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Approximation in LQR Problems for Infinite Dimensional Systems With Unbounded Input Operators*

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Abstract

We present a variational framework based on sesquilinear forms for Galerkin approximation techniques for state feedback control in problems governed by infinite dimensional dynamical systems. Both parabolic and second order in time, hyperbolic partial differential equations with unbounded input and unbounded observation operators are included as special cases of our treatment.

1 Introduction

In this paper we discuss the linear quadratic regulator (LQR) problem for a class of (essentially parabolic) unbounded input or boundary control problems. A variational framework using sesquilinear forms is developed to treat Dirichlet and Neuman boundary control problems for parabolic equations and strongly damped elastic systems. Using such a framework, convergence of Galerkin approximations to solutions of Riccati equations is also established. The boundary control problem for parabolic systems has been studied extensively over the last two decades, inspired by the monograph of J.L. Lions [21] (e.g., see [1, 8, 11, 27, 16] and the references cited there). In a series of papers [19, 23, 24], Lasiecka and Triggiani obtain existence and regularity results for solutions to the operator Riccati equations that appear in the linear quadratic regulator problem. The main tool in their treatment is the theory of analytic semigroups. They have recently

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developed an approximation theory for Riccati solutions in a similar spirit [25].

Our approach is very close in spirit to the studies by Lions [21] and Sorrine [30] and extends the approximation theory developed by Banks and Kunisch [6] to the case of unbounded input control operators. We employ the Gelfand triple formulation ($V \subset H = H^* \subset V^*$) of elliptic operators [21, 31, 29, 32] using coercive sesquilinear forms defined on V. It follows from [31] that the $V-$coercive form defines a generator of an analytic semigroup on H. The control forces through boundary or pointwise actuators are represented by an input sesquilinear form defined on $U \times V$ where U is the input space. Then the approximation methods are derived by restricting these sesquilinear forms onto finite dimensional subspaces of V.

The objective of this paper is to explore to what extent one can develop the sesquilinear form formulation to study unbounded input control problems and to establish a computationally feasible approximation theory for linear quadratic regulator problems. These investigations were motivated by [6] where our treatment originated, by [3] in which a similar treatment has been developed within the context of parameter estimation problems, and by the fact that many boundary control problems can be treated using the sesquilinear forms. Some of our results can be found in the existing literature, but the approach here is new and, in some cases, greatly simplifies the technical details required to establish the results.

The results given here extend to the unbounded case the approximation theory developed in [13, 6, 15] in which the input and output operators are assumed to be bounded. The case of unbounded input operator has also been discussed in [24] and [15].

The outline of the paper is as follows. In Section 2 we discuss the basic results for abstract parabolic control systems described by sesquilinear forms and state the linear quadratic regulator problem we consider in this paper. Also, several examples that can be readily treated by our formulation are discussed. In Section 3 we summarize results for the optimal control problem and Riccati equations. In Section 4 Galerkin approximations to the LQR problem are formulated and a convergence theory is established for the case where the output operator appearing in the quadratic cost is bounded on H. In Section 5 we deal with the case when the output operator is not bounded. Damped second order systems are treated in Section 6.

A few of the theorems and lemma are stated here without proofs. Detailed proofs can be found in the complete paper [4] (also, see a companion paper [15]).

2 Sesquilinear Forms and Parabolic Control Systems

Assume V and H are complex Hilbert spaces and $V \subset H$ with continuous dense injection. Let V^* denote the topological (conjugate) dual space of V. We identify H with its dual, so that $V \hookrightarrow H = H^* \hookrightarrow V^*$ in a Gelfand triple [32]. The duality product $\langle \phi, \psi \rangle_{V^*,V}$ on $V^* \times V$ is the unique extension by continuity of the scalar product $\langle \phi, \psi \rangle_H$ of H defined on $H \times V$. Consider a sesquilinear form σ on V (i.e., $\sigma : V \times V \to \mathbb{C}$) such that

$$|\sigma(\phi, \psi)| \leq C||\phi||_V ||\psi||_V \quad \text{for } \phi, \psi \in V \tag{2.1}$$

$$Re\ \sigma(\phi, \phi) \geq \omega ||\phi||_V^2 - \rho ||\phi||_H^2 \quad \text{for } \phi \in V \tag{2.2}$$

where $\omega > 0$. It then follows from [31, Lemma 3.6.1] that if $\mathcal{A} \in \mathcal{L}(V, V^*)$ is defined by

$$\sigma(\phi, \psi) = \langle -\mathcal{A}\phi, \psi \rangle_{V^*,V} \quad \text{for all } \phi, \psi \in V, \tag{2.3}$$

then for $\text{Re}\lambda \geq \rho,\ \lambda \neq \rho$,

$$||(\lambda I - \mathcal{A})^{-1}\phi||_V \leq \frac{||\phi||_{V^*}}{\omega} \quad \text{for } \phi \in V^* \tag{2.4}$$

$$||(\lambda I - \mathcal{A})^{-1}\phi||_H \leq \frac{M_0}{|\lambda - \rho|} ||\phi||_H \quad \text{for } \ \phi \in H \tag{2.5}$$

$$||(\lambda I - \mathcal{A})^{-1}\phi||_{V^*} \leq \frac{M_0}{|\lambda - \rho|} ||\phi||_{V^*} \quad \text{for } \phi \in V^* \tag{2.6}$$

where $M_0 = 1 + C/\omega$. The dual or adjoint operator $\mathcal{A}^* \in \mathcal{L}(V, V^*)$ defined by

$$\sigma(\phi, \psi) = \langle \phi, -\mathcal{A}^*\psi \rangle_{V,V^*} \quad \text{for all } \phi, \psi \in V \tag{2.7}$$

also satisfies the estimates (2.4)$\sim$(2.6). Moreover, (2.6) for $\mathcal{A}^*$ implies that for $Re\ \lambda \geq \rho,\ \lambda \neq \rho$,

$$||(\lambda I - \mathcal{A})^{-1}\phi||_V \leq \frac{M_0}{|\lambda - \rho|} ||\phi||_V \quad \text{for } \phi \in V. \tag{2.8}$$

In fact, for $\phi \in V$ we have

$$|\langle (\lambda I - \mathcal{A})^{-1}\phi, \psi \rangle_{V,V^*}| = |\langle \phi, (\lambda I - \mathcal{A}^*)^{-1}\psi \rangle_{V,V^*}|$$

$$\leq ||\phi||_V \frac{M_0}{|\lambda - \rho|} ||\psi||_{V^*} \quad \text{for all } \psi \in V^*.$$

This implies (2.8). It thus follows from [26, Chap. 2, Theorem 5.2] (see also [31]) that $\mathcal{A}$ generates an analytic semigroup $S(t)$ on H, V and V^*

where $\mathrm{dom}_H(\mathcal{A}) = \{\phi \in V : |\sigma(\phi,\psi)| \le k_\phi \|\psi\|_H$ for all $\psi \in V\}$. Following conventional notation (see [31]) we will not distinguish between the semigroups in V, H, V^* since each involves either a restriction or extension of one of the others. From (2.1) one can show that $\mathrm{dom}_{V^*}(\mathcal{A})$, the domain of the infinitesimal generator $\mathcal{A}$ of the semigroup $S(t)$ defined on V^*, satisfies

$$\mathrm{dom}_{V^*}(\mathcal{A}) = \{\phi \in V : \mathcal{A}\phi \in V^*\} = V. \tag{2.9}$$

In addition, we have that $\mathcal{A}^*$ generates the adjoint semigroup $S^*(t)$ on H [31, 26] and moreover the dual of the semigroup $S(t)$ defined on V^* equals $S^*(t)|_V$, the restriction of $S^*(t)$ on V; i.e.,

$$(S(t) \text{ on } V^*)^* = S^*(t)|_V. \tag{2.10}$$

It follows from [18, Theorem IX.1.19] that if $t \to f(t) \in V^*$ is continuously differentiable and $\phi \in V = \mathrm{dom}_{V^*}(\mathcal{A})$, then for $t \ge 0$ the function $t \to z(t)$ defined by

$$z(t) = S(t)\phi + \int_0^t S(t-s)f(s)ds \tag{2.11}$$

satisfies $z(t) \in V$, is continuously differentiable in V^*, and satisfies

$$\frac{d}{dt}z(t) = \mathcal{A}z(t) + f(t) \quad \text{in } V^*. \tag{2.12}$$

Now, from [21], [31], [32], we have

Theorem 2.1 *For $\phi \in H$ and $f \in L^2(0,T;V^*)$, (2.12) has a unique solution $z \in L^2(0,T;V) \cap H^1(0,T;V^*)$ given by (2.11) and for some positive constant M_1*

$$||z(t)||_H^2 \le M_1(||\phi||_H^2 + \int_0^t ||f(s)||_{V^*}^2 ds),\ t \in [0,T] \tag{2.13}$$

$$\int_0^T ||z(s)||_V^2 ds \le M_1(||\phi||_H^2 + \int_0^T ||f(s)||_{V^*}^2 ds). \tag{2.14}$$

From Thm. III.1.2 in [21], we have that $t \to z(t) \in H$ is continuous. Thus from (2.11) and (2.13), for all $f \in L^2(0,T;V^*)$, the function $t \to \int_0^t S(t-s)f(s)ds \in H$ is continuous and

$$||\int_0^t S(t-s)f(s)ds||_H^2 \le M_1 \int_0^t ||f(s)||_{V^*}^2 ds \tag{2.15}$$

$$||\int_0^t S^*(t-s)f(s)ds||_H^2 \le M_1 \int_0^t ||f(s)||_{V^*}^2 ds. \tag{2.16}$$

Moreover, we have (again from standard results and arguments) eg. see [4]

Theorem 2.2 *Let us define the sesquilinear forms σ_0 and σ_1 on V by*

$$\left.\begin{aligned}\sigma_0(\phi,\psi) &= \frac{\sigma(\phi,\psi)+\overline{\sigma(\psi,\phi)}}{2}\\ \sigma_1(\phi,\psi) &= \frac{\sigma(\phi,\psi)-\overline{\sigma(\psi,\phi)}}{2}\end{aligned}\right\} \quad \text{for } \phi,\psi\in V.$$

Then $\sigma = \sigma_0 + \sigma_1$, with σ_0 symmetric and σ_1 skew-symmetric, and $\sigma_0(\phi,\phi) = Re\sigma(\phi,\phi)$ for $\phi \in V$. Define $A_1 \in \mathcal{L}(V, V^*)$ by

$$\langle A_1\phi,\psi\rangle_{V^*,V} = \sigma_1(\phi,\psi) \qquad \text{for all } \phi,\psi\in V$$

and

$$dom_H(A_1) = \{\phi\in V : |\sigma_1(\phi,\psi)| \le k_\phi\|\psi\|_H \text{ for all } \psi\in V\}.$$

Assume

$$dom_H(A_1) \supset\ dom_H(\mathcal{A}). \tag{2.17}$$

Then if $\phi = 0$ and $f \in L^2(0,T;H)$ in (2.11), we have $t \to z(t) \in H^1(0,T;H)\cap C(0,T;V)$ and there exists a constant $M_2 > 0$ such that

$$\int_0^t ||\dot z(s)||_H^2 ds + ||z(t)||_V^2 \le M_2 \int_0^t ||f(s)||_H^2 ds, \quad t\in[0,T]. \tag{2.18}$$

The condition (2.17) is satisfied if we assume that for all $\phi,\psi\in V$

$$|\sigma_1(\phi,\psi)| \le K\|\phi\|_V\|\psi\|_H \tag{2.18a}$$

for some $K > 0$ since in this case $\mathrm{dom}_H(A_1) = V$ (it is also satisfied under the milder condition (5.9) given in Section 5).

Assume U and Y are Hilbert spaces ($U^* = U$ and $Y^* = Y$). Consider the finite horizon optimal control problem: Minimize the quadratic functional

$$J(0,T;u,z_0) = \int_0^T (||Cz(t)||_Y^2 + ||u(t)||_U^2)dt + \langle z(T), Gz(T)\rangle \tag{2.19}$$

over $u \in L^2(0,T;U)$ subject to

$$\begin{aligned}\frac{d}{dt}z(t) &= \mathcal{A}z(t) + Bu(t)\\ z(0) &= z_0 \in H.\end{aligned} \tag{2.20}$$

Here we assume $u(t)$ is a U-valued control function, the input map $B \in \mathcal{L}(U, V^*)$, the observation map $C \in \mathcal{L}(V, Y)$, and $G \in \mathcal{L}(H)$ is self-adjoint, nonnegative definite. By Theorem 2.1, the minimization problem for (2.19) is well-posed; i.e., given initial datum $z_0 \in H$ and $u \in L^2(0, T; U)$, the cost $J(u, z_0)$ is finite. We will analyze the solution of (2.19)-(2.20) and its infinite horizon analogue in terms of Riccati operators in Section 3 and show convergence of Galerkin approximations of solutions to Riccati equations in Section 4.

Let us conclude this section by discussing some motivating examples of some importance in applications.

Example 1 (Parabolic systems with boundary control)

Let Ω be either a polygonal domain in $\Re^n$, $n \leq 3$ or a bounded domain of $\Re^n$, $n \leq 3$ with C^2-boundary , . We consider the control problem for (2.19) with Neuman boundary control system:

$$\frac{\partial}{\partial t} z = \triangledown(a \triangledown z) + b \triangledown z + cz \ \text{ in } \ \Omega,$$

$$z(0) = \phi \in L^2(\Omega) \tag{2.21}$$

$$a \frac{\partial}{\partial \nu} z|_{,} \ = u(t)$$

where $z = z(t) \in L^2(\Omega), u = u(t) \in L^2(,\ )$ and $\frac{\partial}{\partial \nu}$ is the normal derivative on , . To write this in abstract form (2.20), without loss of generality we use the real Hilbert spaces $H = L^2(\Omega)$, $V = H^1(\Omega), U = L^2(,\ )$ and define a real valued sesquilinear form σ on V (the complexification of spaces and forms is readily done in the usual way) by

$$\sigma(\phi, \psi) = \int_\Omega (a(x) \triangledown \phi \triangledown \psi - b(x) \triangledown \phi \psi - c(x) \phi \psi) dx \tag{2.22}$$

where $a, b, c \in L^\infty(\Omega)$ and $a(x) \geq \omega > 0$ a.e. in Ω. We consider the input map $B : U \to H$ given by

$$\langle Bu, \psi \rangle = \langle u, \psi|_{,} \rangle_{L^2(,\ )} \quad \text{for all } \psi \in H^1(\Omega). \tag{2.23}$$

As above we define $\mathcal{A} \in \mathcal{L}(V, V^*)$ via $\langle -\mathcal{A}\phi, \psi \rangle = \sigma(\phi, \psi)$ and take $\operatorname{dom}_H(\mathcal{A}) = \{\phi \in V | \mathcal{A}\phi \in H\}$. In the general case, $\operatorname{dom}_H(\mathcal{A})$ is difficult to characterize precisely. But if we define σ_1 as in Theorem 2.2 (i.e., as the skew-symmetric part of σ), then one can readily argue that (2.18a) and hence (2.17), holds in the case that $b \cdot \hat{\nu} = 0$ on , . Moreover, in the special case that $b = 0$ and a possesses sufficient smoothness (e.g., $a \in C^2(\Omega)$), it can be shown that

$$\operatorname{dom}_H(\mathcal{A}) = \operatorname{dom}_H(\mathcal{A}^*) = \{\phi \in H^2(\Omega) \mid \frac{\partial \phi}{\partial \nu}|_{,} \ = 0\}. \tag{2.24}$$

In any case, we have for $\phi \in \ \mathrm{dom}_H(\mathcal{A})$ that $\mathcal{A}\phi = A\phi$ where A is formally given by

$$A\phi = \bigtriangledown(a \bigtriangledown \phi) + b \bigtriangledown \phi + c\phi. \tag{2.25}$$

We next consider the Dirichlet boundary control problem consisting of minimizing (2.19) subject to

$$\frac{\partial}{\partial t} z(t,\cdot) = \triangle z(t,\cdot) \text{ in } \Omega$$

(2.26)

$$z(t) = u(t) \text{ on } ,$$

where $\triangle$ denotes the Laplacian and $U = L^2(,\ )$. To cast this problem in the context of (2.20), we take $V = L^2(\Omega), H = H^{-1}(\Omega) = H_0^1(\Omega)^*$ in a Gelfand triple setting $V \hookrightarrow H \hookrightarrow V^*$ with $H^{-1}(\Omega)$ the pivot space. We thus identify H and H^* through the Riesz map and but of course do not identify $V = L^2(\Omega)$ and V^*. As usual, the duality pairing $\langle\cdot,\cdot\rangle_{V^*,V}$ is the extension by continuity of the inner product $\langle\cdot,\cdot\rangle_H = \langle\cdot,\cdot\rangle_{H^{-1}(\Omega)}$ from $H \times V = H^{-1}(\Omega) \times L^2(\Omega)$ to $V^* \times V$. We recall that $R = (-\triangle)^{-1}$ defines an isometric isomorphism from $H^{-1}(\Omega)$ onto $H_0^1(\Omega)$ (see [32], [14]) and moreover for $\chi \in H^{-1}(\Omega)$ and $\phi \in L^2(\Omega)$ we have

$$\langle \phi, \chi\rangle_{H^{-1}(\Omega)} = \langle \phi, (-\triangle)^{-1}\chi\rangle_{L^2(\Omega)}. \tag{2.27}$$

We define σ on $V = L^2(\Omega)$ by

$$\sigma(\phi,\psi) = \int_\Omega \phi\psi\, dx = \langle\phi,\psi\rangle_{L^2(\Omega)}$$

and $B \in \mathcal{L}(U, V^*)$ by

$$\langle Bu, \chi\rangle_{V^*,V} = \langle -u, \frac{\partial}{\partial\nu}(-\triangle)^{-1}\chi|,\ \rangle_{L^2(,\ )} \tag{2.28}$$

for any $\chi \in V = L^2(\Omega)$. Note that for $\chi \in L^2(\Omega)$ we have $\psi = (-\triangle)^{-1}\chi$ is in $H^2(\Omega) \cap H_0^1(\Omega)$ so that $\frac{\partial\psi}{\partial\nu} \in H^{\frac{1}{2}}(,\ )$ (see [32, p. 127]). Thus we may consider the abstract system

$$\langle \frac{\partial z(t)}{\partial t}, \chi\rangle_{V^*,V} + \sigma(z(t),\chi) = \langle Bu(t), \chi\rangle_{V^*,V} \tag{2.29}$$

which is in the form (2.20). To see that this is an abstract formulation of (2.26), we consider (2.26) in the case where $z \in C(0,T;H^2(\Omega)) \cap C^1(0,T;L^2(\Omega))$. Then by Green's formula ([14, p. 53] [32]) for all $\psi \in H^2(\Omega) \cap H_0^1(\Omega)$ we have

$$\langle \triangle z, \psi\rangle_{L^2(\Omega)} = \langle z, \triangle\psi\rangle_{L^2(\Omega)} - \langle z, \frac{\partial\psi}{\partial\nu}|,\ \rangle_{L^2(,\ )}.$$

Using (2.26) in this relationship we find

$$\langle \frac{\partial z}{\partial t}, \psi \rangle_{L^2(\Omega)} = \langle z, \triangle \psi \rangle_{L^2(\Omega)} - \langle u, \frac{\partial \psi}{\partial \nu}|_{\Gamma} \rangle_{L^2(\Gamma)}.$$

If we choose $\psi = (-\triangle)^{-1}\chi$ for χ arbitrary in $L^2(\Omega)$, this last equation can be written as

$$\langle \frac{\partial z}{\partial t}, (-\triangle)^{-1}\chi \rangle_{L^2(\Omega)} = \langle -z, \chi \rangle_{L^2(\Omega)} + \langle -u, \frac{\partial}{\partial \nu}(-\triangle)^{-1}\chi|_{\Gamma} \rangle_{L^2(\Gamma)},$$

or, in view of (2.27) and (2.28),

$$\langle \frac{\partial z}{\partial t}, \chi \rangle_{H^{-1}(\Omega)} + \langle z, \chi \rangle_{L^2(\Omega)} = \langle Bu, \chi \rangle_{V^*,V}$$

which is just (2.29).

As a special case of the system in (2.21), for the one dimensional case, one can also formulate the Dirichlet boundary control problem as a Neumann problem. For simplicity of discussion, we consider the heat equation

$$\frac{\partial}{\partial t} z(t,x) = \frac{\partial^2}{\partial x^2} z(t,x) \quad \text{ on } \Omega = (0,1) \tag{2.30}$$

with boundary condition $z(t,0) = u(t), \;\; z(t,1) = 0$. We define $y(t,\cdot) \in H^1(\Omega)/\mathcal{R} = \{\phi \in H^1(\Omega) : \int_\Omega \phi dx = 0\}$ satisfying $\frac{\partial}{\partial x} y(t,x) = z(t,x)$. Then (2.30) can be written as

$$\frac{\partial}{\partial t} y(t,x) = \frac{\partial^2}{\partial x^2} y(t,x)$$

with

$$\frac{\partial}{\partial x} y(t,0) = u(t) \text{ and } \frac{\partial}{\partial x} y(t,1) = 0.$$

If we choose $V = H^1(0,1)/\mathcal{R}, \quad H = L^2(0,1)/\mathcal{R}, U = \mathcal{R}^1, Y = L^2(0,1)$, and $C = \frac{\partial}{\partial x}$, then this example can be cast into an abstract Neuman boundary control problem as above.

Example 2 (Second order systems with damping)

We consider second order equations for w in the Hilbert space H_0

$$\frac{d^2}{dt^2} w(t) + 2\gamma A^{1/2} \frac{d}{dt} w(t) + Aw(t) = Bu(t) \tag{2.31}$$

$$\frac{d^2}{dt^2} w(t) + 2\gamma A \frac{d}{dt} w(t) + Aw(t) = Bu(t) \tag{2.32}$$

for $w(t) \in H_0$ with initial data $w(0) = \phi \in \; V_0 \equiv \mathrm{dom}(A^{1/2})$ and $\frac{d}{dt} w(0) = \psi \in H_0$. Here A is a positive definite self-adjoint operator on H_0, and

$\gamma > 0$ is the damping coefficient. Equation (2.31) has often been studied, for example, in [9], and leads to the so-called "structural damping" model. Equation (2.32) has been studied in [28, 3] and numerous other places, and is known as a model with Kelvin-Voigt damping. Special cases of these equations include an Euler-Bernoulli beam equation in which A is defined by

$$A\phi = \frac{d^4}{dx^4}\phi \quad \text{in } L^2(0,1) \tag{2.33}$$

with the appropriate boundary conditions. In general, $A^{1/2}$ is not necessarily a differential operator. But if we consider the boundary condition (corresponding to hinged ends)

$$\phi(0) = \phi(1) = \phi''(0) = \phi''(1) = 0, \tag{2.34}$$

then $\mathrm{dom}(A) = \{\phi \in H^4(0,1) | \phi(0) = \phi(1) = \phi''(0) = \phi''(1) = 0\}$ and, moreover, $V_0 = H^2(0,1) \cap H_0^1(0,1)$ with

$$A^{1/2}\phi = -\frac{d^2}{dx^2}\phi \quad \text{for } \phi \in V_0. \tag{2.35}$$

First we consider the system defined by (2.32). Let $H = V_0 \times H_0$, $V = V_0 \times V_0$, and define a sesquilinear form σ on V by

$$\sigma((\phi_1,\phi_2),(\psi_1,\psi_2)) = \langle A\phi_1, \psi_2\rangle - \langle\langle \phi_2, \psi_1\rangle\rangle + 2\gamma\langle A\phi_2, \psi_2\rangle, \tag{2.36}$$

where $\langle\langle\ ,\ \rangle\rangle$ is the inner product in V_0 and A is the usual extension in $\mathcal{L}(V_0, V_0^*)$ given by $\langle A\phi, \psi\rangle = \langle A\phi, \psi\rangle_{V_0^*,V_0} = \langle A^{1/2}\phi, A^{1/2}\psi\rangle_{H_0}$ for $\phi, \psi \in V_0$. Then

$$|\sigma(\phi,\psi)| \quad \leq 2(1+\gamma)\|\phi\|_V\|\psi\|_V \quad \text{for } \phi = (\phi_1,\phi_2),\ \psi = (\psi_1,\psi_2) \in V,$$

and

$$Re\sigma(\phi,\phi) \geq 2\gamma\|\phi\|_V^2 - 2\gamma\|\phi\|_H^2,$$

where for $\phi = (\phi_1,\phi_2) \in V$

$$\|\phi\|_V^2 = \|\phi_1\|_{V_0}^2 + \|\phi_2\|_{V_0}^2,$$

and

$$\|\phi\|_H^2 = \|\phi_1\|_{V_0}^2 + \|\phi_2\|_{H_0}^2.$$

Hence (2.1) and (2.2) are satisfied for the sesquilinear form σ defined by (2.36), and

$$\mathcal{A} = \begin{bmatrix} 0 & I \\ -A & -2\gamma A \end{bmatrix} \tag{2.37}$$

generates an analytic semigroup on H, where

$$\mathrm{dom}_H(\mathcal{A}) = \{(\phi,\psi)\in H : \psi\in V_0 \text{ and } A\phi+2\gamma A\psi\in H_0\}\,. \tag{2.38}$$

Note that $\mathrm{dom}_H(\mathcal{A})$ can be equivalently defined by

$$\{(\phi,\psi)\in H : \psi\in V_0 \text{ and } \phi+2\gamma\psi\in \mathrm{dom}_{H_0}(A)\}\,.$$

We may also consider the differential operator A of (2.33) on $H_0 = L_2(0,1)$ with the boundary conditions (corresponding to fixed-free ends or a cantilevered beam)

$$\phi(0)=\phi'(0)=\phi''(1)=\phi'''(1)=0.$$

Then,

$$\begin{aligned} V_0 = \mathrm{dom}(A^{1/2}) &= \{\phi\in H^2(0,1):\phi(0)=\phi'(0)=0\},\\ \langle A\phi,\psi\rangle_{V_0^*,V_0} &= \textstyle\int_0^1 \phi''\psi''dx \text{ for } \phi,\psi\in V_0 \end{aligned}$$

with

$$\|\phi\|^2{}_{\mathrm{dom}(A^{1/2})} = \int_0^1 |\phi''|^2dx,$$

even though $A^{1/2}$ is not a simple differential operator as in (2.35).

With (2.32) we may consider, for example, the input operator B defined by

$$\langle Bu,\psi\rangle = \sum_{i=1}^m u_i\int_0^1 \chi_i(x)\psi''(x)dx \text{ for } u=\mathrm{col}(u_1,\ldots,u_m)\in\Re^m = U \tag{2.39}$$

where $\chi_i\in L^2(0,1)$ represents the support of an ith moment control (such controllers are realized using piezoceramic patch actuators). For example, if $\chi(x)$ is the characteristic function of the interval $[0,x_0], x_0\in(0,1)$, then the control input will have the form

$$w''(t,x_0^+)-w''(t,x_0^-)=u(t) \tag{2.40}$$

for an undamped system, or more generally,

$$M(t,x_0^+)-M(t,x_0^-)=u(t)$$

when the moment is given by $M(t,x)=\frac{\partial^2}{\partial x^2}w(t,x)+2\gamma\frac{\partial^3}{\partial t\partial x^2}w(t,x)$ for a Kelvin-Voigt damped beam.

Next we consider the system defined by (2.31). It is not difficult to show that if $w(0)\in \mathrm{dom}(A)$ and $\dot w(0)\in \mathrm{dom}(A^{1/2})$ and $u\equiv 0$, then (2.31) has the solution $w \in C^2(0,T;H_0) \cap C^1(0,T;\ \mathrm{dom}(A^{1/2})) \cap C(0,T;\ \mathrm{dom}(A))$ (e.g., see [9]). Let

$$z_1(t) = \sqrt{1-\gamma^2}A^{1/2}w(t)$$

and

$$z_2(t) = \frac{d}{dt}w(t) + \gamma A^{1/2}w(t).$$

Then for $w(0) \in\ \mathrm{dom}(A)$ and $\dot{w}(0) \in\ \mathrm{dom}(A^{1/2})$

$$\begin{cases} \frac{d}{dt}z_1 + \gamma A^{1/2}z_1 & = \sqrt{1-\gamma^2}A^{1/2}z_2 \\ \frac{d}{dt}z_2 + \gamma A^{1/2}z_2 & = -\sqrt{1-\gamma^2}A^{1/2}z_1\,. \end{cases} \tag{2.41}$$

Let $H = H_0 \times H_0$ and $V = \mathrm{dom}(A^{1/4}) \times \mathrm{dom}(A^{1/4})$, and define a sesquilinear form σ on V by

$$\begin{aligned} \sigma((\phi_1,\phi_2),\psi_1,\psi_2)) = {} & \gamma\langle A^{1/2}\phi_1,\psi_1\rangle - \sqrt{1-\gamma^2}\langle A^{1/2}\phi_2,\psi_1\rangle \\ & + \sqrt{1-\gamma^2}\langle A^{1/2}\phi_1,\psi_2\rangle + \gamma\langle A^{1/2}\phi_2,\psi_2\rangle. \end{aligned} \tag{2.42}$$

Then, σ satisfies (2.1) and (2.2) and yields an abstract system equivalent to (2.41). If A is the differential operator given by (2.33) with the boundary conditions (2.34), then

$$\begin{aligned} \mathrm{dom}(A^{1/4}) & = \{\phi \in H^1(0,1) : \phi(0) = \phi(1) = 0\} \\ \langle A^{1/2}\phi,\psi\rangle & = \int_0^1 \phi'\psi' dx \quad \text{for } \phi,\psi \in \mathrm{dom}(A^{1/4}), \end{aligned}$$

and

$$\|\phi\|^2_{\mathrm{dom}(A^{1/4})} = \int_0^1 |\phi'|^2 dx.$$

For this system we may also consider the input operator defined by (2.39) with $\chi_i \in H^1(0,1)$. If $\chi(x) = -(x - x_0)$ on $[0, x_0]$ and zero otherwise, then the control input is of the form

$$w'''(t,x_0^+) - w'''(t,x_0^-) = u(t),$$

which is shear control.

3 Riccati Equations

First we formulate the LQR problem for (2.19) using an operator theoretic framework as in [1], [13], [17]. Let $\mathcal{L}$ be the bounded linear operator mapping from $L_2(0,T;U)$ into $L_2(0,T;Y)$ defined by

$$(\mathcal{L}u)(t) = C\int_0^t S(t-s)Bu(s)ds, \quad t \in (0,T), \tag{3.1}$$

and $\mathcal{M}$ be the bounded linear operator mapping from H into $L_2(0,T;Y)$ defined by

$$(\mathcal{M}\phi)(t) = CS(t)\phi, \quad t \in (0,T). \tag{3.2}$$

Boundedness of the operators $\mathcal{L}$ and $\mathcal{M}$ follows from Theorem 2.1, the boundedness assumptions $B \in \mathcal{L}(U,V^*)$ and $C \in \mathcal{L}(V,Y)$, and properties of the semigroup $S(t)$. For a given $T > 0$, define the bounded linear operators $\mathcal{L}_T : L_2(0,T;U) \to H$ and $\mathcal{M}_T : H \to H$ by

$$\mathcal{L}_T u = G^{1/2}\int_0^T S(T-s)Bu(s)ds \tag{3.3}$$

and

$$\mathcal{M}_T\phi = G^{1/2}S(T)\phi. \tag{3.4}$$

Then the cost functional (2.19) can be written as

$$J(u,z_0) = \|\mathcal{L}u + \mathcal{M}z_0\|^2_{L_2(0,T;Y)} + \|u\|^2_{L_2(0,T;U)} + \|\mathcal{L}_T u + \mathcal{M}_T z_0\|^2_H$$

and using standard arguments (e.g., see [27],[17]) we have

Theorem 3.1 *Assume the sesquilinear form σ satisfies (2.1) and (2.2). Then for $z_0 \in H$, the optimal control that minimizes (2.19) is given by*

$$u_T^* = -(I + \mathcal{L}^*\mathcal{L} + \mathcal{L}_T^*\mathcal{L}_T)^{-1}(\mathcal{L}^*\mathcal{M} + \mathcal{L}_T^*\mathcal{M}_T)z_0 \tag{3.5}$$

and the self-adjoint operator Π_T on H, defined by

$$\Pi_T = (\mathcal{M}^*, \mathcal{M}_T^*)\left[\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} \mathcal{L} \\ \mathcal{L}_T \end{bmatrix}\,[\mathcal{L}^*\ \mathcal{L}_T^*]\right]^{-1}\begin{pmatrix} \mathcal{M} \\ \mathcal{M}_T \end{pmatrix} \tag{3.6}$$

satisfies

$$\langle \Pi_T z_0, z_0\rangle = J(0,T;u^*,z_0) = \min_u J(u,z_0). \tag{3.7}$$

Moreover, if we assume $C \in \mathcal{L}(H,Y)$ and $G \in \mathcal{L}(V^*,V)$, and either (i) Y is finite dimensional or (ii) σ satisfies (2.17), in addition to (2.1)-(2.2), then $\Pi_T \in \mathcal{L}(V^*,V)$ and the optimal control u^* is given by

$$u^*(t) = -B^*\Pi_T(t)z(t) \tag{3.8}$$

where $\Pi_T(t) \equiv \Pi_{T-t}$ satisfies the differential Riccati equation

$$\left(\frac{d}{dt}\Pi_T(t) + \mathcal{A}^*\Pi_T(t) + \Pi_T(t)\mathcal{A} - \Pi_T(t)BB^*\Pi_T(t) + C^*C\right)\phi = 0 \tag{3.9}$$

for all $\phi \in V$ and $z(t)$ is the corresponding optimal trajectory.

Remark 3.2
(1) The condition (i) or (ii) appearing in Theorem 3.1 is used only to argue (e.g., [4])

$$\|CS(t)\phi\|_{L^2(0,T;Y)} \leq c\|\phi\|_{V^*} \quad \text{for all } \phi \in H, \tag{3.10}$$

which is (H2) of [27] if we identify W and V of [27] with the spaces H and V^*, respectively, of our notation here. The condition (H3) of [27] follows from (2.4) and (2.9). We note that (3.10) implies the operator $\mathcal{M}$ of (3.2) can readily be continuously extended to $V^* \to L_2(0,T;Y)$.

From (2.15) we have

$$\| \int_0^t S(t-s)Bu(s)ds\|_H \leq b\|u\|_{L^2(0,T;U)}, \tag{3.11}$$

which is (H1) of [27]. Thus under the assumptions in Theorem 3.1 the problem can be treated using the Pritchard-Salamon framework [27] (e.g., see Thm 2.2 of [17]).
(2) (See [1]) In the general case (i.e., $B \in \mathcal{L}(U,V^*), C \in \mathcal{L}(V,Y)$ and $G \in \mathcal{L}(H)$), the function $t \to \Pi_T(t)$ satisfies the implicit integral Riccati equation

$$\Pi_T(t)x = S^*(T-t)GU_T(T,t)x + \int_t^T S^*(s-t)C^*CU_T(s,t)xds \text{ for } x \in H, \tag{3.12}$$

where the evolution operator $U_T(s,t)$ is defined by

$$U_T(t,s)x = S(t-s)x + \int_s^t S(t-\xi)Bu^*_{T-s}(\xi - s; x)d\xi \quad \text{for } x \in H, \tag{3.13}$$

and the optimal control u^* is given by

$$u^*_T(t) = -B^*\Pi_T(t)U_T(t,0)z_0 \quad \text{for } z_0 \in H. \tag{3.14}$$

Next we consider the problem of minimizing the quadratic cost functional

$$J(u;z) = \int_0^\infty (||Cz(t)||^2_Y + ||u(t)||^2_U)dt \tag{3.15}$$

subject to (2.20). The following lemma plays an important role in the infinite time interval problem and hence we shall sketch a proof.

Lemma 3.3 *A semigroup $S(t)$ generated via a sesquilinear form σ satisfying (2.1), (2.2) is (uniformly) exponentially stable on H if and only if $S(t)$ and $S^*(t)$ are exponentially stable on H, V and V^*.*

Proof: We first assume that $S(t)$ is exponentially stable on H. Then we have $\|S(t)z_0\|_H \leq Me^{-\alpha t}\|z_0\|_H$ for some $\alpha > 0$, $M \geq 1$ and any $z_0 \in H$. Using (2.2) and (2.12) for $z(t) = S(t)z_0$, $z_0 \in V$, we obtain

$$\frac{1}{2}\|z(t)\|_H^2 + \omega \int_0^t \|z(s)\|_V^2 \, ds \leq \rho \int_0^t \|z(s)\|_H^2 ds + \frac{1}{2}\|z_0\|_H^2.$$

From this inequality and the exponential (in H) bound we find

$$\int_0^\infty \|z(s)\|_V^2 ds \leq \frac{1}{\omega}\left(\frac{M^2\rho}{2\alpha} + \frac{1}{2}\right)\|z_0\|_H^2$$

and an application of Datko's lemma [10] yields exponential stability of $S(t)$ on V. Since $\|S(t)\|_{\mathcal{L}(V,V)} = \|S^*(t)\|_{\mathcal{L}(V^*,V^*)}$, we also obtain exponential stability of $S^*(t)$ on V^*.

Since $S^*(t)$ is also exponentially stable on H, similar arguments beginning with the bound for $S^*(t)$ on H yield that $S^*(t)$ is exponentially stable on V and $S(t)$ is exponentially stable on V^*.

To argue the converse statement in the lemma, suppose that $S(t)$ is exponentially stable on V and recall that for $\phi \in H$ and $t > 0$, $S(t)\phi \in \mathrm{dom}_H(\mathcal{A})$ where $\mathrm{dom}_H(\mathcal{A})$ is dense and continuously embedded in V. For $\phi \in H$ and $t > 1$ we have

$$\|S(t)\phi\|_H \leq k\|S(t)\phi\|_V \leq k\|S(t-1)\|_{\mathcal{L}(V)}\|S(1)\phi\|_V$$

so that exponential stability of $S(t)$ on V implies exponential stability on H.

For the case when C is bounded (i.e., $C \in \mathcal{L}(H,Y)$), we have

Theorem 3.4 *Assume $C \in \mathcal{L}(H,Y)$ and either Y is finite dimensional, or σ satisfies (2.17), in addition to (2.1) and (2.2). If $(\mathcal{A}, B)$ is stabilizable, i.e., there exists an operator $K \in \mathcal{L}(H,U)$ such that $\mathcal{A} - BK$ generates an exponentially stable semigroup on H and $(\mathcal{A}, C)$ is detectable, i.e., there exists $F \in \mathcal{L}(Y, V^*)$ such that $\mathcal{A} - FC$ generates an exponentially stable semigroup on V^*, then the algebraic Riccati equation*

$$(\mathcal{A}^*\Pi + \Pi\mathcal{A} - \Pi BB^*\Pi + C^*C)x = 0 \quad \textit{for all } x \in V$$

has a unique nonnegative solution $\Pi \in \mathcal{L}(V^*, V)$, $\mathcal{A} - BB^*\Pi$ generates an exponentially stable semigroup $U(t)$ on H, V and V^*, and the optimal solution that minimizes (3.15) is given by

$$u^*(t) = -B^*\Pi U(t)z_0.$$

Proof: Note that for $K \in \mathcal{L}(H,U)$

$$\sigma_K(\phi,\psi) = \sigma(\phi,\psi) + \langle K\phi, B^*\psi\rangle_U \quad \text{for all } \phi,\psi \in V$$

defines a continuous sesquilinear form on V where

$$\begin{aligned} Re\sigma_K(\phi,\phi) &\geq \frac{1}{2}\omega||\phi||_V^2 - (\rho + \frac{1}{2\omega}||B||^2_{\mathcal{L}(U,V^*)}||K||^2_{\mathcal{L}(H,U)})||\phi||_H^2 \\ &= \frac{1}{2}\omega||\phi||_V^2 - \rho_K||\phi||_H^2. \end{aligned}$$

Thus, by Lemma 3.3, the stabilizability of $(\mathcal{A}, B)$ implies that there exists an $K \in \mathcal{L}(V^*, U)$ such that $\mathcal{A} - BK$ generates an exponentially stable semigroup on V^*, and similarly, the detectability of $(\mathcal{A}, C)$ implies that there exists an $F \in \mathcal{L}(Y, V^*)$ such that $\mathcal{A} - FC$ generates an exponentially stable semigroup on V^*. The theorem now follows from [27, Theorems 3.3 and 3.4 and Remark 3.5]. Recall that the spaces W and V of [27] are chosen as H and V^*, respectively, in our framework here. We note that the space Z of (H3) in [27] is here taken to be $Z = \mathrm{dom}_{V^*}(\mathcal{A}) = V$ endowed with the graph norm corresponding to $\mathcal{A}$; the condition (H3) follows then from (2.4). Since $B^*\Pi \in \mathcal{L}(H, U)$ it follows from the discussions in Section 2 that $\mathcal{A} - BB^*\Pi$ generates an analytic semigroup $U(t)$ on H, V and V^*. Since $\mathrm{dom}_{V^*}(\mathcal{A} - BB^*\Pi) = V$, the exponential stability of $U(t)$ on V^* implies that on H and V.

The case where C is unbounded ($C \in \mathcal{L}(V, Y)$) will be discussed in Section 5.

4 Galerkin Approximation

We turn now to the approximation results that are the main focus of this paper. Let V^N be a sequence of finite dimensional subspaces of $V \subset H$. Define $A^N : V^N \to V^N$ by

$$\langle -A^N\phi, \psi\rangle = \sigma(\phi, \psi) \quad \text{for all } \phi, \psi \in V^N, \tag{4.1}$$

that is, A^N is defined via restriction of σ to $V^N \times V^N$. For given $B \in \mathcal{L}(U, V^*)$, we define $B^N \in \mathcal{L}(U, V^N)$ by

$$\langle B^N u, \psi\rangle = \langle u, B^*\psi\rangle \quad \text{for all } \psi \in V^N, \tag{4.2}$$

and C^N denotes the restriction of C onto V^N. We assume the approximation condition:

For any $\phi \in V$, there exists a sequence ϕ^N in V^N such that $\|\phi^N - \phi\|_V \to 0$ as $N \to \infty$. $\quad (H1)$

4.1 Convergence of Galerkin approximations

The following lemma is standard in the literature (e.g., see Chapter III in Lions [21]).

Lemma 4.1 *Suppose (H1) is satisfied and let $f \in L^2(0,T;V^*)$ and $z_0 \in H$. If $z(t) \in V$ is defined by (2.11) with $\phi = z_0$ and $z^N(t) \in V^N, t \geq 0$ satisfies*

$$\frac{d}{dt}\langle z^N(t), \psi\rangle + \sigma(z^N(t), \psi) = \langle f(t), \psi\rangle_{V^*,V} \quad \text{for all } \psi \in V^N \tag{4.3}$$

$$z^N(0) = P^N z_0,$$

then the error function $e^N(t) = z^N(t) - z(t)$ satisfies

$$\|e^N(t)\|_H \to 0 \text{ and } \int_0^t \|e^N(s)\|_V^2 ds \to 0 \text{ as } N \to \infty$$

uniformly in $t \in [0,T]$.

In this lemma P^N denotes the usual orthogonal projection of H onto V^N; i.e., for $\phi \in H$

$$P^N\phi \in V^N \text{ and } \langle P^N\phi, \psi\rangle = \langle \phi, \psi\rangle \text{ for all } \psi \in V^N.$$

This projection can readily be extended to $P^N \in \mathcal{L}(V^*, V^N)$ by replacing $\langle \phi, \psi\rangle$ in the definition by $\langle \phi, \psi\rangle_{V^*,V}$ for all $\phi \in V^*$. Then the solution $z^N(t)$ of (4.3) can be written as

$$z^N(t) = e^{tA^N}P^N z_0 + \int_0^t e^{(t-s)A^N}P^N f(s)ds.$$

It follows from (H1) that $P^N\phi \to \phi$ in H for all $\phi \in H$. In addition, we have

Lemma 4.2 *For $Re\lambda \geq \rho$ and $\phi \in V^*$*

$$\|(\lambda I - A^N)^{-1}P^N\phi\|_V \leq \frac{\|\phi\|_{V^*}}{\omega} \tag{4.4}$$

$$\|(\lambda I - A^N)^{-1}P^N\phi\|_H \leq \frac{M_0}{|\lambda - \rho|}\|\phi\|_H \tag{4.5}$$

$$\|(\lambda I - A^N)^{-1}P^N\phi\|_V \leq \frac{M_1}{|\lambda - \rho|^{\frac{1}{2}}}\|\phi\|_H \tag{4.6}$$

where $M_0 = 1 + M/\omega$ and $M_1 = ((1 + M\omega)/\omega)^{\frac{1}{2}}$.

Moreover, for every $\phi \in V^*$

$$(\lambda I - A^N)^{-1}P^N\phi \to (\lambda I - A)^{-1}\phi \quad \text{in } V.$$

Proof: First we observe that the sesquilinear form σ restricted to V^N defines a continuous sesquilinear form on V^N satisfying (2.2) for all $u \in V^N$. Hence for $\phi \in V^*$ and $Re\lambda \geq \rho$, $u^N = (\lambda I - A^N)^{-1} P^N \phi \in V^N$ exists and satisfies

$$\lambda \langle u^N, \psi \rangle + \sigma(u^N, \psi) = \langle \phi, \psi \rangle_{V^*,V} \quad \text{for all } \psi \in V^N.$$

Thus, the first three estimates follow from exactly the same arguments as in Lemma 3.6.1 of [31]. Since for all $\phi \in V^*$, $u = (\lambda I - \mathcal{A})^{-1} \phi \in V$ satisfies

$$\lambda \langle u, \psi \rangle + \sigma(u, \psi) = \langle \phi, \psi \rangle_{V^*,V} \quad \text{for all } \psi \in V,$$

we obtain

$$\lambda \langle u - u^N, \psi \rangle + \sigma(u - u^N, \psi) = 0 \ \text{ for all } \psi \in V^N.$$

By (H1), there exists a sequence $\hat{u}^N \in V^N$ such that $\|\hat{u}^N - u\|_V \to 0$ as $N \to \infty$, and

$$\begin{aligned} \lambda \langle \hat{u}^N - u^N, \psi \rangle \quad & + \sigma(\hat{u}^N - u^N, \psi) \\ & = \lambda \langle \hat{u}^N - u, \psi \rangle + \sigma(\hat{u}^N - u, \psi). \end{aligned}$$

Choosing $\psi = \hat{u}^N - u^N \in V^N$, we then find from (2.1) and (2.2) that for $Re\lambda \geq \rho$,

$$\|\hat{u}^N - u^N\|_V \leq \frac{c}{\omega} \|u - \hat{u}^N\|_V$$

for some positive constant c. From this the desired convergence follows immediately.

Using the Trotter-Kato theorem [26] and the estimates (4.5) and (4.6) one can readily establish the following results (see [3] for proofs).

Lemma 4.3 *For all $\phi \in H$*

$$\begin{cases} \|e^{tA^N} P^N \phi - S(t)\phi\|_H \to 0 & \text{as } N \to \infty \\ \\ \|e^{tA^{N^*}} P^N \phi - S^*(t)\phi\|_H \to 0 & \text{as } N \to \infty \end{cases} \tag{i}$$

where the convergence is uniform on bounded t-intervals. Moreover, for all $\phi \in H$

$$\|e^{tA^N} P^N \phi\|_V \leq \frac{c}{t^{\frac{1}{2}}} \|\phi\|_H, \ \|e^{tA^{N^*}} P^N \phi\|_V \leq \frac{c}{t^{\frac{1}{2}}} \|\phi\|_H \tag{ii}$$

and for $t > 0$

$$\begin{cases} \|e^{tA^N} P^N \phi - S(t)\phi\|_V \to 0 & \text{as } N \to \infty, \\ \|e^{tA^{N^*}} P^N \phi - S^*(t)\phi\|_V \to 0 & \text{as } N \to \infty. \end{cases} \qquad (iii)$$

Using Lemma 4.3 we have an approximation result in the bounded output case corresponding to Theorem 2.2.

Lemma 4.4 *If $C \in \mathcal{L}(H, Y)$ and the sesquilinear form σ satisfies (2.17), in addition to (2.1) and (2.2), then*

$$\mathcal{M}^N \phi \to \mathcal{M}\phi \text{ in } L_2(0, T; Y) \quad \text{for all } \phi \in V^*$$

and

$$\mathcal{M}^{N^*} y \to \mathcal{M}^* y \text{ in } V \quad \text{for all } y \in L_2(0, T; Y)$$

where for $\phi \in V^*$, $\mathcal{M}\phi$ is given by the continuous extension to V^* (see (3.10)) of the operator in (3.2) and $\mathcal{M}^N \in \mathcal{L}(V^*, L_2(0, T; Y))$ is defined by

$$(\mathcal{M}^N \phi)(t) = Ce^{tA^N} P^N \phi \quad \text{for } \phi \in V^*.$$

The arguments behind these results are tedious but straightforward once one uses the estimates of theorems 2.1 and 2.2 (which requires (2.17)) to establish uniform bounds for $\mathcal{M}^N$ and $\mathcal{M}^{N*}$. The details can be found in [4].

4.2 Convergence of Riccati solutions

We first consider the finite horizon optimal control problem for (2.19)-(2.20). Corresponding to a given approximation scheme as defined via (4.1), (4.2), we formulate the Nth approximate problem in V^N: Minimize

$$\begin{aligned} J^N(0, T; u) &= \textstyle\int_0^t (\|Cz^N(t)\|_Y^2 + \|u(t)\|_U^2) dt \\ &\quad + \langle Gz^N(T), z^N(T)\rangle \end{aligned} \qquad (4.8)$$

subject to

$$\begin{aligned} \frac{d}{dt} z^N(t) &= A^N z^N(t) + B^N u(t), \quad t > 0 \\ z^N(0) &= P^N z_0. \end{aligned} \qquad (4.9)$$

Then we may obtain the following convergence results using Theorem 3.1 and Lemmas 4.1-4.3 in the general case of unbounded output.

Theorem 4.5 *Suppose the sesquilinear form σ satisfies (2.1) and (2.2) and $\mathcal{A}$ is defined by (2.3), $B \in \mathcal{L}(U, V^*), C \in \mathcal{L}(V, Y)$ and $G \in \mathcal{L}(H)$ is nonnegative. Suppose the approximation scheme satisfies (H1). Then the optimal control u_T^N to the Nth approximate problem for (4.8)-(4.9) converges strongly to u_T^*, the optimal control for (2.19)-(2.20) in $L_2(0,T;U)$ for all $z_0 \in H$. Moreover, if $\Pi_T^N(t), t \leq T$ is the solution to the Riccati equation in V^N:*

$$\frac{d}{dt}\Pi_T^N(t) + A^{N^*}\Pi_T^N(t) + \Pi_T^N(t)A^N - \Pi_T^N(t)B^N B^{N^*}\Pi_T^N(t) + C^{N^*}C^N = 0 \tag{4.10}$$

and $\Pi_T^N(T) = P^N G P^N$, then for all $\phi \in H$, $\Pi_T^N(t)P^N\phi$ converges strongly to $\Pi_T(t)\phi$ in H, uniformly in $t \in [0,T]$, where the optimal control $u_T^N(t), t \leq T$ is given by

$$u_T^N(t) = -B^{N^*}\Pi_T^N(t)z^N(t). \tag{4.11}$$

Proof: The arguments are quite straight forward and are similar to those given in [17]. We define for the finite dimensional problem analogues $\mathcal{L}^N, \mathcal{M}^N, \mathcal{L}_T^N, \mathcal{M}_T^N, u_T^{N^*}, \Pi_T^N$ of the operators in (3.1)-(3.6). Use of lemmas 4.1 and 4.3 then yield at once that these operators converge in the appropriate topologies to the corresponding operators in (3.1)-(3.6).

Under the stronger assumptions (including bounded output) in Theorem 3.1, we obtain stronger convergence properties for Π_T^N. The arguments are quite similar to those just outlined, except we use the stronger results for the extended operators in Lemma 4.4 in this case.

Theorem 4.6 *Consider the case $C \in \mathcal{L}(H, Y)$. Assume either (i) Y is finite dimensional or (ii) the sesquilinear form σ satisfies (2.17) in addition to (2.1) and (2.2), is satisfied. Then $\Pi_T^N(t)$ converges to $\Pi_T(t)$ in $\mathcal{L}(V^*, V)$, uniformly on $[0,T]$.*

We turn next to the infinite horizon problem involving (3.15). We assume

(H2) the injection $i : V \to H$ is compact.

Then we have the uniform stabilizability result which is the unbounded input analogue of the fundamental results of [6].

Lemma 4.7 *Suppose $(\mathcal{A}, B)$ is stabilizable and (H2) holds. Then there exists a sequence of operators $K^N \in \mathcal{L}(V^N, U)$ and positive constants $M_1 \geq 1$ and ω_1 independent of N and a positive integer N_0 such that for all $N \geq N_0$*

$$\|e^{t(A^N - B^N K^N)}P^N\phi\|_H \leq M_1 e^{-\omega_1 t}\|\phi\|_H, \ t > 0.$$

Proof: Since $(\mathcal{A}, B)$ is stabilizable, there exists an operator $K \in \mathcal{L}(H, U)$ such that $\mathcal{A} - BK$ generates a uniformly exponentially stable semigroup on H. Let K^N be the restriction of K onto V^N. Consider, as in the proof of Theorem 3.4, the corresponding sesquilinear form σ_K on V

$$\sigma_K(\phi, \psi) = \sigma(\phi, \psi) + \langle K\phi, B^*\psi \rangle \text{ for } \phi, \psi \in V. \tag{4.12}$$

Then σ_K is continuous and for $\phi \in V$

$$\begin{aligned} Re\sigma_K(\phi, \phi) &\geq Re\sigma(\phi, \phi) - \|K\|\|B^*\|\|\phi\|_H\|\phi\|_V \\ &\geq \frac{\omega}{2}\|\phi\|_V^2 - \rho_K\|\phi\|_H^2 \end{aligned} \tag{4.13}$$

where $\rho_K = \rho + \|B\|^2\|K\|^2/2\omega$. Thus, Lemma 4.3 applies to $A_K^N = A^N - B^N K^N$ where $\langle -A_K^N\phi, \psi \rangle = \sigma_K(\phi, \psi)$ for all $\phi, \psi \in V^N$, so that

$$\begin{aligned} e^{tA_K^N} P^N \phi &\to T(t)\phi \text{ in } H \\ e^{tA_K^{N^*}} P^N \phi &\to T^*(t)\phi \text{ in } H \end{aligned}$$

for every $\phi \in H$ and $t \geq 0$ and

$$\|e^{tA_K^N} P^N \phi\|_V \leq \frac{\tilde{M}}{t^{\frac{1}{2}}}\|\phi\|_H, \quad t > 0,$$

where $T(t),\ t \geq 0$ is the semigroup generated by $\mathcal{A} - BK$. Since the embedding i is compact, for a fixed $t > 0$ the set $S = \cup_{N \geq 1}(e^{tA_K^N}P^N - T(t))B$ is relatively compact in H, where B is the unit sphere of H. It then follows from Proposition 3.7 in [7, p. 126] that

$$\|e^{tA_K^N}P^N - T(t)\|_{\mathcal{L}(H)} \to 0 \text{ as } N \to \infty.$$

Since $T(t), t \geq 0$ is uniformly exponentially stable, there exists a $t_0 > 0$ such that $\|T(t_0)\| \leq \frac{1}{2}$. Hence for N sufficiently large $\|e^{t_0 A_K^N}P^N\|_{\mathcal{L}(H)} < 1$ from which the desired results follows using standard semigroup arguments (e.g., see the proof of Theorem 3.1.1 in [31]).

Combining Lemma 4.7 with arguments in [6] and [17], one obtains the main result of this section concerning convergence for the bounded output $C \in \mathcal{L}(H, Y)$, unbounded input, $B \in \mathcal{L}(U, V^*)$, case.

Theorem 4.8 *Assume the conditions of Theorem 4.6 hold and (H1) and (H2) are satisfied. Suppose $(\mathcal{A}, B)$ is stabilizable and $(\mathcal{A}, C)$ is detectable. Then for N sufficiently large, there exists a unique nonnegative self-adjoint solution $\Pi^N \in \mathcal{L}(V^*, V)$ to the algebraic Riccati equation in V^N*

$$A^{N^*}\Pi^N + \Pi^N A^N - \Pi^N B^N B^{N^*}\Pi^N + C^{N^*}C^N = 0, \tag{4.14}$$

there exists constants $M_2 \geq 1$ *and* $\omega_2 > 0$ *such that*

$$\|e^{t(A^N - B^N B^{N^*} \Pi^N)} P^N \phi\|_H \leq M_2 e^{-\omega_2 t} \|\phi\|_H, \tag{4.15}$$

and $\Pi^N P^N \phi \xrightarrow{s} \Pi\phi$ *in* V *for every* $\phi \in V^*$. *Moreover, we have*

$$\|B^{N^*} \Pi^N P^N - B^* \Pi\|_{\mathcal{L}(H,U)} \to 0$$

as $N \to \infty$. *The feedback system operator* $\mathcal{A} - BB^{N^*}\Pi^N$ *(i.e., the approximate feedback controls in the original infinite dimensional system) generates an exponentially stable analytic semigroup on* H *and for every* $\phi \in H$

$$J(-B^{N^*}\Pi^N z(\cdot); z_0) - J(u^*; z_0) \leq \varepsilon(N) \|\phi\|_H^2$$

where $\varepsilon(N) \to 0$ *as* $N \to \infty$.

Proof: The arguments are essentially a repeat of those for theorems 2.2 and 3.1 of [6] and Theorem 2.6 of [17] where we use the appropriate norms and topologies at each step. For the sake of completeness we give a brief outline, referring the reader to [4] for details. We first consider σ_K and K^N as in Lemma 4.7. Using the inverse Laplace transform (see [31] or [29]) to obtain a representation for analytic semigroups in terms of resolvents and standard contour arguments (again see [29]) along with the bounds (4.6), one readily obtains bounds

$$\|e^{tA_K^N} P^N \phi\|_V \leq \frac{M}{t^{1/2}} e^{-w_1 t} \|\phi\|_H \tag{4.16}$$

$$\|e^{tA_K^N} P^N \phi\|_H \leq \frac{M}{t^{1/2}} e^{-w_1 t} \|\phi\|_{V^*} \tag{4.17}$$

for $\phi \in V^*$. These can then be used to argue

$$J^N(u^{N^*}; \phi) \leq \alpha \|\phi\|_{V^*}^2$$

for $N \geq N_0$ and $\alpha > 0$ independent of N. This is the result of Step 1 in the proof of Theorem 2.6 in the Appendix of [17] and guarantees (Theorem 3.3 of [27]) existence of the desired solutions to (4.14) for N sufficiently large.

Detectability and arguments similar to those in Lemma 4.7 yield bounds similar to those in (4.16), (4.17) for $e^{t(A^N - G^N C^N)} P^N$ and hence uniform detectability. Following arguments similar to those in Step 2 of the proof of Theorem 2.6 in [17] (relying on Young's inequality and Datko's lemma), one can obtain the bound (4.15). In fact, one also obtains a bound in which the right side of (4.15) is replaced by

$$\frac{M}{t^{1/2}} e^{-\frac{w_2 t}{2}} \|\phi\|_{V^*}.$$

Then with standard estimates (see the Appendix in [6] and (A.6), (A.7) of [17])

$$\|\Pi^N P^N - \Pi_T^N(0)P^N\|_{\mathcal{L}(V^*,V)} \leq \frac{\nu M^2}{T} e^{-w_2 T}$$

$$\|\Pi - \Pi_T(0)\|_{\mathcal{L}(V^*,V)} \leq \nu \frac{M^2}{T} e^{-w_2 T},$$

the convergence of Lemma 4.6 and the triangle inequality, we are able to conclude that $\|\Pi^N P^N - \Pi\|_{\mathcal{L}(V^*,V)} \to 0$. Moreover, since the injection $i^* : H \to V^*$ is compact, we find

$$\begin{aligned} \|B^{N^*}\Pi^N P^N - B^*\Pi\|_{\mathcal{L}(H,U)} &\leq \|B^*(\Pi^N P^N - \Pi)\|_{\mathcal{L}(H,U)} \\ &\leq \|B^*\| \, \|\Pi^N P^N - \Pi\|_{\mathcal{L}(H,V)} \to 0 \end{aligned}$$

as $N \to \infty$.

5 Case of Unbounded C

In this section we consider the case $C \in \mathcal{L}(V, Y)$ and $B \in \mathcal{L}(U, V^*)$. Let A_0 be a self-adjoint operator on H defined by

$$\begin{aligned} \mathrm{dom}(A_0) &= \{u \in V : |\sigma_0(u,v)| \leq k_u \|v\|_H \text{ for all } v \in V\}, \text{ and} \\ \langle A_0 u, v\rangle &= \sigma_0(u,v) + \rho\langle u,v\rangle_H \text{ for all } u, v \in V, \end{aligned} \tag{5.1}$$

where σ_0 is the symmetric part of σ (see Theorem 2.2) and

$$\sigma_0(u,u) + \rho\langle u,u\rangle_H \geq \omega\|u\|_V^2 \text{ for } u \in V.$$

Let $\Lambda = A_0^{1/2}$ with $\mathrm{dom}(\Lambda) = V$ and for $0 \leq \theta \leq 1$, V_θ denote (see [22]) the intermediate space : $V_\theta = [V, H]_\theta$ $=\mathrm{dom}(\Lambda^{1-\theta})$. Then we have $[V, V^*]_{1/2} = H$ and the interpolation inequality

$$\begin{aligned} \|\phi\|_{V_\theta} &\leq c\|\phi\|_H^\theta \|\phi\|_V^{1-\theta} \leq c\|\phi\|_{V^*}^{\frac{\theta}{2}} \|\phi\|_V^{\frac{\theta}{2}} \|\phi\|_V^{1-\theta} \\ &\leq c\|\phi\|_{V^*}^{\frac{\theta}{2}} \|\phi\|_V^{1-\frac{\theta}{2}} = c\|\phi\|_{V^*}^{\alpha_0} \|\phi\|_V^{1-\alpha_0}, \end{aligned} \tag{5.2}$$

where $\alpha_0 = \frac{\theta}{2}$. We assume throughout this section

(H3) $\|B^*\phi\|_U \leq b\|\phi\|_{V_\theta}$ with $\theta < 1$ for some $b > 0$ and let $\alpha_0 = \frac{\theta}{2}$.

Since $\rho \in \mathbb{C}$ is not in $sp(\mathcal{A})$, there exist $\sigma > 0, M > 0$ and U_0, a neighborhood of ρ such that

$$\rho(\mathcal{A}) \supset \Sigma^+ = \{\lambda : 0 \leq |\arg(\lambda - \rho)| < \frac{\pi}{2} + \sigma\} \cup U_0$$

and

$$\|(\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(V^*)} \leq \frac{M}{1 + |\lambda - \rho|} \text{ for } \lambda \in \Sigma^+$$

(e.g., see [31]). It then follows from [26, p. 69-75] that

$$(\rho I - \mathcal{A})^{-\beta} = \frac{1}{,\,(\beta)} \int_0^\infty t^{\beta-1} S_\rho(t) dt \qquad \text{for } \beta \geq 0$$

where , $(\cdot)$ denotes the gamma function and $S_\rho(t) = e^{-\rho t} S(t), t \geq 0$. Since $\rho \in \rho(\mathcal{A})$, there exist constants $\delta > 0$ and $M_1 \geq 1$ such that

$$\begin{aligned} \|S_\rho(t)\|_{\mathcal{L}(V^*)} &\leq M_1 e^{-\delta t} \\ \|S_\rho(t)\|_{\mathcal{L}(V^*, V)} &\leq \tfrac{M_1}{t} e^{-\delta t}. \end{aligned} \tag{5.3}$$

Then for $x \in V = \mathrm{dom}_{V^*}(\mathcal{A}^*)$

$$\begin{aligned} \|(\rho I - \mathcal{A}^*)^{\alpha-1} x\|_{V_\theta} &\leq \tfrac{1}{,\,(1-\alpha)} \int_0^\infty t^{-\alpha} \|S_\rho^*(t) x\|_{V_\theta} dt \\ &\leq \tfrac{c}{,\,(1-\alpha)} \int_0^\infty t^{-\alpha} \|S_\rho^*(t) x\|_{V^*}^{\alpha_0} \|S_\rho^*(t) x\|_V^{1-\alpha_0} \\ &\leq \tfrac{cM_1}{,\,(1-\alpha)} \int_0^\infty t^{-\alpha+\alpha_0-1} e^{-\delta t} \|x\|_{V^*} dt. \end{aligned}$$

Hence if $0 < \alpha < \alpha_0$, then there exists $M_\alpha > 0$ such that

$$\|(\rho I - \mathcal{A}^*)^{\alpha-1} x\|_{V_\theta} \leq M_\alpha \|x\|_{V^*} \text{ for all } x \in V.$$

Since V is dense in V^* and V_θ is closed in V^*, this implies $\mathrm{dom}((\rho I - \mathcal{A}^*)^{1-\alpha}) \subset V_\theta$ and, therefore,

$$B^*(\rho I - \mathcal{A}^*)^{1-\alpha} \in \mathcal{L}(V^*, U). \tag{5.4}$$

Under the assumption (H3), using the same arguments as in Theorem 1 [121], we have the following results on the solution $\Pi_T(\cdot)$ to the Riccati equation (3.12).

Theorem 5.1 *Assume (H3) holds. Let $\Pi_T(\cdot)$ be the solution to the integral Riccati equation (3.12) with $G = 0$ and $U_T(t,s)$ be the evolution operator defined by (3.13). Then for each $T > 0$, there exists a constant $M > 0$ such that*

$$\|U_T(t,s)\|_{\mathcal{L}(V)} \leq M e^{\rho(t-s)}$$

and for $\gamma < 1$, there exists a constant M_γ such that

$$\|(\rho I - \mathcal{A}^*)^\gamma \Pi_T(t) x\|_{V^*} \leq M_\gamma \|x\|_V \quad \text{for } 0 \leq t \leq T. \tag{5.5}$$

Moreover, if we assume

$$\begin{array}{l} \textit{for any } z \in H \textit{ there exists a control } u \in L_2(0,\infty;U) \\ \textit{such that } J(u;z) < \infty, \end{array} \tag{$H4$}$$

then the constants M, M_γ in the above are uniform in T.

By Theorem 5.1 and (5.4),

$$B^*\Pi_T(t) \in \mathcal{L}(V,H) \ \text{ and } (B^*\Pi_T(t))^* = \Pi_T(t)B. \tag{5.6}$$

Now it is not difficult to argue (e.g., see [11]) that for $x \in V, \Pi_T(t)x, t \le T$ is given by

$$\Pi_T(t)x = \int_t^T S(s-t)(C^*C - \Pi_T(s)BB^*\Pi_T(s))S(s-t)x ds. \tag{5.7a}$$

Since for $x \in \mathrm{dom}_H(\mathcal{A})$, $t \to S(t)x \in V$ is strongly differentiable in $L_2(0,T;V)$ with derivative $S(t)Ax$ (see Theorem 2.1), $\Pi_T(t)x, t \le T$ satisfies the differential Riccati equation

$$\begin{aligned} \langle \tfrac{d}{dt}\Pi_T(t)x, y\rangle \quad & + \langle \mathcal{A}x, \Pi_T(t)y\rangle + \langle \Pi_T(t)x, \mathcal{A}y\rangle \\ & - \langle B^*\Pi_T(t)x, B^*\Pi_T(t)y\rangle + \langle Cx, Cy\rangle = 0 \end{aligned}$$

for all $x, y \in \mathrm{dom}_H(\mathcal{A})$ and $\Pi_T(T) = 0$. The following theorem follows from (5.5).

Theorem 5.2 *If (H3), (H4) hold and let $\Pi^\infty = s - \lim \Pi_T(0)$ as $T \to \infty$ in $\mathcal{L}(H)$, then for $\phi \in V$ and $\gamma < 1$*

$$(\rho I - \mathcal{A}^*)^\gamma \Pi_T(0)\phi \xrightarrow{w} (\rho I - \mathcal{A}^*)^\gamma \Pi^\infty \phi \textit{ in } V^*.$$

Moreover, if (H2) is satisfied, the convergence of above becomes strong and Π^∞ satisfies the algebraic Riccati equation

$$\begin{aligned} \langle \Pi^\infty x, \mathcal{A}y\rangle \quad & + \langle \mathcal{A}x, \Pi^\infty y\rangle \\ & - \langle B^*\Pi^\infty x, B^*\Pi^\infty y\rangle + \langle Cx, Cy\rangle = 0 \end{aligned} \tag{5.8}$$

for all $x, y \in dom_H(\mathcal{A})$.

Under the stronger assumption (below (5.11)) the sesquilinear form σ satisfies (2.16) and $\Pi^\infty \in \mathcal{L}(V)$; i.e.,

Theorem 5.3 *Assume that the sesquilinear form σ satisfies*

$$|\sigma_1(\phi,\psi)| \le \tilde{M} \|\phi\|_{V_{\bar\theta}} \|\psi\|_V \tag{5.9}$$

for some $\tilde{M} > 0$ and $0 \le \bar{\theta} < 1$ for all $\phi, \psi \in V$, in addition to (2.1) and (2.2) and that (H2) and (H3) hold. Let $\Pi_T(t), t \le T$ be the solution to the integral Riccati equation (3.12) with $G = 0$. Then, for each $x \in V$, $\Pi_T(t)x \in C([0,T];V) \cap C^1([0,T];V^*)$ and satisfies the Riccati equation

$$\begin{aligned}\langle \tfrac{d}{dt}\Pi_T(t)x, y\rangle &= \sigma(x, \Pi_T(t)y) + \sigma(y, \Pi_T(t)x) \\ &+\langle B^*\Pi_T(t)x, B^*\Pi_T(t)y\rangle - \langle Cx, Cy\rangle \\ &\text{for all } x, y \in V.\end{aligned} \tag{5.10}$$

Moreover, if (H4) holds, then $\Pi^\infty (= s - \lim \Pi_T(0)) \in \mathcal{L}(V)$ and satisfies the algebraic Riccati equation

$$\begin{aligned}\sigma(x, \Pi^\infty y) \quad &+\sigma(y, \Pi^\infty x) + \langle B^*\Pi^\infty x, B^*\Pi^\infty y\rangle \\ &-\langle Cx, Cy\rangle = 0 \quad \text{for all } x, y \in V.\end{aligned} \tag{5.11}$$

Then we have the optimal feedback solution:

Theorem 5.4 *Assume the sesquilinear form σ satisfies (5.9) in addition to (2.1) and (2.2), and (H2) and (H4) hold. Then, if $(\mathcal{A}, B)$ is stabilizable and $(\mathcal{A}, C)$ is detectable, then the algebraic Riccati equation (5.11) has a unique non-negative solution $\Pi = \Pi^* \in \mathcal{L}(V)$ and $\mathcal{A} - BB^*\Pi$ generates a stable analytic semigroup $T(t), t \ge 0$ on H. The optimal control u^* for (3.15) is given by*

$$u^*(t) = -B^*\Pi T(t)z, \quad t \ge 0 \text{ for } z \in H.$$

Next we consider the Galerkin approximation of $\Pi^\infty \in \mathcal{L}(V)$. Let V^N, A^N, B^N, C^N and the projection P^N be defined as in Section 4. We will assume the following (stronger) approximation condition:

$$\text{there exists a constant } \tilde{c} > 0 \text{ such that} \|P^N\phi\|_V \le \tilde{c}\|\phi\|_V \text{ for all } \phi \in V. \tag{$H5$}$$

While the results below can be shown by some elaborate arguments without assuming (H5), the condition (H5) is a standard assumption in finite element approximations (e.g., see [1]). Then, if (H1) and (H5) are satisfied,

$$\begin{aligned}&P^N\phi \to \phi \text{ in } V \text{ for } \phi \in V \text{ and} \\ &P^N\phi \to \phi \text{ in } V^* \text{ for } \phi \in V^*,\end{aligned} \tag{5.12}$$

and moreover for $Re\lambda \ge \rho$

$$\|(\lambda I - A^N)^{-1}P^N\phi\|_{V^*} \le \frac{\tilde{c}M_0}{|\lambda - \rho|}\|\phi\|_{V^*}. \tag{5.13}$$

In fact, if for $\phi \in V^*$, $u^N = (\lambda I - A^N)^{-1} P^N \phi$, then

$$\lambda \langle u^N, \chi \rangle + \sigma(u^N, \chi) = \langle \phi, \chi \rangle \quad \text{for all } \chi \in V^N.$$

Thus,

$$\begin{aligned} |\lambda| \, \|\langle u^N, \chi \rangle\| &\leq \|\phi\|_{V^*} \|\chi\|_V + c\|u^N\|_V \|\chi\|_V \\ &\leq M_0 \|\phi\|_{V^*} \|\chi\|_V \end{aligned}$$

and choosing $\chi = P^N \psi$ for all $\psi \in V$, this inequality yields (5.13).

Theorem 5.5 *Suppose the assumptions in Theorem 5.4 hold and the approximation conditions (H1) and (H5) are satisfied. Then, for N sufficiently large, there exists a unique non-negative solution $\Pi^N = \Pi^{N^*} \in \mathcal{L}(V) \cap \mathcal{L}(H)$ to the algebraic Riccati equation in V^N*

$$\begin{aligned} \sigma(x, \Pi^N y) \quad &+ \sigma(y, \Pi^N x) + \langle B^* \Pi^N x, B^* \Pi^N y \rangle \\ &- \langle Cx, Cy \rangle = 0 \quad \textit{for all } x, y \in V^N, \end{aligned} \tag{5.14}$$

and there exist constants $\tilde{M} \geq 1$ and $\tilde{\omega} > 0$ such that

$$\|e^{t(\mathcal{A}^N - B^N B^{N^*} \Pi^N)} P^N \phi\|_H \leq \tilde{M} e^{-\tilde{\omega} t} \|\phi\|_H, \ t \geq 0.$$

Moreover,

$$\Pi^N P^N \phi \xrightarrow{s} \Pi^\infty \phi \quad \textit{in } H \quad \textit{for each } \phi \in H$$

$$\Pi^N P^N \phi \xrightarrow{w} \Pi^\infty \phi \quad \textit{in } V \quad \textit{for each } \phi \in V$$

and $B^* \Pi^N P^N \xrightarrow{s} B^* \Pi^\infty$ in $\mathcal{L}(V, U)$ where Π^∞ is the unique non-negative solution to (5.11). Finally, for N sufficiently large, $\mathcal{A} - BB^{N^*} \Pi^N$ generates a stable analytic semigroup on H.

6 Second Order Systems

Let $V_0 \hookrightarrow H_0 \hookrightarrow V_0^*$ be a Gelfand triple as discussed in Section 2. We consider general second order systems in the context of sesquilinear forms using the approach in Section 3 of [3]. Consider then the second order system:

$$\frac{d^2}{dt^2} w(t) + D_0 \frac{d}{dt} w(t) + A_0 w(t) = B_0 u(t) \ \text{ in } V_0^*, \tag{6.1}$$

where $B_0 \in \mathcal{L}(U, V_0^*)$ and $A_0, D_0 \in \mathcal{L}(V_0, V_0^*)$ are defined by

$$a(\phi, \psi) = < A_0 \phi, \psi >_{V_0^*, V_0}, \tag{6.2}$$

$$b(\phi, \psi) = < D_0 \phi, \psi >_{V_0^*, V_0}, \tag{6.3}$$

for $\phi, \psi \in V_0$ and a and b are continuous symmetric sesquilinear forms on $V_0 \times V_0$. We assume that a is V_0-coercive (i.e., satisfies an equality of the form (2.2) wherein without loss of generality we take $\omega = 1$ and $\rho = 0$) and b is nonnegative. Thus V_0 can be equipped with the equivalent norm $\|\phi\|_{V_0} = \sqrt{a(\phi, \phi)}$. Let $V = V_0 \times V_0$, $H = V_0 \times H_0$ and define the sesquilinear form σ on $V \times V$ by

$$\sigma((\phi_1, \phi_2), (\psi_1, \psi_2)) = -a(\phi_2, \psi_1) + a(\phi_1, \psi_2) + b(\phi_2, \psi_2). \tag{6.4}$$

We again have a solution semigroup on H for the systems (6.1) written in first order form (e.g., see [3]). The generator $\mathcal{A}$ in this case is given by

$$\mathcal{A} = \begin{bmatrix} 0 & I \\ -A_0 & -D_0 \end{bmatrix} \tag{6.5}$$

with $\mathrm{dom}(\mathcal{A}) = \{(\phi_1, \phi_2) \in H : \phi_2 \in V_0 \text{ and } A_0\phi_1 + D_0\phi_2 \in H_0\}$. The system described by (6.1)-(6.5) requires some separate analysis since some of the conditions assumed in section 3 through 5 above are not satisfied. However as will be outlined below, the results in sections 3-5 are still valid for control problems governed by (6.1) under appropriate conditions on $\mathcal{A}$ and $B = \mathrm{col}\ [0, B_0]$.

6.1 Case of bounded C

It has been shown in [3] that $\mathcal{A}$ generates a C_0-semigroup on H and that if b is V_0-coercive, then $\mathcal{A}$ generates an analytic semigroup $S(t), t \geq 0$. In fact, the latter can be shown readily by observing that the sesquilinear form σ defined by (6.4) satisfies the conditions (2.1)-(2.2). Moreover, we actually have from standard estimates (e.g., see [5]):

Lemma 6.1 *Assume that b is V_0-coercive. Then there exists a positive constant M (which depends only on T) such that*

$$\| \int_0^t S(t-s) f(s) ds \|_{C(0,T;V)} \leq M \|f\|_{L^2(0,T;H)}.$$

It follows directly from Lemma 6.1 that for any $C \in \mathcal{L}(H, Y)$, the following condition holds: there exists $c > 0$ such that

$$\| \int_0^t S^*(t-s) C^* y(s) ds \|_V \leq c \|y\|_{L_2(0,t;Y)} \tag{6.6}$$

for all $t \in [0, T]$ and $y \in L_2(0, T; Y)$. From duality (see [27]) it follows then that (3.10) holds and indeed the conditions of [27] are satisfied (see

Remark 3.2(1)). Thus if b is V_0-coercive, we have that the conclusions of Theorems 3.1 and 3.4 hold for the second order system problem governed by (6.1)-(6.5). Moreover, it is not difficult to show that the conclusions of Lemma 4.4 for the Galerkin approximation holds for this case. Thus the conclusions of Theorem 4.6 applies to this case for approximations of solutions to the differential Riccati equation analogues of (4.10).

In the case of the infinite horizon problem and associated algebraic Riccati equation, to obtain Theorem 4.8 and Lemma 4.7 we assumed (H2): V is compactly embedded into H. This condition (H2) is not true for the second order system problem formulated here. But we have a result corresponding to Lemma 4.7 for these problems.

Lemma 6.2 *Assume that V_0 is compactly embedded into H_0 and $b = \gamma a + \bar{b}$, for some $\gamma > 0$ where the continuous sesquilinear form $\bar{b}$ on $V_0 \times V_0$ satisfies for some $\bar{\rho} \in \mathcal{R}$*

$$Re\ \bar{b}(\phi,\phi) \geq -\frac{\gamma}{2}\|\phi\|_{V_0}^2 - \bar{\rho}\|\phi\|_{H_0}^2$$

for all $\phi \in V_0$, and $A_0^{-1}\bar{D}_0$ is a compact operator on V_0 with $\bar{D}_0 \in \mathcal{L}(V_0, V_0^*)$ defined by $< \bar{D}_0\phi, \psi >_{V_0^*,V_0} = \bar{b}(\phi,\psi)$. Then if for some $\nu \in \mathcal{R}$ and $M \geq 1$

$$\|S(t)\|_{\mathcal{L}(H)} \leq Me^{\nu t}, \qquad t \geq 0$$

then for any $\varepsilon > 0$ there exists an integer N_ε such that for $N \geq N_\varepsilon$

$$\|S^N(t)P^N\|_{\mathcal{L}(H)} \leq \tilde{M}e^{(\nu+\varepsilon)t}, \qquad t \geq 0$$

for some constant $\tilde{M} > 0$ independent of N, where $S^N(t) = e^{A^N t}$ with A^N defined as in (4.1).

Proof: From (6.4) we have for $z = (\phi_1, \phi_2) \in V$

$$\text{Re}\ \sigma(z,z) \geq \frac{\gamma}{2}\|\phi_2\|_{V_0}^2 - \bar{\rho}\|\phi\|_{H_0}^2.$$

Thus σ satisfies (2.1)-(2.2) with $\omega = \frac{\gamma}{2}$ and the semigroup e^{tA^N} on V^N is represented by

$$e^{tA^N}P^N = \frac{1}{2\pi i}\int_{\Gamma} e^{\lambda t}(\lambda I - A^N)^{-1}P^N d\lambda$$

where Γ is the integration path (shifted by $\bar{\rho}$) as described in [29, Theorem 6.A]. Hence it suffices to argue that $\|(\lambda I - A^N)^{-1}P^N\|$ is uniformly bounded on $\{\text{Re}\ \lambda \geq \nu + \varepsilon\}$ and in N. Consider the resolvent equation

$$\lambda z_1 - z_2 = f \in V_0 \tag{6.7}$$

$$\lambda z_2 + D_0 z_2 + A_0 z_1 = g \in H. \tag{6.8}$$

From (6.7), $z_2 = \lambda z_1 - f$. It thus follows from (6.8) that

$$(\lambda^2 + \lambda D_0 + A_0)z_1 = g + \lambda f + D_0 f \quad \text{in } V_0^*,$$

where by the assumption $D_0 = \gamma A_0 + \bar{D}_0$. Equivalently,

$$\begin{aligned}(I + \frac{\lambda^2}{\lambda\gamma+1}A_0^{-1} + \frac{\lambda}{\lambda\gamma+1}A_0^{-1}\bar{D}_0)z_1 \\ = \frac{\gamma f}{\lambda\gamma+1} + \frac{A_0^{-1}}{\lambda\gamma+1}(g + \lambda f + \bar{D}_0 f).\end{aligned}$$

Thus, if $\mathrm{Re}\lambda > \nu$, then $(I + \frac{\lambda^2}{\lambda\gamma+1}A_0^{-1} + \frac{\lambda}{\lambda\gamma+1}A_0^{-1}\bar{D}_0)^{-1}$ exists and is bounded. Similarly we have

$$(\lambda I - A^N)(z_1^N, z_2^N) = (P_{V_0}^N f, P_{H_0}^N g)$$

is equivalent to

$$\begin{aligned}(I + \frac{\lambda^2}{\lambda\gamma+1}(A_0^N)^{-1} + \frac{\lambda}{\lambda\gamma+1}(A_0^N)^{-1}\bar{D}_0^N)z_1^N \\ = \frac{\gamma f^N}{\lambda\gamma+1} + \frac{(A_0^N)^{-1}}{\lambda\gamma+1}(g^N + \lambda f^N + \bar{D}_0^N f^N)\end{aligned} \tag{6.9}$$

and $z_2^N = \lambda z_1^N - f^N$ where $f^N = P_{V_0}^N f, g^N = P_{H_0}^N g$ and $P_{V_0}^N$ and $P_{H_0}^N$ are the orthogonal projections onto V^N of V_0 and H_0, respectively. Here for $z^N \in V^N$

$$\begin{aligned}(I + \frac{\lambda^2}{\lambda\gamma+1}(A_0^N)^{-1} + \frac{\lambda(A_0^N)^{-1}}{\lambda\gamma+1}\bar{D}_0^N)z_1^N \\ = (I + \frac{\lambda^2}{\lambda\gamma+1}A_0^{-1} + \frac{\lambda A_0^{-1}}{\lambda\gamma+1}\bar{D}_0)z_1^N \\ + \frac{\lambda^2}{\lambda\gamma+1}(A_0^{-1} - (A_0^N)^{-1})P_{H_0}^N z_1^N \\ + \frac{\lambda}{\lambda\gamma+1}(A_0^{-1}\bar{D}_0 - (A_0^N)^{-1}\bar{D}_0)z_1^N.\end{aligned}$$

Since $(A_0^N)^{-1}\bar{D}_0^N = P_{V_0}^N A_0^{-1}\bar{D}_0$ and V_0 is compactly embedded into H_0

$$\|(A_0^N)^{-1}P_{H_0}^N - A_0^{-1}\|_{\mathcal{L}(H_0)} \text{ and } \|(A_0^N)^{-1}\bar{D}_0^N - A_0^{-1}\bar{D}_0\|_{\mathcal{L}(H_0)} \to 0$$

as $N \to \infty$. Hence for N sufficiently large

$$(I + \frac{\lambda^2}{\lambda\gamma+1}(A_0^N)^{-1} + \frac{\lambda}{\lambda\gamma+1}(A_0^N)^{-1}\bar{D}_0^N)^{-1}P_{V_0}^N$$

exists and is uniformly bounded in any compact set in $\{\text{Re } \lambda > \nu\}$. It thus follows from (6.9) that there exists an integer N_ε such that for $N \geq N_\varepsilon, (\lambda I - A^N)^{-1} P^N$ is uniformly bounded on $\{\text{Re } \lambda \geq \nu + \varepsilon\}$, which completes the proof.

It thus follows from Lemma 6.2 that under the assumptions in Lemma 6.2, the assumptions (H1) and stabilizability and detectability of Theorem 4.8, the conclusions of Theorem 4.8 hold for the case when $\mathcal{A}$ is defined by (6.5), $B = \text{col }[0, B_0]$, and $C \in \mathcal{L}(H, Y)$.

6.2 Cases of unbounded C

For the case when $C \in \mathcal{L}(V, Y)$ we assume that there exists $0 \leq \theta < 1$ such that

$$\|B_0^* \phi\|_U \leq b \|\phi\|_{V_0^\theta} \text{ for } \phi \in V, \tag{6.9}$$

where V_0^θ is the intermediate space $= [V_0, H_0]_\theta, 0 \leq \theta \leq 1$ defined as in Section 5. Since $V^\theta = [V, H]_\theta = V_0 \times V_0^\theta, 0 \leq \theta \leq 1$ the assumption (H3) is satisfied. Assuming the finite cost condition (H4), it follows from Theorem 5.2 that there exists an operator $\Pi \in \mathcal{L}(H) \cap \mathcal{L}(V, V^\beta)$ (for all $0 \leq \beta < 1$) such that $\Pi_T(0)\phi \to \Pi\phi$ weakly in V^β for all $\phi \in V$. In the case of the second order system (6.1)-(6.3), the assumption (H2) is not valid. However, $P_2 V = V_0$ is compactly embedded into $P_2 H = H_0$ and $B^* P_2 = B^*$ where P_2 is the projection of H onto the subspace $\{0\} \times H_0$. This fact can be used to show that $B^* \Pi_T(0)\phi \to B^* \Pi \phi$ strongly in U for each $\phi \in V$ and that Π satisfies the algebraic Riccati equation

$$\mathcal{A}^* \Pi + \Pi \mathcal{A} - \Pi B B^* \Pi + C^* C = 0 \tag{6.10}$$

in the sense of Theorem 5.2 (i.e., see (5.8)).

The condition (5.9) is not satisfied for the second order system defined by (6.4)-(6.5) since

$$\sigma_1(\phi, \psi) = -a(\phi_2, \psi_1) + a(\phi_1, \psi_2)$$

for $\phi = (\phi_1, \phi_2),\ \psi = (\psi_1, \psi_2) \in V$. Hence, theorems 5.3-5.5 cannot be applied directly to the second order system (6.1)-(6.3). In what follows we employ the structure of solutions to the Riccati equation to establish the results that correspond to theorems 5.3-5.5 for the second order system. Since the adjoint operator $\mathcal{A}^*$ of $\mathcal{A}$ is given by

$$\mathcal{A}^* = \begin{bmatrix} 0 & -I \\ A_0 & -D_0^* \end{bmatrix},$$

the Riccati equation (6.10) can be written in terms of the operator matrix $\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{bmatrix}$ on $V_0 \times H_0$ as

$$\begin{aligned} -\Pi_{21} - \Pi_{12}A_0 - \Pi_{12}B_0B_0^*\Pi_{21} + Q_{11} = 0 \\ A_0\Pi_{11} - D_0^*\Pi_{21} - \Pi_{22}A_0 - \Pi_{22}B_0B_0^*\Pi_{21} + Q_{21} = 0 \\ A_0\Pi_{12} + \Pi_{21} - D_0^*\Pi_{22} - \Pi_{22}D_0 - \Pi_{22}B_0B_0^*\Pi_{22} + Q_{22} = 0 \end{aligned} \tag{6.11}$$

where $Q = C^*C = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}$ on $V_0 \times H_0$. Since Π is self-adjoint on $H = V_0 \times H_0$ and H is equipped with the norm $\sqrt{\langle A_0\phi_1, \phi_1\rangle + |\phi_2|^2_{H_0}}$, $\phi = (\phi_1, \phi_2) \in H$, it follows that $\Pi_{21} \in \mathcal{L}(V_0, H_0)$, $\Pi_{21}^* = A_0\Pi_{12} \in \mathcal{L}(H_0, V_0)$ and $\Pi_{11}^* = A_0\Pi_{11} \in \mathcal{L}(V_0, V_0^*)$. Note that the third equation of (6.11) is the Lyapunov equation for the self adjoint operator Π_{22} on H_0, given $\Pi_{21} \in \mathcal{L}(V_0, H_0)$; and that $Q_{22} + A_0\Pi_{12} + \Pi_{21} - \Pi_{22}BB^*\Pi_{22} \in \mathcal{L}(V_0, V_0^*)$ is bounded and symmetric.

As in Lemma 6.2, we have assumed that V_0 is compactly embedded into H_0 and that the sesquilinear form b in (6.3) is given by $b = \gamma a + \bar{b}$ (equivalently, $D_0 = \gamma A_0 + \bar{D}_0$) where $\bar{b}$ now satisfies

$$|\bar{b}(\phi, \psi)| \le \bar{M}\|\phi\|_{V_0^{\bar{\theta}}}\|\psi\|_{V_0}, \qquad 0 \le \bar{\theta} < 1 \tag{6.12}$$

for all $\phi, \psi \in V_0$. It then follows from Theorem 5.3 that $\Pi_{22} \in \mathcal{L}(V_0, V_0)$. Moreover, premultiplying the second equation of (6.11) by $A_0^{-\frac{1}{2}}$ (recall that A_0 is self-adjoint) we obtain

$$\begin{aligned} (-\gamma I - A_0^{-\frac{1}{2}}(\bar{D}_0^* - \Pi_{22}B_0B_0^*)A_0^{-\frac{1}{2}})A_0^{\frac{1}{2}}\Pi_{21} \\ = A_0^{\frac{1}{2}}\Pi_{11} + A_0^{-\frac{1}{2}}Q_{21} \end{aligned}$$

where $A_0^{-\frac{1}{2}}(\bar{D}_0^* - \Pi_{22}B_0B_0^*)A_0^{-\frac{1}{2}} \in \mathcal{L}(H_0)$ is compact and

$$A_0^{\frac{1}{2}}\Pi_{11} + A_0^{-\frac{1}{2}}Q_{21} \in \mathcal{L}(V_0, H_0).$$

Thus $A_0^{\frac{1}{2}}\Pi_{21} \in \mathcal{L}(V_0, H_0)$ and therefore $\Pi \in \mathcal{L}(H) \cap \mathcal{L}(V)$. Furthermore, a similar argument using the partitioning of Riccati solutions can be applied to establish the corresponding result to Theorem 5.5.

The case of the structural damping (2.31) can be treated directly using the results in sections 3-5.

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