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Forward/Backward Periodic Realizations of Nonproper Rational Matrices*

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Abstract

In many control problems, periodic controllers are usually represented by nonproper rational transfer matrices. The study of the model requires implementing the controllers as periodic linear systems. In this paper we study the problem of realizing a periodic collection of nonproper rational matrices as a discrete-time forward/backward periodic linear system.

Key words: singular discrete time-varying linear periodic systems, nonproper rational matrices, minimal periodic realizations

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1 Introduction

The realization theory of discrete-time invariant linear systems has been developed using two different approaches: the time domain approach and the frequency domain or z domain approach. In the time domain approach, the existence of realizations has been characterized by means of the Markov parameters that define the input-output invariant application. This characterization has been oriented in different ways, mainly by means of recurrence equations that are satisfied by the Markov parameters and using the Hankel matrix associated with the collection of Markov parameters [17] [8]. In the frequency domain approach, the existence of an invariant realization is equivalent to the proper rational character of the associated

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transfer function matrix [1] [16]. Moreover, it is well known that in general a nonproper rational matrix always represents a transfer function matrix associated with a singular invariant linear system [7] [3].

Discrete-time periodic linear systems are important and useful in system modeling and seem to be of real interest for generalizing the above results. The existence of periodic realizations in the time domain and frequency domain approaches is studied in [6]. The first part of [6] presents a characterization of the existence of a periodic realization associated with an input-output periodic application based on a certain recurrence equation satisfied by the periodic Markov parameters. The second part of [6], studies the problem in the frequency domain and gives a necessary and sufficient condition for the existence of periodic realizations associated with a periodic collection of proper rational matrices. In [11] realization and minimality theory for discrete-time finite dimensional linear systems with time-varying state spaces is developed in the case of the input-output model. In [14], [5] the minimal periodic realization problem is studied and it is shown that there exists a periodic system associated with a proper rational matrix. In this work we solve the periodic realization problem in the general case of nonproper rational matrices.

When generalized periodic models, i.e. discrete-time singular periodic linear systems are considered, we know that this kind of system can be approached in the frequency domain by a periodic collection of nonproper rational matrices. The main goal of this paper is to solve the converse problem. We characterize when a periodic collection of nonproper rational matrices has a singular periodic realization and study the existence of discrete time-varying forward/backward minimal periodic realizations.

The main tools we use are based on the Invariant Formulations of discrete-time periodic linear systems and discrete-time singular forward/ backward periodic linear systems. In general the study of a discrete-time singular invariant linear system is based on the decomposition of the system into two subsystems: a discrete-time forward invariant linear system and a discrete-time backward invariant linear system. The properties of the initial singular system are given in terms of the properties of the forward and the backward subsystems [7].

The structure of the paper is the following. First, we introduce the concept of discrete time-varying forward/backward periodic linear systems and their associated forward and backward invariant systems. In section 3 we define the forward/backward periodic realization associated with a periodic collection of nonproper rational matrices. In section 4 we construct a forward/backward periodic linear system that realizes the periodic collection of rational matrices, proving that a periodic collection of rational matrices $\{H_s(z),\ s \in \mathcal{Z}\}$, $H_{s+N}(z) = H_s(z) \in \Re^{pN\times mN}(z)$ is realizable if and only if $H_{s+1}(z) = R_p(z)H_s(z)R_m^{-1}(z)$, $s \in \mathcal{Z}$, where $R_j(z)$, $j = p, m$, are given below. In section 5 we present some definitions and results related with reachability and observability properties of forward/backward periodic systems. Finally, in section 6 we prove that there exists a minimal and c-minimal discrete time-varying forward/backward periodic linear realization of a given periodic collection of nonproper rational matrices if and only if the matrices of the collection satisfy the same recurrence equation of section 4.

2 Discrete Time-Varying Forward/Backward Periodic Linear Systems

Consider the discrete time-varying periodic linear system

$$\begin{bmatrix} I & 0 \\ 0 & A_2(k) \end{bmatrix} x(k+1) = \begin{bmatrix} A_1(k) & 0 \\ 0 & I \end{bmatrix} x(k) + \begin{bmatrix} B_1(k) \\ B_2(k) \end{bmatrix} u(k), \quad (2.1)$$

$$y(k) = [C_1(k), \quad C_2(k)]\, x(k), \quad (2.2)$$

where, $A_1(k+N) = A_1(k) \in \Re^{n_1(k+1)\times n_1(k)}$, $A_2(k+N) = A_2(k) \in \Re^{n_2(k)\times n_2(k+1)}$, $B_1(k+N) = B_1(k) \in \Re^{n_1(k+1)\times m}$, $B_2(k+N) = B_2(k) \in \Re^{n_2(k)\times m}$, $C_i(k+N) = C_i(k) \in \Re^{p\times n_i(k)}$, $k \in \mathcal{Z}$, $m, p, N \in \mathcal{Z}^+$ and $n_i(k+N) = n_i(k) \in \mathcal{Z}^+$, $i = 1, 2$. We refer to it as a forward/backward periodic singular system. Note that in this system, the dimension of the state space $X(k) = X_1(k) \oplus X_2(k)$ is time-varying and given by $\dim X_1(k) = n_1(k)$ and $\dim X_2(k) = n_2(k)$, $k \in \mathcal{Z}$. From the special structure of this system we consider the periodic systems given by

$$\begin{aligned} x_1(k+1) &= A_1(k)\, x_1(k) + B_1(k) u(k), && (2.3) \\ y_1(k) &= C_1(k) x_1(k), && (2.4) \end{aligned}$$

and

$$\begin{aligned} A_2(k) x_2(k+1) &= x_2(k) + B_2(k) u(k), && (2.5) \\ y_2(k) &= C_2(k) x_2(k), && (2.6) \end{aligned}$$

denoted by

$$\Sigma^F \equiv (I,\, A_1(\cdot),\, B_1(\cdot),\, C_1(\cdot); X_1(\cdot))_N$$

and

$$\Sigma^B \equiv (A_2(\cdot),\, I,\, B_2(\cdot), C_2(\cdot); X_2(\cdot))_N.$$

System (2.1)-(2.2) will be denoted by $\Sigma^{F/B}$. Systems (2.3)-(2.4) and (2.5)-(2.6) are called the Discrete-Time Forward and Backward Periodic Linear Subsystems of (2.1)-(2.2), respectively. Note that the three systems have the same inputs, and the states and outputs are such that $x(k) = \text{col}[x_1(k),\ x_2(k)]$, $y(k) = y_1(k) + y_2(k)$.

It is well known [15] that, for any $s \in \mathcal{Z}$, there exists a forward invariant linear system associated with periodic system (2.3)-(2.4)

$$\begin{aligned} x_{1,s}(k+1) &= A_{1,s}x_{1,s}(k) + B_{1,s}u_s(k), && (2.7)\\ y_{1,s}(k) &= C_{1,s}x_{1,s}(k) + D_{1,s}u_s(k),\ k \in \mathcal{Z} && (2.8)\end{aligned}$$

with

$$\begin{aligned} x_{1,s}(k) &= x_1(s+kN),\\ u_s(k) &= \text{col}[u(s+kN),\, u(s+kN+1),\, \dots,\, u(s+kN+N-1)],\\ y_{1,s}(k) &= \text{col}[y_1(s+kN),\, y_1(s+kN+1),\, \dots,\, y_1(s+kN+N-1)],\end{aligned}$$

and

$$\begin{aligned} A_{1,s} &= \phi_{A_1}(s+N,s) \in \Re^{n_1(s)\times n_1(s)},\\ B_{1,s} &= \text{row}[\phi_{A_1}(s+N,s+j+1)B_1(s+j)]_{j=0}^{N-1} \in \Re^{n_1(s)\times mN},\\ C_{1,s} &= \text{col}[C_1(s+j)\phi_{A_1}(s+j,s)]_{j=0}^{N-1} \in \Re^{pN\times n_1(s)},\\ D_{1,s} &= [D_{ij}^{1,s}] \in \Re^{pN\times mN}, D_{ij}^{1,s} \in \Re^{p\times m},\quad i,j = 1,2,\dots,N,\\ D_{ij}^{1,s} &= \begin{cases} 0, & \text{if } i \le j,\\ C_1(s+i-1)\phi_{A_1}(s+i-1,s+j)B_1(s+j-1), & \text{if } i > j.\end{cases}\end{aligned}$$

Note that $D_{1,s}$ is a strictly lower block triangular matrix and $A_{1,s}$ is the forward monodromy matrix of (2.3)-(2.4) at time s, $\phi_{A_1}(k,k_0) = A_1(k-1)A_1(k-2)\dots A_1(k_0)$, $k > k_0$, $\phi_{A_1}(k_0,k_0) = I$. Systems (2.7)-(2.8) define the Invariant Formulation of forward periodic system (2.3)-(2.4). We denote invariant system (2.7)-(2.8) by $\Sigma_s^F \equiv (I,\, A_{1,s},\, B_{1,s},\, C_{1,s},\, D_{1,s})$. From the periodicity of system (2.3)-(2.4) we deduce that $\Sigma_{s+N}^F \equiv \Sigma_s^F$, $s \in \mathcal{Z}$.

Following the same approach we construct invariant linear systems associated with a backward periodic system (2.5)-(2.6). For each $s \in \mathcal{Z}$, we consider the associated backward invariant linear system

$$\begin{aligned} A_{2,s}x_{2,s}(k+1) &= x_{2,s}(k) + B_{2,s}u_s(k), && (2.9)\\ y_{2,s}(k) &= C_{2,s}x_{2,s}(k+1) + D_{2,s}u_s(k),\quad k \in \mathcal{Z}, && (2.10)\end{aligned}$$

with

$$\begin{aligned} x_{2,s}(k) &= x_2(s+kN),\\ u_s(k) &= \text{col}[u(s+kN),\, u(s+kN+1),\, \dots,\, u(s+kN+N-1)],\\ y_{2,s}(k) &= \text{col}[y_2(s+kN),\, y_2(s+kN+1),\, \dots,\, y_2(s+kN+N-1)],\end{aligned}$$

and

$$\begin{aligned}
A_{2,s} &= \psi_{A_2}(s, s+N) \in \Re^{n_2(s)\times n_2(s)},\\
B_{2,s} &= \mathrm{row}[\psi_{A_2}(s, s+j)B_2(s+j)]_{j=0}^{N-1} \in \Re^{n_2(s)\times mN},\\
C_{2,s} &= \mathrm{col}[C_2(s+j)\psi_{A_2}(s+j, s+N)]_{j=0}^{N-1} \in \Re^{pN\times n_2(s)},\\
D_{2,s} &= [D_{ij}^{2,s}] \in \Re^{pN\times mN}, D_{ij}^{2,s} \in \Re^{p\times m},\quad i,j = 1,2,\ldots,N,\\
D_{ij}^{2,s} &= \begin{cases} -C_2(s+i-1)\psi_{A_2}(s+i-1, s+j-1)B_2(s+j-1), & \text{if } i \le j,\\ 0, & \text{if } i > j.\end{cases}
\end{aligned}$$

Note that $D_{2,s}$ is an upper block triangular matrix and $A_{2,s}$, is the backward monodromy matrix of (2.5)-(2.6) at time s, $\psi_{A_2}(k,k_0) = A_2(k)A_2(k+1)\ldots A_2(k_0-1)$, $k < k_0$, $\psi_{A_2}(k_0,k_0) = I$. Systems (2.9)-(2.10) define the Invariant Formulation of the backward periodic system (2.5)-(2.6). We denote system (2.9)-(2.10) by $\Sigma_s^B \equiv (A_{2,s}, I, B_{2,s}, C_{2,s}, D_{2,s})$. From periodicity, systems (2.9)-(2.10) satisfy that $\Sigma_{s+N}^B \equiv \Sigma_s^B$, $s \in \mathcal{Z}$.

3 Transfer Matrices of Discrete-Time Forward/Backward Periodic Linear Systems

Given forward periodic system (2.3)-(2.4), the transfer matrix of associated invariant system (2.7)-(2.8) is given by $F_s(z) = C_{1,s}(zI - A_{1,s})^{-1}B_{1,s} + D_{1,s}$, $s \in \mathcal{Z}$. Note that $F_{s+N}(z) = F_s(z)$ is a periodic collection of proper rational matrices, with strictly lower block triangular polynomial parts given by $D_{1,s}$. In addition to [12] these matrices satisfy the relation $F_{s+1}(z) = R_p(z)F_s(z)R_m^{-1}(z)$, where $R_p(z)$ and $R_m(z)$ are given by

$$R_j(z) = \begin{bmatrix} 0 & I_{(N-1)j} \\ zI_j & 0 \end{bmatrix} \in \Re^{jN\times jN},\quad j = p, m. \tag{3.1}$$

In the case of backward periodic system (2.5)-(2.6), if the monodromy matrix $A_{2,s} = \psi_{A_2}(s, s+N)$, $s = 0, 1, \ldots, N-1$ is nilpotent, the transfer matrix of associated invariant system (2.9)-(2.10) is given by the polynomial matrix $G_s(z) = zC_{2,s}(zA_{2,s} - I)^{-1}B_{2,s} + D_{2,s}$, $s \in \mathcal{Z}$. Note that $G_{s+N}(z) = G_s(z)$ is a periodic collection of polynomial matrices, with upper block triangular independent terms given by $D_{2,s}$. These matrices also satisfy the relation $G_{s+1}(z) = R_p(z)G_s(z)R_m^{-1}(z)$, where $R_p(z)$ and $R_m(z)$ are given in (3.1).

In this way we introduce the concept of forward (backward) periodic realization of a periodic collection of rational (polynomial) matrices.

Definition 1 ([6])

a) Periodic system $(I,\, E(\cdot),\, F(\cdot),\, G(\cdot); X(\cdot))_N$ *is a forward periodic realization of a periodic collection of rational matrices* $\{H_s(z),\ s \in \mathcal{Z}\}$, $H_{s+N}(z) = H_s(z) \in \Re^{pN\times mN}(z)$, *if* $H_s(z) = G_s(zI - E_s)^{-1}F_s + J_s$, $s \in \mathcal{Z}$, *where* $(I,\, E_s,\, F_s,\, G_s,\, J_s)$, *is the associated invariant system at time* s.

b) Periodic system $(E(\cdot),\, I,\, F(\cdot),\, G(\cdot); X(\cdot))_N$ *is a backward periodic realization of a periodic collection of polynomial matrices*

$$\{H_s(z),\ s \in \mathcal{Z}\}\,, H_{s+N}(z) = H_s(z)\ \in \Re^{pN\times mN}[z]$$

if

$$H_s(z)\ =\ zG_s(z\,E_s - I)^{-1}F_s + J_s, \quad s \in \mathcal{Z},$$

where $(E_s,\, I,\, F_s,\, G_s,\, J_s)$ *is the associated invariant system at time* s.

Now we introduce the concept of discrete-time singular forward/backward periodic realization of a periodic collection of nonproper rational matrices.

Definition 2 *The discrete time-varying singular forward/backward periodic linear system* $\Sigma^{F/B}$*, given by (2.1)-(2.2), realizes the periodic collection of rational matrices* $\{H_s(z),\ s \in \mathcal{Z}\}$, $H_{s+N}(z) = H_s(z) \in \Re^{pN\times mN}(z)$, $N \in \mathcal{Z}^+$*, if*

$$H_s(z) \ = \ [C_{1,s}, zC_{2,s}] \begin{bmatrix} (zI - A_{1,s}) & 0 \\ 0 & (zA_{2,s} - I) \end{bmatrix}^{-1} \begin{bmatrix} B_{1,s} \\ B_{2,s} \end{bmatrix} + \\ + D_{1,s} + D_{2,s}, \quad s \in \mathcal{Z}$$

where $(I,\, A_{1,s},\, B_{1,s},\, C_{1,s},\, D_{1,s})$ *and* $(A_{2,s},\, I,\, B_{2,s},\, C_{2,s},\, D_{2,s})$ *are the forward and backward invariant systems associated with the forward and backward periodic sub systems* Σ^F *and* Σ^B*, given by (2.3)-(2.4) and (2.5)-(2.6), respectively.*

In the above definition it is assumed that matrices $A_{2,s}$, $s \in \mathcal{Z}$ are nilpotent.

4 Existence of Singular Forward/Backward Periodic Realizations

Let

$$\{H_s(z),\, s \in \mathcal{Z}\}\,, \quad H_{s+N}(z) = H_s(z) \in \Re^{pN\times mN}(z), \quad N \in \mathcal{Z}^+ \tag{4.1}$$

be a periodic sequence of rational matrices satisfying

$$H_{s+1}(z) \;=\; R_p(z) H_s(z) R_m^{-1}(z), \tag{4.2}$$

where matrices $R_p(z)$, $R_m(z)$ are given in expression (3.1).

Consider the decomposition $H_0(z) \;=\; F_0(z) + P_0(z)$, where $F_0(z)$ is a proper rational matrix whose polynomial part is strictly lower block triangular and $P_0(z)$ is a polynomial matrix whose independent term is upper block triangular. Then, from (4.2), $H_s(z) \;=\; F_s(z) + P_s(z)$, $s \in \mathcal{Z}$, where $F_s(z)$ is a proper rational matrix whose polynomial part is also strictly lower block triangular and $P_s(z)$ is a polynomial matrix with upper block triangular independent term, and such that

$$\begin{aligned} F_{s+1}(z) \;&=\; R_p(z) F_s(z) R_m^{-1}(z), & (4.3)\\ P_{s+1}(z) \;&=\; R_p(z) P_s(z) R_m^{-1}(z), s \in \mathcal{Z}. & (4.4) \end{aligned}$$

Consider the periodic collection of proper rational matrices $\{F_s(z),$ $s \in \mathcal{Z}\}$. We know from [6] that there exists a forward periodic realization $(I,\, A_1(\cdot),\, B_1(\cdot),\, C_1(\cdot);\, X_1(\cdot))_N$ of $\{F_s(z),\;\; s \in \mathcal{Z}\}$.

On the other hand, for each $s \in \mathcal{Z}$, $P_s(z) \;=\; \tilde{P}_s(z) + D_s^2$, $\tilde{P}_s(z) \;=\; \sum_{i=1}^{l+1} L_i^s z^i$, is a polynomial matrix with upper block triangular independent term D_s^2. Thus $-\tilde{P}_s(\frac{1}{z}) \;=\; \frac{1}{z^{l+1}} \tilde{N}_s(z)$ is a strictly proper rational matrix, where $\tilde{N}_s(z)$ is a polynomial matrix of degree least or equal to l. Rational matrix $-\tilde{P}_s(\frac{1}{z})$ has a forward canonical invariant realization $(I,\, A_{2,s},\, B_{2,s},\, C_{2,s},\, 0)$ given by [1]

$$A_{2,s} \;=\; \left[\begin{array}{c|c} 0 & I_{lpN} \\ \hline 0_{pN} & 0 \end{array}\right] \in \Re^{(l+1)pN \times (l+1)pN}, \tag{4.5}$$

$$B_{2,s} \;=\; \mathrm{col}\,[-L_i^s]_{i=1}^{l+1}\,, \quad C_{2,s} \;=\; [\,I_{pN},\, 0_{pN},\, \cdots,\, 0_{pN}]\,. \tag{4.6}$$

Thus,

$$-\tilde{P}_s(\frac{1}{z}) = C_{2,s}(zI - A_{2,s})^{-1} B_{2,s}.$$

This implies that

$$\tilde{P}_s(z) \;=\; z C_{2,s}(z A_{2,s} - I)^{-1} B_{2,s}.$$

As $P_s(z) = \tilde{P}_s(z) + D_s^2$, the backward invariant linear system

$$\begin{aligned} A_{2,s} x(k+1) \;&=\; x(k) + B_{2,s} u(k), & (4.7)\\ y(k) \;&=\; C_{2,s} x(k+1) + D_{2,s} u(k), & (4.8) \end{aligned}$$

where $A_{2,s}$, $C_{2,s}$, $B_{2,s}$ are given in (4.5), (4.6) and $D_{2,s} = D_s^2$, is a backward invariant realization of the matrix $P_s(z)$. Note that matrix $A_{2,s}$ given by (4.5) is a nilpotent matrix with order of nilpotence $l+1$. So we only have a finite number of nonzero Markov parameters associated with backward invariant realization $(A_{2,s}, I, B_{2,s}, C_{2,s}, D_{2,s})$. Consider these Markov parameters, $V_{2,s}(k) \in \Re^{pN\times mN}$,

$$V_{2,s}(k) = \begin{cases} -C_{2,s}A_{2,s}^{k-1}B_{2,s}, & \text{if} \quad 1 \le k \le l+1 \\ D_{2,s}, & \text{if} \quad k = 0. \end{cases} \tag{4.9}$$

These parameters satisfy

$$P_s(z) = -zC_{2,s}(zA_{2,s} - I)^{-1}B_{2,s} + D_{2,s} = \sum_{k=1}^{l+1} V_{2,s}(k)z^k + D_{2,s}.$$

Applying to the backward case the same technique described in [6] we define the following periodic collection of Markov parameters

$$\{V_{2,s}(k,j), \quad k \ge 1, \quad j = 0,1,\dots k-1\} \subset \Re^{p\times m}, \quad s \in \mathcal{Z}. \tag{4.10}$$

Given a partition of $V_{2,s}(k)$ into blocks of size $p \times m$

$$V_{2,s}(k) = ([V_{2,s}(k)]_{\alpha,\beta}), \qquad [V_{2,s}(k)]_{\alpha,\beta} \in \Re^{p\times m}, \qquad \alpha, \beta = 1, 2, \dots, N,$$

and $k = pN + \gamma$, $k \ge 1$, we define $V_{2,s}(k,j)$ in the following way:

1. If $\gamma = 0$ ($k = pN$, $\quad p > 0$), $j = \tau N + \theta$, $0 \le \tau < p$, $0 \le \theta \le N-1$, then
$$V_{2,s}(k,j) = [V_{2,s}(\tau)]_{1,\theta+1}.$$
2. Consider $\gamma = 1, 2, \dots, N-1$.
   (a) If $j = 0, 1, \dots, \gamma - 1$, then
   $$V_{2,s}(k,j) = [V_{2,s}(0)]_{N-\gamma+1, N-\gamma+j+1}.$$
   (b) If $j = (\tau - 1)N + \gamma + \theta$, $\theta = 0, 1, \dots, N-1$ ($1 \le \tau \le p$, $p \ge 1$), then
   $$V_{2,s}(k,j) = [V_{2,s}(\tau)]_{N-\gamma+1, \theta+1}.$$

As in [6], we obtain the following proposition.

Proposition 3 *The backward periodic linear system* $(A_2(\cdot), I, B_2(\cdot), C_2(\cdot); X_2(\cdot))_N$, *where*

$$A_2(s) = \left[\begin{array}{c|c} 0 & I_{((l+1)N-1)p} \\ \hline 0_p & 0 \end{array}\right] \in \Re^{(l+1)pN\times(l+1)pN}, \tag{4.11}$$

$$B_2(s) = \mathrm{col}\left[-V_{2,s+1}(j, j-1)\right]_{j=1}^{(l+1)N}, \tag{4.12}$$
$$C_2(s) = \left[I_p, 0_p, \cdots, 0_p\right], \quad s \in \mathcal{Z} \tag{4.13}$$

is a periodic realization of the Markov sequences (4.10).

Moreover, the backward invariant system associated with $(A_2(\cdot), I, B_2(\cdot), C_2(\cdot); X_2(\cdot))_N$ at time s, is a backward invariant realization of the Markov parameters $V_{2,s}(k)$ given in (4.9). So from definition 1, periodic system $(A_2(\cdot), I, B_2(\cdot), C_2(\cdot); X_2(\cdot))_N$ is a backward periodic realization of the periodic collection of polynomial matrices $\{P_s(z),\, s \in \mathcal{Z}\}$.

The next theorem resumes the above discussion.

Theorem 4 *A periodic collection of, in general, nonproper rational matrices*

$$\{H_s(z),\, s \in \mathcal{Z}\}, \qquad H_{s+N}(z) = H_s(z) \in \Re^{pN\times mN}(z), \quad N \in \mathcal{Z}^+,$$

is realized by a discrete time-varying singular forward/backward N-periodic linear system such that its backward subsystem has a nilpotent monodromy matrix at any time, if and only if the relation $H_{s+1}(z) = R_p(z)H_s(z)R_m^{-1}(z)$ is satisfied, where $R_p(z)$ and $R_m(z)$ are given in (3.1).

Next, we study the problem of the existence of minimal discrete time-varying linear forward/backward periodic realizations of a periodic collection of rational matrices. If a forward/backward discrete-time linear periodic system is a realization of a periodic collection of rational matrices, then the monodromy matrices $\psi_{A_2}(s, s+N)$, $s = 0, 1, 2, \ldots, N-1$ of the backward subsystem are nilpotent. We assume nilpotence as an implicit characteristic of the monodromy matrices of backward discrete-time linear periodic systems.

5 Solutions, Reachability and Observability of Forward/Backward Discrete-Time Linear Periodic Systems

Consider the forward/backward discrete-time linear periodic system (2.1)-(2.2), in a finite set of times $k_0, k_0+1, \ldots, k_f \in \mathcal{Z}$. The forward system (2.3)-(2.4) and the backward system (2.5)-(2.6) will be restricted to the same discrete-time interval $[k_0, k_f]_{\mathcal{Z}}$. Equation (2.3) is a forward recurrence equation whose state $x_1(k)$ is determined uniquely by initial state $x_1(k_0)$ and the forward sequence of inputs $u(j)$, $j = k_0,\, k_0+1, \ldots, k-1$

$$x_1(k) = \phi_{A_1}(k, k_0)x_1(k_0) + \sum_{j=k_0}^{k-1} \phi_{A_1}(k, j+1)B_1(j)u(j). \tag{5.1}$$

On the other hand, equation (2.5) is a backward recurrence equation whose state $x_2(k)$ is determined uniquely by the terminal state $x_2(k_f)$ and the backward sequence of inputs $u(j)$, $j = k_f - 1,\ k_f - 2, \ldots, k$

$$x_2(k) = \psi_{A_2}(k, k_f)x_2(k_f) + \sum_{j=k}^{k_f-1} \psi_{A_2}(k,j)B_2(j)u(j). \qquad (5.2)$$

A pair formed by initial state $x_1(k_0)$ and terminal state $x_2(k_f)$ for systems (2.3)-(2.4) and (2.5)-(2.6), respectively, will be called a complete initial/terminal condition. The state $x(k)$ of singular system (2.1)-(2.2), at time $k \in [k_0, k_f]_{\mathcal{Z}}$, is determined by a complete initial/terminal condition, $x_1(k_0)$ / $x_2(k_f)$, and by a sequence of inputs $u(j)$, $j = k_0,\ k_0+1, \ldots, k_f-1$,

$$\begin{aligned} x(k) \quad = \quad & \begin{bmatrix} I_{n_1(k)} \\ 0 \end{bmatrix} (\phi_{A_1}(k,k_0)x_1(k_0) + \sum_{j=k_0}^{k-1} \phi_A(k, j+1)B_1(j)u(j)) + \\ & \begin{bmatrix} 0 \\ I_{n_2(k)} \end{bmatrix} (\psi_{A_2}(k,k_f)x_2(k_f) + \sum_{j=k}^{k_f-1} \psi_{A_2}(k,j)B_2(j)u(j)). \quad (5.3) \end{aligned}$$

5.1 Reachability

Definition 5

- *Given system (2.1)-(2.2), the state $w \in X(k)$ will be called reachable at time k in the interval $[k_0, k_f]_{\mathcal{Z}}$ if there exists a set of inputs $u(j) \in \Re^{m(j)}$, $j = k_0, k_0+1, \ldots, k_f-1$, such that, if $x_1(k_0) = 0$ and $x_2(k_f) = 0$, then $x(k) = w$.*

- *The state $w \in X(k)$ will be called reachable at time $k \in \mathcal{Z}$ if there exist $k_0,\ k_f \in \mathcal{Z}$, $k_0 < k < k_f$ such that w is reachable at time k in the interval $[k_0, k_f]_{\mathcal{Z}}$.*

- *System (2.1)-(2.2) is completely reachable if w is reachable at time k, for every state $w \in X(k)$, and for all $k \in \mathcal{Z}$.*

The backward reachability definition can be introduced in a similar way as the forward reachability condition. Thus the subspaces

$$\mathcal{R}_1(k_0,k) = \text{Im}[B_1(k-1),\ \phi_{A_1}(k,k-1)B_1(k-2),\ \ldots,\ \phi_{A_1}(k,k_0+1)B_1(k_0)],$$

and $\mathcal{R}_1(k) = \bigcup_{k_0<k} \mathcal{R}_1(k_0,k)$ are the subspace of reachability in the interval $[k_0,k]_{\mathcal{Z}}$ and the subspace of reachability at time k of system (2.3)-(2.4). The subspaces

$$\begin{aligned} \mathcal{R}_2(k,k_f) \quad = \quad & \text{Im}[\psi_{A_2}(k,k_f)B_2(k_f),\ \psi_{A_2}(k,k_f-1)B_2(k_f-1), \ldots \\ & \ldots, \psi_{A_2}(k,k+1)B_2(k+1),\ B_2(k)], \end{aligned}$$

and $\mathcal{R}_2(k) = \bigcup_{k_f>k} \mathcal{R}_2(k,k_f)$ will be called the subspace of reachability in the interval $[k,k_f]_{\mathcal{Z}}$ and the subspace of reachability at time k of system (2.5)-(2.6).

System (2.3)-(2.4) is completely reachable if and only if $\dim\mathcal{R}_1(k) = \dim X_1(k)$, for all $k \in \mathcal{Z}$, and system (2.5)-(2.6) is completely reachable if and only if $\dim\mathcal{R}_2(k) = \dim X_2(k)$, for all $k \in \mathcal{Z}$.

Proposition 6 ([10]) *System (2.1)-(2.2) is completely reachable if and only if systems (2.3)-(2.4) and (2.5)-(2.6) are both completely reachable.*

5.2 Observability

There are different concepts of observability for generalized state space models ([18] [4] [2] [7]). Several authors coincide in introducing observability of a singular system, at the interval $[k_0,k_f]_{\mathcal{Z}}$, as the ability to reconstruct a complete initial/terminal condition $\text{col}[x_1(k_0),\, x_2(k_f)]$ from the inputs $u(k)$ and the outputs $y(k)$ of the system. We extend the observability definition for singular invariant systems proposed in [4] and [7] to the case of forward/backward periodic systems.

Definition 7 *System (2.1)-(2.2) is observable in the interval $[k_0,k_f]_{\mathcal{Z}}$ if any state $x(k)$, $k \in [k_0,k_f]_{\mathcal{Z}}$ is uniquely determined by the inputs $u(j)$, $j \in [k_0,k_f[_{\mathcal{Z}}$ and the outputs $y(j)$, $j \in [k_0,k_f]_{\mathcal{Z}}$, of the system.*

We denote by $\mathcal{N}_1(k_0,k)$ the set of unobservable states, in the interval $[k_0,k]_{\mathcal{Z}}$, for system (2.3)-(2.4), i.e. the unobservability subspace in the interval, $[k_0,k]_{\mathcal{Z}}$. Note that

$$\begin{aligned}
\mathcal{N}_1(k_0,k) &= \text{Ker}\begin{bmatrix} C_1(k_0) \\ C_1(k_0+1)\phi_{A_1}(k_0+1,k_0) \\ \vdots \\ C_1(k)\phi_{A_1}(k,k_0) \end{bmatrix} = \\
&= \bigcap_{j=0}^{k-k_0} \text{Ker} C_1(k_0+j)\phi_{A_1}(k_0+j,k_0).
\end{aligned}$$

The set $\mathcal{N}_1(k_0) = \bigcap\limits_{k>k_0} \mathcal{N}_1(k_0,k)$ is called the unobservability subspace at time k_0, of system (2.3)-(2.4). Analogously, we denote by $\mathcal{N}_2(k,k_f)$ the set of unobservable states in the interval $[k,k_f]_{\mathcal{Z}}$, of the system (2.5)-(2.6), i.e.

the unobservability subspace in the interval $[k, k_f]_{\mathcal{Z}}$. In this case

$$\begin{aligned} \mathcal{N}_2(k, k_f) &= \text{Ker} \begin{bmatrix} C_2(k)\psi_{A_2}(k, k_f) \\ C_2(k+1)\psi_{A_2}(k+1, k_f) \\ \vdots \\ C_2(k_f-1)\psi_{A_2}(k_f-1, k_f) \end{bmatrix} = \\ &= \bigcap_{j=1}^{k_f-k} \text{Ker} C_2(k_f-j)\psi_{A_2}(k_f-j, k_f). \end{aligned}$$

The subspace $\mathcal{N}_2(k_f) = \bigcap_{k<k_f} \mathcal{N}_2(k, k_f)$ will be called the unobservability subspace at time k_f of system (2.5)-(2.6). Note that if (2.1)-(2.2) is observable in the interval $[k_0, k_f]_{\mathcal{Z}}$, then

$$\mathcal{N}(k_0, k_f) =$$
$$\text{Ker} \left[\begin{array}{c|c} C_1(k_0) & C_2(k_0)\psi_{A_2}(k_0, k_f) \\ C_1(k_0+1)\phi_{A_1}(k_0+1, k_0) & C_2(k_0+1)\psi_{A_2}(k_0+1, k_f) \\ \vdots & \vdots \\ C_1(k_f-1)\phi_{A_1}(k_f-1, k_0) & C_2(k_f-1)\psi_{A_2}(k_f-1, k_f) \end{array} \right] = \{0\}.$$

$\mathcal{N}(k_0, k_f)$ will be called the unobservability subspace in the interval $[k_0, k_f]$ of system (2.1)-(2.2) and $\mathcal{N}(k_0) = \bigcap_{k>k_0} \mathcal{N}(k_0, k)$ the unobservability subspace at time k_0 of system (2.1)-(2.2).

Definition 8

- *System (2.1)-(2.2) is observable at time $k_0 \in \mathcal{Z}$ if there exists $k_f \in \mathcal{Z}$, $k_f > k_0$ such that $\mathcal{N}(k_0, k) = \{0\}$, for all $k \in \mathcal{Z}$, $k \geq k_f$.*
- *System (2.1)-(2.2) is completely observable if it is observable at time k_0, for all $k_0 \in \mathcal{Z}$.*

Proposition 9 *System (2.1)-(2.2) is completely observable if and only if systems (2.3)-(2.4) and (2.5)-(2.6) are completely observable.*

6 Minimal Forward/Backward Periodic Realizations

In this section we study the problem of the existence of minimal discrete time-varying forward/ backward periodic realizations of a periodic collection of rational matrices. First we introduce the definition and an important result about similarity of forward and backward periodic linear systems.

Definition 10 *Two discrete time-varying forward (backward) periodic linear systems* $(I, A_i(\cdot), B_i(\cdot), C_i(\cdot); X_i(\cdot))_N$, *(*$(A_i(\cdot), I, B_i(\cdot), C_i(\cdot); X_i(\cdot))_N$*),* $i = 1, 2$, *are similar if there exists a periodic succession of invertible matrices* $\{U(k)\}$, $U(k+N) = U(k)$, $U(k) : X_2(k) \to X_1(k)$ *(*$\{V(k)\}$, $V(k+N) = V(k)$, $V(k) : X_2(k) \to X_1(k)$*),* $k \in \mathcal{Z}$, *such that*

$$\begin{aligned} A_2(k) &= (U(k+1))^{-1} A_1(k) U(k) \ \left(A_2(k) = (V(k))^{-1} A_1(k) V(k+1)\right), \\ B_2(k) &= (U(k+1))^{-1} B_1(k) \ \left(B_2(k) = (V(k))^{-1} B_1(k)\right), \\ C_2(k) &= C_1(k) U(k) \ \left(C_2(k) = C_1(k) V(k)\right). \end{aligned}$$

Definition 11 *Two discrete time-varying forward/backward periodic linear systems* $\Sigma_i^{F/B}$, $i = 1, 2$ *are similar if the two forward subsystems* Σ_i^F, $i = 1, 2$, *are similar and the two backward subsystems* Σ_i^B, $i = 1, 2$ *are also similar.*

Proposition 12 ([11], [10]) *If two discrete time-varying forward (backward) linear periodic systems* $(I,\ A_i(\cdot),\ B_i(\cdot),\ C_i(\cdot);\ X_i(\cdot))_N$ *(*$(A_i(\cdot),\ I,\ B_i(\cdot),\ C_i(\cdot);\ X_i(\cdot))_N$*), which are completely reachable and completely observable, realize the same periodic collection of rational (polynomial) matrices, then they are similar.*

Definition 13

a) The discrete time-varying forward (backward) periodic system $(I,\ A(\cdot),\ B(\cdot),\ C(\cdot);\ X(\cdot))_N$ *(*$(A(\cdot),\ I,\ B(\cdot),\ C(\cdot);\ X(\cdot))_N$*), is a minimal forward (backward) periodic realization of a periodic collection of rational (polynomial) matrices* $\{F_s(z),\ s \in \mathcal{Z}\}$, $F_{s+N}(z) = F_s(z) \in \Re^{pN \times mN}(z)$ *(*$\{P_s(z),\ s \in \mathcal{Z}\}$, $P_{s+N}(z) = P_s(z) \in \Re^{pN \times mN}[z]$*), if, for any other forward (backward) periodic realization,* $(I,\ \tilde{A}(\cdot),\ \tilde{B}(\cdot),\ \tilde{C}(\cdot);\ \tilde{X}(\cdot))_N$ *(*$(\tilde{A}(\cdot),\ I,\ \tilde{B}(\cdot),\ \tilde{C}(\cdot);\ \tilde{X}(\cdot))_N$*), of the same periodic collection of rational (polynomial) matrices, it is true that* $dim X(k) \leq dim \tilde{X}(k)$, *for all* $k \in \mathcal{Z}$.

b) The discrete-time linear forward (backward) periodic system $(I,\ A(\cdot),\ B(\cdot),\ C(\cdot))_N$ *(*$(A(\cdot),\ I,\ B(\cdot),\ C(\cdot))_N$*, with constant dimension* n *is a c-minimal forward (backward) periodic realization of a periodic collection of rational (polynomial) matrices*

$$\begin{aligned} \{F_s(z),\ s \in \mathcal{Z}\}, F_{s+N}(z) &= F_s(z) \in \Re^{pN \times mN}(z)(\{P_s(z),\ s \in \mathcal{Z}\}, \\ P_{s+N}(z) &= P_s(z) \in \Re^{pN \times mN}[z]), N \in \mathcal{Z}^+, \end{aligned}$$

if, the dimension of any other constant dimension forward (backward) periodic realization of the same periodic collection of rational (polynomial) matrices is greater than or equal to n.

a') The discrete time-varying forward/backward periodic linear system $\Sigma^{F/B}$ with state space $X(k)$, $k \in \mathcal{Z}$, of variable dimension, is a minimal forward/backward periodic realization of a periodic collection of rational matrices $\{H_s(z),\, s \in \mathcal{Z}\}$, $H_{s+N}(z) = H_s(z) \in \Re^{pN\times mN}(z)$, $N \in \mathcal{Z}^+$, if, for any other forward/backward periodic realization, $\tilde{\Sigma}^{F/B}$, of the same periodic collection, with variable state space $\tilde{X}(k)$, $k \in \mathcal{Z}$, it is true that $dimX(k) \leq dim\tilde{X}(k)$, for all $k \in \mathcal{Z}$.

b') The discrete-time forward/backward periodic linear system $\Sigma^{F/B}$ with constant dimension n is a a c-minimal forward/backward periodic realization of a periodic collection of rational matrices, if the dimension of any other constant dimension forward/backward periodic realization, $\tilde{\Sigma}^{F/B}$, of the same collection, is greater than or equal to n.

Remark 1 *For discrete time-varying periodic systems a minimal periodic realization is a periodic system which has state spaces of minimal dimension at each time. In the context of periodic systems with state space of constant dimension a c-minimal periodic realization is a periodic system with state space of minimal constant dimension.*

Let $\{H_s(z),\, s \in \mathcal{Z}\}$, $H_{s+N}(z) = H_s(z) \in \Re^{pN\times mN}(z)$, $N \in \mathcal{Z}^+$ be a periodic collection of rational matrices. We consider the decomposition $H_s(z) = F_s(z)+P_s(z)$, $s \in \mathcal{Z}$ where $F_s(z)$ is a proper rational matrix whose polynomial part is a strictly lower block triangular matrix and $P_s(z)$ is a polynomial matrix whose independent term is an upper block triangular matrix. It is easy to deduce the next proposition.

Proposition 14 ([10]) *The forward/backward periodic linear system $\Sigma^{F/B}$ is a minimal forward/backward periodic realization of the periodic collection of rational matrices $\{H_s(z),\, s \in \mathcal{Z}\}$, if and only if its forward and backward periodic subsystems, Σ^F and Σ^B, are minimal forward and minimal backward periodic realizations of the periodic collections of rational matrices $\{F_s(z),\, s \in \mathcal{Z}\}$ and polynomial matrices $\{P_s(z),\, s \in \mathcal{Z}\}$, respectively.*

The next result gives a solution to the problem of reducing realizations to reachable and observable realizations with smaller dimension.

Theorem 15 *If $\Sigma^{F/B}$ is a discrete time-varying forward/backward periodic linear system, with variable state space $X(k)$, that realizes a periodic collection of rational matrices*

$$\{H_s(z),\, s \in \mathcal{Z}\}, \qquad H_{s+N}(z) = H_s(z) \in \Re^{pN\times mN}(z), \quad N \in \mathcal{Z}^+,$$

then there exists a completely reachable and completely observable discrete time-varying forward/ backward periodic linear system $\Sigma_0^{F/B}$, with variable state space $X_0(k)$, that realizes the same periodic collection of rational matrices, such that $dimX_0(k) \leq dimX(k)$, for all $k \in \mathcal{Z}$.

Proof: Suppose that $\Sigma^{F/B}$ realizes the periodic collection of rational matrices

$$\{H_s(z),\, s \in \mathcal{Z}\}, \qquad H_{s+N}(z) = H_s(z) \in \Re^{pN\times mN}(z), \quad N \in \mathcal{Z}^+.$$

For any $s \in \mathcal{Z}$, let $H_s(z) = F_s(z) + P_s(z)$ be a decomposition where $F_s(z)$ is a proper rational matrix with strictly lower block triangular polynomial part and $P_s(z)$ is a polynomial matrix whose independent term is an upper block triangular matrix.

Consider the direct sum $X(k) = X_1(k) \oplus X_2(k)$, $k \in \mathcal{Z}$, where $X_1(k)$, $X_2(k)$, are the state spaces of the forward subsystem, $\Sigma^F \equiv (I,\, A_1(\cdot),\, B_1(\cdot),\, C_1(\cdot); X_1(\cdot))_N$ and the backward subsystem, $\Sigma^B \equiv (A_2(\cdot),\, I,\, B_2(\cdot),\, C_2(\cdot); X_2(\cdot))_N$. From [9] we know that Σ^F realizes the periodic collection of rational matrices $\{F_s(z),\, s \in \mathcal{Z}\}$, $F_{s+N}(z) = F_s(z) \in \Re^{pN\times mN}(z)$, $N \in \mathcal{Z}^+$, and Σ^B realizes the periodic collection of polynomial matrices

$$\{P_s(z),\, s \in \mathcal{Z}\}, P_{s+N}(z) = P_s(z) \in \Re^{pN\times mN}[z], N \in \mathcal{Z}^+.$$

In [11] it is proved that an appropriate decomposition of $X_1(k)$ allows us to obtain a completely reachable and completely observable realization, Σ_0^F, of the periodic collection of rational matrices $\{F_s(z),\, s \in \mathcal{Z}\}$. In the backward case, an analogous decomposition of $X_2(k)$ gives rise to a completely reachable and completely observable realization, Σ_0^B, of the periodic collection of polynomial matrices $\{P_s(z),\, s \in \mathcal{Z}\}$.

The discrete time-varying forward/backward periodic linear system $\Sigma_0^{F/B}$, defined by Σ_0^F and Σ_0^B, is a completely reachable and completely observable forward/backward periodic realization of the periodic collection of rational matrices $\{H_s(z),\, s \in \mathcal{Z}\}$. □

As a consequence, from a discrete time-varying forward/backward periodic linear system $\Sigma^{F/B}$, with variable state space $X(k)$, that realizes a periodic collection of rational matrices, we always can obtain a discrete time-varying linear forward/backward periodic system whose state space has smaller or equal dimension.

Next we characterize minimal forward/backward periodic realizations in terms of reachability and observability properties.

Theorem 16 *The discrete time-varying forward/backward periodic linear system $\Sigma^{F/B}$, is a minimal periodic realization of the periodic collection of rational matrices*

$$\{H_s(z),\, s \in \mathcal{Z}\}, \qquad H_{s+N}(z) = H_s(z) \in \Re^{pN\times mN}(z), \quad N \in \mathcal{Z}^+,$$

if and only if it is completely reachable and completely observable.

Proof: From theorem 15, we deduce that a minimal forward/backward periodic realization is completely reachable and completely observable.

To prove the converse result, we apply proposition 14. From [11] we know that the converse result is true in the case of the forward subsystem Σ^F. So it is sufficient to prove it for the backward case. Suppose that $\Sigma^B \equiv (A_2(\cdot), I, B_2(\cdot), C_2(\cdot); X_2(\cdot))_N$ is a nonminimal completely reachable and completely observable backward periodic realization of the periodic collection of polynomial matrices $\{P_s(z), s \in \mathcal{Z}\} \subset \Re^{pN\times mN}[z]$, $P_{s+N}(z) = P_s(z)$. Then there exists a realization $\tilde{\Sigma}^B \equiv (\tilde{A}_2(\cdot), I, \tilde{B}_2(\cdot), \tilde{C}_2(\cdot); \tilde{X}_2(\cdot))_N$ of the same periodic collection of rational matrices, such that $\dim\tilde{X}_2(k) \leq \dim X_2(k)$ and $\dim\tilde{X}_2(k_0) < \dim X_2(k_0)$ for some $k_0 \in \mathcal{Z}$. By theorem 15, from $\tilde{\Sigma}^B$ we construct a completely reachable and completely observable realization $\tilde{\Sigma}^B_0 \equiv (\tilde{A}^0_2(\cdot), I, \tilde{B}^0_2(\cdot), \tilde{C}^0_2(\cdot); \tilde{X}^0_2(\cdot))_N$ of the periodic collection of polynomial matrices with $\dim\tilde{X}^0_2(k) \leq \dim\tilde{X}_2(k)$ for all $k \in \mathcal{Z}$. By proposition 12, $\tilde{\Sigma}^B_0$ and Σ^B are similar. So, $\dim X_2(k) = \dim\tilde{X}^0_2(k)$, $k \in \mathcal{Z}$. This contradicts that $\dim\tilde{X}_2(k_0) < \dim X_2(k_0)$ and the theorem is proved □.

Corollary 17 *Given a periodic collection of nonproper rational matrices* $\{H_s(z),\ s \in \mathcal{Z}\} \subset \Re^{pN\times mN}(z)$, $H_{s+n}(z) = H_s(z)$, *there exists a minimal forward/backward periodic realization* $\Sigma^{F/B}_0$ *of this periodic collection if and only if*

$$H_{s+1}(z) = R_p(z) H_s(z) R_m^{-1}(z), \tag{6.1}$$

where $R_p(z)$ *and* $R_m(z)$ *are given in (3.1).*

Proof: From theorem 4 we know that, given a periodic collection of rational matrices $\{H_s(z),\ \ s \in \mathcal{Z}\} \subset \Re^{pN\times mN}(z)$, $H_{s+n}(z) = H_s(z)$, there exists a discrete time-varying forward/backward periodic realization $\Sigma^{F/B}$ of this periodic collection if and only if the recurrence equation (6.1) is true. By means of the method described in theorem 15, from $\Sigma^{F/B}$ we obtain a completely reachable and completely observable system $\Sigma^{F/B}_0$ that realizes the same periodic collection of rational matrices. By theorem 16, realization $\Sigma^{F/B}_0$ is minimal □.

Now we consider the c-minimal realization problem. To absorb this problem, we are going to present some consequences of theorem 16 that relate the minimality of forward and backward periodic systems with the minimality of its associated invariant forward and backward systems.

Proposition 18

a) *Discrete time-varying forward (backward) periodic system* $\Sigma^F \equiv (I, A_1(\cdot), B_1(\cdot), C_1(\cdot); X_1(\cdot))_N$ ($\Sigma^B \equiv (A_2(\cdot), I, B_2(\cdot), C_2(\cdot); X_2(\cdot))_N$) *is completely reachable and completely observable at time* $s \in \mathcal{Z}$ *if and only if its associated invariant forward (backward) system* $\Sigma^F_s \equiv (I, A_{1,s}, B_{1,s}, C_{1,s}, D_{1,s})$ ($\Sigma^B_s \equiv (A_{2,s}, I, B_{2,s}, C_{2,s}, D_{2,s})$), *is minimal.*

b) *Discrete time-varying forward (backward) periodic system* Σ^F (Σ^B) *is minimal if and only if its associated invariant forward (backward) system,* Σ_s^F (Σ_s^B), $s = 0, 1, \ldots, N-1$ *are minimal.*

c) *Discrete time-varying forward/backward periodic system* $\Sigma^{F/B}$, *is minimal if and only if the forward and backward invariant systems* Σ_s^F, Σ_s^B, *are both minimal, for each* $s = 0, 1, \ldots, N-1$.

The following result can be proved by direct computation and is fundamental to characterize the existence of c-minimal realizations.

Proposition 19 *Let* $(I,\, A(\cdot),\, B(\cdot),\, C(\cdot); X(\cdot))_N$ *be a discrete time-varying forward periodic realization of a periodic collection of rational matrices and* $\{\mu_j,\ j \in \mathcal{Z}\}$, $\mu_{j+N} = \mu_j$, *a periodic succession of non-negative integers. Consider* $\nu_0 = \max\{\dim X(j) + \mu_j\}$, $s_0 \in \mathcal{Z}$ *such that* $\dim X(s_0) + \mu_{s_0} = \nu_0$ *and* $\tau_\alpha = \dim X(s_0+\alpha)$, $\alpha \in \mathcal{Z}$. *Consider the discrete time-varying periodic system* $(I,\, \tilde{A}(\cdot),\, \tilde{B}(\cdot),\, \tilde{C}(\cdot); \tilde{X}(\cdot))_N$ τ_0 *defined by*

$$
\begin{aligned}
\tilde{A}(s_0+\alpha) &= \begin{bmatrix} 0 & 0 \\ 0 & A(s_0+\alpha) \end{bmatrix} \in \Re^{(\nu_0-\mu_{\alpha+1})\times(\nu_0-\mu_\alpha)}, \\
\tilde{B}(s_0+\alpha) &= \begin{bmatrix} 0 \\ B(s_0+\alpha) \end{bmatrix} \in \Re^{(\nu_0-\mu_{\alpha+1})\times m}, \\
\tilde{C}(s_0+\alpha) &= \begin{bmatrix} 0 & C(s_0+\alpha) \end{bmatrix} \in \Re^{p\times(\nu_0-\mu_\alpha)},
\end{aligned}
$$

where $A(s_0+\alpha) \in \Re^{\tau_{\alpha+1}\times\tau_\alpha}$, $B(s_0+\alpha) \in \Re^{\tau_{\alpha+1}\times m}$ *and* $C(s_0+\alpha) \in \Re^{p\times\tau_\alpha}$. *Then,* $(I,\, \tilde{A}(\cdot),\, \tilde{B}(\cdot),\, \tilde{C}(\cdot); \tilde{X}(\cdot))_N$ *realizes the same periodic collection of rational matrices.*

The next theorem characterizes c-minimal realizations.

Theorem 20 *A forward/backward periodic realization,* $\Sigma^{F/B}$, *of a periodic collection of rational matrices is c-minimal if and only if there exists some* $s_0 \in \{0,\, 1,\, \ldots,\, N-1\}$ *such that the associated invariant forward and backward systems* $\Sigma_{s_0}^F$ *and* $\Sigma_{s_0}^B$ *are both completely reachable and completely observable.*

Proof: Consider the c-minimal forward/backward periodic realization $\Sigma^{F/B}$, with constant dimension τ, of the periodic collection of rational matrices $\{H_s(z),\, s \in \mathcal{Z}\} \subset \Re^{pN\times mN}(z)$, $H_{s+n}(z) = H_s(z)$. Let

$$
\begin{aligned}
\Sigma^F &\equiv (I, A_1(\cdot), B_1(\cdot), C_1(\cdot); X_1(\cdot))_N, \\
\Sigma^B &\equiv (A_2(\cdot), I, B_2(\cdot), C_2(\cdot); X_2(\cdot))_N,
\end{aligned}
$$

be the forward and backward periodic subsystems of $\Sigma^{F/B}$ where $\dim X_1(k) + \dim X_2(k) = \tau$, $k \in \mathcal{Z}$. Consider the associated invariant forward and backward systems Σ_s^F, Σ_s^B, $s \in \mathcal{Z}$. Suppose that there is not any $s \in \mathcal{Z}$, such that Σ_s^F, Σ_s^B are both completely reachable and completely observable. By theorem 15, there exists a discrete time-varying forward/backward periodic system $\tilde{\Sigma}^{F/B}$ such that the corresponding forward and backward periodic subsystems

$$\begin{aligned} \tilde{\Sigma}^F &\equiv (I, \tilde{A}_1(\cdot), \tilde{B}_1(\cdot), \tilde{C}_1(\cdot); \tilde{X}_1(\cdot))_N, \\ \tilde{\Sigma}^B &\equiv (\tilde{A}_2(\cdot), I, \tilde{B}_2(\cdot), \tilde{C}_2(\cdot); \tilde{X}_2(\cdot))_N, \end{aligned}$$

are both completely reachable and completely observable. We have that $\dim\tilde{X}_1(k) \leq \dim X_1(k)$ and $\dim\tilde{X}_2(k) \leq \dim X_2(k)$, $k \in \mathcal{Z}$. Let

$$\tau_0 = \max_{k \in \mathcal{Z}} \left\{ \dim\tilde{X}_1(k) + \dim\tilde{X}_2(k) \right\},$$

$\tau_0 \leq \tau$ and $s_0 \in \mathcal{Z}$ such that $\tau_0 = \left\{ \dim\tilde{X}_1(s_0) + \dim\tilde{X}_2(s_0) \right\}$. We define $\tau_\alpha^{(i)} = \dim\tilde{X}_i(s_0 + \alpha)$, $i = 1, 2$. The associated invariant forward and backward systems $\tilde{\Sigma}_{s_0}^F$, $\tilde{\Sigma}_{s_0}^B$ are both completely reachable and completely observable. Consider the forward/backward periodic system with constant dimension τ_0, $\hat{\Sigma}^{F/B}$, with backward periodic subsystem given by $\hat{\Sigma}^B = \tilde{\Sigma}^B$ and forward periodic subsystem $\hat{\Sigma}^F \equiv (I,\ \hat{A}_1(\cdot),\ \hat{B}_1(\cdot),\ \hat{C}_1(\cdot);\ \hat{X}_1(\cdot))_N$, defined by

$$\begin{aligned} \hat{A}_1(s_0 + \alpha) &= \begin{bmatrix} 0 & 0 \\ 0 & \tilde{A}_1(s_0 + \alpha) \end{bmatrix} \in \Re^{(\tau_0 - \tau_{\alpha+1}^{(2)}) \times (\tau_0 - \tau_\alpha^{(2)})}, \\ \hat{B}_1(s_0 + \alpha) &= \begin{bmatrix} 0 \\ \tilde{B}_1(s_0 + \alpha) \end{bmatrix} \in \Re^{(\tau_0 - \tau_{\alpha+1}^{(2)}) \times m}, \\ \hat{C}_1(s_0 + \alpha) &= \begin{bmatrix} 0 & \tilde{C}_1(s_0 + \alpha) \end{bmatrix} \in \Re^{p \times (\tau_0 - \tau_\alpha^{(2)})}. \end{aligned}$$

By proposition 19, $\Sigma^{F/B}$ and $\hat{\Sigma}^{F/B}$ are forward/backward periodic realizations, with constant dimension τ and τ_0, respectively, of the periodic collection of rational matrices $\{H_s(z),\ s \in \mathcal{Z}\}$. Note that $\tau_0 \leq \tau$. As $\Sigma^{F/B}$ is a c-minimal realization; we have that $\tau = \tau_0$. Then $\dim X_1(s_0) = \dim\tilde{X}_1(s_0)$ and $\dim X_2(s_0) = \dim\tilde{X}_2(s_0)$. As $\Sigma_{s_0}^F$ and $\tilde{\Sigma}_{s_0}^F$, realize the same rational matrix, we deduce that $\Sigma_{s_0}^F$ is completely reachable and completely observable. In the same way we prove that $\Sigma_{s_0}^B$ is also completely reachable and completely observable. This contradiction completes the proof of the necessary condition.

Conversely, consider the completely reachable and completely observable associated invariant forward and backward systems $\Sigma_{s_0}^F$, $\Sigma_{s_0}^B$. Note that $\Sigma^F_{s_0}$, is a minimal realization of a rational matrix $F_{s_0}(z)$ and $\Sigma^B_{s_0}$ is a minimal realization of a polynomial matrix $P_{s_0}(z)$, $H_{s_0}(z) = F_{s_0}(z) + P_{s_0}(z)$. Denote by $\tau_0^{(1)}$, $\tau_0^{(2)}$ the dimension of systems $\Sigma^F_{s_0}$, $\Sigma^B_{s_0}$, respectively. We have that $\tau_0^{(1)} + \tau_0^{(2)} = \tau$, where τ is the constant dimension of $\Sigma^{F/B}$. Suppose that $\Sigma^{F/B}$ is not a c-minimal periodic realization of the periodic collection of rational matrices $\{H_s(z),\, s \in \mathcal{Z}\}$. Then there exists a forward/backward periodic realization, $\tilde{\Sigma}^{F/B}$, with constant dimension less than τ. The associated invariant forward and backward systems $\tilde{\Sigma}^F_{s_0}$, $\tilde{\Sigma}^B_{s_0}$, realize the rational matrix $F_{s_0}(z)$ and the polynomial matrix $P_{s_0}(z)$, respectively. If we denote by $\gamma_0^{(1)}$, $\gamma_0^{(2)}$ the dimension of these systems, then $\gamma_0^{(1)} + \gamma_0^{(2)} < \tau$. So $\Sigma^F_{s_0}$ and $\Sigma^B_{s_0}$ cannot be minimal realizations. This contradiction completes the proof.□

Corollary 21 *Consider a c-minimal forward/backward periodic realization, $\Sigma^{F/B}$, of a periodic collection of rational matrices, such that its periodic forward subsystem Σ^F and periodic backward subsystem Σ^B have constant dimension. Then, periodic subsystems Σ^F and Σ^B are c-minimal.*

In general, the converse implication of the above corollary is not true. The following theorem is a consequence of the previous results and characterizes the existence of minimal and c-minimal realizations.

Theorem 22 *Let*

$$\{H_s(z),\, s \in \mathcal{Z}\}, \qquad H_{s+N}(z) = H_s(z) \in \Re^{pN\times mN}(z), \quad N \in \mathcal{Z}^+, \tag{6.2}$$

be a periodic collection of rational matrices. Then, the following statements are equivalent:

(i) There exists a discrete time-varying minimal forward/backward periodic system $\Sigma^{F/B}$, with variable state space $X(k)$, $k \in \mathcal{Z}$, that realizes the periodic collection (6.2).

(ii) There exists a c-minimal forward/backward periodic realization $\tilde{\Sigma}^{F/B}$ that realizes the periodic collection (6.2).

(iii) The matrices of the periodic collection (6.2) satisfy the recurrence equation

$$H_{s+1}(z) = R_p(z)H_s(z)R_m^{-1}(z)$$

where $R_p(z)$ and $R_m(z)$ are given by

$$R_j(z) = \begin{bmatrix} 0 & I_{(N-1)j} \\ zI_j & 0 \end{bmatrix} \in \Re^{jN\times jN}, \quad j = p, m.$$

References

[1] R. W. Brockett. *Finite Dimensional Linear Systems.* New York: John Wiley, 1970.

[2] A. Banaszuk, M. Bocieki and K.M. Przyluski. Remarks on observability of implicit linear discrete-time systems, *Automatica*, **26** (1990), 421-423.

[3] M. A. Christodoulou. Realization of singular systems via Markov parameters and moments, *Int. J. of Control*, **45(1)** (1987), 237-245.

[4] D. Cobb. Controllability, observability, and duality in singular systems, *IEEE Trans. on Aut. Control*, **29(12)** (1984), 1076-1082.

[5] P. Colaneri and S. Longhi. The minimal realization problem for discrete-time periodic systems, *Automatica*, (to appear).

[6] C. Coll, R. Bru, V. Hernández and E. Sánchez, Discrete-time linear periodic realizations in the frequency domain, *Linear Algebra and its Applications*, **203** (1994), 421-423.

[7] L. Dai. *Singular Control Systems.* New York: Springer Verlag, 1989.

[8] D. F. Delchamps. *State-Space and Input-Output Linear Systems.* New York: Springer Verlag, 1988.

[9] V.D. Estruch, V. Hernández and E. Sánchez. Singular discrete-time periodic realizations in the frequency domain, *Tech. Rep. Dep. Matemática Aplicada, Universidad Politécnica de Valencia*, n. 2/93, 1993.

[10] V.D. Estruch, V. Hernández, E. Sánchez and C. Coll. Minimal forward/backward periodic realization of a periodic collection of rational matrices, *Tech. Rep. Dep. Matemática Aplicada, Universidad Politécnica de Valencia*, n. 6/94, 1994.

[11] I. Gohberg, M. A. Kaashoek, and L. Lerer. Minimality and realization of discrete time-varying systems, *Operator Theory: Advances and Applications*, **56** (1992), 261-296.

[12] O. M. Grasselli and S. Longhi. Zeros and poles of linear periodic multivariable discrete-time systems, *Circuits Systems Signal Process*, **7(3)** (1988), 361-380.

[13] T. Kailath. *Linear Systems.* Englewood Cliffs, NJ: Prentice Hall, 1980.

[14] C.A. Lin and C.W. King. Minimal periodic realizations of transfer matrices, *IEEE Trans on Aut. Control,* **38(3)** (1993), 462-466.

[15] R.A. Meyer and C.S. Burrus. A unified analysis of multirate and periodically time-varying digital filters, *IEEE Trans. Circuits Systems*, **22(3)** (1975), 162-168.

[16] H. Rosenbrock. *State-Space Multivariable Theory.* Nelson, 1970.

[17] E.D. Sontag. *Mathematical Control Theory.* New York: Springer Verlag, 1991.

[18] E.L. Yip and R.F. Sincovec. Solvability, controllability and observability of continuous descriptor systems, *IEEE Trans. on Aut. Control*, **26(3)** (1981), 702-707.

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