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Robust Adaptive Control of Discrete-Time Systems with Arbitrary Rate of Variations*

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Abstract

In this paper an indirect adaptive regulator is constructed for single-input single-output linear time-varying discrete-time systems. The plant parameters are modeled as a linear combination of known bounded functions which may be fast varying with unknown slowly-varying coefficients that are confined to a convex region. The unknown coefficients are estimated using a gradient type estimator with a projection operation. It is assumed that the system is uniformly N-step reachable, where N is an integer greater than or equal to the system dimension. It is then shown that the resulting closed-loop system is globally stable if the rate of unknown parameter variations and the normalized model errors are sufficiently small.

Key words: robust adaptive control time-varying systems parameter estimation

1 Introduction

Much of the work that has been done on linear time-varying systems is based on the "frozen-time" approach where a time-varying plant is viewed as a collection of parameterized linear time-invariant systems. This method is obviously adequate only for systems with slowly-varying parameters (e.g, [1], [2], [3], [5], [6], [9], [10], [15]). Many systems, however, contain some rapid parameter variations. Several results have treated the class of systems possessing a stable inverse (e.g, [16], [14]). On the other hand, many systems arising in practice are not stably invertible. For instance, the linearized system dynamics of air-to-air missiles contain rapidly-varying coefficients due to missile acceleration and high velocity. In this case, some

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results concerning the regulation of this class of systems have been established.

Kamen et al. [11] proposed a regulator scheme based on the generalized Kleinman's method [13] for multi-input multi-output linear discrete-time systems. This regulator is designed under the assumption that the plant is uniformly N-step reachable, where N is greater or equal to the system dimension. In adaptive control framework, it is assumed that the plant parameters are expressed in terms of a linear combination of known bounded functions that are allowed to be fast varying and unknown constant coefficients. The coefficients are estimated using a RLS estimator with a projection operation and a variable forgetting factor. Using the generalized Kleinman's regulator with the unknown coefficients replaced with the estimates, an indirect adaptive regulator was proposed.

In an input-output framework, Ioannou and Tsakalis [8] considered continuous-time systems that are modeled by polynomial differential operators with time-varying coefficients. They proposed a new pole-placement regulator structure that is able to overcome the non-commutativity problem associated with the time-varying polynomial differential operators. This regulator scheme is based on the assumption of uniform coprimeness of the plant operators. Ioannou and Tsakalis [8] then considered an indirect adaptive pole-placement regulator. It is assumed that the time-varying parameters are modeled in terms of a known part which may be rapidly varying and an unknown slowly-varying part which is estimated using a gradient estimator with σ-modification and a normalizing signal.

In this paper, we consider a single-input/single-output discrete-time systems modeled by a linear time-varying difference equation that includes an error term to incorporate model errors and/or disturbances. The discrete-time model may be a result of sampling the input and output signals of a continuous-time system; that is, the model corresponds to a sampled-data format. The system parameters are modeled as in [11], except that the unknown coefficients, which are confined to a convex region, are assumed to be slowly varying. It is further assumed that for all possible values in the convex set, the observable realization of the system model is N-step uniformly reachable. The purpose of this paper is to design a robust indirect adaptive pole-placement regulator for discrete-time systems without requiring any excitation and without requiring the nominal model to be stably invertible. This paper is to some extent a discrete-time version of the work presented in [8], except that in [8] model errors or disturbances are not considered in the system model.

This paper is outlined as follows: after some preliminaries in the following section, the problem statement is formulated and the assumptions are specified in Section 3. The estimation algorithm and the pole-placement regulator are discussed in Section 4 and Section 5 respectively. Then the adaptive pole-placement regulator is presented, and the main result is given in Section 6. It is shown that the adaptive regulator results in global stability (i.e, all signals are bounded for any initial conditions) if the mean of the rate of variations of the unknown coefficients is sufficiently small and if the model errors are sufficiently small. A simulation example is also given in Section 6.

2 Preliminaries

With $\mathbf{Z}$ equal to the set of integers and $\Re$ equal to set of real numbers, let $\Re^{\mathbf{Z}}$ denote the $\Re$-linear space of all functions from $\mathbf{Z}$ into $\Re$. Moreover, let $\mathcal{S}$ denote the subspace of $\Re^{\mathbf{Z}}$ that consists of all functions with support bounded to the left; that is, for every $y(k) \in \mathcal{S}$ there exists an integer k_0 which may depend on $y(k)$ such that $y(k) = 0$ for all $k < k_0$.

Now, let q denote the *left − shift operator* on $\mathcal{S}$ defined by $q(y(k)) = y(k+1)$. Also let q^{-1} denote the *right − shift operator* on $\mathcal{S}$ given by $q^{-1}(y(k)) = y(k-1)$. It can be verified that q^{-1} is the left and right inverse of the operator q on the space $\mathcal{S}$; that is, for any $y(k) \in \mathcal{S}$,

$$q(q^{-1}(y(k)) = q^{-1}(q(y(k))) = y(k).$$

Moreover, let $P(q,k)$ denote a polynomial in q defined by

$$P(q,k) = \sum_{i=0}^{n} a_i(k) q^i \tag{2.1}$$

where the $a_i(k)$ belong to $\Re^{\mathbf{Z}}$. The degree of the polynomial $P(q,k)$ is the largest value i such that the coefficient $a_i(k)$ is nonzero for at least one value of k. In the case where the degree of $P(q,k)$ is equal to n, $a_n(k)$ is called the leading coefficient of $P(q,k)$, and if the leading coefficient is the identity function, $P(q,k)$ is said to be *monic*.

The polynomial $P(q,k)$ defines a linear operator from $\mathcal{S}$ to $\mathcal{S}$, which for every $y(k) \in \mathcal{S}$ defines an element $z(k) \in \mathcal{S}$ given by

$$z(k) = P(q,k)y(k) = \sum_{i=0}^{n} a_i(k) y(k+i).$$

Given a monic polynomial operator $P(q,k)$, the operator denoted by $P^{-1}(q,k)$ is defined as

$$P^{-1}(q,k)z(k) = h^T \sum_{i=-\infty}^{k-1} \Phi(k, i+1) g z(i) \tag{2.2}$$

where Φ is the state transition matrix of the following state model associated with the equation $P(q,k)y(k) = z(k)$:

$$x(k+1) = F(k)x(k) + gz(k); \qquad y(k) = h^T x(k) \tag{2.3}$$

where "T" denotes the transpose operation. The state vector $x(k)$ is given by

$$x(k) = [y(k+n-1) \cdots y(k)], \tag{2.4}$$

$g = [1, 0, \cdots, 0]^T$, $h = [0, \cdots, 1]^T$, and $F(k)$ is a $n \times n$ matrix which is given by

$$F(k) = \begin{bmatrix} -a_{n-1}(k) & \cdots & \cdots & -a_0(k) \\ 1 & 0 & \cdots & \\ & \ddots & & \mathbf{0} \\ \mathbf{0} & & 1 & 0 \end{bmatrix}.$$

The state transition matrix Φ is defined as

$$\Phi(k,k_0) = \begin{cases} F(k-1)F(k-2)...F(k_0) & \text{if } k > k_0 \\ I & \text{if } k = k_0 \\ \text{not defined} & \text{if } k < k_0 \end{cases}. \tag{2.5}$$

Having defined the operator $P^{-1}(q,k)$, we have the following theorem:

Theorem 1 *Given a monic polynomial operator $P(q,k)$, the operator $P^{-1}(q,k)$ is the left inverse and the right inverse operator for $P(q,k)$ on the space $\mathcal{S}$; that is, for any $y(k) \in \mathcal{S}$:*

(i) $P^{-1}(q,k)P(q,k)(y(k)) = y(k)$

(ii) $P(q,k)P^{-1}(q,k)(y(k)) = y(k)$.

For proof of Theorem 1, see Appendix A.

A definition and a proposition associated with the inverse operator which will be used in subsequent discussion are given next.

Definition 1 *Given a monic operator $P(q,k)$, the inverse operator $P^{-1}(q,k)$ defined in (2.2) is said to be exponentially stable if the state transition matrix $\Phi(k,k_0)$ satisfies*

$$\|\Phi(k,k_0)\| \le d\mu^{k-k_0} \tag{2.6}$$

where $\|.\|$ is the Euclidean norm, $0 \le \mu < 1$, and d is a positive real number.

Proposition 1 *Given the monic polynomial operators* $Q(q,k)$ *and* $P(q,k)$*, the inverse operators* $Q^{-1}(q,k)$*,* $P^{-1}(q,k)$*, and* $(P(q,k)Q(q,k))^{-1}$ *satisfy the following equality*

$$(P(q,k)Q(q,k))^{-1}(z(k)) = Q^{-1}(q,k)P^{-1}(q,k)(z(k))$$

for any $z(k) \in \mathcal{S}$*.*

Proof: Consider the monic polynomial operator $P(q,k)Q(q,k)$ which maps $y(k)$ into $z(k)$, both elements of $\mathcal{S}$, in the following way

$$P(q,k)Q(q,k)y(k) = z(k).$$

Since $P(q,k)Q(q,k)$ is monic, there exists a left inverse operator $(P(q,k)$ $Q(q,k))^{-1}$ such that

$$y(k) = (P(q,k)Q(q,k))^{-1}z(k). \tag{2.7}$$

Also given that $P(q,k)$ and $Q(q,k)$ are monic, there exist $P^{-1}(q,k)$ and $Q^{-1}(q,k)$ such that

$$Q(q,k)y(k) = P^{-1}(q,k)z(k)$$

and

$$y(k) = Q^{-1}(q,k)P^{-1}(q,k)z(k). \tag{2.8}$$

Equating (2.7) and (2.8), the desired result is obtained. □

Now, given a monic polynomial $A(q,k)$ with the degree of $A(q,k)$ equal to n, and a polynomial $B(q,k)$ with the degree of $B(q,k)$ equal to $n-1$, consider the linear time-varying discrete-time system specified by the input/output difference equation

$$A(q,k)y(k) = B(q,k)u(k) + \eta(k) \tag{2.9}$$

where $y(k) \in \mathcal{S}$ is the output, $u(k) \in \mathcal{S}$ is the control input, $\eta(k)$ is an unknown signal which consists of model errors and/or disturbances and $A(q,k)$, $B(q,k)$ are given by

$$A(q,k) = q^n + \sum_{i=0}^{n-1} a_i(k+n)q^i, \quad a_i(k+n) \in \Re^{\mathbf{Z}}$$

$$B(q,k) = \sum_{i=0}^{n-1} b_i(k+n)q^i, \quad b_i(k+n) \in \Re^{\mathbf{Z}}.$$

Equation (2.9) can be written as the following ARMA model:

$$y(k) + \sum_{i=1}^{n} a_{n-i}(k) y(k-i) = \sum_{i=1}^{n} b_{n-i}(k) u(k-i) + \eta(k). \tag{2.10}$$

Moreover, the system (2.10) can be expressed in the following observable canonical form:

$$x(k+1) = A(k)x(k) + b(k)u(k); \qquad y(k) = c^T x(k) \tag{2.11}$$

where $b(k) = [b_{n-1}(k+n), \cdots, b_0(k+1)]^T$, $c = [0, \cdots, 1]^T$, and $A(k)$ is given by

$$A(k) \quad = \quad \begin{bmatrix} -a_{n-1}(k+n-1) & 0 & \cdots & & 0 \\ -a_{n-2}(k+n-2) & 1 & & \mathbf{0} & \\ \vdots & & \ddots & & \\ -a_0(k+1) & 0 & & 1 & 0 \end{bmatrix}.$$

Given the system defined by (2.9) and (2.10), the objective is to design an adaptive regulator (in the case where the coefficients of the polynomials $A(q,k)$ and $B(q,k)$ contain unknown time-varying parameters) so that the resulting closed-loop system is globally stable. In order to design the regulator, we will make use of the notion of the time-varying resultant [7].

The resultant matrix is a classical tool for determining the existence of a common factor of two time-invariant polynomials. In the time-varying case, Hwang [7] related the issue of coprimeness of two time-varying polynomials and the Bezout identity to the resultant matrix. In this regard, it was shown that the existence of a Bezout identity is sufficient but not necessary for the coprimeness of two polynomials. This is due to the fact that discrete-time systems can be reachable in a number of steps which is greater than the system dimension.

Given the polynomials $A(q,k)$ and $B(q,k)$ defined above, we can write

$$A(q,k) = q^n + q^{n-1} a_{n-1}(k+1) + \cdots + a_0(k+n)$$

$$B(q,k) = q^{n-1} b_{n-1}(k+1) + \cdots + b_0(k+n).$$

Then for any positive integer $N \geq n$, the $(N+n-1) \times (2N-1)$ resultant matrix is given by

$$\Gamma_N(k) = \begin{bmatrix} a_0(k+n) & 0 & \cdots & b_0(k+n) & 0 & \cdots \\ \vdots & \ddots & & \vdots & \ddots & \\ a_{n-1}(k+1) & & q^{2-N}a_0(k+n) & b_{n-1}(k+1) & & \\ 1 & & & 0 & & q^{1-N}b_0(k+n) \\ & \ddots & & & \ddots & \\ \vdots & & q^{2-N}a_{n-1}(k+1) & \vdots & & \\ 0 & \cdots & 1 & 0 & \cdots & q^{1-N}b_{n-1}(k+1) \end{bmatrix} \tag{2.12}$$

Hwang [7] has established that the system (2.11) is N-step reachable if and only if the rank of the resultant matrix (2.12) is equal to $N+n-1$.

In the next section the problem statement is formulated and the assumptions required to solve the adaptive control problem are given.

3 Problem Formulation

Again consider the discrete-time single-input single-output time-varying system given by (2.10). It is assumed that the order of the nominal model is known apriori, where the nominal model is given by (2.10) with $\eta(k)=0$. As in [11], the time-varying coefficients in (2.10) are modeled in terms of linear combinations of known time-varying functions $f_j(k)$ with unknown slowly-varying coefficients $a_{i,j}(k)$ and $b_{i,j}(k)$, except that in [11] the unknown part is assumed to be constant. The plant coefficients $a_i(k)$, $b_i(k)$ can then be expressed as

$$a_i(k) = \sum_{j=1}^{s} a_{i,j}(k) f_j(k), \quad b_i(k) = \sum_{j=1}^{s} b_{i,j}(k) f_j(k) \tag{3.13}$$

where i is an integer ranging from 0 to $n-1$ and s is a known positive integer. It should be emphasized that in this development, the known functions $f_j(k)$ are allowed to vary arbitrarily fast.

Consider the scaled s-element row vectors $\lambda_i(k)$, $\omega_i(k)$, and $h_i(k)$ defined as

$$\begin{aligned} \lambda_i(k) &= \tau^{i-n+1}[a_{i,1}(k) \cdots a_{i,s}(k)] \\ \omega_i(k) &= \tau^{i-n+1}[b_{i,1}(k) \cdots b_{i,s}(k)] \\ h_i(k) &= \tau^{n-1-i}[f_1(k) \cdots f_s(k)], \quad i = 0, 1, ..., n-1 \end{aligned} \tag{3.14}$$

where $0 < \tau < 1$.

By defining the $2ns$-element column vector

$$\theta^T(k) = [\ \lambda_{n-1}(k) \quad \cdots \quad \lambda_0(k) \quad \omega_{n-1}(k) \quad \cdots \quad \omega_0(k)\ ], \tag{3.15}$$

the plant parameters can be written in a compact way as

$$[\ a_{n-1}(k) \ \cdots \ a_0(k) \ \ b_{n-1}(k) \ \cdots \ b_0(k)\ ]^T = H(k)\theta(k) \tag{3.16}$$

where the $2n \times 2ns$ matrix $H(k)$ is given by

$$H(k) = \begin{bmatrix} h_{n-1}(k) & 0 & \cdots & & & \\ & \ddots & & \mathbf{0} & & \\ & & h_0(k) & & & \\ \mathbf{0} & & & h_{n-1}(k) & & \\ & & & & \ddots & \\ & & & & & h_0(k) \end{bmatrix}. \tag{3.17}$$

Moreover, by defining the $2ns$-element weighted regression vector

$$\begin{aligned} \phi^T(k) \quad = \quad & [-h_{n-1}(k)y(k-1)\cdots - h_0(k)y(k-n) \\ & h_{n-1}(k)u(k-1)\cdots h_0(k)u(k-n)]. \end{aligned}$$

Equation (2.10) can be written in the standard form

$$y(k) = \phi^T(k)\theta(k) + \eta(k). \tag{3.18}$$

The weighted regression vector $\phi(k)$ defined above differs from the standard one by the decaying terms τ^i, $0 \le i \le n-1$. By appropriately selecting the value τ, it will be shown that under the following assumptions, there is an adaptive regulator for the system (2.9) or (2.10) which results in global stability.

The parameter vector $\theta(k)$ is an element of the space $(\Re^{\mathbf{Z}})^{2ns}$ of 2ns-element column vectors over $\Re^{\mathbf{Z}}$. The following constraint on the parameter variations is imposed:

A.1 There exists an integer k_0 and a known, bounded convex subset $\mathcal{H}$ of the $2ns$-dimensional Euclidean space $\Re^{2ns}$ such that $\theta(k) \in \mathcal{H}$ for all $k > k_0$.

It is worth noting that the assumption that $\theta(k)$ belongs to $\mathcal{H}$ for all $k > k_0$ implies that there exists nonnegative constants v_0 and v_1 such that

$$\sum_{i=k_0}^{k-1} \|\theta(i+1) - \theta(i)\| \le v_0 + v_1(k-k_0) \qquad \forall \ \ k > k_0. \tag{3.19}$$

Moreover the following assumption on the resultant matrix is required:

A.2 There exists a positive integer $N \geq n$ such that for any possible parameter trajectory θ with $\theta(k) \in \mathcal{H}$ for all $k > k_0$,

$$|det(,\ _{N,\theta}(k),\ {}^{T}_{N,\theta}(k))| > \varepsilon, \tag{3.20}$$

where ε is some real constant strictly greater than zero, det denotes the determinant, and $,\ _{N,\theta}(k)$ is the resultant given by (2.12) with the $a_i(k)$ and $b_i(k)$ defined by (3.16), where $i = 0, \cdots, n-1$.

To test for the condition given by (3.20), it is necessary to know what the possible parameter trajectories $\theta(k)$ are. In practice, such knowledge is usually not available a priori; however, (3.20) can be tested using particular "sample trajectories" such as ramp or sine waveforms with vector values belonging to $\mathcal{H}$. Although this does not guarantee that (3.20) is satisfied for all possible trajectories, based on a number of simulation examples we have run, the use of sample trajectories appears to be a good indicator as to when (3.20) is satisfied in general.

From Assumption **A.2**, it can be deduced that $,\ _{N,\theta}(k)$ has a rank $N + n - 1$ and thus the system is N-step reachable [7]. Further, since it is required that $(|det(,\ _{N,\theta}(k),\ {}^{T}_{N,\theta}(k))| > \varepsilon)$, a uniformity condition is imposed. Therefore **A.2** implies that the system (2.11) is uniformly N-step reachable for all possible vectors $\theta(k) \in \mathcal{H}$.

Regarding the error term $\eta(k)$ in (2.10) or (3.18), the following assumption is required

A.3 The error term $\eta(k)$ is bounded by

$$|\eta(k)| \leq \alpha\|\phi(k)\| + \rho(k), \tag{3.21}$$

where α is an unknown nonnegative constant and $\rho(k)$ is an unknown nonnegative bounded function.

Now given the plant defined by (2.9) or (3.18) with the assumptions **A.1**-**A.3**, the objective is to design a pole-placement regulator using polynomial operators of the form

$$u(k) = C(q,k)D^{-1}(q,k)(r(k) - y(k)) \tag{3.22}$$

where $r(k)$ is an arbitrary bounded reference and $C(q,k)$, $D(q,k)$ are given by

$$C(q,k) = \sum_{i=0}^{N-1} c_i(k)q^i \qquad D(q,k) = q^{N-1} + \sum_{i=0}^{N-2} d_i(k)q^i$$

so that the resulting closed-loop system is globally stable (i.e, all signals are bounded for any initial conditions). Again N is a positive integer which is greater or equal to the system dimension n.

The following section discusses the parameter estimator.

4 The Parameter Estimator

Consider the plant given by (3.18). The parameter estimation scheme that will be used to generate an estimate $\hat{\theta}(k)$ of the parameter vector $\theta(k)$ is given by

$$\theta^*(k) = \hat{\theta}(k-1) - \frac{\phi(k)(\hat{y}(k) - y(k))}{\gamma(k-1) + \phi^T(k)\phi(k)} \tag{4.23}$$

$$\hat{\theta}(k) = \mathcal{P}[\theta^*(k)] \tag{4.24}$$

where $\mathcal{P}$ denotes a projection operator that ensures that $\hat{\theta}(k)$ lies within $\mathcal{H}$, and $\hat{y}(k)$ is the predicted output given by

$$\hat{y}(k) = \phi^T(k)\hat{\theta}(k-1).$$

For details on a projection satisfying (4.24), see [4].

In (4.23), $\gamma(k-1)$ is a normalization term (a design parameter) which is assumed to satisfy the condition

$$\gamma(k-1) \geq c, \qquad \forall k,$$

for some real number $c > 0$.

Defining the output prediction error

$$e(k) = \hat{y}(k) - y(k) = \phi^T(k)(\hat{\theta}(k-1) - \theta(k)) - \eta(k)$$

and the normalized output prediction error,

$$\bar{e}(k) = \frac{e(k)}{\sqrt{\gamma(k-1) + \phi^T(k)\phi(k)}} \tag{4.25}$$

we have a key property of the parameter estimator (4.23), (4.24) which is given in the following result:

vspace3mm

Theorem 2 *Given the plant (*3.18*), suppose that Assumptions* **A**.**1** *and* **A**.**2** *hold, so that there exist* v_0, v_1, α*, and* $\rho(k)$ *for which (*3.19*) and (*3.21*) are satisfied.*

Define

$$D = \sup_{\theta_1, \theta_2 \in \mathcal{H}} \|\theta_1 - \theta_2\|$$

$$\epsilon = \alpha + \sup_k \left[\frac{\rho(k)}{\sqrt{\gamma(k-1) + \phi^T(k)\phi(k)}} \right].$$

Then for all $k \geq k_0$

$$\sum_{i=k_0}^{k-1} \bar{e}^2(i) \leq (D^2+3Dv_0+2\epsilon v_0)+(2D\epsilon+2\epsilon^2+3Dv_1+2\epsilon v_1)(k-k_0). \quad (4.26)$$

The proof of Theorem 2 is given in [9].

This property of the parameter estimation process allows us to prove global stability of the adaptive control system.

In order to illustrate the design method, we first assume that the plant parameters are known for all $k > k_0$, for some integer k_0. Then, we extend the design approach to the case when the slowly-varying parameters are unknown.

5 The Control Law

Most of the existing pole-placement regulators for time-varying discrete-time systems have been derived using the frozen-time approach. More precisely, by considering the frozen-time method, commutativity of polynomials holds and thus a regulator of the form (3.22) can be expressed as

$$u(k) = D^{-1}(q,k)C(q,k)(r(k) - y(k)).$$

Although this regulator can be easily realized, it guarantees closed-loop stability only if the plant is slowly time varying.

Now the objective is to design a pole-placement regulator (3.22) so that the operator $[A(q,k)D(q,k) + B(q,k)C(q,k)]$ is equal to a specified constant Hurwitz polynomial whose zeros can be assigned. The fundamental problem that immediately arises in this framework is associated with the polynomial operators. Namely, since these operators are not commutative (i.e., $C(q,k)D^{-1}(q,k) \neq D^{-1}(q,k)C(q,k)$), the approach taken above is evidently not applicable. Therefore the design issue does not only involve the proof of existence of the regulator, but also involves the question as to whether the regulator is implementable.

In the continuous-time case, Ioannou and Tsakalis [8] have proposed a new regulator structure that achieves stabilization regardless of the rate of time variations. On the other hand, in the discrete-time case, Kamen [12] gives a cascade realization of the regulator (3.22) as

$$\begin{aligned} C(q,k)D^{-1}(q,k) &= (C(q,k)q^{-N+1})(q^{N-1}D^{-1}(q,k)) \\ &= (C(q,k)q^{-N+1})(D(q,k)q^{-N+1})^{-1}. \end{aligned}$$

So the control signal can be obtained as

$$
\begin{aligned}
u(k) &= (C(q,k)q^{-N+1})v(k) \\
&= (\sum_{i=0}^{N-1} c_i(k)q^{i-N+1})v(k) \\
&= \sum_{i=0}^{N-1} c_i(k)v(k+i-N+1)
\end{aligned}
\tag{5.27}
$$

where $v(k)$ is computed from the following equation

$$
\begin{aligned}
v(k) &= (D(q,k)q^{-N+1})^{-1}(r(k)-y(k)) \\
&= (1+\sum_{i=0}^{N-2} d_i(k)q^{i-N+1})^{-1}(r(k)-y(k)) \\
&= -(\sum_{i=0}^{N-2} d_i(k)v(k+i-N+1)) + r(k) - y(k).
\end{aligned}
\tag{5.28}
$$

It should be emphasized that this cascade realization in the discrete-time case can not be obtained in continuous time.

The next theorem asserts that the regulator (3.22) in fact exists. It is believed that the following result and constructions are new.

Theorem 3 *Consider the nominal model of the plant* (2.9) *which satisfies the assumption* **A.3**. *Then the coefficients of the operators* $C(q,k)$ *and* $D(q,k)$ *in* (3.22) *can always be selected such that* $[A(q,k)D(q,k)+B(q,k)C(q,k)]$ *is equal to any desired Hurwitz time-invariant polynomial of degree* $N+n-1$.

Proof: Given the nominal model of the system (2.9) and the pole-placement regulator (3.22), the output signal $y(k)$ can be expressed in terms of the closed-loop operator applied to the reference input as

$$
\begin{aligned}
y(k) &= A^{-1}(q,k)B(q,k)u(k) \\
&= A^{-1}(q,k)B(q,k)C(q,k)D^{-1}(q,k)(r(k)-y(k)).
\end{aligned}
\tag{5.29}
$$

Equation (5.29) can be written as

$$
\begin{aligned}
y(k) &= (1-[1+A^{-1}(q,k)B(q,k)C(q,k)D^{-1}(q,k)]^{-1})r(k) \\
&= r(k) - [1+A^{-1}(q,k)B(q,k)C(q,k)D^{-1}(q,k)]^{-1} \\
&\quad A^{-1}(q,k)A(q,k)r(k).
\end{aligned}
$$

Using Proposition 1, the above equation becomes

$$
y(k) = (1-[A(q,k)+B(q,k)C(q,k)D^{-1}(q,k)]^{-1}A(q,k))r(k)
$$

$$
\begin{aligned}
&= \quad r(k) - D(q,k)[A(q,k)D(q,k) + B(q,k)C(q,k)]^{-1} \\
&\quad\;\; A(q,k)r(k). \qquad (5.30)
\end{aligned}
$$

From (5.30), it can be concluded that the inverse operator associated with the closed-loop system is given by

$$[A(q,k)D(q,k) + B(q,k)C(q,k)]^{-1}.$$

It is the goal of the regulator to make the inverse operator of (5.30) equal to the inverse of a Hurwitz polynomial $(R(q))^{-1}$, or

$$R(q) = A(q,k)D(q,k) + B(q,k)C(q,k). \qquad (5.31)$$

Equation (5.31) is a time-varying version of the Bezout identity which can be expressed in terms of the resultant matrix defined in (2.12) as

$$, _N(k)X(k) = Z(k) \qquad (5.32)$$

where the vector $X(k)$ contains the adjustable parameters

$$
\begin{aligned}
X(k) \quad &= \quad [d_0(k) \cdots d_{N-2}(k-N+2) \\
&\quad\;\; c_0(k) \cdots c_{N-1}(k-N+1)]^T \qquad (5.33)
\end{aligned}
$$

for which equation (5.32) is solved. The vector $Z(k)$ consists of the coefficients of the operator $(R(q) - A(q,k)q^{N-1})$. So if $R(q)$ is given by

$$R(q) = q^{N+n-1} + r_{N+n-2}q^{N+n-2} + \cdots + r_0,$$

the vector $Z(k)$ can be written as

$$Z(k) = [\ r_0 - a_0(k+n-N+1) \quad \cdots \quad r_{N+n-2} - a_{n-1}(k-N+2)\ ]^T.$$

In order to solve (5.32) for the vector $X(k)$, the notion of pseudo-inverse is used; that is,

$$X(k) = , ^T_N(k)(, _N(k), ^T_N(k))^{-1}Z(k). \qquad (5.34)$$

Using assumption **A.2**, $(, _N(k), ^T_N(k))^{-1}$ exists and so the regulator parameters always exist. This concludes the proof. □

Note that the vector $X(k)$ given by (5.33) contains the regulator parameters at times prior to and at time k. However, from equations (5.27) and (5.28), the regulator parameters at time k should be available. To resolve this problem, we consider the equation

$$, _N(k+N-1)X(k+N-1) = Z(k+N-1). \qquad (5.35)$$

By solving for $X(k+N-1)$, the parameter $c_{N-1}(k)$ is obtained and all the other parameters are stored for future iterations. For initial computation at time k_0, where k_0 is an integer, equation (5.35) should be solved at times k_0 until $k_0 + N - 1$ to obtain all the regulator parameters at time k_0. It should be noted that in order to obtain the controller parameters at time k, the resultant matrix (2.12) and the vector $Z(k)$ should be computed at time $k + N - 1$. This implies that the plant parameters are to be known in the future up to time $k + n + N - 1$.

In order to highlight the fundamental difference between the design technique developed in this section and the frozen-time approach, the following example is considered.

Example 5.1:

Consider the second order system

$$(q^2 + qa_1(k) + a_0(k))y(k) = (qb_1(k) + b_0(k))u(k)$$

and the pole-placement regulator

$$u(k) = (qc_1(k) + c_0(k))(q + d_0(k))^{-1}(r(k) - y(k)).$$

As mentioned above, the objective is to solve the following Bezout identity for the coefficients $d_0(k)$, $c_0(k)$, and $c_1(k)$:

$$R(q) = (q^2 + qa_1(k) + a_0(k))(q + d_0(k)) + (qb_1(k) + b_0(k))(qc_1(k) + c_0(k)) \tag{5.36}$$

where $R(q)$ is a Hurwitz polynomial given by

$$R(q) = q^3 + r_2q^2 + r_1q + r_0.$$

Following the definition of the operator q in Section 2, and by defining

$$n(k) = a_0(k)b_1(k)b_1(k-1) - b_0(k)a_1(k)b_1(k-1) + b_0(k)b_0(k-1), \tag{5.37}$$

the controller coefficients are given by

$$\begin{aligned} d_0(k) \quad = \quad & \frac{1}{n(k)}\{r_0b_1^2(k) - b_0(k)b_1(k)(r_1 - a_0(k)) + b_0^2(k)(r_2 - a_1(k)) \\ & +(r_0b_1(k) - r_1b_0(k))(b_1(k-1) - b_1(k)) + b_0(k)r_2 \\ & (b_0(k-1) - b_0(k)) + b_0(k)(b_1(k-1)a_0(k-1) \\ & -b_1(k)a_0(k) + a_1(k)b_0(k) - a_1(k-1)b_0(k-1))\} \end{aligned} \tag{5.38}$$

$$\begin{aligned} c_0(k) \quad = \quad & \frac{1}{n(k)}\{-r_0a_1(k)b_1(k) + r_0b_0(k) + a_0(k)b_1(k)(r_1 - a_0(k)) \\ & -a_0(k)b_0(k)(r_2 - a_1(k)) + a_0(k)[r_1(b_1(k-1) - b_1(k)) \\ & +r_2(b_0(k) - b_0(k-1)) + b_1(k)a_0(k) - b_1(k-1)a_0(k-1) \\ & +b_0(k-1)a_1(k-1) - b_0(k)a_1(k)] + r_0a_1(k)(b_1(k) - b_1(k-1)) \\ & +r_0(b_0(k-1) - b_0(k))\} \end{aligned} \tag{5.39}$$

$$\begin{aligned} c_1(k) &= \frac{1}{n(k)}\{-r_0 b_1(k) + b_0(k)(r_1 - a_1(k)) + (a_0(k)b_1(k) - a_1(k)b_0(k)) \\ &\quad (r_2 - a_1(k)) + (b_0(k) + a_0(k)b_1(k) - a_1(k)b_0(k)) \\ &\quad (a_1(k) - a_1(k-1))\} \end{aligned} \tag{5.40}$$

Now by considering the Bezout identity (5.36) pointwise; that is setting $qa_1(k) = a_1(k)q$, and by defining

$$n'(k) = a_0(k)b_1(k)b_1(k) - b_0(k)a_1(k)b_1(k) + b_0^2(k), \tag{5.41}$$

the controller coefficients are now given by

$$\begin{aligned} d_0(k) &= \frac{1}{n'(k)}\{r_0 b_1^2(k) - b_0(k)b_1(k)(r_1 - a_0(k)) \\ &\quad + b_0^2(k)(r_2 - a_1(k))\} \end{aligned} \tag{5.42}$$

$$\begin{aligned} c_0(k) &= \frac{1}{n'(k)}\{-r_0 a_1(k)b_1(k) + r_0 b_0(k) + a_0(k)b_1(k)(r_1 - a_0(k)) \\ &\quad - a_0(k)b_0(k)(r_2 - a_1(k))\} \end{aligned} \tag{5.43}$$

$$\begin{aligned} c_1(k) &= \frac{1}{n'(k)}\{-r_0 b_1(k) + b_0(k)(r_1 - a_1(k)) + (a_0(k)b_1(k) \\ &\quad - a_1(k)b_0(k))(r_2 - a_1(k))\}. \end{aligned} \tag{5.44}$$

First, it should be noted that $n(k)$ in (5.37) can be written in terms of $n'(k)$ in (5.41) as

$$\begin{aligned} n(k) &= n'(k) + (a_0(k)b_1(k) - a_1(k)b_0(k))(b_1(k-1) - b_1(k)) \\ &\quad + b_0(k)(b_0(k-1) - b_0(k)). \end{aligned} \tag{5.45}$$

Moreover, consider the controller coefficients given by (5.38), (5.39), (5.40) (solution using the method described above) and (5.42), (5.43), (5.44) (solution using the frozen-time approach). It should be noted that (5.38), (5.39), (5.40) contain the same terms as in (5.42), (5.43), (5.44) respectively, plus some extra terms which are functions of parameter rate of variations. Therefore in the case where the parameters are slowly varying, the controller coefficients generated by either method are approximately identical and the control objective is met. However, if the parameter variations are arbitrarily fast, only the approach discussed in this section will yield satisfactory performance since the design method takes into account the rate of parameter variations. This explains why the frozen-time approach is applicable only to slowly-varying systems.

6 The Adaptive Control Law

In this section, we consider the design of the adaptive pole-placement regulator in the case where the slowly-varying part of the plant parameters are unknown. Then the main result of this paper is given.

The adaptive control law is implemented using the certainty equivalence principle. In other words, the unknown vector $\theta(k)$ is replaced with its estimate $\hat{\theta}(k-1)$ and so the plant parameter estimates are given by

$$\begin{bmatrix} \hat{a}_{n-1}(k) & \cdots & \hat{a}_0(k) & \hat{b}_{n-1}(k) & \cdots & \hat{b}_0(k) \end{bmatrix}^T = H(k)\hat{\theta}(k-1) \quad (6.46)$$

where the matrix $H(k)$ is given by (3.17).

In order to obtain the adaptive controller parameters, the vector $\hat{\theta}(k-1)$ is frozen up to time $k+n+N-1$ in the future. Because the unknown parameters are assumed to be slowly varying, freezing the estimates in the future will not destabilize the closed-loop system.

The objective is to design the following adaptive regulator

$$u(k) = \hat{C}(q,k)\hat{D}^{-1}(q,k)(r(k)-y(k)) \quad (6.47)$$

such that the following equality is satisfied

$$\hat{A}(q,k)\hat{D}(q,k) + \hat{B}(q,k)\hat{C}(q,k) = R(q) \quad (6.48)$$

where again $R(q)$ is a Hurwitz polynomial, and $\hat{A}(q,k)$, $\hat{B}(q,k)$ are given by

$$\begin{aligned} \hat{A}(q,k) &= q^n + q^{n-1}\hat{a}_{n-1}(k+1) + \cdots + \hat{a}_0(k+n) \\ \hat{B}(q,k) &= q^{n-1}\hat{b}_{n-1}(k+1) + \cdots + \hat{b}_0(k+n) \end{aligned} \quad (6.49)$$

and $\hat{C}(q,k)$, $\hat{D}(q,k)$ are given by

$$\hat{C}(q,k) = \sum_{i=0}^{N-1} \hat{c}_i(k)q^i \qquad \hat{D}(q,k) = q^{N-1} + \sum_{i=0}^{N-2} \hat{d}_i(k)q^i.$$

The time-varying Bezout identity (6.48) can be solved using the resultant matrix $\hat{\Gamma}_N(k)$ which is given by (2.12) except that the estimates $\hat{a}_i(k)$, $\hat{b}_i(k)$ are used instead of the coefficients $a_i(k)$, $b_i(k)$.

As in Equation (5.35), the regulator coefficients can then be given by

$$X(k+N-1) = \hat{\Gamma}_N^T(k+N-1)(\hat{\Gamma}_N(k+N-1)\hat{\Gamma}_N^T(k+N-1)^{-1}\hat{Z}(k+N-1)$$

and so using (5.27), (5.28), the adaptive regulator is given by

$$u(k) = \sum_{i=0}^{N-1} \hat{c}_i(k)v(k+i-N+1) \quad (6.50)$$

$$v(k) = -(\sum_{i=0}^{N-2} \hat{d}_i(k)v(k+i-N+1)) + r(k) - y(k). \tag{6.51}$$

In the next theorem, the main result is given.

Theorem 4 *Given the plant (*2.9*), (*3.18*) with the assumptions* **A.1***,* **A.2***, and* **A.3***, consider the adaptive regulator* (6.47) (6.50)*,* (6.51) *which consists of the parameter estimator* (4.23)*,* (4.24) *and the control law* (3.22)*. There exist real numbers $v_1^* > 0$ and $\alpha^* > 0$ such that if v_1 in (*3.19*) satisfies $v_1 \leq v_1^*$ and α in (*3.21*) satisfies $\alpha \leq \alpha^*$, then for a sufficiently large normalization factor $\gamma(k-1)$ and a bounded reference input $r(k)$, the resulting closed-loop system is globally stable (i.e., all signals are bounded for any initial conditions).*

The proof is given in Appendix B.

Example 6.1:

In this example, the performance of the adaptive pole-placement regulator discussed in this section is investigated. For comparison reasons, the adaptive regulator considered in [11], which is based on the generalized Kleinman's method [13], is also evaluated. For convenience, the adaptive regulator discussed in this section is denoted by Regulator 1, and the one considered in [11] is referred to as Regulator 2.

Consider the second-order system defined for $k \geq 0$ and given by

$$y(k) = a_1(k)y(k-1) + a_0(k)y(k-2) + b_1u(k-1) + b_0u(k-2) + \eta(k) \tag{6.52}$$

where

$$a_1(k) = a_{11}(k) + a_{12}cos(2k) \qquad a_0(k) = a_{01} + a_{02}sin(2k)$$

with a_{12}, a_{01}, a_{02}, b_1, b_0 equal to $3, -2, 1, 1, 3$ respectively and $a_{11}(k)$ is shown in Figure 1. It should be noted that in the above equations the coefficients $a_{11}(k)$, a_{12}, a_{01}, a_{02}, b_1, b_0 constitute the unknown part and the functions $cos(2k)$ and $sin(2k)$ are the known rapidly-varying terms. Moreover $\eta(k)$ is given by

$$\eta(k) = 0.008\phi(k) + 0.5sin(k). \tag{6.53}$$

Before considering the adaptive case, we assume that all the coefficients of the nominal model of (6.52) ($\eta(k) = 0$) are known. As shown in Figure 2, a pole-placement regulator based on the frozen-time approach does not achieve stabilization of the closed-loop system. On the other hand, as shown in Figure 3, Regulator 1 and Regulator 2 both satisfy the regulation objective.

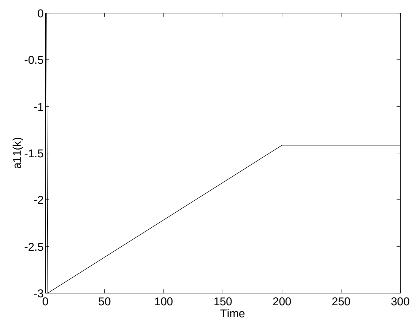

Figure 1: Time-varying coefficient

Now consider the nominal model of (6.52). With initial conditions $y(1) = 1$, $y(2) = 3$ and with initial estimates set to $\hat{\theta}(0) = [-4.5\; 4.5\; -3\; 1.5\; 1.5\; 4.5]$, reflecting a 50 percent error of the initial true parameters, the system is simulated using Regulator 1 and Regulator 2 and the output response is shown in Figure 4. Note that both regulators yield the same performance in this case.

Consider the system (6.52) where $\eta(k)$ is given by (6.53). As shown in Figure 5, the regulation response of Regulator 1 and Regulator 2 is similar. However, if the rate of variations of the coefficient $a_{11}(k)$ is increased as shown in Figure 6, Regulator 1 (Figure 7(a)) yields a much better performance than Regulator 2 (Figure 7(b)).

The conclusion which can be drawn from these simulations is that while Regulator 1 performs like Regulator 2 in the case of low rate of parameter variations, Regulator 1 outperforms Regulator 2 in the case of relatively high rate of parameter variations.

7 Conclusion

In this paper, we considered the adaptive pole-placement regulation of time-varying discrete-time non-stably invertible systems whose parameters are allowed to vary arbitrarily fast. In this case, regulators that are based on the frozen-time approach will not be applicable. To handle the case of rapidly-varying parameters, the control law must be based on the nominal model viewed as a time-varying system, not as a parameterized time-invariant system. This approach is taken in designing a pole-placement regulator in the framework of polynomial difference operators. Unlike the work of Ioannou and Tsakalis [8] in the continuous-time case, it is not required that the plant difference polynomials are coprime. Since we are

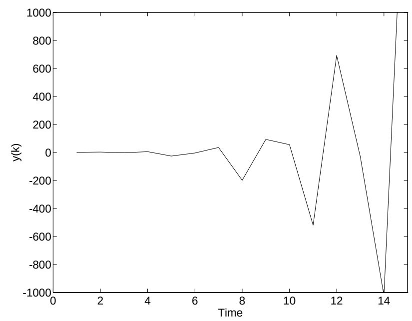

Figure 2: Pole-placement regulator based on the frozen-time approach

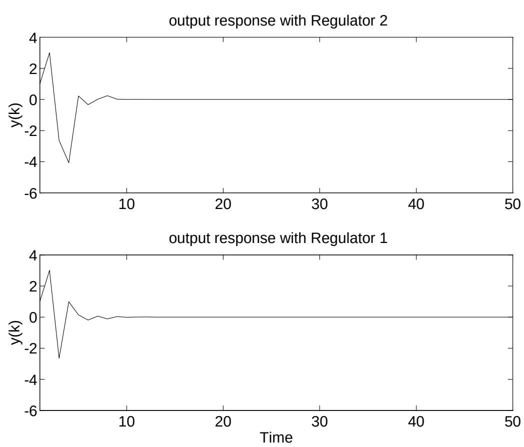

Figure 3: The output response with known system parameters

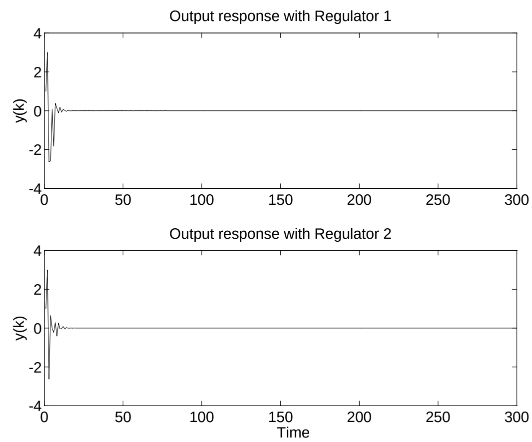

Figure 4: The output response

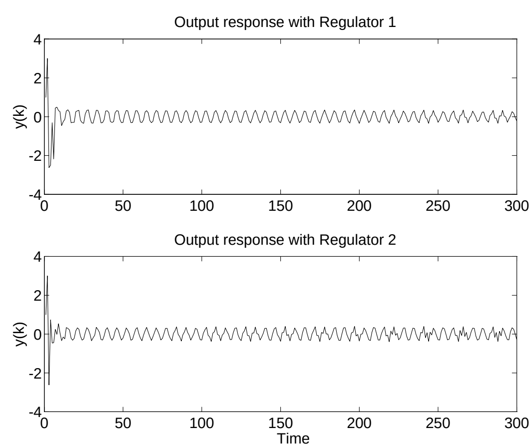

Figure 5: The output response with uncertainties

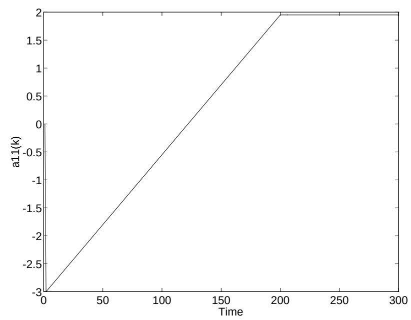

Figure 6: Time-varying coefficient

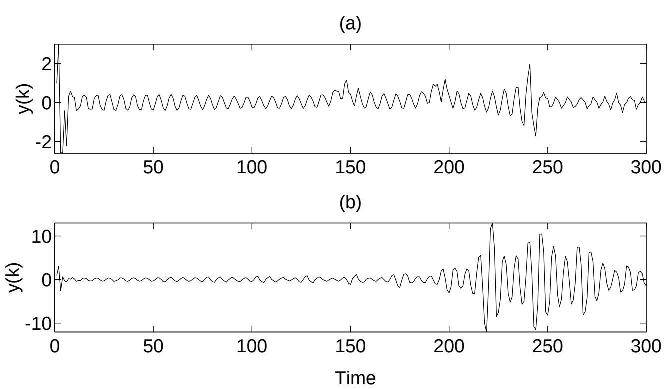

Figure 7: The output response with uncertainties

not assuming coprimeness, it does not appear to be possible to extend the approach in [8] to the discrete-time case considered here. This work is based on the weaker assumption that the system is N-step reachable for all parameter values in a convex region. Moreover, the plant coefficients consist of a known fast-varying part and unknown slowly-varying parameters which are to be estimated. The unknown part is required to be slowly varying. In Section 6, it was shown that the adaptive regulator guarantees global stability of all signals in the closed-loop system if the unknown parameter variations and the normalized model errors are sufficiently small regardless of the overall rate of variations. This result is a contribution to the adaptive control field in that the class of systems that can be treated is much larger than the class considered in the literature. Through a simulation example, the adaptive regulator discussed in this paper is compared with an existing regulator which is known as the generalized Kleinman's method [11]. Simulations have shown that the adaptive pole-placement regulator performs better in general than the adaptive regulator that is based on the generalized Kleinman's method, especially if the rate of unknown parameter variations is relatively high.

This paper addresses the regulation problem of time-varying discrete-time systems. A fundamental open problem is whether it is possible to design adaptive controllers that can achieve tracking for the general class of systems considered in this paper.

A Proof of Theorem 1

Consider first the state-space model (2.3). We can write the left side of (2.2) as

$$\begin{aligned} h^T \sum_{i=-\infty}^{k-1} \Phi(k,i+1)gz(i) &= h^T F(k-1)(\sum_{i=-\infty}^{k-2} \Phi(k-1,i+1)gz(i)) \\ &= [0\cdots 10](\sum_{i=-\infty}^{k-2} \Phi(k-1,i+1)gz(i)). \end{aligned}$$

Repeating $n-1$ times, it follows that

$$h^T \sum_{i=-\infty}^{k-1} \Phi(k,i+1)gz(i) = g^T(\sum_{i=-\infty}^{k-n} \Phi(k-n+1,i+1)gz(i)). \tag{A.1}$$

To prove Theorem 1, the mathematical induction argument is used.
(i) Consider $z(k) = P(q,k)y(k)$ an element of $\mathcal{S}$, the objective is to establish

the equality $y(k) = P^{-1}(q,k)z(k)$ or

$$y(k) = h^T \sum_{i=k_0}^{k-1} \Phi(k, i+1) g z(i) \quad \forall k \geq k_0 + n.$$

First, it should be verified that

$$y(k_0+n) = h^T \sum_{i=k_0}^{k_0+n-1} \Phi(k_0+n, i+1) g z(i).$$

Using $(A.1)$, it follows

$$\begin{aligned} h^T \sum_{i=k_0}^{k_0+n-1} \Phi(k_0+n, i+1) g z(i) &= g^T \sum_{i=k_0}^{k_0} \Phi(k_0+1, i+1) g z(i) \\ &= g^T g z(k_0) = z(k_0). \end{aligned}$$

Knowing that $z(k_0) = P(q,k_0)y(k_0)$ and the initial state vector $x(k_0) = 0$, it follows that $z(k_0) = y(k_0+n)$.
Now assume that

$$y(k) = h^T \sum_{i=k_0}^{k-1} \Phi(k, i+1) g z(i) = h^T x(k),$$

the goal is to establish that

$$y(k+1) = h^T \sum_{i=k_0}^{k} \Phi(k+1, i+1) g z(i). \tag{A.2}$$

The left side of $(A.2)$ can be written as

$$\begin{aligned} h^T \sum_{i=k_0}^{k} \Phi(k+1, i+1) g z(i) &= h^T F(k)(F(k-1)...F(k_0) g z(k_0-1) \\ &\quad + \cdots + g z(k-1)) + h^T g z(k) \\ &= h^T x(k+1) = y(k+1) \end{aligned}$$

and this concludes the proof for the first part.

(ii) Now consider $\psi(k) = P^{-1}(q,k)y(k)$ which belongs to $\mathcal{S}$, the objective is to show that

$$P(q,k)\psi(k) = y(k) \tag{A.3}$$

for all $k \geq k_0$.

It should be first be verified that $(A.3)$ holds for the initial condition at time k_0 for which it follows

$$P(q,k_0)\psi(k_0) = \psi(k_0+n) + a_{n-1}(k_0)\psi(k_0+n-1) + \cdots + a_0(k_0)\psi(k_0).$$

Again since the initial state $x(k_0) = [\psi(k_0+n-1)\cdots\psi(k_0)]^T$ is set equal to zero, we get

$$\begin{aligned} P(q,k_0)\psi(k_0) &= \psi(k_0+n) \\ &= h^T \sum_{i=k_0}^{k_0+n-1} \Phi(k_0+n, i+1) g y(i). \end{aligned} \qquad \text{(A.4)}$$

Using $(A.1)$, $(A.4)$ becomes

$$\begin{aligned} P(q,k_0)\psi(k_0) &= g^T \sum_{i=k_0}^{k_0} \Phi(k_0+1, i+1) g y(i) \\ &= g^T g y(k_0) = y(k_0). \end{aligned}$$

Now assume that $P(q,k)\psi(k) = y(k)$, the goal is to check whether

$$P(q,k+1)\psi(k+1) = y(k+1).$$

Consider the following equation

$$\begin{aligned} P(q,k+1)\psi(k+1) &= \psi(k+n+1) + a_{n-1}(k+1)\psi(k+n) + \cdots \\ &\quad + a_0(k+1)\psi(k+1). \end{aligned} \qquad \text{(A.5)}$$

Inserting $(A.3)$ in $(A.5)$, it follows that

$$\begin{aligned} P(q,k+1)\psi(k+1) &= \psi(k+n+1) + (a_{n-2}(k+1) - a_{n-1}(k+1) \\ &\quad a_{n-1}(k))\psi(k+n-1) + (a_{n-3}(k+1) \\ &\quad - a_{n-1}(k+1)a_{n-2}(k))\psi(k+n-2) + \cdots \\ &\quad + (a_0(k+1) - a_{n-1}(k+1)a_1(k))\psi(k+1) \\ &\quad - a_{n-1}(k+1)a_0(k)\psi(k) \\ &\quad + a_{n-1}(k+1)y(k). \end{aligned} \qquad \text{(A.6)}$$

Using $(A.1)$, it follows that

$$\psi(k+n+1) = g^T \sum_{i=k_0}^{k+1} \Phi(k+2, i+1) g y(i)$$

$$\psi(k+n-1) = g^T \sum_{i=k_0}^{k-1} \Phi(k, i+1) g y(i)$$

$$\vdots$$

$$\psi(k) = [0 \cdots 01] \sum_{i=k_0}^{k-1} \Phi(k, i+1) g y(i).$$

Using the above equalities, $(A.6)$ becomes

$$\begin{aligned} P(q, k+1)\psi(k+1) \quad = \quad & g^T F(k+1) F(k) (\sum_{i=k_0}^{k-1} \Phi(k, i+1) g y(i)) \\ & + g^T F(k+1) g y(k) + y(k+1) \\ & + [a_{n-2}(k+1) - a_{n-1}(k+1) a_{n-1}(k) \\ & \cdots - a_{n-1}(k+1) a_0(k)] \\ & .(\sum_{i=k_0}^{k-1} \Phi(k, i+1) g y(i)) + a_{n-1}(k+1) y(k)). \end{aligned} \tag{A.7}$$

Knowing that

$$g^T F(k+1) F(k) = [a_{n-2}(k+1) + a_{n-1}(k+1) a_{n-1}(k) \ \cdots a_{n-1}(k+1) a_0(k)]$$

and

$$g^T F(k+1) g = -a_{n-1}(k+1),$$

$(A.7)$ becomes

$$P(q, k+1)\psi(k+1) = y(k+1)$$

and this proves the theorem. □

B Proof of Theorem 4

Recall that the output prediction $\hat{y}(k)$ is given by

$$\hat{y}(k) = \phi^T(k) \hat{\theta}(k-1). \tag{B.1}$$

Equation $(B.1)$ can also be expressed in terms of the polynomial difference operator as

$$\hat{A}(q, k) \hat{y}(k) = \hat{B}(q, k) u(k) + \hat{N}(q, k) e(k) \tag{B.2}$$

where again $e(k)$ is the output prediction error, $\hat{A}(q, k)$, $\hat{B}(q, k)$ are given by (6.49), and $\hat{N}(q, k)$ is given by

$$\hat{N}(q, k) = \hat{A}(q, k) - q^n.$$

Given that $e(k) = \hat{y}(k) - y(k)$, it follows that

$$\hat{A}(q,k)y(k) = \hat{A}(q,k)\hat{y}(k) - \hat{A}(q,k)e(k). \tag{B.3}$$

Using $(B.2)$, $(B.3)$ becomes

$$\begin{aligned} \hat{A}(q,k)y(k) &= \hat{B}(q,k)u(k) + (\hat{N}(q,k) - \hat{A}(q,k))e(k) \\ &= \hat{B}(q,k)u(k) - q^n e(k). \end{aligned} \tag{B.4}$$

Now given the adaptive regulator (6.47), $u(k)$ and $y(k)$ can be expressed in terms of the inverse Hurwitz polynomial $R^{-1}(q)$ as

$$\begin{aligned} u(k) &= \hat{C}(q,k)\hat{D}^{-1}(q,k)(r(k) - y(k)) \\ &= \hat{C}(q,k)R^{-1}(q)[\hat{A}(q,k)\hat{D}(q,k) + \hat{B}(q,k)\hat{C}(q,k)] \\ &\quad \hat{D}^{-1}(q,k)(r(k) - y(k)) \\ &= \hat{C}(q,k)R^{-1}(q)\hat{A}(q,k)(r(k) - y(k)) \\ &\quad +\hat{C}(q,k)R^{-1}(q)\hat{B}(q,k)\hat{C}(q,k)\hat{D}^{-1}(q,k)(r(k) - y(k)) \\ &= \hat{C}(q,k)R^{-1}(q)\hat{A}(q,k)r(k) - \hat{C}(q,k)R^{-1}(q) \\ &\quad (\hat{A}(q,k)y(k) - \hat{B}(q,k)u(k)). \end{aligned} \tag{B.5}$$

Using $(B.4)$ in $(B.5)$, it follows that

$$u(k) = \hat{C}(q,k)R^{-1}(q)(\hat{A}(q,k)r(k) + q^n e(k)). \tag{B.6}$$

Moreover, using $(B.4)$, $y(k)$ can be expressed as

$$y(k) = \hat{A}^{-1}(q,k)(\hat{B}(q,k)u(k) - q^n e(k)). \tag{B.7}$$

Inserting (6.47), $(B.7)$ becomes

$$\begin{aligned} y(k) &= \hat{A}^{-1}(q,k)(\hat{B}(q,k)\hat{C}(q,k)\hat{D}^{-1}(q,k)(r(k) - y(k)) - q^n e(k)) \\ &= r(k) - [1 + \hat{A}^{-1}(q,k)\hat{B}(q,k)\hat{C}(q,k)\hat{D}^{-1}(q,k)]^{-1} r(k) \\ &\quad [1 + \hat{A}^{-1}(q,k)\hat{B}(q,k)\hat{C}(q,k)\hat{D}^{-1}(q,k)]^{-1}\hat{A}^{-1} q^n e(k) \\ &= r(k) - \hat{D}(q,k)[\hat{A}(q,k)\hat{D}(q,k) + \hat{B}(q,k)\hat{C}(q,k)]^{-1}\hat{A}(q,k)r(k) \\ &\quad -\hat{D}(q,k)[\hat{A}(q,k)\hat{D}(q,k) + \hat{B}(q,k)\hat{C}(q,k)]^{-1} q^n e(k) \\ &= r(k) - \hat{D}(q,k)R^{-1}(q)\hat{A}(q,k)r(k) - \hat{D}(q,k)R^{-1}(q)q^n e(k). \end{aligned} \tag{B.8}$$

Now consider the signal $z(k) \in \mathcal{S}$ which is given by

$$z(k) = \hat{D}(q,k)x(k) \tag{B.9}$$

where $x(k)$ is defined by

$$x(k) = R^{-1}(q)q^n e(k), \tag{B.10}$$

$(B.10)$ can be written as

$$x(k+N+n-1) = -r_{N+n-2}x(k+N+n-2)-\cdots-r_0x(k)+e(k+n) \tag{B.11}$$

or

$$x(k+N-1) = -r_{N+n-2}x(k+N-2) - \cdots - r_0x(k-n) + e(k). \tag{B.12}$$

By defining the state $\zeta(k)$ as

$$\zeta(k) = [x(k+N-2)\cdots x(k-n)]^T$$

$(B.12)$ can be expressed as

$$\zeta(k+1) = A\zeta(k) + be(k)$$

where $b = [1 \quad 0\cdots 0]^T$ and A is given by

$$A = \begin{bmatrix} -\alpha_{N+n-2} & \cdots & \cdots & -\alpha_0 \\ 1 & 0 & \cdots & \\ & \ddots & & \mathbf{0} \\ \mathbf{0} & & 1 & 0 \end{bmatrix}.$$

Using $(B.9)$, $z(k)$ can be given by

$$z(k) = h^T(k)\zeta(k+1) \tag{B.13}$$

where $h(k) = [\ 1 \quad d_{N-2}(k) \quad \cdots \quad d_0(k) \quad 0 \quad \cdots \quad 0\ ]^T$, with the number of zeros in $h(k)$ is $n-1$.

Now given that $R(q)$ is a Hurwitz polynomial and since $r(k)$ is a bounded signal for all $k \geq k_0$ for some integer k_0, it follows that there exist β_0, β_1 such that

$$|y(k)|, |u(k)| \leq \beta_0 + \beta_1 \sum_{i=k_0}^{k} \mu^{k-i}|e(i)| \tag{B.14}$$

where μ is the dominant pole of $R(q)$.

Using Shwartz inequality, $(B.14)$ becomes

$$y^2(k), u^2(k) \leq \beta 2 + \beta 3 \sum_{i=k_0}^{k} \mu^{k-i}e^2(i). \tag{B.15}$$

Now, consider a function $v(k)$ which includes the signals $u(k)$, $y(k)$. By proving that $v(k)$ is exponentially stable, the boundedness of all signals in the closed-loop system is established. This proof technique is inspired by the work of Ioannou and Tsakalis [8], although details in this context differ from that in [8]. Specifically, let $v(k)$ be chosen as

$$v(k) = k_1 g(k) + m(k) \tag{B.16}$$

where

$$g(k) = u^2(k) + y^2(k)$$

and $m(k)$ is defined as

$$m(k) = \gamma(k-1) + \kappa \sum_{i=k_0}^{k} \mu^{k-i}(u^2(i) + y^2(i)) \tag{B.17}$$

where κ satisfies the following equation

$$\kappa = \sup_k [\sum_{k_0=1}^{s} f_{k_0}^2]. \tag{B.18}$$

Inserting $(B.17)$ and $(B.15)$ in $(B.16)$, it follows that

$$\begin{aligned} v(k) \quad \leq \quad & k_1(\beta_4 + \beta_5 \sum_{i=k_0}^{k} \mu^{k-i} e^2(i)) \\ & + \gamma(k-1) + \kappa \sum_{i=k_0}^{k} \mu^{k-i}(u^2(i) + y^2(i)). \end{aligned} \tag{B.19}$$

Given that the function $\gamma(k-1)$ is known to be bounded for all $k \geq k_0$, $(B.19)$ can be written as

$$\begin{aligned} v(k) \quad & \leq \quad \beta_6 + \sum_{i=k_0}^{k} \mu^{k-i}[k_1 \beta_5 e^2(i) + \kappa(u^2(i) + y^2(i))] \\ & \leq \quad \beta_6 + \sum_{i=k_0}^{k-1} \mu^{k-i-1}[k_1 \beta_5 \frac{e^2(i)}{m(i)} + \frac{\kappa}{k_1}][k_1 g(i) + m(i)]. \end{aligned} \tag{B.20}$$

At this point, the objective is to introduce the normalized output prediction error $\bar{e}(k)$ in the above inequality. In order to achieve that, consider the function $m(i)$ defined by $(B.17)$ which can be written as

$$\begin{aligned} m(i) \quad = \quad & \gamma(i-1) + \kappa(\sum_{r=i-n}^{i-1} \mu^{i-1-r}(u^2(r) + y^2(r)) \\ & + \kappa(u^2(i) + y^2(i)) + \sum_{r=k_0}^{i-n-1} \mu^{i-n-1-r}(u^2(r) + y^2(r))) \end{aligned}$$

where again n is the order of the model (2.10). If the parameter τ, introduced in (3.14), is chosen such that $0 < \tau \leq \sqrt{\mu}$ and if κ satisfies $(B.18)$, it follows that

$$m(i) \geq \gamma(i-1) + \phi^T(i)\phi(i). \tag{B.21}$$

Therefore, equation $(B.20)$ can be written as

$$v(k) \leq \beta_6 + \sum_{i=k_0}^{k} \mu^{k-i}[k_1\beta_5 \frac{e^2(i)}{\gamma(i-1)+\phi^T(i)\phi(i)} + \frac{\kappa}{k_1}]v(i). \tag{B.22}$$

Multiplying both sides in $(B.22)$ by μ^{k_0-k-1}, the above inequality becomes

$$\mu^{k_0-k-1}v(k) \leq \beta_6 + \sum_{i=k_0}^{k} [k_1\beta_5\bar{e}^2(i) + \frac{\kappa}{k_1}]v(i)\mu^{k_0-i-1}. \tag{B.23}$$

Using Bellman Gronwall lemma, $(B.23)$ is expressed as

$$\mu^{k_0-k-1}v(k) \leq \beta_6 \prod_{i=k_0}^{k} [1 + \beta_5 k_1 \bar{e}^2(i) + \frac{\kappa}{k_1}]. \tag{B.24}$$

Multiplying through by μ^{k-k_0+1}

$$v(k) \leq \beta_6 \prod_{i=k_0}^{k} \mu[1 + \beta_5 k_1 \bar{e}^2(i) + \frac{\kappa}{k_1}]. \tag{B.25}$$

Now given the following inequality

$$\prod_{i=k_0}^{k-1} a_i \leq [\frac{1}{k-k_0+1}\sum_{i=k_0}^{k} a_i]^{k-k_0+1},$$

inequality $(B.25)$ becomes

$$v(k) \leq \beta_6\mu[1 + \frac{\kappa}{k_1} + \frac{k_1\beta_5}{k-k_0+1}\sum_{i=k_0}^{k}\bar{e}^2(i)]^{k-k_0+1}.$$

Given inequality (4.26) in Theorem 2, it follows

$$\begin{aligned} v(k) \quad \leq \quad & \beta_6[\mu + \mu\frac{\kappa}{k_1} + \mu\frac{k_1\beta_5}{k-k_0+1}((D^2 + 3Dv_0 + 2\epsilon v_0) \\ & +(2D\epsilon + 2\epsilon^2 + 3Dv_1 + 2\epsilon v_1)(k-k_0+1))]^{k-k_0+1}. \end{aligned} \tag{B.26}$$

For sufficiently small v_1 and α, there exists some k_1 such that as k goes to infinity, the following inequality is obtained

$$\mu + \mu\frac{\kappa}{k_1} + \mu k_1\beta_5[2D\epsilon + 2\epsilon^2 + 3Dv_1 + 2\epsilon v_1] < 1$$

and thus $v(k)$ is asymptotically stable.

From $(B.16)$, it follows that the signals $y(k)$ and $u(k)$ are globally stable. This implies that the regression vector $\phi(k)$ is bounded and since the parameter error $\tilde{\theta}(k)$ is also bounded, it can be concluded that the output prediction error $e(k)$ is bounded. This concludes the proof.□

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