burgers' model as a special case, for which a parallel entire theory can be established. Such generalizations do present certain nontrival difficulties which the authors address in a forthcoming paper devoted to a general class of convective reaction diffusion equations on bounded domains in n-dimensional space. Another approach to generalizing these results, which the authors have not considered but clearly would work, is to state the entire problem in terms of abstract operators in Hilbert space. In this case the family of closed loop operators A_k defined below could be replaced by a family of positive self-adjoint operators with compact resolvent which are holomorphic in the parameter k. In addition, somewhat more general nonlinear terms can be considered, but in general the norm of the nonlinear term must, in some sense, be bounded by the norm of some fractional power of the operators A_k. Carrying out such an abstract project might be an interesting objective for future work.

2 Statement of Main Results

Consider the controlled viscous Burgers' system

$$\begin{aligned} & z_t(x,t) + \epsilon A_0 z(x,t) = F(z(x,t)) + f(x,t), \qquad x \in (0,1),\ t>0 \\ & -z_x(0,t) = u_0(t), \quad z_x(1,t) = u_1(t), \\ & z(x,0) = \phi(x), \\ & y_0(t) = z(0,t), \quad y_1(t) = z(1,t) \end{aligned} \tag{2.1}$$

where $\epsilon > 0$ is a fixed viscosity, $u_0(t)$, $u_1(t)$ are boundary inputs, $y_0(t)$, $y_1(t)$ are boundary outputs, $f(t) \in L^2(0,1)$ is an external forcing term modeling an unknown disturbance, and A_0 is the unbounded selfadjoint operator defined by the differential operator

$$A = -\frac{d^2}{dx^2} \tag{2.2}$$

with dense domain in $L^2(0,1)$

$$D(A_0) = \{f \in H^2(0,1) \,:\, f'(0) = f'(1) = 0\}, \tag{2.3}$$

$(' = d/dx)$ and

$$F(z) = -z'z. \tag{2.4}$$

Formally introducing proportional error feedback in the form

$$u_0 = -k_0 y_0, \quad u_1 = -k_1 y_1, \tag{2.5}$$

with feedback gains $k_0,\ k_1 \in \mathbb{R}$, we obtain the closed loop Burgers' system

$$\begin{aligned} &z_t(x,t) + \epsilon A_k z(x,t) = F(z(x,t)) + f(x,t), \quad x \in (0,1),\ t > 0 \\ &-z_x(0,t) + k_0 z(0,t) = 0, \\ &z_x(1,t) + k_1 z(1,t) = 0, \\ &z(x,0) = \phi(x), \end{aligned} \tag{2.6}$$

where $A_k = A$, $k = (k_0, k_1)$, with domain

$$\mathcal{D}(A_k) = \{f \in H^2(0,1) :\ f'(0) - k_0 f(0) = 0,\ f'(1) + k_1 f(1) = 0\}. \tag{2.7}$$

The operator A_k, for all real k_0 and k_1, is a selfadjoint operator, which is strictly positive, with A_k^{-1} compact for $k_0, k_1 \geq 0$ and $k_0 + k_1 > 0$ (see, e.g. [17], pages 146, 148 and 157 and also [13]). We note that for $k_0 = k_1 = 0$, the uncontrolled system (2.1) is obtained and, in this paper, we restrict the gain parameters by $k_0 \geq 0$ and $k_1 \geq 0$ and $k_0 + k_1 > 0$.

In analogy with the lumped nonlinear case [11], we define the zero dynamics for the controlled Burgers' system to be the system obtained by constraining the output to be identically zero in time. This corresponds to the system

$$\begin{aligned} &z_t(x,t) + \epsilon A_\infty z(x,t) = F(z(x,t)) + f(x,t), \quad x \in (0,1),\ t > 0, \\ &z(1,t) = 0, \quad z(0,t) = 0, \\ &z(x,0) = \phi(x), \end{aligned} \tag{2.8}$$

where $A_\infty = A$ is positive self-adjoint with domain

$$\mathcal{D}(A_\infty) = \{f \in H^2(0,1) :\ f(0) = 0,\ f(1) = 0\}. \tag{2.9}$$

Evidently, the zero dynamics also can be formally identified with the boundary value system obtained as the feedback gain parameters tend to infinity. For this reason, it is natural to expect that trajectories of the closed loop system converge in some sense to the trajectories of the zero dynamics as the feedback gains tend to infinity. For zero forcing term, (2.8) is the classical Burgers' equation with Dirichlet boundary conditions which is well-known to be exponentially stable on $H^1(0,1)$. Indeed, in the series of papers [3], [4] optimal control methods are used to develop feedback strategies which enhance stability on this dense subspace of $L^2(0,1)$. In the language of lumped nonlinear control we would refer to the unforced system as being "minimum phase." On the other hand, the unforced open-loop system, which corresponds to the Burgers' equation with Neumann boundary conditions, is easily shown not to be asymptotically stable. Indeed, the open-loop dynamics has a one dimensional center manifold, corresponding to the constant solutions. As predicted by the center manifold theorem, trajectories of solutions, even for small initial data orthogonal to the constants, may approach this center manifold without going to zero. In contrast to the case of Dirichlet or periodic boundary conditions, for which stability can be derived using either Lyapunov or Hopf-Cole methods, a complicating feature arises for boundary conditions which are neither Dirichlet nor periodic. Nonetheless, inspired by geometric methods for nonlinear control, we observe that the feedback laws (2.5) locally asymptotically stabilize the open loop Burgers' equation given by Neumann boundary conditions in $H^1(0,1)$, [8]. Our analytic techniques also allow for a preliminary analysis of the robustness of these schemes with respect to the effects of small disturbances, modeled as forcing terms.

The unforced linearization about zero of (2.6) is the controlled heat equation

$$
\begin{aligned}
&z_t(x,t) + \epsilon A_k z(x,t) = 0, \quad x \in (0,1),\ t>0,\\
&-z_x(0,t) + k_0 z(0,t) = 0,\\
&z_x(1,t) + k_1 z(1,t) = 0,\\
&z(x,0) = \phi(x).
\end{aligned} \tag{2.10}
$$

For $k_0 = k_1 = 0$, neither (2.6) nor the linearization about zero of (2.6) is asymptotically stable, but for $k_0 + k_1 > 0$ the linearization (2.10) is asymptotically stable. As a consequence of the spectral theorem for unbounded selfadjoint operators, it is easy to see that $(-\epsilon A_k)$ generates a holomorphic semigroup $S_k(t)$ which can be represented explicitly in terms of its orthonormal eigenfunctions (described below), as

$$
S_k(t)f = \sum_{j=1}^{\infty} e^{-\epsilon\lambda_j(k)t} f_j^k \psi_j^k, \ \ f_j^k = \int_0^1 f(x)\psi_j^k(x)\,dx.
$$

A straightforward calculation based on the explicit eigenfunctions and eigenvalues given in (2.12)-(2.16) shows that this semigroup is exponentially stable in $L^2(0,1)$; i.e.,

$$||S_k(t)|| \le e^{-\epsilon\lambda_1(k)t}, \tag{2.11}$$

where $\lambda_1(k)$ denotes the first eigenvalue of A_k.

Thus the system (2.10) has solution $z(t) = \exp(-A_k t)\phi$ satisfying

$$\|z(t)\| = \|S_k(t)\phi\| \le e^{-\epsilon\lambda_1 t}\|\phi\|.$$

Here and below we use the same notation $\|\cdot\|$ for the norm in $L^2(0,1)$

$$\|\phi\| = \left(\int_0^1 |\phi(x)|^2\,dx\right)^{1/2},$$

and also for the operator norm for a bounded operator on $L^2(0,1)$.

The spectrum of A_k is readily obtained by noting that a basis of solutions (for all λ in the complex plane) of the equation

$$y''(x) + \lambda y(x) = 0$$

is given by

$$y_1(x) = \frac{\sin(\mu x)}{\mu}, \quad y_2(x) = \cos(\mu x)$$

where $\lambda = \mu^2$ and $\Re(\mu) \ge 0$. Thus every eigenfunction can be written as a linear combination of these basis functions. Applying the boundary conditions in (2.7) to a linear combination of these basis functions and computing the determinant of the resulting coefficient matrix, we obtain the characteristic equation

$$\left(1 - \frac{\mu^2}{k_0 k_1}\right)\frac{\sin(\mu)}{\mu} + \left(\frac{1}{k_0} + \frac{1}{k_1}\right)\cos(\mu) = 0. \tag{2.12}$$

This equation has infinitely many zeros $\{\mu_j(k)\}_{j=1}^{\infty}$ satisfying

$$(j-1)\pi \le \mu_j(k) \le j\pi, \; j = 1, 2, \cdots \tag{2.13}$$

providing the closed loop eigenvalues

$$\lambda_j(k) = \mu_j(k)^2.$$

For $k_0, k_1 \ge 0$, $k_0 + k_1 > 0$, the above inequalities are strict and from (2.12) it is easy to see that

$$(\lambda_j(k) - (j\pi)^2) \to 0, \; k_0, k_1 \to \infty, \; j = 1, 2, \cdots. \tag{2.14}$$

Corresponding to the eigenvalues $\lambda_j(k) = \mu_j^2(k)$, there is a complete orthonormal system of eigenfunctions in $L^2(0,1)$ given by

$$\psi_j^k(x) = \kappa_j(k)\sin\left(\mu_j(k)x + \theta_j(k)\right), \quad j = 1, 2, \cdots, \tag{2.15}$$

where in (2.15)

$$\sin(\theta_j(k)) = \frac{\mu_j(k)}{\sqrt{k_0^2 + \mu_j(k)^2}}, \quad \cos(\theta_j(k)) = \frac{k_0}{\sqrt{k_0^2 + \mu_j(k)^2}},$$

and $\kappa_j(k)$ is a normalization constant given by

$$\kappa_j(k) = \sqrt{\frac{2a_0a_1}{a_0a_1 + (1/k_0 + 1/k_1)c}} \tag{2.16}$$

with

$$a_0 = 1 + \mu_j(k)^2/k_0^2, \quad a_1 = 1 + \mu_j(k)^2/k_1^2, \quad c = 1 + \mu_j(k)^2/(k_0k_1).$$

By direct calculation or a purely formal limit calculation using the above formulas, we obtain the eigenvalues and eigenfunctions for the operator A_∞ defined in (2.9) corresponding to $k_0 = k_1 = \infty$

$$\lambda_j(\infty) = (j\pi)^2\,, \ \psi_j^\infty(x) = \sqrt{2}\sin(j\pi x). \tag{2.17}$$

This formal limit procedure can also be proved rigorously using the root locus techniques developed in [5] together with the fact that the operators A_k form a holomorphic family in the sense of Kato, (see Example 1.15, page 374, [17]).

An immediate consequence of the spectral representation theorem is that the operators A_k define an infinite scale of Hilbert spaces $\mathcal{H}_k^\alpha$ $(\alpha \in \mathbb{R})$. For each $\alpha \geq 0$ the space $\mathcal{H}_k^\alpha$ consists of vectors $\phi \in \mathcal{H}_k^0 = L^2(0,1)$ such that

$$\|\phi\|_{\alpha,k} = \left(\sum_{j=1}^{\infty} \lambda_j^\alpha(k)(\phi,\psi_j^k)^2\right)^{1/2} < \infty. \tag{2.18}$$

These same spaces can also be described in a different way. Namely, the space $\mathcal{H}_k^\alpha$ is the domain of the operator $A_k^{\alpha/2}$ with inner product space given by

$$(\phi,\psi)_\alpha = \left(A_k^{\alpha/2}\phi, A_k^{\alpha/2}\psi\right), \tag{2.19}$$

which is the same as (2.18) for $\psi = \phi$. The operator $A_k^{\alpha/2}$ is defined on $\mathcal{H}_k^\alpha$ by the formula

$$A_k^{\alpha/2}\phi = \sum_{j=1}^{\infty} \lambda_j(k)^{\alpha/2}(\phi,\psi_j^k)\psi_j^k. \tag{2.20}$$

Lemma 2.1 *For $k = (k_0,k_1) > 0$ or $k = (\infty,\infty)$, the spaces $\mathcal{H}_k^\alpha$ have the following properties:*

1. *If* $\beta > \alpha$, *then* $\mathcal{H}_k^\beta \subset \mathcal{H}_k^\alpha$ *and*

$$\|\phi\|_{\alpha,k} \le \lambda_1(k)^{(\alpha-\beta)/2}\|\phi\|_{\beta,k} \tag{2.21}$$

   for all $\phi \in \mathcal{H}_k^\beta$;

2. $\mathcal{H}_k^\beta$ *is dense in* $\mathcal{H}_k^\alpha$; *and*

3. *The embedding* $\mathcal{H}_k^\beta \subset \mathcal{H}_k^\alpha$ *is compact.*

The proof of this lemma is easily established using elementary $L^2(0,1)$ estimates and the formulas in (2.12)-(2.23). For example, as a consequence of (2.13) and (2.18), for $\phi \in \mathcal{H}_k^\beta$, the first statement of the lemma can be established as follows:

$$\begin{aligned}
\|\phi\|_{\alpha,k}^2 &= \sum_{j=1}^\infty \lambda_j(k)^\alpha (\phi, \psi_j^k)^2 \\
&= \lambda_1(k)^\alpha \sum_{j=1}^\infty \left(\frac{\lambda_j(k)}{\lambda_1(k)}\right)^\alpha (\phi, \psi_j^k)^2 \\
&\le \lambda_1(k)^\alpha \sum_{j=1}^\infty \left(\frac{\lambda_j(k)}{\lambda_1(k)}\right)^\beta (\phi, \psi_j^k)^2 \\
&= \lambda_1(k)^{(\alpha-\beta)} \sum_{j=1}^\infty \lambda_j(k)^\alpha (\phi, \psi_j^k)^2 \\
&= \lambda_1(k)^{(\alpha-\beta)} \|\phi\|_{\beta,k}^2.
\end{aligned}$$

Using the fact that $\mathcal{D}(A_k)$ is a core in $\mathcal{D}(A_k^{1/2}) = \mathcal{H}_k^1$ (cf, Theorem 3.35, [17]), it is easy to show that the norm in $\mathcal{H}_k^1$ can be written as

$$\|\varphi\|_{1,k}^2 = \begin{cases} \|\varphi_x\| + k_0|\varphi(0)|^2 + k_1|\varphi(1)|^2, \\ \quad \text{for } k = (k_0, k_1) \in (0,\infty) \times (0,\infty), \\ \|\varphi_x\|^2 + k_0|\varphi(0)|^2, \\ \quad \text{for } k_0 \in (0,\infty),\ k_1 = \infty, \\ \|\varphi_x\|^2 + k_1|\varphi(1)|^2, \\ \quad \text{for } k_0 = \infty,\ k_1 \in (0,\infty) \end{cases} \tag{2.22}$$

and

$$\|\varphi\|_{1,\infty}^2 = \|\varphi_x\|^2,\ \text{for } k = (\infty,\infty). \tag{2.23}$$

Let $H^1(0,1)$ denote the usual Sobolev space with norm

$$\|\varphi\|_{H^1(0,1)}^2 = \|\varphi_x\|^2 + \|\varphi\|^2.$$

Then, as sets, the spaces $\mathcal{H}^1_k$ are given explicitly by

$$\mathcal{H}^1_k = \begin{cases} H^1(0,1), \text{ for } k = (k_0, k_1) \in (0,\infty) \times (0,\infty), \\ H^1(0,1) \cap \{\varphi : \varphi(1) = 0\}, \text{ for } k_0 \in (0,\infty), \; k_1 = \infty, \\ H^1(0,1) \cap \{\varphi : \varphi(0) = 0\}, \text{ for } k_0 = \infty, k_1 \in (0,\infty), \end{cases} \tag{2.24}$$

and

$$\mathcal{H}^1_\infty = H^1(0,1) \cap \{\varphi : \varphi(0) = 0, \varphi(1) = 0\}, \text{for } k_0 = k_1 = \infty. \tag{2.25}$$

We emphasize that

$$\mathcal{H}^1_\infty = \mathcal{D}(A^{1/2}_\infty) = H^1_0(0,1), \tag{2.26}$$

where $H^1_0(0,1)$ is the standard notation for the Sobolev space of functions in $H^1(0,1)$ which vanish at x equal 0 and 1.

Definition 2.1 *The "zero dynamics" subspace is the natural domain of $A^{1/2}_\infty$, i.e., $\mathcal{H}^1_\infty = H^1_0(0,1)$.*

The results in (2.24) are well known but can easily be obtained from several different points of view. One point of view is the notion of interpolation spaces (see, for example, [1], pages 111-115) and fractional powers of dissipative operators. The basic ideas used in [1] can also be found in several other references but the main idea is generally based on the work of P. Grisvard, [15].

An immediate consequence of part 1 of Lemma 2.1, with $\alpha = 0$, $\beta = 1$, and the representation of the norm in $\mathcal{H}^1_k$ given in (2.22) is the following lemma which will be used repeatedly.

Lemma 2.2 *For any $k = (k_0, k_1) \in (0,\infty] \times (0,\infty]$ and $\varphi \in \mathcal{H}^1_k$, we have*

$$\|\varphi\|^2 \le \lambda_1(k)^{-1} \|\varphi\|^2_{1,k}, \tag{2.27}$$

where $0 < \lambda_1(k) < \pi^2$ is the first eigenvalue of A_k, $\|\cdot\|$ is the norm in $L^2(0,1)$ and $\|\cdot\|_{1,k}$ is defined in (2.22). It follows that for all $\varphi \in \mathcal{H}^1_k$,

$$\|\varphi\|^2_{H^1(0,1)} \le \left(1 + \lambda_1(k)^{-1}\right) \|\varphi\|^2_{1,k}. \tag{2.28}$$

We will also need the following well known relationship between the $L^\infty(0,1)$ norm

$$\|z\|_{L^\infty(0,1)} = \operatorname{ess} \sup_{x \in [0,1]} |z(x)|$$

and the norm in $H^1(0,1)$. In general such results follow from the classical Sobolev embedding theorem but this special case can easily be established using elementary calculus, the Cauchy-Schwartz inequality and Lemma 2.2.

Lemma 2.3 *For $z \in H^1(0,1)$ we have the estimate*

$$\|z\|_{L^\infty(0,1)} \leq \sqrt{2}\|z\|_{H^1(0,1)},$$

and, hence, for $z \in \mathcal{H}^1_k$,

$$\|z\|_{L^\infty(0,1)} \leq c\|z\|_{1,k}$$

where

$$c = \sqrt{2}(1+\lambda_1(k)^{-1})^{1/2}. \tag{2.29}$$

From (2.22), for all finite $k = (k_0, k_1) > 0$ and $\phi \in \mathcal{H}^1_\infty$, we have the norm equality

$$\|\phi\|_{1,k} = \|\phi\|_{1,\infty}. \tag{2.30}$$

Furthermore, we note that for all finite $k = (k_0, k_1) > 0$ and $\phi \in \mathcal{H}^1_k$, with $\|\phi\|_{1,k} \leq \rho$, it follows from (2.22) that

$$|\phi(0)| \leq \frac{\rho}{\sqrt{k_0}}, \quad |\phi(1)| \leq \frac{\rho}{\sqrt{k_1}}.$$

So, as k_0 and k_1 tend to infinity, we must have

$$\phi(0) = \phi(1) = 0.$$

This implies

$$\bigcap_{k>0} \{\phi \in \mathcal{H}^1_k \,:\, \|\phi\|_{1,k} \leq \rho\} = \{\phi \in \mathcal{H}^1_\infty \,:\, \|\phi\|_{1,\infty} \leq \rho\}. \tag{2.31}$$

We are now in a position to state the main results of the paper. First in Theorem 2.1 we state a global in time existence and uniqueness result for small initial data and forcing terms. The proof of this result can be obtained using the same methods found, for example, in [3, 4], [8], [16], [18]. In particular, (cf., Theorem 3.3.3, 3.3.4 and 5.1.1 in [16]) the proof is based on assuming the initial data to be small and obtaining certain estimates from the variation of parameters formula which ensure that the solution will stay small for all time. In our case, how small the initial data and forcing term must be in order to guarantee global existence of a solution depends on the values of the gain parameters k_0 and k_1. This is due to the fact that for $k_0 = k_1 = 0$ the linearization about zero possesses a zero eigenvalue and certain estimates no longer hold, e.g., the generalized Poincare inequality (cf. Lemma 2.2 part 1). Since the primary goal in this paper is to consider the behavior of solutions for large values of k_0 and k_1 – the high gain limit – the existence Theorem 2.1 is not stated in most generality. Rather, we first assume that k_0 and k_1 are sufficiently large, i.e., greater than a certain value $\widetilde{k}$. With this assumption we can state the existence Theorem 2.1 for a fixed ball of initial data in the zero dynamics subspace $\mathcal{H}^1_\infty$ and the same forcing term f. This result is valid for all $(k_0, k_1) \geq \widetilde{k}$ for (2.6) and for $k_0 = k_1 = \infty$ in (2.8).

It is easy to see from (2.12) that the eigenvalues $\lambda_j(k)$ are monotone increasing functions of k_0 and k_1 and as mentioned in (2.14),

$$\lim_{k_0, k_1 \to \infty} \lambda_1(k) = \pi^2.$$

In general, it is well known that multivariable root locus analysis, even for finite dimensional linear systems, can be quite complicated and that typically the way in which the gain parameters tend to infinity can be critical. Nevertheless, for the closed loop system described in (2.10), the root locus analysis is quite simple. Indeed, for gains (k_0, k_1) restricted to $[0, \infty] \times [0, \infty]$ the closed loop poles are real and nonnegative and eigenfunctions defined in (2.15)-(2.16) vary continuously with respect to the gains. That is, it is possible to pass to the limit, as either k_0 or k_1 tends to infinity first, and then the other, and the resulting limit is the same. In order to avoid the necessity of considering all the resulting special cases that arise in dealing with the cases in which either k_0 or k_1 becomes infinite while the other is finite, we will make the following assumption on our passage to the high gain limit.

Assumption 2.1

1. *We assume that* $(k_0, k_1) = k \geq \widetilde{k}$ *where* $\widetilde{k}$ *has been chosen so that*

$$\lambda_1 \equiv \lambda_1(\widetilde{k}) > \frac{3\pi^2}{4}. \tag{2.32}$$

2. *Throughout the paper we use the slight abuse of notation* $k \geq \widetilde{k}$ *where* $k = (k_0, k_1)$ *is a pair and* $\widetilde{k}$ *is a number. This is to be interpreted as meaning that* $k_0, k_1 \geq \widetilde{k}$.

3. *Finally, in using the notation*

$$\lim_{k_0, k_1 \to \infty} g(k_0, k_1) = L$$

in a normed space with norm $\|\cdot\|$, *it is understood that given* $\delta > 0$, *there is a* $\widetilde{k} > 0$ *so that*

$$\|g(k_0, k_1) - L\| < \delta$$

for $k_0, k_1 \geq \widetilde{k}$ *with* k_0 *and* k_1 *restricted to a sector*

$$0 < \theta < \tan^{-1}\left(\frac{k_1}{k_0}\right) \leq \frac{\pi}{2} - \theta$$

for some $\theta \in (0, \pi/2)$.

Note that the last assumption ensures that in passing to the limit in k_0 and k_1 it is not possible for one to become infinite while the other remains finite.

The main result of the paper concerning convergence of trajectories is contained in Theorem 2.2. The proof of this result is given in the appendix in a series of lemmas, several of which are of independent interest.

Next we state a compactness result, Theorem 2.3, for the trajectories of solutions to (2.6) in the Sobolev space $\mathcal{H}^1_k$. This result can be obtained with appropriate modification of Theorem 3.3.6 in [16] together with the compact embedding in part 3 of Lemma 2.1. So once again the proof will be omitted.

Next in Theorem 2.4 we present a continuous dependence result in $\mathcal{H}^1_k$ which also shows that for fixed gain values, different initial values and the same forcing term, the difference of the trajectories converge to zero exponentially as $t \to \infty$.

These last two results are needed in the proof of our final Theorem 2.6 concerning the convergence of local attractors in the high gain limit. Before stating Theorem 2.6 we describe how our closed loop system (2.6) and zero dynamics (2.8) generate nonlinear semigroups in a bounded complete metric phase space in $H^1(0,1)$. This provides us with the standard setup to define certain nonlinear dynamical systems and discuss the existence of local attractors. Namely, in Theorem 2.5 we state the existence of a local attractor for the system (2.6) in $\mathcal{H}^1_k$.

Theorem 2.1 *Let ϵ be a fixed positive viscosity, $k = (k_0, k_1) \geq \widetilde{k}$ and $\rho = \rho_k > 0$ be any positive number satisfying*

$$\rho < \frac{\epsilon}{4}. \tag{2.33}$$

Assume that $t \to f(\cdot, t) : (0,\infty) \to L^2(0,1)$ is locally Hölder continuous with $f \in L^\infty([0,\infty), L^2(0,1))$ such that

$$\operatorname{ess} \sup_{t\in[0,\infty)} \|f(\cdot,t)\| \leq \frac{\epsilon\rho}{3}. \tag{2.34}$$

Then for every initial condition $\phi \in H^1(0,1)$ with

$$\|\phi\|_{1,k} \leq \rho/4 \tag{2.35}$$

there is a unique solution $z_k \in C([0,\infty), H^1(0,1)) \cap C^1([0,\infty), L^2(0,1))$ of (2.6) and, moreover, this solution satisfies

$$\|z_k(t)\|_{1,k} \leq \rho, \text{ for all } t \geq 0. \tag{2.36}$$

Furthermore, for every initial condition $\phi \in \mathcal{H}^1_\infty = H^1_0(0,1)$ with

$$\|\phi\|_{1,\infty} \leq \rho/4 \tag{2.37}$$

there is a unique solution $z_\infty \in C([0,\infty), H^1(0,1)) \cap C^1([0,\infty), L^2(0,1))$ *of (2.8) which satisfies*

$$\|z_\infty(t)\|_{1,\infty} \le \rho, \text{ for all } t \ge 0. \tag{2.38}$$

The main result of the paper is contained in the following theorem.

Theorem 2.2 *Let* z_k *and* z_∞ *denote the solutions of (2.6) and (2.8), respectively for the same initial data* $\phi \in \mathcal{H}^1_\infty$ *satisfying (2.35) and forcing term* f *satisfying (2.34).*

Then

$$\lim_{(k_0,k_1)\to\infty} \sup_{t\ge 0} \|z_k(\cdot,t) - z_\infty(\cdot,t)\|_{1,k} = 0 \tag{2.39}$$

and

$$\lim_{(k_0,k_1)\to\infty} \sup_{t\ge 0} \sup_{x\in[0,1]} |z_k(x,t) - z_\infty(x,t)| = 0; \tag{2.40}$$

i.e., the trajectories converge uniformly in space and time.

We note that, due to Lemma 2.3, (2.40) follows from (2.39).

The following result shows that for fixed $k = (k_0, k_1)$, the trajectories of solutions given in Theorem 2.1 lie in a fixed compact set in $\mathcal{H}^1_k$.

Theorem 2.3 *Assume the hypotheses in Theorem 2.1 on* ϕ, f *and* $k = (k_0, k_1)$. *Then for* $1 < \alpha < 2$, *there is a continuous bounded function* $C(\|\phi\|, \alpha, k) > 0$ *such that*

$$\|z_k(t)\|_{\alpha,k} \le C(\|\phi\|, \alpha, k) \tag{2.41}$$

for all $t \in [0,\infty)$, *where* $z_k(t)$ *is the solution of Theorem 2.1 for* $k_0, k_1 \in [\widetilde{k}, \infty]$. *Thus due to the compact embedding of* $\mathcal{H}^\alpha_k$ *in* $\mathcal{H}^1_k$, *the set*

$$\{z_k(t)\}_{t\ge 0}$$

lies in a fixed compact set in $\mathcal{H}^1_k$.

Theorem 2.4 *Assume the hypotheses of Theorem 2.1 and let* z^1_k, z^2_k *denote the solutions given in Theorem 2.1 for initial data* ϕ^1 *and* ϕ^2 *satisfying*

$$\|\phi^j\|_{1,k} \le \frac{\rho}{4}, \ j = 1, 2, \tag{2.42}$$

and with the same forcing term $f = f(x)$ *(independent of* t*) satisfying*

$$\|f\| \le \frac{\epsilon\rho}{3}. \tag{2.43}$$

Then for all $t \geq 0$ we have

$$\|z_k^1(t) - z_k^2(t)\|_{1,k} \leq e^{-\epsilon\pi^2 t/8}\|\phi^1 - \phi^2\|_{1,k} \tag{2.44}$$

and hence by (2.42)

$$\|z_k^1(t) - z_k^2(t)\|_{1,k} \leq \frac{\rho}{2} e^{-\epsilon\pi^2 t/8} \tag{2.45}$$

We now assume that the forcing term $f \in L^2(0,1)$ is independent of time. Let

$$B_0^k = \{\phi \in \mathcal{H}_k^1 \,:\, \|\phi\|_{1,k} < \rho/4\},$$
$$B_1^k = \{\phi \in \mathcal{H}_k^1 \,:\, \|\phi\|_{1,k} < \rho\},$$

then we can define the mappings

$$T_t^k(\phi) = z_k(t)$$

where $z_k(t)$ is the solution to (2.6) for k finite or infinite and for initial data $\phi \in B_0^k$ given in Theorem 2.1. By Theorem 2.1 $T_t^k(\phi)$ is in B_1^k for all $t \geq 0$. That is,

$$T_t^k(B_0^k) \subset B_1^k, \quad \forall\, t \geq 0.$$

Consider now the set

$$\gamma_k(B_0^k) = \bigcup_{t \geq 0} T_t^k(B_0^k).$$

T_t^k can be naturally extended to $\gamma_k(B_0^k)$; i.e., if $\phi \in \gamma_k(B_0^k)$, then there is a $\phi_0 \in B_0^k$ and $t_0 \geq 0$ such that $\phi = T_{t_0}^k(\phi_0)$ and for $t \geq 0$ it makes sense to define $T_t^k(\phi) = T_{t_0+t}^k(\phi_0)$. The family $\{T_t^k,\ t \geq 0\}$ becomes a semigroup of transformations on $\gamma_k(B_0^k)$ which can be extended to $\overline{\gamma_k(B_0^k)}$ by continuity. Thus, in this way, we obtain a nonlinear semigroup $\{T_t^k,\ t \geq 0\}$ with phase space

$$B^k = \overline{\gamma_k(B_0^k)}.$$

We see that $\gamma_k(B_0^k) \subset B_1^k$; so, the whole phase space of our dynamical system is bounded. From Theorem 2.3 we see that all the mappings T_t^k with $t > 0$ are compact; i.e., if $\widetilde{B} \subset B^k$ then $T_t^k(\widetilde{B})$ is relatively compact in $\mathcal{H}_k^1$.

Theorem 2.5 *Define*

$$\mathcal{A}_k = \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} T_t^k(B_0^k)}. \tag{2.46}$$

Then $\mathcal{A}_k$ is a local attractor:

1. *$\mathcal{A}_k$ is nonempty, connected and compact.*

2. *$\mathcal{A}_k$ attracts all bounded subsets of B^k:*

$$\lim_{t\to\infty} d_k\left(T_t^k(\widetilde{B}), \mathcal{A}_k\right) = 0, \quad \text{for all} \quad \widetilde{B} \subset B^k.$$

   Here $d_k(C,D) = \sup_{c\in C} \inf_{d\in D} \|c-d\|_{1,k}$.

3. *$\mathcal{A}_k$ is invariant: $T_t^k(\mathcal{A}_k) = \mathcal{A}_k$ for all $t \geq 0$.*

The proof of statements 1-3 of Theorem 2.5 are simple consequences of Theorems 2.1, 2.3 and the general results on attractors found, for example, in [20]-[23].

An alternative characterization of $\mathcal{A}_k$ is given by the set of all convergent sequences of the form $T_{t_j}^k(\phi_j)$ for $\phi_j \in B_0^k$ and $t_j \uparrow \infty$ (see [21, 22, 23]). We will use this formulation to show that the attractors $\mathcal{A}_k$ converge to $\mathcal{A}_\infty$ as $k \to \infty$. As a consequence of the results in [2] it follows that the local attractor for the zero dynamics consists of a single point; i.e., $\mathcal{A}_\infty = z^\infty$ with $z^\infty \in H_0^1(0,1)$. Indeed, based on the work in [2] or by straightforward application of the Hopf-Cole transformation, it can be shown that there is a unique stationary solution that is globally attractive for any positive value of the viscosity ϵ and for any forcing term f.

Theorem 2.6

$$\lim_{k_0,k_1\to\infty} d(\mathcal{A}_k, \mathcal{A}_\infty) = 0, \tag{2.47}$$

where

$$d(\mathcal{A}_k, \mathcal{A}_\infty) = \sup_{\phi\in\mathcal{A}_k} \|\phi - z^\infty\|_{H^1(0,1)}.$$

The proof of Theorem 2.6 is given in the appendix.

Remark 1 In this section we were able to define a family of nonlinear semigroups depending on the gain parameters k_0 and k_1 and make several statements concerning properties of the nonlinear semigroups. The proofs of Theorems 2.1, 2.3 and 2.5 can be obtained using modifications of proofs that can be found elsewhere (see, e.g., [16, 18, 20, 21, 22, 23]) and so to conserve space we have not included these proofs.

While the proof of Theorem 2.1 is not included, we will provide some remarks that might be helpful in understanding how the results in [16] can be modified to handle the present case.

A straightforward modification of the uniqueness and existence results in Theorem 3.3.3, 3.3.4 and 5.1.1 of [16] show that our assumption on ρ is not best possible. For example, for $k_0 \geq 0$, $k_1 \geq 0$ with $k_0 + k_1 > 0$ and for initial data $\phi \in \mathcal{H}_k^1$ with

$$\|\phi\|_{1,k} \leq \frac{\rho}{4}$$

we need only take ρ a positive number satisfying

$$c\mathcal{C}_k\rho \leq \frac{1}{2} \tag{2.48}$$

with c defined in (2.29), and

$$\mathcal{C}_k = \sqrt{\frac{2\pi}{\epsilon^2 e \lambda_1(k)}} \tag{2.49}$$

is defined in (4.8).

Under Assumption 2.1 for $k \geq \widetilde{k}$, we have

$$\frac{3\pi^2}{4} \leq \lambda_1(k) \leq \pi^2$$

and it is easy to show that

$$c\mathcal{C}_k \leq \frac{1}{\epsilon}$$

so that

$$\rho \leq \frac{\epsilon}{2}$$

implies (2.48). We have chosen the extra smallness condition in (2.33) since it is needed in the proof of Theorem 2.4. Similarly, we need

$$\mathcal{C}_k \sup_{t\geq 0} \|f(t)\| \leq \frac{\rho}{4} \tag{2.50}$$

which again is satisfied for all $k \geq \widetilde{k}$ if

$$\sup_{t\geq 0} \|f(t)\| \leq \frac{\epsilon\rho}{3}.$$

The proofs of Theorems 2.2, 2.4 and 2.6 are included in the Appendix following the next section on numerical simulation.

3 Numerical Simulations

In this section we present a numerical example with approximate solutions obtained using a Galerkin method based on the eigenfunctions of the linearization about zero of the spatial part of Burgers' equation. In this example we consider initial data $\phi(x) = x^2(1-x)^2$, forcing term $f \equiv .5(.5-x)$.

The error referred to in the table below is the maximum variation between the closed loop and the zero dynamics solutions on a uniform grid of $[0,1] \times [0,5]$ with mesh size $\Delta x = .01$ and $\Delta t = .01$. In Table 1 we present the maximum deviation between the zero dynamics computed using an eight node approximation and the closed loop solution generated using $N = 2$, 4 and 6 nodes. In this example we have set $\epsilon = (60)^{-1/2}$ and the gains to be equal, $k_0 = k_1 \equiv k$. The values of k are varied from .05 to 1000. For the forcing term in this example, it can be shown (see [14]) that for $k_0 = k_1 > .008$ the closed loop system has a single stationary solution. This helps to explain why the approximate solutions depicted in Figure 3 - Figure 6 approach a single equilibrium. This data was generated using a program developed in MATLAB.

$$w_t - \epsilon w_{xx} + w w_x = .5\,(.5 - x)$$

$$-w_x(0,t) + kw(0,t) = 0$$

$$w_x(1,t) + kw(1,t) = 0$$

$$w(x,0) = x^2(1-x)^2,$$

k	N=2	N=4	N=6
.05	3.2898e-001	3.073e-001	3.072e-001
.5	1.949e-001	1.897e-001	1.899e-001
1	1.356e-001	1.356e-001	1.358e-001
10	2.732e-002	3.005e-002	3.083e-002
50	1.131e-002	6.482e-003	6.887e-003
100	1.033e-002	3.632e-003	3.438e-003
1000	9.509e-003	2.581e-003	7.623e-004

Table 1. Maximum Deviations From Zero Dynamics.
N: number of Galerkin nodes, $k = k_0 = k_1$: gain values

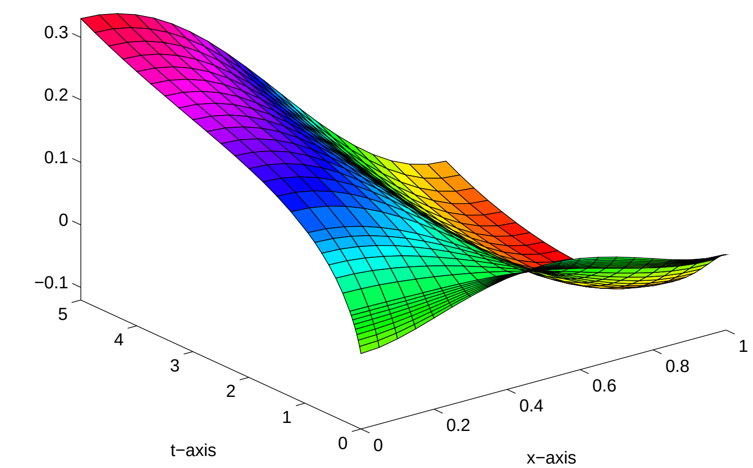

Figure 1. Open Loop Dynamics: $k = 0$

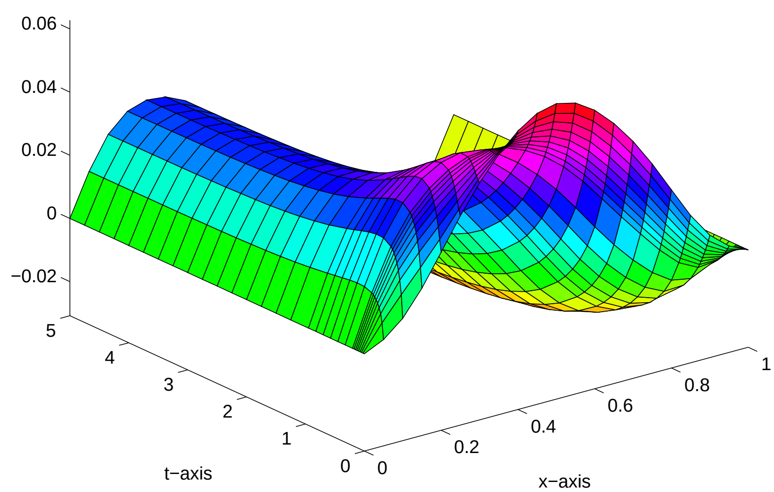

Figure 2. Zero Dynamics: $k = \infty$

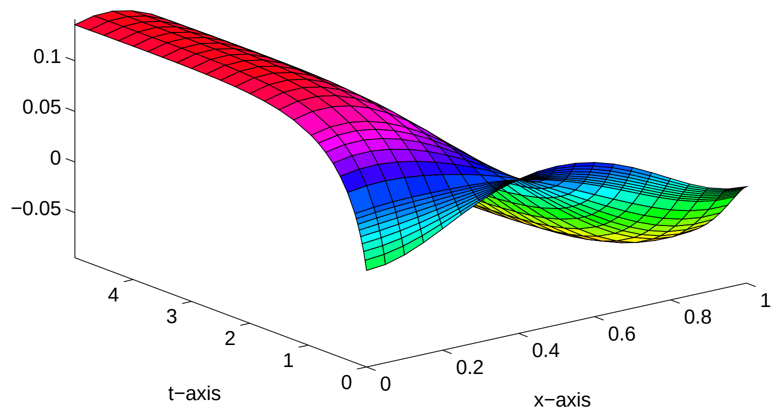

Figure 3. Closed Loop Dynamics: $k = 1$
max deviation from zero dynamics: .136

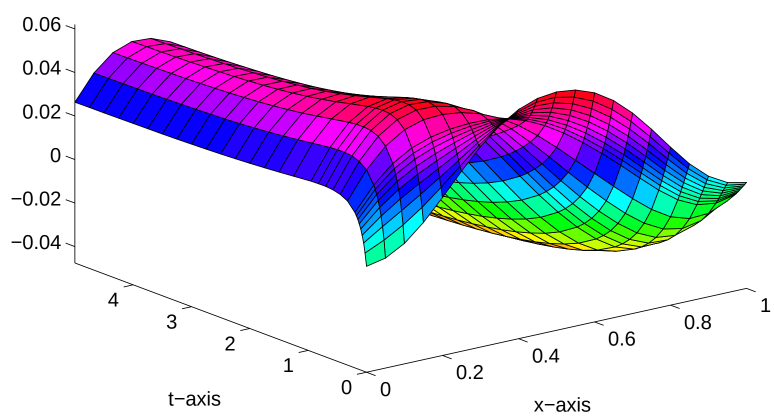

Figure 4. Closed Loop Dynamics: $k = 10$
max deviation from zero dynamics: .03

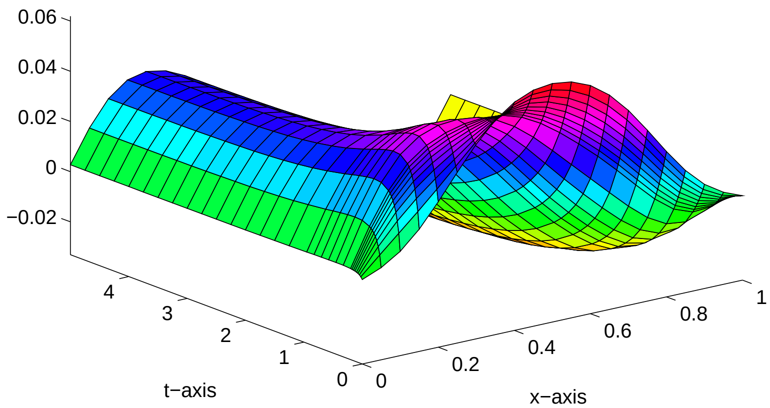

Figure 5. Closed Loop Dynamics: $k = 100$
max deviation from zero dynamics: .003

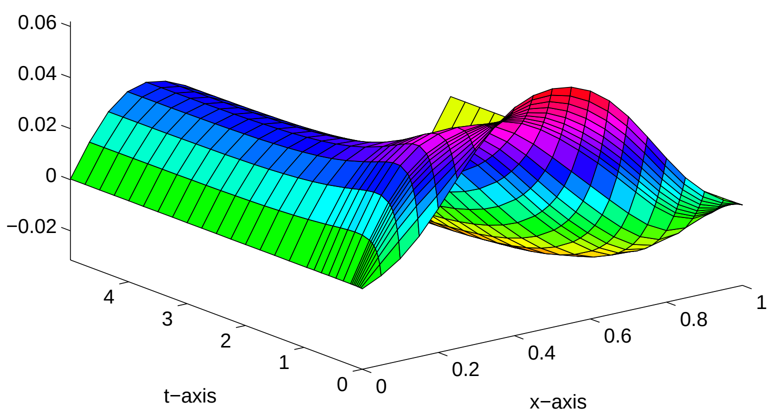

Figure 6. Closed Loop Dynamics: $k = 1000$
max deviation from zero dynamics: .0007

4 Appendix

Throughout all the proofs of Theorems 2.1 - 2.6 the following estimates are used time and time again. First it is easy to see that the nonlinear term F is locally Lipschitz from $\mathcal{H}_k^1$ to $L^2(0,1)$, i.e., for $\|z_1\|_{1,k}$, $\|z_2\|_{1,k} < M$, there exists a constant, C depending only on M, and k but otherwise independent of z_1, z_2 such that

$$\|F(z_1) - F(z_2)\| \le C\|z_1 - z_2\|_{1,k}.$$

In particular, for z_1, z_2 in $\mathcal{H}_k^1$ with $\|z_1\|_{1,k}$, $\|z_2\|_{1,k} < M$

$$\begin{aligned} \|F(z_1) - F(z_2)\| &= \|z_1 z_1' - z_2 z_2'\| \qquad (4.1)\\ &\le \|z_1 z_1' - z_1 z_2' + z_1 z_2' - z_2 z_2'\| \\ &\le \|z_1 z_1' - z_1 z_2'\| + \|z_1 z_2' - z_2 z_2'\| \\ &\le \|z_1\|_\infty \|z_1' - z_2'\| + \|z_2'\| \|z_1 - z_2\|_\infty \\ &\le c\left(\|z_1\|_{1,k}\|z_1 - z_2\|_{1,k} + \|z_2\|_{1,k}\|z_1 - z_2\|_{1,k}\right) \\ &= c\left(\left(\|z_1\|_{1,k} + \|z_2\|_{1,k}\right)\|z_1 - z_2\|_{1,k}\right. \\ &\le C\|z_1 - z_2\|_{1,k}, \end{aligned}$$

where

$$C = 2cM.$$

As a special case of the above proof we note that

Lemma 4.1 *For $z \in \mathcal{H}_k^1$ and $F(z) = -zz_x$,*

$$\|F(z)\| \le c\left(\|z\|_{1,k}\right)^2, \qquad (4.2)$$

where

$$c = \sqrt{2}(1 + \lambda_1^{-1})^{1/2}$$

is defined in (2.29).

Proof: This result follows immediately from the inequalities

$$\begin{aligned} \|F(z)\| &= \|zz_x\| \le \|z\|_\infty \|z_x\| \\ &\le \sqrt{2}\|z\|_{H^1(0,1)}\|z\|_{1,k} \le c\|z\|_{1,k}^2. \end{aligned}$$

□

Lemma 4.2 *The following estimates hold*

1. *If $g \in \mathcal{H}_k^\alpha$, then for all $t > 0$, we have*
$$\|S_k(t)g\|_{\alpha,k} \le e^{-\epsilon\lambda_1 t}\|g\|_{\alpha,k}. \tag{4.3}$$

2. *For $\phi \in L^2(0,1)$ and all $t > 0$*
$$\|S_k(t)\phi\|_{\alpha,k} \le C_\alpha \frac{1}{t^{\alpha/2}} e^{-\epsilon\lambda_1 t/2}\|\phi\|, \tag{4.4}$$
where
$$C_\alpha = \left(\frac{\alpha}{\epsilon}\right)^{\alpha/2} e^{-\alpha/2}. \tag{4.5}$$
In the particular case $\alpha = 1$ we have
$$\|S_k(t)\phi\|_{1,k} \le C_1 \frac{1}{t^{1/2}} e^{-\epsilon\lambda_1 t/2}\|\phi\|, \tag{4.6}$$
with
$$C_1 = \frac{1}{\sqrt{e\epsilon}}. \tag{4.7}$$

3. *Using the estimate in 2 we can, by a straightforward integration, obtain*
$$\int_0^\infty C_1 t^{-1/2} e^{-\epsilon\lambda_1(k)t}\, d\tau = C_1\sqrt{\frac{2\pi}{\epsilon\lambda_1(k)}} \equiv \mathcal{C}_k. \tag{4.8}$$

Remark 2 We notice that $S_k(t) : L^2(0,1) \to L^2(0,1)$ is compact for all $t > 0$ from (4.6) and the compactness of the embedding $\mathcal{H}_k^1 \subset L^2(0,1)$ (Lemma 2.1). This is also correct for $S_\infty(t)$ and the estimate for $k_0 = k_1 = \infty$ can be obtained exactly as above.

Proof:

For part 1) we have for $g \in \mathcal{H}_k^\alpha$,
$$\|S_k(t)g\|_{\alpha,k} = \|A_k^{\alpha/2}(S_k(t)g)\| = \|S_k(t)A_k^{\alpha/2}(g)\|$$
$$\le \|S_k(t)\|\|A_k^{\alpha/2}g\| = \|S_k(t)\|\|g\|_{\alpha,k} \le e^{-\epsilon\lambda_1 t}\|g\|_{\alpha,k}.$$

As for part 2), with $\phi \in L^2(0,1)$ one computes
$$\|S_k(t)\phi\|_{\alpha,k} = S_k(t)\phi\|$$
$$= \quad \|\sum_{j=1}^\infty \lambda_j^{\alpha/2} e^{-\epsilon\lambda_j(k)t} < \phi, \psi_j^k > \psi_j^k\|$$

$$= \left(\sum_{j=1}^{\infty} \lambda_j^{\alpha}(k) e^{-2\epsilon\lambda_j(k)t} | < \phi, \psi_j^k > |^2 \right)^{1/2}$$

$$\leq \sup_{j\geq 1, t>0} \left(\lambda_j^{\alpha/2}(k) e^{-\epsilon\lambda_j(k)t} \right) \left(\sum_{j=1}^{\infty} | < \phi, \psi_j^k > |^2 \right)^{1/2}$$

$$= \sup_{j\geq 1, t>0} \left(\lambda_j^{\alpha/2}(k) e^{-\epsilon\lambda_j(k)t} \right) \|\phi\|.$$

We now proceed to estimate

$$\sup_{j\geq 1, t>0} \left(\lambda_j^{\alpha/2}(k) e^{-\epsilon\lambda_j(k)t} \right).$$

To this end define

$$p(\xi) = \xi^{\alpha/2} e^{-\epsilon\xi t}$$

and we have

$$\frac{dp}{d\xi}(\xi) = \xi^{\alpha/2-1} \left(\frac{\alpha}{2} - \xi\epsilon t \right) e^{-\epsilon\xi t}$$

with a single positive critical value at

$$\xi_0 = \frac{\alpha}{2\epsilon t}.$$

Note that p is increasing for $\xi < \xi_0$ and decreasing for $\xi > \xi_0$. Also recall that we need $\xi \geq \lambda_1(k)$. Thus we have two cases: 1) $\xi_0 < \lambda_1(k)$ and 2) $\xi_0 > \lambda_1(k)$.

A straightforward calculation shows that

1. for $\xi_0 < \lambda_1(k)$, the max of p occurs at $\xi = \lambda_1(k)$. Also $\xi_0 < \lambda_1(k)$ implies

$$t > \frac{\alpha}{2\epsilon\lambda_1(k)} \equiv t_c$$

   and in this case

$$\sup_{j\geq 1} \left(\lambda_j^{\alpha/2}(k) e^{-\epsilon\lambda_j(k)t} \right) \leq \lambda_1^{\alpha/2}(k) e^{-\epsilon\lambda_1(k)t}.$$

2. for $\xi_0 > \lambda_1(k)$, the max of p occurs at ξ_0 and $\xi_0 > \lambda_1$ implies

$$t < \frac{\alpha}{2\epsilon\lambda_1(k)} \equiv t_c$$

   and in this case we have

$$\sup_{j\geq 1} \left(\lambda_j^{\alpha/2}(k) e^{-\epsilon\lambda_j(k)t} \right) \leq p(\xi_0) \leq \left(\frac{\alpha}{2\epsilon t} \right)^{\alpha/2} e^{-\alpha/2}.$$

Combining these results we can say that for all $t > 0$

$$\sup_{j\geq 1}\left(\lambda_j^{\alpha/2}(k)e^{-\epsilon\lambda_1(k)t}\right) \leq \begin{cases} \left(\frac{\alpha}{2\epsilon t}\right)^{\alpha/2} e^{-\alpha/2} & \text{for } t < \frac{\alpha}{2\epsilon\lambda_1(k)} \\ \lambda_1^{\alpha/2}(k)e^{-\epsilon\lambda_1(k)t} & \text{for } t > \frac{\alpha}{2\epsilon\lambda_1(k)} \end{cases}.$$

We now look for a constant C_α such that

$$\sup_{j\geq 1}\left(\lambda_j^{\alpha/2}(k)e^{-\epsilon\lambda_1(k)t}\right) \leq C_\alpha \frac{e^{-\epsilon\lambda_1(k)t/2}}{t^{\alpha/2}}, \ \forall\ t > 0. \tag{4.9}$$

For $t > \dfrac{\alpha}{2\epsilon\lambda_1(k)}$ we seek a constant C_+ such that

$$\lambda_1^{\alpha/2}(k)t^{\alpha/2}e^{-\epsilon\lambda_1(k)t/2} \leq C_+.$$

Let $s = \lambda_1 t$, and define $h_+(s) = s^{\alpha/2}\exp(-\epsilon s)$. Then we need to find the maximum of h_+ on $s > \alpha/(2\epsilon)$. Now $h_+(0) = 0$ and $h_+(\infty) = 0$ and $h_+(s) > 0$ for $s > 0$ and it has only one critical value at $s = \alpha/\epsilon$ which gives a maximum value. From this we obtain the desired constant C_+, namely,

$$C_+ = \left(\frac{\alpha}{\epsilon}\right)^{\alpha/2} e^{-\alpha/2}.$$

For $t < \dfrac{\alpha}{2\epsilon\lambda_1}(k)$ we seek a constant C_- such that

$$\left(\frac{\alpha}{\epsilon}\right)^{\alpha/2} e^{-\alpha/2}e^{\epsilon\lambda_1(k)t/2} \leq C_-.$$

In this case we see that the function

$$h_-(t) = \left(\frac{\alpha}{\epsilon}\right)^{\alpha/2} e^{-\alpha/2}e^{\epsilon\lambda_1(k)t/2}$$

is a monotone increasing function of t so the maximum on the interval $[0, t_c]$ occurs at t_c so that

$$h_-(t_c) = \left(\frac{\alpha}{\epsilon}\right)^{\alpha/2} e^{-\alpha/2}e^{\alpha/4} = \left(\frac{\alpha}{\epsilon}\right)^{\alpha/2} e^{-\alpha/4}$$

and

$$C_- = \left(\frac{\alpha}{2\epsilon}\right)^{\alpha/2} e^{-\alpha/4}$$

works.

Now take

$$C_\alpha = \max(C_+, C_-)$$

and we have

$$\max(C_+, C_-) = \left(\frac{\alpha}{\epsilon}\right)^{\alpha/2} e^{-\alpha/4} \max\left[\left(\frac{1}{2}\right)^{\alpha/2}, e^{-\alpha/4}\right].$$

But for $\alpha > 0$ we have

$$\left(\frac{1}{2}\right)^{\alpha/2} < e^{-\alpha/4}$$

so that

$$C_\alpha = \max(C_+, C_-) = \left(\frac{\alpha}{\epsilon}\right)^{\alpha/2} e^{-\alpha/2}.$$

From this estimate we now have

$$\|S_k(t)\phi\|_{\alpha,k} \le C_\alpha \frac{e^{-\epsilon\lambda_1(k)t/2}}{t^{\alpha/2}} \|\phi\|.$$

□

These results together with our assumption on the forcing term $f(t)$ in (2.6) and (2.8) allows us to obtain Theorem 2.1 based on the local existence result found in [16] or [18]. That is, we now can apply Theorem 6.3.1, page 196 of [18], or Theorems 3.3.3, 3.3.4 and 5.1.1 of [16] for small enough initial data in $\mathcal{H}_k^1$.

Proof of Theorem 2.4: If we write out the variation of parameters formula for the solutions z^1 and z^2,

$$z_k^j(t) = S_k(t)\phi^j + \int_0^t S_k(t-\tau)(F(z_k^j(\tau)) + f(\tau))\, d\tau,$$

take the difference, apply the $\|\cdot\|_{1,k}$ norm to both sides and multiply both sides by $e^{\epsilon\beta t/2}$ with $\beta = \lambda_1(k)/4$ to obtain

$$\begin{aligned}
&e^{\epsilon\beta t/2}\|z_k^1(t) - z_k^2(t)\|_{1,k} \le e^{\beta t/2}\|S_k(t)(\phi^1 - \phi^2)\|_{1,k} + \\
&e^{\epsilon\beta t/2}\int_0^t \|S_k(t-\tau)(F(z_k^1(\tau)) - F(z_k^2(\tau)))\|_{1,k}\, d\tau, \\
&\le e^{-\epsilon(\lambda_1(k)-\beta)t/2}\|\phi^1 - \phi^2\|_{1,k} + \qquad (4.10)\\
&C_1 \int_0^t (t-\tau)^{-1/2} e^{-\epsilon(\lambda_1(k)-\beta)(t-\tau)/2} e^{\epsilon\beta\tau/2}\|F(z_k^1(\tau)) - F(z_k^2(\tau))\|\, d\tau \\
&\le \|\phi^1 - \phi^2\|_{1,k} + 2c\rho C_1 \int_0^t (t-\tau)^{-1/2} e^{-\epsilon(\lambda_1(k)-\beta)(t-\tau)/2} \\
&\quad e^{\epsilon\beta\tau/2}\|z_k^1(\tau) - z_k^2(\tau)\|\, d\tau,
\end{aligned}$$

where C_1 is defined in (4.7), the terms involving $f(t)$ have canceled and we have used (4.1) with $M = \rho$.

Defining

$$\omega(T) = \sup_{0\le\tau\le T} e^{\epsilon\beta t/2}\|z_k^1(\tau) - z_k^2(\tau)\|_{1,k},$$

we then have for all $0 < t < T$

$$e^{\epsilon\beta t/2}\|z_k^1(t) - z_k^2(t)\|_{1,k} \le \|\phi^1 - \phi^2\|_{1,k} + \\ 2\omega(T)c\rho C_1 \int_0^t (t-\tau)^{-1/2} e^{-\epsilon(\lambda_1(k)-\beta)(t-\tau)/2}\, d\tau.$$

Now we note that

$$\int_0^\infty C_1(t-\tau)^{-1/2}e^{-\epsilon(\lambda_1(k)-\beta)(t-\tau)/2}\, d\tau = C_1\sqrt{\frac{2\pi}{\epsilon(\lambda_1(k)-\beta)}} \equiv \mathcal{C}_{k,\beta},$$

so we can write

$$e^{\epsilon\beta t/2}\|z_k^1(t) - z_k^2(t)\|_{1,k} \le \|\phi^1 - \phi^2\|_{1,k} + 2\omega(T)c\mathcal{C}_{k,\beta}\rho.$$

Under the Assumption 2.1 with $k = (k_0, k_1) \ge \widetilde{k}$ we have

$$c\mathcal{C}_{k,\beta} \le \frac{1}{\epsilon}.$$

We now have

$$e^{\epsilon\beta t/2}\|z_k^1(t) - z_k^2(t)\|_{1,k} \le \|\phi^1 - \phi^2\|_{1,k} + \omega(T)\frac{2\rho}{\epsilon} \\ \le \frac{\omega(T)}{2} + \|\phi^1 - \phi^2\|_{1,k},$$

since we assumed in (2.33) that

$$\rho \le \frac{\epsilon}{4}.$$

We can now sup on $0 \le t \le T$ to obtain

$$\|z_k^1(t) - z_k^2(t)\|_{1,k} \le e^{-\epsilon\beta t/2}\|\phi^1 - \phi^2\|_{1,k}.$$

□

Proof of Theorem 2.2

Recall once again that a classical solution of (2.6) satisfies the variation of parameters formula for $0 < k_0 + k_1 < \infty$

$$z_k(t) = S_k(t)\phi + \int_0^t S_k(t-\tau)(F(z_k(\tau)) + f(\tau))\, d\tau,$$

as does the solution of the zero dynamics (2.8)

$$z_\infty(t) = S_\infty(t)\phi + \int_0^t S_\infty(t-\tau)(F(z_\infty(\tau)) + f(\tau))\,d\tau.$$

Also we have the pointwise estimate

$$|z_k(x,t) - z_\infty(x,t)| \le c\|z_k - z_\infty\|_{1,k},$$

with $c = \sqrt{2}(1+\lambda_1(k)^{-1})^{1/2}$ defined in (2.29).

So to compute a pointwise estimate for the difference of the solutions, we need only consider the $\mathcal{H}_k^1$ norm of the difference.

$$\begin{aligned}
&\|z_k(\cdot,t) - z_\infty(\cdot,t)\|_{1,k} \le \qquad (4.11)\\
\le\ & \left\|A_k^{1/2}(z_k(t) - z_\infty(t))\right\| \le \left\|A_k^{1/2}\left[S_k(t)\phi - S_\infty(t)\phi\right]\right\| +\\
&+ \int_0^t \left\|A_k^{1/2}\left[S_k(t-\tau)F(z_k(\tau)) - S_\infty(t-\tau)F(z_\infty(\tau))\right]\right\| d\tau,\\
&+ \int_0^t \left\|A_k^{1/2}\left[S_k(t-\tau) - S_\infty(t-\tau)\right]f(\tau)\right\| d\tau,\\
\le\ & \|A_k^{1/2}[S_k(t) - S_\infty(t)]\phi\|\\
+\ & \|\int_0^t \left[A_k^{1/2}S_k(t-\tau)F(z_k(\tau)) - A_\infty^{1/2}S_\infty(t-\tau)F(z_\infty(\tau))\right]d\tau\|\\
+\ & \left\|\int_0^t \left[A_k^{1/2} - A_\infty^{1/2}\right]S_\infty(t-\tau)F(z_\infty(\tau))\,d\tau\right\|\\
&+\left\|\int_0^t A_k^{1/2}\left[S_k(t-\tau) - S_\infty(t-\tau)\right]f(\tau)\,d\tau\right\|\\
\equiv\ & I + II + III + IV. \qquad (4.12)
\end{aligned}$$

For the term II we have the estimates

$$\begin{aligned}
II \ \le\ & \int_0^t \left\|A_k^{1/2}S_k(t-\tau)(F(z_k(\tau)) - F(z_\infty(\tau)))\right\| d\tau \qquad (4.13)\\
+\ & \left\|\int_0^t [A_k^{1/2}S_k(t-\tau) - A_\infty^{1/2}S_\infty(t-\tau)]F(z_\infty(\tau))\,d\tau\right\|\\
\equiv\ & IIa + IIb.
\end{aligned}$$

For the term IIa we can apply (4.1) to obtain the estimate

$$\|F(z_k) - F(z_\infty)\| \le 2c\rho \|z_k - z_\infty\|_{1,k}. \tag{4.14}$$

Now using (4.4), (4.7), (4.8) and (4.14) we obtain

$$\begin{aligned} &IIa \le \\ &2c\rho C_1 \int_0^t (t-\tau)^{-1/2} e^{-\epsilon\lambda_1(k)(t-\tau)/2} \|z_k(\tau) - z_\infty(\tau)\|_{1,k}\, d\tau \quad (4.15) \\ &\le 2c\rho \mathcal{C}_k \sup_{0<\tau<t} \|z_k(\tau) - z_\infty(\tau)\|_{1,k} \\ &< (1/2) \sup_{0<\tau<t} \|z_k(\tau) - z_\infty(\tau)\|_{1,k} \end{aligned}$$

where the last inequality follows from our choice of ρ in (2.33). Now take $t_0 > 0$ and define

$$\omega(t_0) = \sup_{0<\tau<t_0} \|z_k(\tau) - z_\infty(\tau)\|_{1,k}.$$

Then we have for all $0 \le t \le t_0$

$$\|z_k(\tau) - z_\infty(\tau)\|_{1,k} \le \frac{1}{2}\omega(t_0) + I + IIb + III + IV \tag{4.16}$$

and we need to show that the last four terms on the right can be made arbitrarily small independent of t for k sufficiently large. The term I is considered in detail in Lemmas 4.3 to 4.6. The term IIb is considered in Lemma 4.8, the term III is examined in Lemma 4.7 and finally the last term IV is estimated in Lemma 4.9.

As a consequence of the main theorem of [5] (see also [17] Theorem 1.14 and Example 1.15, page 374), we have that the negative selfadjoint operators $-\epsilon A_k$ form a holomorphic family in $k_0, k_1 \in [0,\infty]$ with $k_0 + k_1 > 0$ in the sense of Kato ([17], (Theorem 2.25, page 206 and Theorem 1.3 and Example 1.4, page 367). Therefore, defining

$$R_k(\lambda) = (\lambda I + A_k)^{-1}$$

for any $\lambda \notin (0,\infty)$, we have

$$\|R_k(\lambda) - R_\infty(\lambda)\| \to 0, \ k \to \infty.$$

In one form or another, most of the following results repose on this strong statement concerning the fact that the resolvents converge in the uniform operator topology as k_0 and k_1 go to infinity. In addition to this, we now recall various properties of the resolvents $R_k(\lambda)$, $R_\infty(\lambda)$ and the semigroups $S_k(t)$, $S_\infty(t)$ that will be used in the proofs of the main results.

Lemma 4.3 *1. For every* $f \in L^2(0,1)$

$$\|sR_k(s)f - f\| \to 0, \;\; s \to +\infty, \tag{4.17}$$

and

$$\|sR_k(s)\| \leq 1.$$

We can also see that

$$\|s^2 R_k(s)^2 f - f\| \to 0, \;\; s \to +\infty. \tag{4.18}$$

2. For all $t \geq 0$ *and* $s > 0$

$$\|R_k(s)\left(S_\infty(t) - S_k(t)\right)R_\infty(s)\| \leq te^{-\epsilon\lambda_1(k)t}\|R_k(s) - R_\infty(s)\|.$$

3. For all $t \geq 0$ *and* $s > 0$

$$[S_k(t) - S_\infty(t)]R_\infty(s)^2 = R_k(s)[S_k(t) - S_\infty(t)]R_\infty(s) \tag{4.19}$$

$$+[R_k(s) - R_\infty(s)]S_\infty(t)R_\infty(s) - S_k(t)[R_k(s) - R_\infty(s)]R_\infty(s).$$

4. We have

$$\|[S_k(t) - S_\infty(t)]R_\infty^2(s)\| \leq (t+2)e^{-\epsilon\lambda_1(k)t}\|R_k(s) - R_\infty(s)\|.$$

Proof:

Part 1 is a well-known consequence of the fact that $-A_k$ generates a contraction semigroup (cf. [18]). Part 2 can be found in [17] (page 501, Theorem 2.14). Part 3 is simple algebra obtained by adding and subtracting certain terms. Part 4 follows from parts 1 through 3 and the fact that $\lambda_1(k) < \lambda_1(\infty)$. □

Lemma 4.4 *For every* $\delta > 0$ *we can find* K *for which* $k_0, k_1 > K$ *implies*

$$\|S_k(t) - S_\infty(t)\| \leq \delta e^{-\epsilon\lambda_1 t/2}, \;\; \text{for all } t \geq 0. \tag{4.20}$$

Proof:

Take $g \in L^2(0,1)$ such that $\|g\| \leq 1$, then

$$\begin{aligned} \|[S_k(t) - S_\infty(t)]g\| &\leq \|[S_k(t) - S_\infty(t)](g - s^2 R_\infty(s)^2 g(s))\| \\ &+ s^2\|[S_k(t) - S_\infty(t)]R_\infty^2(s)g\| \\ &\equiv \widetilde{I} + \widetilde{II}. \end{aligned} \tag{4.21}$$

Now from Lemma 4.3, part 1 we can take an $s>1$ so that

$$\|g-s^2R_\infty(s)^2g\| \le \frac{\delta}{4}$$

so by (2.11)

$$\widetilde{I} \le (\|S_k(t)\| + \|S_\infty(t)\|)\,\frac{\delta}{4} \le e^{-\epsilon\lambda_1 t}\frac{\delta}{2}.$$

Now by Lemma 4.3 part 4

$$\begin{aligned}\widetilde{II} &= s^2\|[S_k(t)-S_\infty(t)]R_\infty^2(s)g\| \\ &\le s^2(t+2)e^{-\epsilon\lambda_1 t/2}\|R_k(s)-R_\infty(s)\| \qquad (4.22)\\ &\le s^2Ce^{-\epsilon\lambda_1 t/2}\|R_k(s)-R_\infty(s)\| \\ &\le \frac{\delta}{2}e^{-\epsilon\lambda_1 t/2},\end{aligned}$$

where

$$C=\max_{t\ge 0}(t+2)e^{-\epsilon\lambda_1 t/2} < \infty.$$

Here, for s fixed, in the last inequality, we have chosen K so that $k_0, k_1 > K$ implies

$$\|R_k(s)-R_\infty(s)\| \le \frac{\delta}{2s^2C}.$$

Finally, taking the sup over all $\|g\|\le 1$ we obtain

$$\|S_k(t)-S_\infty(t)\| \le \delta e^{-\epsilon\lambda_1 t/2}.$$

□

Lemma 4.5 *For every $\psi\in L^2(0,1)$, the bounded operators*

$$B_k = A_k^{1/2}A_\infty^{-1/2} \qquad (4.23)$$

converge to the identity in $L^2(0,1)$ in the strong operator topology as $k=(k_0,k_1)$ tend to infinity. Further, if $C\subset L^2(0,1)$ is a relatively compact set, then

$$\sup_{\psi\in C}\|[B_k-I]\psi\| \to 0, \quad k_0,k_1\to\infty.$$

Proof:

The first part of the proof follows from the norm resolvent convergence of the resolvents. See, for example, the proof of Theorem 3.13, page 459 of [17]. The second part is a simple general fact which can be found for example in [12], Theorem 3.2 page 124. □

With these Lemmas we can bound the first term I.

Lemma 4.6 *For $\phi \in \mathcal{H}_k^1$ and $\delta > 0$, there exists a K such that for $k_0, k_1 > K$*

$$I = \|A_k^{1/2}\,[S_k(t) - S_\infty(t)]\,\phi\| \le \delta \tag{4.24}$$

for all $t \ge 0$.

Proof:

For each fixed $t > 0$ a simple Banach algebra argument can be employed to establish this result (cf. [19] Theorem VIII.20, page 286). But to obtain the result uniformly in $t \ge 0$ a bit more work is required.

Take $\delta > 0$ sufficiently small and any $t \ge 0$ to obtain

$$\begin{aligned}
& \|A_k^{1/2}[S_k(t) - S_\infty(t)]\phi\| \\
\le\ & \|[A_k^{1/2}S_k(t) - A_\infty^{1/2}S_\infty(t)]\phi\| \\
+\ & \|[A_k^{1/2} - A_\infty^{1/2}]S_\infty(t)]\phi\| \qquad (4.25)\\
=\ & \|[S_k(t)A_k^{1/2}A_\infty^{-1/2} - S_\infty(t)]A_\infty^{1/2}\phi\| \\
+\ & \|[A_k^{1/2}A_\infty^{-1/2} - I]A_\infty^{1/2}S_\infty(t)]\phi\| \\
\le\ & \|[S_k(t)[A_k^{1/2}A_\infty^{-1/2} - I]A_\infty^{1/2}\phi\| \\
+\ & \|[S_k(t) - S_\infty(t)]A_\infty^{1/2}\phi\| \\
+\ & \|[A_k^{1/2}A_\infty^{-1/2} - I]A_\infty^{1/2}S_\infty(t)]\phi\| \\
\equiv\ & T_1 + T_2 + T_3.
\end{aligned}$$

Now define $\psi = A_\infty^{1/2}\phi$. For the first term we have from Lemma 4.5 that there is a $K_1 > 0$ so that for $k > K_1$

$$T_1 \le \|[A_k^{1/2}A_\infty^{-1/2} - I]\psi\|\|S_k(t)\| \le \delta/3e^{-\epsilon\lambda_1 t}. \tag{4.26}$$

For the second term by Lemma 4.4 there is a $K_2 > 0$ so that for $k > K_2$ we have

$$T_2 = \|[S_k(t) - S_\infty(t)]\psi\| \le \delta/3e^{-\epsilon\lambda_1 t/2}. \tag{4.27}$$

For the third term we note that the set

$$\{S_\infty(t)\psi\}_{t\ge 0}$$

is a relatively compact set in $L^2(0,1)$ for $\psi \in L^2(0,1)$. Indeed, for any $\delta > 0$ the set $\{S_k(t)\phi\}_{t\ge\delta}$ is compact due to Remark 4.1. On the other hand, the set $\{S_k(t)\phi\}_{0\le t\le\delta}$ is compact as a continuous image of a compact interval $[0,\delta]$. Hence by Lemma 4.5, we again have a $K_3>0$ so that for $k>K_3$

$$T_3 \le \sup_{t\ge 0} \|[A_k^{1/2}A_\infty^{-1/2}-I]S_\infty(t)\psi\| \le \delta/3. \tag{4.28}$$

Combining (4.26), (4.27) and (4.28) with $K=\max(K_1,K_2,K_3)$ the result follows. □

We have left to consider the remaining terms IIb, III and IV in (4.16):

$$IIb = \left\| \int_0^t \left[A_k^{1/2}S_k(t-\tau) - A_\infty^{1/2}S_\infty(t-\tau)\right] F(z_\infty(\tau))\, d\tau \right\|,$$

$$III = \left\| \int_0^t \left[A_k^{1/2} - A_\infty^{1/2}\right] S_\infty(t-\tau)F(z_\infty(\tau))\, d\tau \right\|,$$

$$IV = \left\| \int_0^t A_k^{1/2}\left[S_k(t-\tau) - S_\infty(t-\tau)\right] f(\tau)\, d\tau \right\|.$$

Recall from (4.2) and (2.36) that

$$\|F(z_\infty(\tau))\| \le c\rho^2, \quad \text{for all} \quad t>0. \tag{4.29}$$

Lemma 4.7 *For any $\delta>0$ there exists a $K>0$ so that for $k_0,k_1>K$ we have*

$$III \le \delta$$

for all $t\ge 0$.

Proof: First note that with B_k defined in (4.23) we have

$$III = \left\| [B_k - I] \int_0^t A_\infty^{1/2} S_\infty(t-\tau)F(z_\infty(\tau))\, d\tau \right\|. \tag{4.30}$$

Now we show that the set

$$\mathcal{S} = \bigcup_{t\ge 0} \left\{ \int_0^t A_\infty^{1/2} S_\infty(t-\tau)F(z_\infty(\tau))\, d\tau \right\}$$

is a relatively compact set in $L^2(0,1)$ and hence the result follows from Lemma 4.5. Take $\alpha = 1+\gamma$ where $(1+\gamma)/2<1$. Now use (4.29) and part 2 of Lemma 4.5 to show that the set $\mathcal{S}$ is bounded in $\mathcal{H}_k^\gamma$ (here we refer to the Hilbert scale generated by A_∞) and is therefore a relatively compact set in $L^2(0,1)$. We have

$$\left\| \int_0^t A_\infty^{1/2} S_\infty(t-\tau) F(z_\infty(\tau))\, d\tau \right\|_{\gamma,k} \tag{4.31}$$

$$= \left\| \int_0^t A_\infty^{1/2+\gamma/2} S_\infty(t-\tau) F(z_\infty(\tau))\, d\tau \right\|$$
$$= \left\| \int_0^t A_\infty^{\alpha/2} S_\infty(t-\tau) F(z_\infty(\tau))\, d\tau \right\|$$

$$\leq c\rho^2 \int_0^t \|A_\infty^{\alpha/2} S_\infty(t-\tau)\|\, d\tau$$

$$\leq c\rho^2 \mathcal{C}_\infty < \infty,$$

where $\mathcal{C}_\infty$ is defined in (2.49) with $k=\infty$ (see also (4.8)). □

For the term IIb we extend a result found in Simon and Reed (cf. [19] Theorem VIII.20, page 286).

Lemma 4.8 *For any $\delta > 0$ there exists a $K > 0$ so that for $k_0, k_1 > K$ we have*

$$IIb \leq \delta$$

for all $t \geq 0$.

Proof:

We first reduce our calculations to a compact time interval. To this end, we consider the following

$$\left\| \int_0^t [A_k^{1/2} S_k(t-\tau) - A_\infty^{1/2} S_\infty(t-\tau)] F(z_\infty(\tau))\, d\tau \right\|$$

$$= \left\| \int_0^t [A_k^{1/2} S_k(\tau) - A_\infty^{1/2} S_\infty(\tau)] F(z_\infty(t-\tau))\, d\tau \right\|$$

$$\leq c\rho^2 \int_0^t \left\| [A_k^{1/2} S_k(\tau) - A_\infty^{1/2} S_\infty(\tau)] \right\|\, d\tau \tag{4.32}$$

$$\leq c\rho^2 \int_0^{t_0} \left\| [A_k^{1/2} S_k(\tau) - A_\infty^{1/2} S_\infty(\tau)] \right\|\, d\tau$$

$$+ c\rho^2 \int_{t_0}^{t_1} \left\| [A_k^{1/2} S_k(\tau) - A_\infty^{1/2} S_\infty(\tau)] \right\|\, d\tau$$

$$+c\rho^2 \int_{t_1}^{\infty} \left\| [A_k^{1/2} S_k(\tau) - A_\infty^{1/2} S_\infty(\tau)] \right\| \, d\tau.$$

For the first term, after the last equality above, we have

$$\int_0^{t_0} \left\| [A_k^{1/2} S_k(\tau) - A_\infty^{1/2} S_\infty(\tau)] \right\| \, d\tau \tag{4.33}$$

$$\leq \int_0^{t_0} \left(\left\| [A_k^{1/2} S_k(\tau) \right\| + \left\| A_\infty^{1/2} S_\infty(\tau)] \right\| \right) \, d\tau$$

$$\leq 2C_1 \int_0^{t_0} \frac{e^{-\epsilon \lambda_1 s}}{\sqrt{s}} \, ds$$

$$\leq 2C_1 \int_0^{t_0} s^{-1/2} \, ds \leq 2C_1 \sqrt{t_0} \leq \delta/3$$

for $t_0 < (\delta/12C_1)^2$ with C_1 defined in (4.7).

For the last term above

$$\int_{t_1}^{\infty} \left\| [A_k^{1/2} S_k(\tau) - A_\infty^{1/2} S_\infty(\tau)] \right\| \, d\tau \tag{4.34}$$

$$\leq \int_{t_1}^{\infty} \left(\left\| [A_k^{1/2} S_k(\tau) \right\| + \left\| A_\infty^{1/2} S_\infty(\tau)] \right\| \right) \, d\tau$$

$$\leq 2C_1 \int_{t_1}^{\infty} \frac{e^{-\epsilon \lambda_1 s}}{\sqrt{s}} \, ds$$

$$\leq \frac{2C_1}{\sqrt{t_1}} \int_0^{\infty} e^{-\epsilon \lambda_1 s} \, ds$$

$$\leq \frac{2C_1}{\sqrt{t_1} \epsilon \lambda_1} \leq \delta/3$$

for $t_1 > (6C_1/(\epsilon\lambda_1\delta))^2$.

With t_0 and t_1 chosen above, we need only consider the integral over the fixed compact interval $[t_0, t_1]$. Thus we need to estimate the expression

$$\int_{t_0}^{t_1} \left\| \left[A_k^{1/2} S_k(\tau) - A_\infty^{1/2} S_\infty(\tau) \right] \right\| d\tau.$$

Let

$$g(\lambda, t) = \begin{cases} \lambda^{1/2} e^{-\epsilon \lambda t}, & \lambda > 0 \\ 0 & \lambda \leq 0 \end{cases}$$

The function g is a uniformly continuous function on $\mathbb{R} \times [t_0, t_1]$ and vanishes at infinity in λ. Furthermore, by the spectral theorem we have

$$g(A_k, t) = A_k^{1/2} S_k(t),$$

$$g(A_\infty, t) = A_\infty^{1/2} S_\infty(t).$$

Since $[t_0, t_1]$ is compact we can find $\{s_j\}_{j=1}^N$ and disjoint intervals $\{I_j\}$ such that

$$[t_0, t_1] = \bigcup_{j=1}^{N} \overline{I_j}$$

and

$$\sup_{\lambda \in \mathbb{R}, t \in I_j} |g(\lambda, t) - g(\lambda, s_j)| \le \frac{\delta}{9(t_1 - t_0)}$$

for $j = 1, \cdots, N$. Once again by the spectral theorem

$$\sup_{\lambda \in \mathbb{R}, t \in I_j} \left\| A_k^{1/2} S_k(t) - A_k^{1/2} S_k(s_j) \right\| \le \frac{\delta}{9(t_1 - t_0)}$$

for $j = 1, \cdots, N$. A similar expression holds for $k = \infty$.

Finally, and once again as a consequence of the spectral theorem (cf. [19] Theorem VIII.20, page 286) and the norm resolvent convergence as k_0, k_1 go to infinity, there exists a $K > 0$ so that for $k_0, k_1 > K$ and for $j = 1, \cdots, N$

$$\sup_{t \in I_j} \left\| A_k^{1/2} S_k(s_j) - A_\infty^{1/2} S_\infty(s_j) \right\| \le \frac{\delta}{9(t_1 - t_0)}.$$

Combining these results we can compute

$$\int_{t_0}^{t_1} \left\| \left[A_k^{1/2} S_k(\tau) - A_\infty^{1/2} S_\infty(\tau) \right] \right\| d\tau \tag{4.35}$$

$$= \sum_{j=1}^{N} \int_{I_j} \left\| \left[A_k^{1/2} S_k(\tau) - A_\infty^{1/2} S_\infty(\tau) \right] \right\| d\tau.$$

$$\le \sum_{j=1}^{N} \left\{ \int_{I_j} \left\| \left[A_k^{1/2} S_k(\tau) - A_k^{1/2} S_k(s_j) \right] \right\| d\tau \right.$$

$$+ \int_{I_j} \left\| \left[A_k^{1/2} S_k(s_j) - A_\infty^{1/2} S_\infty(s_j) \right] \right\| d\tau$$

$$\left. + \int_{I_j} \left\| \left[A_\infty^{1/2} S_\infty(\tau) - A_\infty^{1/2} S_\infty(s_j) \right] \right\| d\tau \right\}$$

$$\le \frac{\delta}{9(t_1 - t_0)} \left[3(t_1 - t_0) \right] = \frac{\delta}{3}. \tag{4.36}$$

□

Finally, we consider the term IV

$$IV = \left\| \int_0^t A_k^{1/2} \left[S_k(t-\tau) - S_\infty(t-\tau) \right] f(\tau)\, d\tau \right\|.$$

Lemma 4.9 *For any $\delta > 0$ there exists a $k > 0$ so that for $k_0, k_1 > k$ we have*

$$IV \leq \delta$$

for all $t \geq 0$.

Proof: It is easy to see that under the assumption (2.34), so that

$$\operatorname{ess} \sup_{t \in [0,\infty)} \|f(\cdot, t)\| < \infty,$$

we can simply repeat the proofs given in Lemmas 4.7 and 4.8 in the present case. Namely, we estimate IV by

$$\begin{aligned} IV &= \left\| \int_0^t A_k^{1/2} \left[S_k(t-\tau) - S_\infty(t-\tau) \right] f(\tau)\, d\tau \right\| \\ &\leq \int_0^t \left\| \left[A_k^{1/2} S_k(t-\tau) - A_\infty^{1/2} S_\infty(t-\tau) \right] f(\tau) \right\| d\tau \\ &+ \left\| \left[A_k^{1/2} A_\infty^{-1/2} - I \right] \int_0^t A_\infty^{1/2} S_\infty(t-\tau) f(\tau)\, d\tau \right\|. \end{aligned}$$

Following the proof of Lemma 4.7 it can be shown that the set

$$\bigcup_{t \geq 0} \left\{ \int_0^t A_\infty^{1/2} S_\infty(t-\tau) f(\tau))\, d\tau \right\}$$

is a relatively compact set in $L^2(0,1)$. Now using Lemma 4.5 we can conclude that last term can be made small uniformly in $t \geq 0$ for k_0, k_1 sufficiently large. As for the second to last term simply replay the proof of Lemma 4.8 replacing $F(z_\infty(\tau))$ by $f(\tau)$. □

Proof of Theorem 2.6: In order to prove that the attractors $\mathcal{A}_k$ converge to $\mathcal{A}_\infty$ (which consists of a single point), with respect to the semidistance determined by the $H^1(0,1)$ norm, we recall (2.28) and our Assumption 2.1 which implies that

$$\|\phi\|_{H^1(0,1)} \leq C \|\phi\|_{1,k}, \text{ for all } \phi \in H^1(0,1),$$

where the constant C is independent of k. Therefore is suffices to show that

$$\lim_{k_0,k_1\to\infty} \sup_{\phi\in\mathcal{A}_k} \|\phi - z^\infty\|_{1,k} = 0.$$

We suppose that this is not true and arrive at a contradiction. If the attractors $\mathcal{A}_k$ do not converge to the single point z^∞, then there exists a sequence $k_n = (k_0^n, k_1^n)$ with $k_0^n, k_1^n \to \infty$ as n tends to infinity and a $\delta > 0$ such that $d(\mathcal{A}_{k_n}, z^\infty) > \delta$ for every n.

Take $\phi_0 \in H_0^1(0,1)$ such that $\|\phi_0\|_{1,\infty} \leq \rho/4$, then $\phi_0 \in B_0^k$ for all k. Since k_0^n and k_1^n are diverging to infinity, we know by our main result, Theorem 2.2, that there is an n for which

$$\|T_t^{k_n}(\phi_0) - T_t^\infty(\phi_0)\|_{1,k_n} \leq \frac{\delta}{4}, \quad \text{for all } t \geq 0. \tag{4.37}$$

Now fix this k_n and take an arbitrary element $\phi \in \mathcal{A}_{k_n}$. Then by the definition of the ω-limit sets $\mathcal{A}_{k_n}$ (see [22]), there are sequences $\{\phi_j\}_{j=1}^\infty \subset B_0^{k_n}$ (see (2.46) and [22]) and $t_j \uparrow \infty$ such that $T_{t_j}^{k_n}(\phi_j) \to \phi$ as $j \to \infty$. Take $J_1 > 0$ so that for all $j > J_1$,

$$\|\phi - T_{t_j}^{k_n}(\phi_j)\|_{1,k_n} \leq \frac{\delta}{4}. \tag{4.38}$$

By Theorem 2.4, there exists a $J_2 > 0$ such that for all $j > J_2$,

$$\|T_{t_j}^{k_n}(\phi_j) - T_{t_j}^{k_n}(\phi_0)\|_{1,k_n} \leq 2\rho e^{-\epsilon t_j/8} \leq \frac{\delta}{4} \tag{4.39}$$

due to (2.44), (2.45). Here we have used the fact that ϕ_0 and ϕ_j ($j > 0$) are in $B_0^{k_n}$.

Next we note that since $\mathcal{A}_\infty = \{z^\infty\}$ (see [2]), there exists a $J_3 > 0$ such that for all $j > J_3$

$$\|T_{t_j}^\infty(\phi_0) - z^\infty\|_{1,k_n} \leq \frac{\delta}{4}. \tag{4.40}$$

Note that since $T_{t_j}^\infty(\phi_0)$ and z^∞ are in $\mathcal{H}_\infty^1$, the norms $\|\cdot\|_{1,k_n}$ and $\|\cdot\|_{1,\infty}$ coincide, (2.30).

We now combine (4.38)-(4.40) and use the triangle inequality to obtain

$$\begin{aligned}
&\|\phi - z^\infty\|_{1,k_n} \leq \\
&\leq \|\phi - T_{t_j}^{k_n}(\phi_j)\|_{1,k_n} \\
&+\|T_{t_j}^{k_n}(\phi_j) - T_{t_j}^{k_n}(\phi_0)\|_{1,k_n} \\
&+\|T_{t_j}^{k_n}(\phi_0) - T_{t_j}^\infty(\phi_0)\|_{1,k_n} \\
&+\|z^\infty - T_{t_j}^\infty(\phi_0)\|_{1,k_n} \\
&\leq \delta,
\end{aligned}$$

for all $j > \max\{J_1, J_2, J_3\}$. Thus we have arrived at a contradiction and the proof is complete. □

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C.I. BYRNES, D.S. GILLIAM, AND V.I. SHUBOV

Department of Systems Science and Mathematics, Washington University, St. Louis, MO 63130

Department of Mathematics, Texas Tech University, Lubbock, TX, 79409

Department of Mathematics, Texas Tech University, Lubbock, TX, 79409

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