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Internal Approximation Schemes for Optimal Control Problems in Hilbert Spaces*

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Abstract

We consider semi–discrete approximations of optimal control problems for linear distributed parameter dynamical systems, with cost functionals in Bolza or infinite horizon form. We give conditions for the convergence of approximate value functions and prove that the approximate optimal controls are a minimizing sequence for the continuous problem. We also show some concrete applications.

Key words: distributed parameter systems, optimal control, dynamic programming, approximation schemes

AMS Subject Classifications: 49L10, 65J10

1 Introduction

The theory of optimal control of distributed parameter dynamical systems has been mainly developed for linear evolution equations of the form:

$$\begin{cases} y'(t) = Ay(t) + Bu(t) \\ y(0) = x \end{cases} \tag{1.1}$$

posed in some Hilbert space H, and mostly for quadratic costs. The numerical approximation has been studied in the framework of classical optimal control (i.e., by means of Pontryagin's Maximum Principle). Several authors have proved the convergence of optimal solutions obtained by internal approximations of the dynamical system. The case of a Galerkin spectral

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scheme is treated in [15], whereas more general results may be found in [11], [14] (see also [3] for parabolic evolution equations and [18] for boundary control problems).

The basic technique considered in these works to discretize a control problem in some Hilbert space H requires the approximation of equation (1.1) by means of a sequence of ODE systems in suitable subspaces $H_n \subset H$ of increasing dimension. We write here this (semi–discrete) approximation as:

$$\begin{cases} y_n'(t) = A_n y_n(t) + B_n u_n(t) \\ y_n(0) = x_n \end{cases} . \qquad (1.1_n)$$

The typical endpoint of this analysis is the convergence of solutions obtained by approximate Riccati feedback operators. The next step for numerical computation is to solve the new control problem posed in R^n by a numerical scheme for finite–dimensional problems.

On the other hand, in recent years there has been a great development in the theory of Hamilton–Jacobi (HJ) equations in Hilbert spaces (see [BD], [5]). This is partly motivated by the use of more general cost functionals. The Galerkin technique for reducing the problem to finite dimension is used in a Dynamic Programming framework in [2] (chapter 3) for the case of convex hamiltonians and bounded linear terms, and in [5] (part IV), for general form HJ equations with unbounded linear terms.

The aim of this paper is to extend the existing approximation results to problems with non–quadratic cost. Working with Dynamic Programming techniques, we will give a general convergence result (in terms of approximate value functions) for this discretization procedure and prove explicit error estimates, provided the dynamical system satisfies some proper approximability assumptions. We will also show that the convergence of approximate value functions ensures the convergence of approximate optimal solutions obtained by means of open–loop techniques. This result may be seen as a sort of abstract "convergence" for the semi–discretized control problems.

It is worth mentioning that other approximation results have been obtained in [1], [20], [17] via a completely different approach, based on time discretization, but they seem not to be of direct use for computational purposes.

The outline of the paper is the following. Section 2 sets the basic assumptions about the dynamical system and recalls the known results about semi–discrete approximations. Sections 3 and 4 give the main theorems of existence, uniqueness and convergence of discrete value functions, for respectively the infinite horizon and the finite horizon problem, whereas section 5 treats the convergence of approximate optimal solutions. Lastly, in section 6 we discuss the problems related to this approach, and give some concrete application.

2 The Evolution Equation and Its Approximation

Let us consider the dynamical system described by (1.1), where for any t, $y(t)$, x belong to a separable real Hilbert space H, A is a linear operator mapping $D(A) \subset H$ into H, the control $u(t)$ is a real function from $[0, +\infty[$ into a closed bounded subset U of a (possibly finite–dimensional) separable real Hilbert space V, and B is a linear bounded operator mapping $U \subset V$ into H.

For the sake of clarity, in the sequel the solution of (1.1) will be usually denoted by $y(x,t,u)$; $< \cdot, \cdot >$ and $\| \cdot \|$ will denote respectively the scalar product and the related norm in H, whereas $\mathcal{L}(V,H)$ will denote the space of linear bounded operators mapping V into H, and $\mathcal{L}(H) = \mathcal{L}(H,H)$.

We assume that:

$$u \in \mathcal{U} = L^1([0,+\infty[;U). \tag{h1a}$$

$$A \text{ generates a strongly continuous semigroup } T(t) = e^{At}. \tag{h1b}$$

$$\text{There exist two real constants } K',\ \lambda' \text{ such that } \|e^{At}\|_{\mathcal{L}(H)} \le K' e^{\lambda' t} \tag{h1c}$$

where $\mathcal{U}$ denotes the set of admissible controls. Under these assumptions, it is well known (see [19]) that (1.1) admits a unique mild solution $y(t) \in C([0,+\infty[;H)$, given by:

$$y(t) = e^{At}x + \int_0^t e^{A(t-s)} Bu(s) ds. \tag{2.1}$$

We will consider now a finite–dimensional approximation of (1.1) by means of the classical method of semi–discretization (see [22], [12]), following the convergence theory due to P. Lax. Let H_n, V_n be two sequences of vector spaces, and P_n, Π_n be two sequences of projections, and assume:

$$H_n \subset H \quad ; \quad \dim H_n = k_n \tag{h2a}$$

$$V_n \subseteq V \quad ; \quad \dim V_n = h_n \tag{h2b}$$

$P_n : H \to H_n$ and $\Pi_n : V \to V_n$ are linear mappings such that:

$$\lim_{n\to\infty} \|x - P_n x\| = 0 \text{ for any } x \in H \tag{h2c}$$

$$\lim_{n\to\infty} \|u - \Pi_n u\|_V = 0 \text{ for any } u \in V \tag{h2d}$$

$$x_n = P_n x_n \text{ for any } x_n \in H_n,\ u_n = \Pi_n u_n \text{ for any } u_n \in V_n \tag{h2e}$$

$$\Pi_n U \subseteq U \tag{h2f}$$

with $k_n, h_n \to \infty$. Assumption $(h2f)$ is a somewhat less trivial requirement, but it is satisfied, for instance, if U is a closed ball and Π_n is the orthogonal projection into V_n. If V has finite dimension, we will set $V_n \equiv V$, $h_n \equiv \dim V$ and $\Pi_n \equiv I$ (identity operator).

On the other hand, we assume there exist two sequences of approximating operators:

$$A_n : H_n \to H_n$$

$$B_n : V_n \to H_n$$

and consider the (semi–discretized) approximation of (1.1) given by (1.1_n), where for any t, $y_n(t) \in H_n$, $x_n \in H_n$, $u_n \in \mathcal{U}_n = L^1([0, +\infty[; U_n)$, and:

$$u_n(t) = \Pi_n u(t)$$

$$x_n = P_n x.$$

Here and in the sequel $U_n := U \cap V_n$ is the discretization of U obtained imposing the constraints of U on the elements of V_n.

Approximation (1.1_n) is assumed to be consistent. In other words, we assume, in addition to $(h2)$, that there exist $\hat{\mathcal{U}}$ dense in $L^1([0, +\infty[; V)$, $\hat{H}$ dense in H such that, if $(x, u) \in \hat{H} \times \hat{\mathcal{U}}$, then:

$$\lim_{n\to\infty} \|Ay(x,t,u) - A_n P_n y(x,t,u)\| = 0 \qquad (h3a)$$

$$\lim_{n\to\infty} \|Bu(t) - B_n \Pi_n u(t)\| = 0. \qquad (h3b)$$

It is easy to check that $(h3b)$ is a consequence of $(h2c, d)$ if $B_n := P_n B$; therefore, we will usually refer to this definition of B_n in the sequel.

Before recalling the main result about consistent semi–discrete approximations, we give some definitions.

Approximation scheme (1.1_n) is said to be *stable* if there exists a locally bounded function $K(t)$ independent of n such that:

$$\|e^{A_n P_n t}\|_{\mathcal{L}(H)} \le K(t).$$

Approximation scheme (1.1_n) is said to be *convergent* if, for any $u \in L^1([0, +\infty[; V)$, $x \in H$ and $t > 0$:

$$\lim_{n\to\infty} \|y(x,t,u) - y_n(P_n x, t, \Pi_n u)\| = 0. \qquad (2.2)$$

We will need in the sequel to assume a sort of "uniform convergence" of (1.1_n). We will possibly require that

$$\lim_{n\to\infty} \sup_{u_n \in \mathcal{U}_n} \|y(x,t,u_n) - y_n(P_n x, t, u_n)\| = 0. \qquad (2.3)$$

The following classical theorem gives a characterization of convergent schemes (see [12], section 4, and the references therein):

Equivalence Theorem *Assume* $(h1)$–$(h3)$*. Then approximation* (1.1_n) *is convergent if and only if it is stable.*

We are also concerned in sufficient conditions for (2.3) to be satisfied. A partial answer is given by the following proposition.

Proposition 2.1 *Assume* $(h1)$–$(h3)$*. If* (1.1_n) *is convergent and if one of the following conditions is satisfied:*

i) There exists a subspace $S \subset H$ *such that, if* $u \in U$*, then* $Bu \in B_S(R) := \{x \in S \ : \ \|x\|_S \leq R\}$*, and*

$$\xi_n(t) = \sup_{x \in B_S(R)} \left\| e^{At}x - e^{A_n t} P_n x \right\| \to 0; \tag{2.4}$$

ii) V *has finite dimension;*
then (2.3) *is satisfied.*

Proof: We can express $y(x,t,u_n)$, $y_n(P_n x, t, u_n)$ as:

$$y(x,t,u_n) = e^{At}x + \int_0^t e^{A(t-s)} B u_n(s) ds;$$

$$y_n(P_n x, t, u_n) = e^{A_n t} P_n x + \int_0^t e^{A_n(t-s)} P_n B u_n(s) ds.$$

Now, taking the difference, with simple calculations we obtain:

$$\|y(t) - y_n(t)\| \leq$$

$$\leq \|e^{At}x - e^{A_n t} P_n x\| + \int_0^t \left[\left\| e^{A(t-s)} B u_n(s) - e^{A_n(t-s)} P_n B u_n(s) \right\| \right] ds \leq$$

$$\leq \|e^{At}x - e^{A_n t} P_n x\| + \int_0^t \xi_n(t-s) ds.$$

Taking into account the convergence of (1.1_n) and the trivial bound $\xi_n(t) \leq K' e^{\lambda' t} + K(t)$, applying the dominated convergence theorem we get (2.3).

If $V = R^M$, then the terms $Bu(t)$, $B_n u(t)$ read:

$$Bu(t) = \sum_i b_i u_i(t)$$

$$B_n u(t) = \sum_i (P_n b_i) u_i(t)$$

where u_i is the i–th component of the vector u, and the $b_i \in H$ are given. Then, setting $M_u = \sup_u \|u\|_V$, we have:

$$\|y(t) - y_n(t)\| \le$$

$$\le \|e^{At}x - e^{A_n t}P_n x\| +$$

$$+ \int_0^t \left[\left\| e^{A(t-s)} \sum_i b_i u_i(s) - e^{A_n(t-s)} \sum_i (P_n b_i) u_i(s) \right\| \right] ds \le$$

$$\le \|e^{At}x - e^{A_n t}P_n x\| + M_u \int_0^t \sum_i \left\| e^{A(t-s)} b_i - e^{A_n(t-s)}(P_n b_i) \right\| ds.$$

Since all the norms being integrated are bounded and converge to zero, we obtain again (2.3).

■

Remark Usual choices for the spaces H_n, V_n include spaces of finite elements, orthogonal polynomials, sinusoidal functions, with the operators A_n and B_n constructed by variational or collocation techniques. We will give in section 6 some concrete examples of semi–discrete approximations, along with explicit error estimates.

3 The Infinite Horizon Problem

We state our first optimal control problem: Given the evolution equation (1.1), find a control $u(t) \in \mathcal{U}$ minimizing the discounted infinite horizon cost:

$$J_i(x,u) := \int_0^\infty e^{-\lambda t} g(y(x,t,u), u(t)) dt \tag{3.1}$$

assuming $(h1)$, and:

$$\lambda > 0 \tag{h4a}$$

$$g : H \times U \to R$$

$$|g(y,u)| \le M_g \text{ for any } y \in H,\ u \in U \tag{h4b}$$

$$|g(y_1,u_1) - g(y_2,u_2)| \le L_g(\|y_1 - y_2\| + \|u_1 - u_2\|_V)$$

$$\text{for any } y_1, y_2 \in H,\ u_1, u_2 \in U. \tag{h4c}$$

The approximate version of this problem is, given the evolution equation (1.1_n), to find a control $u_n(t) \in \mathcal{U}_n$ minimizing the cost:

$$J_{i,n}(x_n,u_n) := \int_0^\infty e^{-\lambda t} g(y_n(x_n,t,u_n), u_n(t)) dt \tag{3.2}$$

where $x_n := P_n x$.

To carry out a detailed analysis of the infinite horizon case, we start by defining in the usual way the value functions for both the original and the approximate problem:

$$v(x) := \inf_{u \in \mathcal{U}} J_i(x, u)$$

$$v_n(x_n) := \inf_{u_n \in \mathcal{U}_n} J_{i,n}(x_n, u_n).$$

If optimal controls exist, they will be denoted by

$$u^* \in \text{argmin } J_i(x, u)$$

$$u_n^* \in \text{argmin } J_{i,n}(P_n x, u_n)$$

dropping the dependence on the initial state x.

Let now $\{\psi_i\}_{1 \le i \le k_n}$ be a base of the space H_n, $\{\theta_i\}_{1 \le i \le h_n}$ be a base of the space V_n. With respect to these bases, we may write x_n, u_n as:

$$x_n = \sum_i x^i \psi_i \ ; \quad u_n = \sum_j u^j \theta_j$$

and introduce the notations (where $(\cdot)^t$ denotes the transpose of a vector):

$$X = (x^1 \dots x^n)^t \ ; \quad \Psi = (\psi_1 \dots \psi_n)^t$$

$$U = (u^1 \dots u^m)^t$$

(the use of U to denote the vector above should not cause confusions in this context). With a slightly improper notation, we will identify H_n with R^{k_n}, V_n with R^{h_n} and write:

$$v_n(x_n) = v_n(\sum_i x^i \psi_i) = v_n(X)$$

$$g(x_n, u_n) = g(\sum_i x^i \psi_i, \sum_j u^j \theta_j) = g(X, U).$$

Finally, the operators A_n and B_n will be represented by a $k_n \times k_n$ and a $k_n \times h_n$ matrix, still denoted A_n and B_n (both depending, in general, on the kind of discretization chosen for (1.1_n)); moreover, we will use the same notation for both $U_n \subset V_n$ and the corresponding set of R^{h_n}.

We first prove the following result of existence and uniqueness:

Proposition 3.1 *The approximate value function $v_n(x_n)$ defined above is the unique solution of the Hamilton–Jacobi equation:*

$$\lambda v_n(X) + \sup_{U \in U_n} [-(A_n X + B_n U)^t \nabla v_n(X) - g(X, U)] = 0 \tag{3.3}$$

where $X \in R^{k_n}$, $U \in R^{h_n}$.

Proof: We note that A_n may be defined on the whole of H by:

$$A_n x := A_n P_n x$$

and that this is a bounded operator for any n. Therefore, for all $x \in H$, the value function of the approximate problem coincides (see [5], [7]) with the unique viscosity solution in H of the HJ equation:

$$\lambda v_n(x) + \sup_{u_n \in U_n} [- < A_n x + B_n u_n, \nabla v_n(x) > - g(x, u_n)] = 0. \tag{3.4}$$

Now, if we restrict ourselves to the subspace H_n and refer to the notations introduced above, we may rewrite (3.4) as:

$$\lambda v_n(X) + \sup_{U \in U_n} [- < (A_n X + B_n U)^t \Psi, \nabla v_n(X) > - g(X, U)] = 0$$

and to give a more explicit form:

$$< (A_n X + B_n U)^t \Psi, \nabla v_n >= (A_n X + B_n U)^t \begin{pmatrix} < \psi_1, \nabla v_n > \\ \vdots \\ < \psi_n, \nabla v_n > \end{pmatrix} =$$

$$= (A_n X + B_n U)^t \begin{pmatrix} < \psi_1, \psi_1 > & \dots & < \psi_1, \psi_n > \\ \vdots & & \vdots \\ < \psi_n, \psi_1 > & \dots & < \psi_n, \psi_n > \end{pmatrix} D =$$

$$= (A_n X + B_n U)^t M D \tag{3.5}$$

where we have set:

$$M = (m_{ij}) = (< \psi_i, \psi_j >)$$

$$\nabla v_n(X) = \sum_i d^i \psi_i \ ; \quad D = (d^1 \dots d^n)^t.$$

Using (3.5), the final form of (3.4) is therefore:

$$\lambda v_n(X) + \sup_{U \in U_n} [-(A_n X + B_n U)^t M D - g(X, U)] = 0, \tag{3.6}$$

which coincides with (3.3) since MD is the expression of ∇v_n in the canonical base of the dual space H_n^*.

∎

Remark If we introduce the additional notations:

$$y_n = \sum_i y^i \psi_i \ ; \quad Y = (y^1 \dots y^n)^t,$$

we can reformulate (1.1_n) as the following ODE in R^{k_n}:

$$\begin{cases} Y'(t) = A_n Y(t) + B_n U(t) \\ Y(0) = X \end{cases} \tag{3.7}$$

and the value function may be written (with obvious notation) as:

$$v_n(X) = \inf_{U \in \mathcal{U}_n} J_{i,n}(X, U) = \inf_{U \in \mathcal{U}_n} \int_0^\infty e^{-\lambda t} g(Y(X,t,U), U(t)) dt. \tag{3.8}$$

Then, another way of proving Proposition 3.1 is to observe that under the previous assumptions the value function given by (3.8) is the unique viscosity solution (see [16], section 8.4) of equation (3.3).

Remark In principle, the function $g(x_n, u_n)$ as defined above needs not to be approximated. However, all the results below may be easily extended to the case of a sequence of approximating functions $g_n \to g$.

The following theorem gives conditions for the convergence of $v_n(P_n x)$ to $v(x)$.

Theorem 3.2 *Assume* $(h1)$–$(h4)$ *and* (2.3). *Then, for any* $x \in H$, $|v_n(P_n x) - v(x)| \to 0$ *as* $n \to \infty$.

Proof: By the definition of $v(x)$ and $v_n(x_n)$, for any $x \in H$ and $\varepsilon > 0$, it is possible to find two ε–optimal controls $u^\varepsilon \in \mathcal{U}$, $u_n^\varepsilon \in \mathcal{U}_n$ such that:

$$v(x) \le J_i(x, u^\varepsilon) \le v(x) + \varepsilon \tag{3.9}$$

$$v_n(P_n x) \le J_{i,n}(P_n x, u_n^\varepsilon) \le v_n(P_n x) + \varepsilon. \tag{3.10}$$

Since u_n^ε is an admissible control for (1.1), one has:

$$v(x) \le \int_0^\infty e^{-\lambda t} g(y(x,t,u_n^\varepsilon), u_n^\varepsilon(t)) dt.$$

Adding the terms $\pm J_{i,n}(P_n x, u_n^\varepsilon)$, we obtain:

$$v(x) \le \int_0^\infty e^{-\lambda t} [g(y(x,t,u_n^\varepsilon), u_n^\varepsilon(t)) - g(y_n(P_n x, t, u_n^\varepsilon), u_n^\varepsilon(t))] dt +$$

$$+J_{i,n}(P_n x, u_n^\varepsilon) \leq$$

$$\leq \int_0^\infty e^{-\lambda t} \min[L_g \|y(x,t,u_n^\varepsilon) - y_n(P_n x, t, u_n^\varepsilon)\|, 2M_g] dt + v_n(P_n x) + \varepsilon \leq$$

$$\leq \int_0^\infty e^{-\lambda t} \min[L_g \sup_{u_n \in \mathcal{U}_n} \|y(x,t,u_n) - y_n(P_n x, t, u_n)\|, 2M_g] dt + v_n(P_n x) + \varepsilon \tag{3.11}$$

where we have used $(h2e)$, $(h4b)$, (2.3) and (3.10). On the other hand, since $\Pi_n u^\varepsilon$ is admissible for (1.1_n), one has:

$$v_n(P_n x) \leq \int_0^\infty e^{-\lambda t} g(y_n(P_n x, t, \Pi_n u^\varepsilon), \Pi_n u^\varepsilon(t)) dt.$$

Adding the terms $\pm J_i(x, u^\varepsilon)$ and operating the same way as before:

$$v_n(P_n x) \leq \int_0^\infty e^{-\lambda t} \min[L_g(\|y_n(P_n x, t, \Pi_n u^\varepsilon) - y(x,t,u^\varepsilon)\| +$$

$$+\|\Pi_n u^\varepsilon(t) - u^\varepsilon(t)\|), 2M_g] dt + v(x) + \varepsilon. \tag{3.12}$$

From (3.11), (3.12) we get:

$$|v_n(P_n x) - v(x)| \leq \int_0^\infty e^{-\lambda t} \min[L_g(\sup_{u_n \in \mathcal{U}_n} \|y(x,t,u_n) - y_n(P_n x, t, u_n)\| +$$

$$+\|y_n(P_n x, t, \Pi_n u^\varepsilon) - y(x,t,u^\varepsilon)\| + \|\Pi_n u^\varepsilon(t) - u^\varepsilon(t)\|), 2M_g] dt + \varepsilon. \tag{3.13}$$

Since the term being integrated is bounded by $2e^{-\lambda t} M_g$ and convergent to zero for $n \to \infty$, we can use the dominated convergence theorem to show that for n large enough:

$$|v_n(P_n x) - v(x)| \leq 2\varepsilon$$

which completes the proof.

■

We turn now to the problem of giving explicit estimates for $|v_n - v|$. To this end, we prove the following

Lemma 3.3 *Let C_1, λ_1, C_2, λ_2 be real constants such that $\lambda_2 > 0$, $C_2 > C_1 > 0$. Then:*

$$\int_0^\infty \min[C_1 e^{\lambda_1 t}, C_2 e^{-\lambda_2 t}] dt \leq \begin{cases} -\frac{C_1}{\lambda_1} & \text{if } \lambda_1 < 0 \\ \frac{C_1}{\lambda_2}\left(1 + \ln \frac{C_2}{C_1}\right) & \text{if } \lambda_1 = 0 \\ \frac{C_1(\lambda_1+\lambda_2)}{\lambda_1 \lambda_2} \left(\frac{C_2}{C_1}\right)^{\frac{\lambda_1}{\lambda_1+\lambda_2}} & \text{if } \lambda_1 > 0. \end{cases}$$

Proof: The first inequality is trivial since:

$$\min[C_1 e^{\lambda_1 t}, C_2 e^{-\lambda_2 t}] \leq C_1 e^{\lambda_1 t}.$$

To prove the other two inequalities, let us observe that for $\lambda_1 \geq 0$:

$$\int_0^\infty \min[C_1 e^{\lambda_1 t}, C_2 e^{-\lambda_2 t}]dt = C_1 \int_0^{t_0} e^{\lambda_1 t}dt + C_2 \int_{t_0}^\infty e^{-\lambda_2 t}dt \qquad (3.14)$$

where:

$$t_0 = \frac{1}{\lambda_1 + \lambda_2} \ln \frac{C_2}{C_1} \qquad (3.15)$$

is the time such that $C_1 e^{\lambda_1 t_0} = C_2 e^{-\lambda_2 t_0}$ (we note that the assumption $C_2 > C_1$ ensures that $t_0 > 0$). Therefore, when $\lambda_1 = 0$, (3.14) reads:

$$\int_0^\infty \min[C_1, C_2 e^{-\lambda_2 t}]dt = C_1 t_0 + \frac{C_2}{\lambda_2} e^{-\lambda_2 t_0},$$

which proves the second inequality once (3.15) is used for t_0. When $\lambda_1 > 0$ we have:

$$\int_0^\infty \min[C_1 e^{\lambda_1 t}, C_2 e^{-\lambda_2 t}]dt = \frac{C_1}{\lambda_1}\left(e^{\lambda_1 t_0} - 1\right) + \frac{C_2}{\lambda_2} e^{-\lambda_2 t_0} \leq$$

$$\leq \frac{C_1}{\lambda_1} e^{\lambda_1 t_0} + \frac{C_2}{\lambda_2} e^{-\lambda_2 t_0} = \frac{C_1(\lambda_1 + \lambda_2)}{\lambda_1 \lambda_2} e^{\lambda_1 t_0}$$

which proves the third inequality using (3.15).

■

The following theorem gives the main result about the rate of convergence of v_n to v. Here, E will denote a proper bounded subset of H (we will discuss the choice of this set in the examples of section 6).

Theorem 3.4 *Assume (h1)–(h4). Assume moreover that there exist optimal controls u^*, u_n^*. If for any $x \in E$ there exists a sequence $\xi_n \in R_+$, and a constant λ'' such that:*

$$\|y(x,t,u^*) - y_n(P_n x, t, \Pi_n u^*)\| + \|u^*(t) - \Pi_n u^*(t)\|_V \leq \xi_n e^{\lambda'' t} \qquad (3.16a)$$

$$\|y(x,t,u_n^*) - y_n(P_n x, t, u_n^*)\| \leq \xi_n e^{\lambda'' t} \qquad (3.16b)$$

with $\xi_n \to 0$ as $n \to \infty$, then the following estimate holds:

$$|v(x) - v_n(P_n x)| \leq \omega(\xi_n) \qquad (3.17)$$

where $\omega(\xi_n)$ *is defined by:*

$$\omega(\xi_n) = \begin{cases} C_1\xi_n & \text{if} \quad \lambda'' < \lambda \\ C_2\xi_n \ln\frac{1}{\xi_n} & \text{if} \quad \lambda'' = \lambda \\ C_3\xi_n^{\frac{\lambda}{\lambda''}} & \text{if} \quad \lambda'' > \lambda \end{cases}$$

with positive constants C_i $(i = 1, 2, 3)$ *independent of* n*, and for any* $x \in E$.

Proof: By the same technique of theorem 3.2, setting $\varepsilon = 0$ we have:

$$v(x) - v_n(P_nx) \le \int_0^\infty e^{-\lambda t} \min[L_g\|y(x,t,u_n^*) - y_n(P_nx,t,u_n^*)\|, 2M_g]dt, \tag{3.18}$$

$$v_n(P_nx) - v(x) \le$$

$$\le \int_0^\infty e^{-\lambda t} \min[L_g(\|y_n(P_nx,t,\Pi_nu^*) - y(x,t,\Pi_nu^*)\| +$$

$$+ \|u^*(t) - \Pi_nu^*(t)\|_V), 2M_g]dt. \tag{3.19}$$

Using (3.16), we get:

$$|v_n(P_nx) - v(x)| \le \int_0^\infty \min[L_g\xi_n e^{(\lambda''-\lambda)t}, 2M_g e^{-\lambda t}]dt$$

and hence we obtain (3.17) applying lemma 3.3.

■

Remark As we will see in the examples of section 6, estimates in the form (3.16) require further regularity conditions on the optimal controls, as well as on the initial state.

4 The Finite Horizon Problem

Before turning to our second optimal control problem, let τ, T be real numbers such that $0 < \tau < T$, and let initial conditions in (1.1), (1.1_n) be replaced by conditions in $t = \tau$:

$$\begin{cases} y'(t) = Ay(t) + Bu(t) \\ y(\tau) = x \end{cases} \tag{4.1}$$

$$\begin{cases} y_n'(t) = A_ny_n(t) + B_nu_n(t) \\ y_n(\tau) = x_n \end{cases} . \tag{4.1_n}$$

With these definitions, we can formulate in the usual way the finite horizon problem: given the evolution equation (4.1), find a control $u(t) \in \mathcal{U}$ minimizing the finite horizon cost:

$$J_f(\tau, x, u) := \int_\tau^T g(y(x,t,u), u(t))dt + \Phi(y(x,T,u)) \tag{4.2}$$

assuming $(h1)$, and:

$$\begin{gathered} g : H \times U \to R \\ |g(y_1,u_1) - g(y_2,u_2)| \le L_g(\|y_1 - y_2\| + \|u_1 - u_2\|_V) \\ \text{for any } y_1, y_2 \in H,\ u_1, u_2 \in U \end{gathered} \tag{$h5a$}$$

$$\begin{gathered} \Phi : H \to R \\ |\Phi(y_1) - \Phi(y_2)| \le L_\Phi \|y_1 - y_2\| \text{ for any } y_1, y_2 \in H. \end{gathered} \tag{$h5b$}$$

In the approximate version, we consider the evolution equation (4.1_n) under the assumptions $(h2)$, $(h3)$, and we look for a control $u_n(t) \in \mathcal{U}_n$ minimizing the finite horizon cost:

$$J_{f,n}(\tau, x_n, u_n) := \int_\tau^T g(y_n(x_n,t,u_n), u_n(t))dt + \Phi(y_n(x_n,T,u_n)) \tag{4.3}$$

where $x_n := P_n x$.

As for the previous case, we define the value functions of the control problems:

$$v(\tau, x) := \inf_{u \in \mathcal{U}} J_f(\tau, x, u)$$

$$v_n(\tau, x_n) := \inf_{u_n \in \mathcal{U}_n} J_{f,n}(\tau, x_n, u_n)$$

and we denote the optimal controls, if they exist, by

$$u^* \in \text{argmin } J_f(\tau, x, u)$$

$$u_n^* \in \text{argmin } J_{f,n}(\tau, P_n x, u_n)$$

dropping the dependence on τ and x (the context will avoid in the sequel any ambiguity with the optimal controls for the infinite horizon case).

We first show the existence and uniqueness result (where U_n has the same meaning as in Proposition 3.1):

Proposition 4.1 *The approximate value function $v_n(\tau, x_n)$ defined above is the unique solution of the Hamilton–Jacobi equation:*

$$\begin{cases} -\dfrac{\partial}{\partial \tau} v_n(\tau, X) + \sup\limits_{U \in U_n} [-(A_n X + B_n U)^t \nabla v_n(\tau, X) - g(X, U)] = 0 \\ v_n(T, X) = \Phi(X) \end{cases} \tag{4.4}$$

where $X \in R^{k_n}$, $U \in R^{h_n}$ *and* $0 \le \tau \le T$.

Proof: The result may be achieved by repeating the proof of Proposition 3.1 with minor changes, either starting from the HJ equation in H:

$$\begin{cases} -\dfrac{\partial}{\partial \tau} v_n(\tau, x) + \sup_{u_n \in U_n} [- < A_n x + B_n u_n, \nabla v_n(\tau, x) > - g(x, u_n)] = 0 \\ v_n(T, x_n) = \Phi(x_n) \end{cases}$$

or from the evolution equation (3.7) in R^{k_n}.

■

We will prove now the analogous of theorems 3.2 and 3.4 for the finite horizon problem:

Theorem 4.2 *Assume* $(h1)$–$(h3)$, $(h5)$ *and* (2.3). *Then, for any* $\tau \in [0, T]$ *and* $x \in H$, $|v_n(\tau, P_n x) - v(\tau, x)| \to 0$ *for* $n \to \infty$. *If moreover there exist optimal controls* u^*, u_n^* *and* (3.16) *is satisfied for any* $x \in E$, *then the estimate*

$$|v(\tau, x) - v_n(\tau, P_n x)| \le C \xi_n \tag{4.5}$$

holds for any $x \in E$, $\tau \in [0, T]$.

Proof: We will only prove the estimate (4.5), whereas the general case may be obtained with obvious changes following the lines of theorem 3.2.

Operating as in theorem 3.4, for any given $\tau \in [0, T]$, we consider two optimal controls $u^* \in \mathcal{U}, u_n^* \in \mathcal{U}_n$; we obtain:

$$v(\tau, x) = J_f(\tau, x, u^*) \le J_f(\tau, x, u_n^*) \tag{4.6}$$

$$v_n(\tau, P_n x) = J_{f,n}(\tau, P_n x, u_n^*) \le J_{f,n}(\tau, P_n x, \Pi_n u^*). \tag{4.7}$$

From (4.6) one has:

$$v(\tau, x) \le \int_\tau^T g(y(x, t, u_n^*), u_n^*(t)) dt + \Phi(y(x, T, u_n^*)).$$

Adding the terms $\pm J_{f,n}(\tau, P_n x, u_n^*)$, we obtain:

$$v(\tau, x) - v_n(\tau, P_n x) \le L_g T \sup_{t \in [0,T]} \|y(x, t, u_n^*) - y_n(P_n x, t, u_n^*)\| +$$

$$+ L_\Phi \|y(x, T, u_n^*) - y_n(P_n x, T, u_n^*)\|,$$

which yields, using (3.16):

$$v(\tau, x) - v_n(\tau, P_n x) \le (L_g T + L_\Phi) e^{\lambda'' T} \xi_n. \tag{4.8}$$

From (4.7) we have:

$$v_n(\tau, P_n x) \leq \int_\tau^T g(y_n(P_n x, t, \Pi_n u^*), \Pi_n u^*(t))dt + \Phi(y_n(P_n x, T, \Pi_n u^*)).$$

Adding the terms $\pm J_f(\tau, x, u^*)$ we obtain:

$$v_n(\tau, P_n x) - v(\tau, x) \leq$$

$$\leq L_g T \sup_{t\in[0,T]} [\|y_n(P_n x, t, \Pi_n u^*) - y(x, t, u^*)\| + \|u^*(t) - \Pi_n u^*(t)\|_V]+$$

$$+L_\Phi(\|y_n(P_n x, T, \Pi_n u^*) - y(x, T, u^*)\|)$$

and hence:

$$v_n(\tau, P_n x) - v(\tau, x) \leq (L_g T + L_\Phi)e^{\lambda'' T}\xi_n. \tag{4.9}$$

Then, defining the constant C by:

$$C = (L_g T + L_\Phi)e^{\lambda'' T}$$

(4.5) follows from (4.8), (4.9).

■

5 The Approximate Optimal Control

In this section we will always assume that there exist optimal solutions to our control problems. We have already examined the convergence of the value function of the approximate problem to the exact value function. We are now concerned in comparing the optimal costs $v(x)$, $v(\tau, x)$ with the cost related to the evolution of the exact system, when the approximate optimal control u_n^* is used. The result is given by the two following theorems, treating both the infinite and the finite horizon case:

Theorem 5.1 *Under the assumptions of theorem 3.2, let $u_n^*(t)$ be defined as in section 3. Then $|v(x) - J_i(x, u_n^*)| \to 0$ as $n \to \infty$. Moreover, if the assumptions of theorem 3.4 are satisfied, the following estimate:*

$$|v(x) - J_i(x, u_n^*)| \leq 2\omega(\xi_n)$$

holds for any $x \in E$.

Proof: We first note that:

$$|v(x) - J_i(x, u_n^*)| \leq |v(x) - v_n(P_n x)| + |v_n(P_n x) - J_i(x, u_n^*)|. \tag{5.1}$$

By the definition of v_n and u_n^* we have:

$$v_n(P_n x) = \int_0^\infty e^{-\lambda t} g(y_n(P_n x, t, u_n^*), u_n^*(t))dt.$$

Therefore:

$$|v_n(P_n x) - J_i(x, u_n^*)| \le$$
$$\le \int_0^\infty e^{-\lambda t} |g(y_n(P_n x, t, u_n^*), u_n^*(t)) - g(y(x, t, u_n^*), u_n^*(t))|dt$$
$$\le \int_0^\infty e^{-\lambda t} \min[L_g \|y_n(P_n x, t, u_n^*) - y(x, t, u_n^*)\|, 2M_g]dt, \qquad (5.2)$$

which, using (2.3), theorem 3.2 and the dominated convergence theorem, proves the first part of the statement.

If we assume (3.16), then we obtain from (5.2):

$$|v_n(P_n x) - J_i(x, u_n^*)| \le \int_0^\infty \min[e^{(\lambda'' - \lambda)t} \xi_n, 2M_g e^{-\lambda t}]dt \le \omega(\xi_n).$$

The use of the previous estimate and of (3.17) in (5.1) completes the proof.

■

Theorem 5.2 *Under the assumptions of theorem* 5.1*, let* $u_n^*(t)$ *be defined as in section* 4*. Then* $|v(\tau, x) - J_f(\tau, x, u_n^*)| \to 0$ *as* $n \to \infty$*. Moreover, if* (3.16) *is satisfied, the following estimate:*

$$|v(\tau, x) - J_f(\tau, x, u_n^*)| \le 2C\xi_n$$

holds for any $x \in E$*,* $\tau \in [0, T]$*.*

Proof: Follows the same argument as before with minor differences, and therefore will be omitted.

■

Remark The above convergence results hold regardless of the way used to obtain the approximate optimal controls u_n^*. In particular, the optimal solution could be obtained either in feedback form by the Dynamic Programming Principle or in open loop form by Pontryagin's Maximum Principle. In the latter case, the computation of the value function is in fact unnecessary. However, in both cases the numerical computation of the optimal solution introduces a further error term. If we denote by $\hat{u}_n^*(t)$ the numerical approximation (e.g. piecewise constant) of the optimal control for the infinite horizon problem, we may write:

$$|v(x) - J_i(x, \hat{u}_n^*)| \le |v(x) - J_i(x, u_n^*)| + |J_i(x, u_n^*) - J_i(x, \hat{u}_n^*)|$$

whose right–hand side consists of a first term related to the finite–dimensional approximation, and a second term related to the suboptimal numerical solution. The same remark applies to the finite horizon case.

6 General Remarks and Examples

Since $v_n(x_n)$, $v_n(\tau, P_n x)$ are the exact solutions of (3.3) and (4.2), the numerical analysis of the problem requires to approximate the same equation in R^{k_n}. A global estimate for the discretization error could be obtained combining the estimates from theorems 3.4 and 4.2, and the error bounds for the approximation of Hamilton–Jacobi equations in R^N (see [5], [6], [8], [10], [13], [23]). We point out that the choice of a scheme based on the Discrete Dynamic Programming Principle corresponds to a complete discretization of (1.1) (i.e. obtained replacing (1.1_n) by its one–step approximation).

However, the procedure we have outlined here (first semi–discretization of the evolution equation, then numerical solution of the HJ equation in R^{k_n}) is mostly theoretical, due to its very high computational complexity. To see this, suppose (1.1) is a one–dimensional PDE, and it is discretized with $k_n = 10$ degrees of freedom. The discretized state space will then be R^{10}, and solving the HJ equation in this space would require a mesh with a number of points of the order of 10^{10} or more, even for this very coarse approximation of (1.1). One could also note that basically the number of points in this mesh grows exponentially with n, and therefore exponentially with some negative power of the required error $|v(x) - v_n(P_n x)|$.

Nevertheless, theorems 5.1 and 5.2 show that the numerical approximation of the finite–dimensional control problem may be carried out by means of schemes based on Pontryagin's Maximum Principle, and this approach may be satisfactory as well.

It is worth noting that this convergence theory may also be extended to control problems with convex costs, without use of compactness arguments as in [2] (chapter 3). We sketch the few technical adaptations required, taking as an example the infinite horizon cost (3.1) with g defined by:

$$g(y, u) = \gamma(y) + \|u\|_V^2,$$

where $\gamma(\cdot)$ is bounded and Lipschitz continuous, and no constraint is imposed on the control. In this case the natural space of admissible controls is

$$\mathcal{U} = L^2_\lambda([0, +\infty[; V) = \left\{ u : [0, +\infty[\to V \text{ s.t. } \int_0^\infty e^{-\lambda t} \|u(t)\|_V^2 dt < \infty \right\},$$

that is, the space of controls which are square integrable with respect to the weight $e^{-\lambda t}$. It is well known that for this class of problems there always exists an optimal control. This is also true for the approximate problems since the convexity is preserved by internal approximations.

If we replace $(h1c)$ and the stability condition with the stronger assumption

$$\|e^{At}\|_{\mathcal{L}(H)}, \|e^{A_n t}\|_{\mathcal{L}(H)} \leq K,$$

then we obtain that $v(x)$ and $v_n(x)$ are Lipschitz continuous with a Lipschitz constant given by $L_v = \frac{L_\gamma K}{\lambda}$. This implies that for any t and any initial state x, the optimal controls satisfy the bound

$$\|u^*(t)\|_V, \|u_n^*(t)\|_V \leq M_u$$

for some positive constant M_u independent of n. Lastly, since $g(y, \cdot)$ is locally Lipschitz continuous, it is possible to apply again theorem 3.2 to prove the convergence of approximate value functions. A similar argument applies to the finite horizon problem as well.

A question arising about the use of a sequence of finite–dimensional operators, concerns the extension of this technique to the approximation of general form Hamilton–Jacobi equations in Hilbert spaces. A partial theory for Galerkin approximations is contained in [2] and [5], but we are not aware of results for other kinds of approximation, and for more general cases of HJ equations.

In some sense, the present results may be seen as approximation results for the Hamilton–Jacobi equations related to the infinite–dimensional control problems considered. However, they rely entirely on the representation of the solution as a value function, and it is not clear how to extend them to the general case. Moreover, the following example due to Crandall and Lions shows that finite–dimensional approximations may be improperly posed.

Example 1 *A Case of Lacking Convergence*

We wish now to discuss a counterexample given in [5] (part III), showing that finite–dimensional approximations may fail to converge to the right solution. Let H be the Hilbert space of square summable doubly infinite sequences. Here, $x = \{x_i\}_{i\neq 0}$ and the inner product and norm are given by:

$$< x, y >= \sum_{i\neq 0} x_i y_i \quad , \quad \|x\| =< x, x >^{1/2} .$$

We consider the HJ equation:

$$\begin{cases} \dfrac{\partial}{\partial \tau} v(\tau, x) + \sup\limits_{u\in U} < u, \nabla v(\tau, x) >= 0 \\ v(0, x) = \Phi(x) = \sum\limits_{i\geq 1} (x_i^2 - x_{-i}^2) \end{cases} \tag{6.1}$$

where $U = \{u \in H : \|u\| \leq 1 \;\text{ and }\; u_i = u_{-i} \;\text{ for }\; i \neq 0\}$. For our purposes, we will note that equation (6.1) is related to the finite horizon control of the evolution equation:

$$\begin{cases} y'(t) = -u(t) \\ y(T-\tau) = x \end{cases}$$

with the cost given by $J(\tau, x) = \Phi(y(T))$. With our notations, $A \equiv 0$, $B = -I$, $g \equiv 0$ and the control space V coincides with H.

If we assume now that the spaces H_n are given by:

$$H_n = \{x_n \in H : x_{n_i} = 0 \;\text{ for }\; i \geq n+1 \;\text{ and }\; i \leq -n-2\} \tag{6.2}$$

and $P_n = \Pi_n$ is the orthogonal projection into H_n ($k_n = \dim\, H_n = 2n+1$ in this case), if v_n is defined as the solution of equation (6.1) restricted to the space H_n, namely:

$$\begin{cases} \dfrac{\partial}{\partial \tau} v_n(\tau, x_n) + \sup\limits_{u_n \in P_n U} < u_n, \nabla v_n(\tau, x_n) >= 0 \\ v_n(0, x_n) = \Phi(x_n) = \sum\limits_{1 \leq i \leq n} (x_{n_i}^2 - x_{n_{-i}}^2) \end{cases}$$

then, Crandall and Lions prove (see [5], part III, example II.1) that the sequence v_n converges to a function which is not the solution of (6.1). However, we will soon show that the definition $U_n := P_n U$ does not match the assumptions of the main convergence theorem. Indeed, in our case:

$$P_n U = \{u \in H_n : \|u\| \leq 1 \;\text{ and }\; u_i = u_{-i} \;\text{ for }\; 0 < i \leq n+1\}$$

$$U \cap H_n = \{u \in H_n : \|u\| \leq 1 \;\text{ and }\; u_i = u_{-i} \;\text{ for }\; i \neq 0\}$$

so that $P_n U \neq U \cap H_n$ and the controls for the approximate problems may not be admissible for the exact one. In fact, the optimal controls u_n^* lie in the difference $(P_n U) \backslash (U \cap H_n)$ and this implies that $v_n(x) < v(x)$. Note that all the assumptions of the convergence theorem including (2.3) are satisfied as soon as we define $U_n := U \cap H_n$.

Example 2 *Control of the Heat Equation, Finite Elements Approximation*

Let us consider the heat equation on a bounded domain, with homogeneous Dirichlet boundary condition and with a distributed source term:

$$\begin{cases} y'(t) = \Delta y(t) + u(t) \\ y(0) = x \end{cases} . \tag{6.3}$$

We set $H = L^2(\Omega)$, $V = H^1(\Omega)$, $A = \Delta$ with $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$, $B = I$ and Ω is assumed to be an open polygonal bounded set of R^N. We recall that Δ generates an analytic semigroup which satisfies the bound:

$$\|e^{At}\|_{\mathcal{L}(H)} \leq e^{-\lambda_1 t} \tag{6.4}$$

where λ_1 is the first eigenvalue of Δ in the domain Ω. In the sequel $\|\cdot\|_0, \|\cdot\|_1, \|\cdot\|_2$ will denote the norms respectively in $L^2(\Omega)$, in $H^1(\Omega)$ and in $H^2(\Omega)$.

We assume an approximation of u and y in (2.4) by means of finite elements of P_1 (that is, piecewise linear functions); we may set $k_n = h_n = n$ (number of degrees of freedom of the discretization) and $V_n = H_n$. The projection P_n may be chosen either as the piecewise linear interpolation (see [22], section 4.4) or as the orthogonal projection. Π_n should be defined as the orthogonal projection to satisfy $(h2f)$.

The discretization error will then be expressed as a function of the discretization step h that in turn can be easily shown to be $h = O(n^{-1/N})$. It is well known from the general theory of finite elements that, under proper assumptions on the regularity of the mesh, for any function $v \in H^k(\Omega)$, $(k = 1, 2)$ there exists a positive constant C independent of h such that:

$$\|P_n v - v\|_0 \leq C h^k \|v\|_k. \tag{6.5}$$

To construct the approximate operators A_n one starts from the weak formulation:

$$\frac{\partial}{\partial t} \sum_j < \psi_i, \psi_j > y^j = \sum_j < \nabla\psi_i, \nabla\psi_j > y^j + \sum_j < \psi_i, \psi_j > u^j$$

for $i = 1, \ldots, n$ (where $\{\psi_i\}_{1 \leq i \leq n}$ is the base of H_n), and setting:

$$M_n = (m_{ij}) = (< \psi_i, \psi_j >)$$

$$R_n = (r_{ij}) = (< \nabla\psi_i, \nabla\psi_j >)$$

the matrix A_n is defined by $A_n = M_n^{-1} R_n$, whereas B_n is the identity matrix I_n.

Then, the numerical analysis of this problem shows that:

a) The scheme is convergent. Moreover, for any $x \in H^1(\Omega)$:

$$\left\| e^{At} x - e^{A_n t} P_n x \right\|_0 \leq C \|x\|_1 h$$

so that the assumption $\|u(t)\|_1 \leq M_u$ ensures by proposition 2.1 that (2.3) is satisfied. Therefore the approximate value functions converge to the exact ones.

b) With more regularity assumptions, explicit error estimates can be given. From the theory of finite elements (see [22], section 7.6), if we assume that:

$$u^*, u_n^* \in C^1([0, +\infty[; D(A)),$$
$$x \in E \subset D(A^2),$$

then we can find a constant C independent of n (or, equivalently, independent of h) such that, for any $t \in [0, +\infty[$:

$$\|y_n(P_n x, t, u_n^*) - y(x, t, u_n^*)\|_0 \le Ch^2 = \xi_n$$

$$\|y_n(P_n x, t, u^*) - y(x, t, u^*)\|_0 + \|u^* - \Pi_n u^*\|_0 \le Ch^2 = \xi_n$$

with $\lambda'' = 0$, so that by theorems 3.4 and 4.2 we obtain the estimates:

$$|v(x) - v_n(P_n x)| \le Ch^2 \le Cn^{-2/N};$$

$$|v(\tau, x) - v_n(\tau, P_n x)| \le Ch^2 \le Cn^{-2/N}.$$

Estimates of the same order apply to the costs of the approximate optimal controls; a faster rate of convergence could be achieved with higher order schemes, under more strict assumptions on the regularity of the problem.

Example 3 *Control of the Wave Equation, Finite Elements Approximation*

Let us now consider the wave equation on a bounded domain Ω, with homogeneous Dirichlet boundary conditions and with a scalar control term:

$$\begin{cases} y_1''(t) = \Delta y_1(t) + b_2 u_2(t) \\ y_1(0) = x_1 \\ y_1'(0) = x_2 \end{cases} \tag{6.6}$$

where Ω is assumed to be an open polygonal bounded set of R^N, $b_2 \in L^2(\Omega)$ is a given function and $u_2 : [0, T[\to U \subset R$. With the notations:

$$y(t) := \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} \quad ; \quad x := \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad ; \quad u(t) := \begin{pmatrix} 0 \\ u_2(t) \end{pmatrix}$$

(6.6) may be rewritten in the form (1.1), with the operators A, B given by:

$$A = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix} \quad , \quad B = \begin{pmatrix} 0 & 0 \\ 0 & b_2 \end{pmatrix}.$$

Equation (6.6) may be studied in the space $H = H_0^1(\Omega) \times L^2(\Omega)$, endowed with the scalar product:

$$< x, y >= \int_\Omega \nabla x_1 \nabla y_1 dx + \int_\Omega x_1 y_1 dx + \int_\Omega x_2 y_2 dx$$

with $D(A) = [H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega)$, whereas the control space V coincides with R.

We recall that A generates a C^0 semigroup e^{At}, which verifies, under the assumption of boundedness for Ω:

$$\|e^{At}\|_{\mathcal{L}(H)} = 1. \tag{6.7}$$

We consider again approximations of $y \in H, b_2 \in L^2$ in (6.6) by means of finite elements of P_1. We have obviously $h_n = 1$, $\Pi_n = I$ and $|u - \Pi_n u| \equiv 0$. The approximation procedure is the same of the previous example, and the numerical analysis shows that:

a) Under the basic assumptions the scheme is convergent; moreover, since $V = R$ has finite dimension, (2.3) always holds by proposition (2.1);

b) Under stronger assumptions, namely:

$$u^*, u_n^* \in C^2([0, T[; R)$$

$$x \in D(A^3)$$

there exist (see [22], section 8.7) two constants C_1, C_2 independent of n such that, for any $t \in [0, +\infty[$:

$$\|y_n(P_n x, t, u_n^*) - y(x, t, u_n^*)\| \le h(C_1 + C_2 t),$$

$$\|y_n(P_n x, t, u^*) - y(x, t, u^*)\| \le h(C_1 + C_2 t).$$

Hence, theorems 3.4 and 4.2 apply with $\xi_n = Ch$ and with any $\lambda'' > 0$, giving the estimates:

$$|v(x) - v_n(P_n x)| \le Ch \le Cn^{-1/N};$$

$$|v(\tau, x) - v_n(\tau, P_n x)| \le Ch \le Cn^{-1/N}.$$

Example 4 *Control of a First Order Wave Equation, Spectral Approximation*

Lastly, we consider a first order hyperbolic equation with a scalar control term, which (for the sake of simplicity) will be stated as a one–dimensional problem as follows:

$$\begin{cases} y'(t) = -\dfrac{\partial}{\partial z} y(t) + bu(t) \quad \text{in } \Omega =]-1,1[\\ y(0) = x \\ y(z=-1,t) = 0 \end{cases} \tag{6.8}$$

where $b \in L^2(]-1,1[)$ is a given function, z denotes the spatial coordinate and $u : [0,T[\to U \subset R$. This problem is well posed in $H = L^2(]-1,1[)$, with $D(A) = \{x \in H^1(]-1,1[) : x(z=-1) = 0\}$; the related semigroup satisfies the bound:

$$\|e^{At}\|_{\mathcal{L}(H)} = 1.$$

The approximation of this evolution equation may be carried out (see [12], section 8) by means of Legendre polynomials, which provide a complete set of orthogonal functions in $L^2(]-1,1[)$. We choose ψ_n as the Legendre polynomial of degree n, and $H_n = span(\psi_1, \ldots, \psi_n)$; the projection operator is defined as:

$$P_n x = \sum_i < \psi_i, x > \psi_i.$$

The weak formulation of this problem reads:

$$\frac{\partial}{\partial t} \sum_j < \psi_i, \psi_j > y^j = \sum_j < \psi_i, \frac{\partial}{\partial z}\psi_j > y^j + < \psi_i, b > u \quad (i = 1, \ldots, n)$$

and setting:

$$M_n = (m_{ij}) = \text{diag}(< \psi_i, \psi_i >)$$

$$R_n = (r_{ij}) = (< \psi_i, \frac{\partial}{\partial z}\psi_j >)$$

the matrix A_n is defined by $A_n = M_n^{-1} R_n$.

The numerical analysis of this approximation scheme shows that the scheme is convergent. Again, the assumption $V = R$ ensures the pointwise convergence of the approximate value functions.

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