Journal of Mathematical Systems, Estimation, and Control
Vol. 7, No. 1, 1997, pp. 1–16

Stabilization of Decentralized Control Systems*

Zhiming Gong   Mohammad Aldeen

Abstract

The problem of stabilization of linear time-invariant systems under general decentralized feedback schemes is considered in this paper. A novel approach to the problem is advised, in which the interactions between the strongly connected subsystems of a system are treated as disturbances. A necessary and sufficient condition for the existence of a decentralized controller which stabilizes a given system is presented.

Keywords: decentralized control, fixed modes, large-scale systems, stabilization, quotient fixed modes

AMS Subject Classifications: 93A14, 93A15, 93B52, 93D15

1 Introduction

For nearly two decades, much attention has been paid to the problem of stabilization of decentralized linear time-invariant systems. Wang and Davison [21] introduced the notion of "fixed mode" and obtained a solution to the decentralized control problem. They concluded that the absence of fixed modes is a necessary and sufficient condition for the existence of a decentralized linear time-invariant system which stabilizes a given system (also see [1],[4],[6],[13]). This result has been the most important contribution to the decentralized control theory so far. Anderson and Moore [2] considered the case where a decentralized time-varying controller is employed. They concluded that the absence of unstable fixed modes of a system is *not* a necessary condition for the existence of a stabilizing decentralized time-varying controller, and in fact fixed modes may be eliminated by time-varying controllers. Wang [20] observed independently the property of time-varying controller in the elimination of fixed modes. It has

*Received September 1, 1994; received in final form November 20, 1995. Summary appeared in Volume 7, Number 1, 1997.

also been shown that fixed modes can be eliminated by vibrational control and by sampling techniques (see [12],[14],[17]).

However, there has been some confusion in the literature as to what kind of fixed modes can actually be eliminated, see, e.g., [17],[18]. Willems [22] clarified the question to certain extent and argued that "structurally fixed modes" caused by the fact that the system is not strongly connected are also fixed modes with respect to time-varying output feedback. It should be noted that "structurally fixed modes" meant by Willems [22] are not the same structurally fixed modes as studied in [15] and [16], where structurally fixed modes were defined through the notions of structured matrices and structurally equivalent systems. Very recently, Khargonekar and Ozguler [11] further clarified the question by using "lifting" technique for periodic systems. They showed that all fixed modes, except those associated with the complementary subsystems ([3],[13]) having zero transfer function matrices, can be eliminated by a periodically time-varying decentralized controller.

Then, the following general question may be asked. What kind of fixed modes can, or cannot, be eliminated by an even larger class of controllers than periodically time-varying decentralized controllers, say general time-varying, nonlinear, vibrational, or any other kind of suitable controllers?

To provide an answer to the question given above, we consider in this paper general decentralized feedback schemes where no restrictions are imposed on the type and structure of the decentralized controllers, as long as the constraint of the decentralized information structure is satisfied. A necessary and sufficient condition for the existence of a decentralized controller which stabilizes a given system is presented in this paper. The condition is stated in a neat term of fixed modes of a quotient system, which were first identified in [9] and [10] and are termed *Quotient Fixed Modes* in this paper. It is shown that the quotient fixed modes are exactly those fixed modes associated with the complementary subsystems having zero transfer function matrices, which were identified by Khargonekar and Ozguler [11]. The result of this paper provides a precise statement of the arguments in [22] and [11] and reveals the further fact that if a fixed modes cannot be eliminated by a decentralized periodically time-varying controller, then there is no way this fixed mode can be eliminated by a decentralized controller, no matter what kind of controllers are used. This provides a complete answer to the question of what kind of fixed modes can, or cannot, be eliminated.

The layout of the paper is as follows. Section 2 introduces concept of time-varying vectors with a degree of exponential stability and gives formal statement of the problem. The main result of this paper is presented in Section 3 and a proof of the main result is given in Section 4. Section 5 is the conclusion.

2 Statement of the problem

Consider a decentralized control system described by

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{N} B_i u_i(t) \tag{2.1a}$$

$$y_i(t) = C_i x(t), \qquad i = 1, 2, \cdots, N \tag{2.1b}$$

where N is the number of the control stations, $x(t) \in R^n$ is the state vector of the system, $u_i(t) \in R^{m_i}$ and $y_i(t) \in R^{r_i}$ are, respectively, the input and the output vectors of the ith control station, and A, B_i and C_i are real constant matrices of appropriate dimensions. Assume that the information available to the ith control station at time t is represented by

$$I_i(t) = \{y_i(\xi), u_i(\zeta) : \xi \in [0, t], \zeta \in [0, t)\}. \tag{2.2}$$

It is also assumed that the constraint of the decentralized information structure is such that the control input $u_i(t)$ at the ith local control station can only be calculated from $I_i(t)$, i.e.,

$$u_i(t) = F_i(I_i(t), t), \qquad i = 1, 2, \cdots, N \tag{2.3}$$

where $F_i(I_i(t), t)$ denotes a function of $I_i(t)$ and time t.

Obviously, the class of the controllers (2.3) includes the decentralized linear time-invariant, time-varying, or even nonlinear controllers. In fact, it is the largest class of decentralized controllers for the system (2.1).

To state the problem studied in this paper, the following definition is required.

Definition 1 *A time-varying vector $v(t)$ is said to be stable with a degree of exponential stability (DES) α, if there exists a bounded scalar function $f(\cdot)$ so that*

$$\|v(t)\| \leq f(t_0) e^{-\alpha(t-t_0)}, \qquad \forall t_0 \geq 0, t \geq t_0 \tag{2.4}$$

where α is a positive real number and $\|v(t)\|$ denotes the Euclidean norm of $v(t)$. In addition, if

$$f(t_0) = a\|v(t)\|, \qquad \forall t_0 \geq 0 \tag{2.5}$$

where a is a constant, then, the time-varying vector $v(t)$ is said to be uniformly stable with a DES α.

Under Definition 1, stable time-varying vectors have the following useful properties. Let $G_1(t)$ and $G_2(t)$ denote two bounded function matrices with appropriate dimensions. If time-varying vectors $v_1(t)$ and $v_2(t)$ are stable with DES α_1 and α_2, respectively, then,

(i) the vector

$$G_1(t)v_1(t)$$

is stable with a DES α_1; and

(ii) the vector

$$G_1(t)v_1(t) + G_2(t)v_2(t)$$

is stable with a DES α=min$\{\alpha_1,\, \alpha_2\}$; and

(iii) the vector

$$\begin{bmatrix} G_1(t)v_1(t) \\ G_2(t)v_2(t) \end{bmatrix}$$

is stable with a DES α=min$\{\alpha_1,\, \alpha_2\}$.

In the sequel, a system is said to be stable with a DES α if the state vector of this system, with zero external input, is stable with a DES α for any possible initial state. When the DES is not concerned, the system is said to be stable for any positive real numbers α.

The problem of stabilization of decentralized linear time-invariant systems can now be stated as follows: Find a decentralized controller (2.3) for the system (2.1) so that the resulting closed-loop system is stable, or stable with a prescribed DES.

3 Main Result

In order to state the main result of this paper, we need to introduce the concept of the digraphs and strongly connectedness of decentralized control systems [3],[9]. Consider the N-channel decentralized control system (2.1), the digraph of this system is defined as a set of N nodes and some directed arcs connecting these nodes. The nodes represent the control stations of the system and directed arcs represent the connections between them. If $C_j(sI-A)^{-1}B_i \neq 0$, then there exist a directed arc from node i to node j $(i, j = 1, 2, \cdots, N)$. If for each distinct pair of i and j, the digraph contains directed paths from node i to node j and from node j to node i, then the digraph is said to be strongly connected, and so is called the system (2.1).

If a digrah is not strongly connected, it can always be decomposed uniquely into a number of strongly connected components, which is the largest strongly connected sub-digraphs, in the sense that whenever an extra node is added to such a strongly connected sub-digraph, it will be no longer strongly connected.

Corresponding to the decomposition of the digraph into a number of strongly connected components, the system equations (2.1) can be written

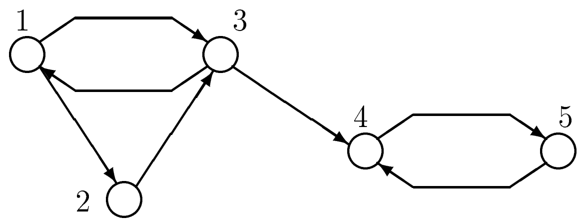

Figure 1: Digraph of a Decentralized Control System

as

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{N^*} B_i^* u_i^*(t) \tag{3.1a}$$

$$y_i^*(t) = C_i^* x(t), \qquad\qquad i = 1, 2, \cdots, N^* \tag{3.1b}$$

where N^* denotes the number of the strongly connected components of the system (2.1), $u_i^*(t) \in R^{m_i^*}$ and $y_i^*(t) \in R^{r_i^*}$ denote the input and the output vectors consisting of all of the inputs and the outputs in the ith strongly connected component, respectively, and B_i^* and C_i^* are the corresponding input and output matrices.

Suppose that the N control stations of the system (2.1) are aggregated into the N^* control stations as described by (3.1). Then the decentralized control system (3.1) with N^* control stations is called *a quotient system* of the system (2.1) [9].

The following example illustrates the concept of quotient systems. Let $i \to j$ stands for the condition $C_j(sI - A)^{-1}B_i \neq 0$. Suppose that a decentralized control system has 5 local control stations and only the following conditions hold: $1 \to 2$, $1 \to 3$, $2 \to 3$, $3 \to 1$, $3 \to 4$, $4 \to 5$, $5 \to 4$. Then, the digraph of the system is as shown in Figure 1. It is easy to see that this digraph has 2 strongly connected components. Therefore, its quotient system has 2 aggregated local control stations. For comparison, signal flow graphs of the original decentralized control system and its quotient system are illustrated in Figure 2 and Figure 3, respectively.

Let Λ^* denote the set of the fixed modes [21] of the quotient system (3.1), i.e.

$$\Lambda^* = \bigcap_{K_i^* \in R^{m_i^* \times r_i^*}} \sigma\left(A + \sum_{i=1}^{N^*} B_i^* K_i^* C_i^*\right) \tag{3.2}$$

where $\sigma(\cdot)$ denotes the set of the eigenvalues of the argument matrix and the intersection takes over all of the matrices $K_i^* \in R^{m_i^* \times r_i^*}$, $i = 1, 2, \cdots, N^*$.

Definition 2 *The fixed modes of the quotient system (3.1) are called* quotient fixed modes *of the system (2.1).*

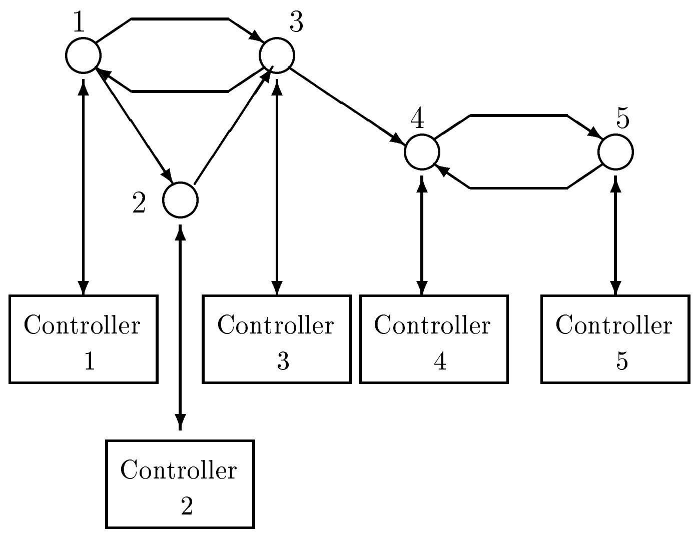

Figure 2: A Decentralized Control System

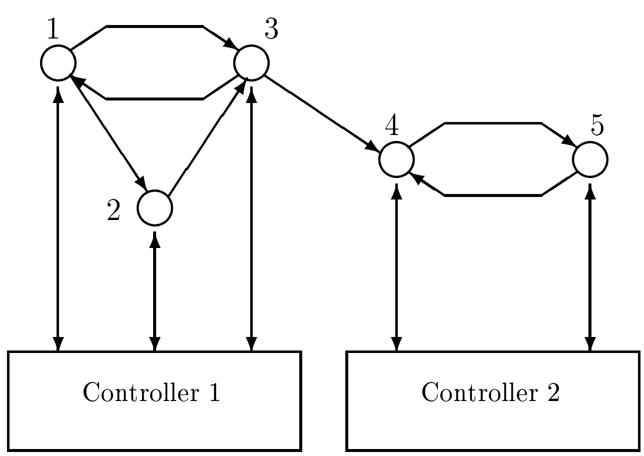

Figure 3: A Quotient System

It is clear that the set of the quotient fixed modes of the system (2.1) is a subset of the fixed modes of this system, i.e. $\Lambda^* \subset \Lambda$, where Λ denotes the set of the fixed modes of the system (2.1) and is given by [21]

$$\Lambda = \bigcap_{K_i \in R^{m_i \times r_i}} \sigma \left(A + \sum_{i=1}^{N} B_i K_i C_i \right). \tag{3.3}$$

The main result of this paper is now stated in the following theorem.

Theorem 1 *Given the system (2.1), there exists a decentralized controller (2.3) for this system so that*

(i) the closed-loop system is stable with any prescribed DES if and only if the system (2.1) has no quotient fixed modes, i.e. $\Lambda^* = \emptyset$; *and*

(ii) the closed-loop system is stable if and only if the system (2.1) has no unstable quotient fixed modes, i.e. $\Lambda^* \subset C^-$, *where* C^- *denotes the left half open region of the complex plane.*

There exists an obvious analogy between Theorem 1 above and the celebrated result by Wang and Davison [21, Theorem 1], which may be stated as follows: Given the system (2.1), there exists a decentralized linear time-invariant (dynamic) controller for this system so that the closed-loop system is stable if and only if the system (2.1) has no unstable fixed modes, i.e. $\Lambda \subset C^-$.

Comparing with Wang and Davison's result [21, Theorem 1] with Theorem 1 of this paper, it is clear that when the considered family of decentralized controllers for the system (2.1) is enlarged from linear time-invariant ones to the most general ones (2.3), the set of fixed modes Λ in Wang and Davison's result [21, Theorem 1] should be replaced by the set of quotient fixed modes Λ^*, an subset of Λ.

Using "lifting" technique, Khargonekar and Ozguler [11, Theorem 2] showed recently that a linear time-invariant system is stabilizable by a decentralized periodically time-varying controller if and only if there exists no unstable "incompleting zeros" in the complementary subsystems which have zero transfer function matrices. Their result gives a precise statement of the argument claimed by Willems [22]. As shown late in this paper, the incompleting zero of the complementary subsystems having zero transfer matrices are exactly the quotient fixed modes defined in this paper. Therefore, Theorem 1 given above extends the results of Willems [22] and Kargonekar and Ozguler [11], and reveals the further fact that if a decentralized linear time-invariant system cannot be stabilized by a decentralized periodically time-varying controller, then, no matter what kind of structures of the controllers are employed, there exists no decentralized controllers which can stabilize the system. In the neat term of quotient fixed modes, Theorem 1 of this paper gives a complete answer to the question on what kind of fixed modes can, and cannot , be eliminated under the constraint of decentralized control.

4 Proof of the Main Result

In this section, a proof of Theorem 1 is presented. We first introduce a lemma.

Lemma 1 *Consider a decentralized control system, which is subject to unmeasurable disturbances and otherwise identical to the system (2.1), described by*

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{N} B_i u_i(t) + d(t) \tag{4.1a}$$

$$y_i(t) = C_i x(t) + g_i(t), \qquad i = 1, 2, \cdots, N \tag{4.1b}$$

where $d(t)$ and $g_i(t)$ $(i = 1, 2, \cdots, N)$ are disturbances and are stable with a DES α. If the system (2.1) is strongly connected, then, there exists a decentralized controller (2.3) for the system (4.1) so that

(i) the state vector of the closed-loop system is stable with a prescribed DES $\delta \leq \alpha$ if the system (2.1) is centralized controllable and observable; and

(ii) the state vector of the closed-loop system is stable if the system (2.1) has no unstable centralized uncontrollable and/or unobservable modes.

Proof of Lemma 1: See the Appendix.

Next, we introduce some preliminary development of the proof of Theorem 1. From the definitions of digraph and quotient systems, it follows that the transfer function matrix of the system (2.1) is of block triangular structure when the control stations of the system are ordered appropriately. Assume that the control stations of the system (2.1) have been ordered appropriately and the state vector has also been chosen appropriately. Then the matrices in the system equations (2.1) and (3.1) can assume the following triangular structure [3]:

$$A = \begin{bmatrix} \tilde{A}_1 & \times & \cdots & \times \\ 0 & \tilde{A}_2 & \cdots & \times \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \tilde{A}_{N^*} \end{bmatrix},$$

$$B = [B_1^* B_2^* \cdots B_{N^*}^*] = \begin{bmatrix} \tilde{B}_1 & \times & \cdots & \times \\ 0 & \tilde{B}_2 & \cdots & \times \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \tilde{B}_{N^*} \end{bmatrix},$$

$$C = \begin{bmatrix} C_1^* \\ C_2^* \\ \vdots \\ C_{N^*}^* \end{bmatrix} = \begin{bmatrix} \tilde{C}_1 & \times & \cdots & \times \\ 0 & \tilde{C}_2 & \cdots & \times \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \tilde{C}_{N^*} \end{bmatrix} \tag{4.2}$$

where $C = [C_1' C_2' \cdots C_N']'$, $B = [B_1 B_2 \cdots B_N]$ and $\tilde{A}_i$, $\tilde{B}_i$ and $\tilde{C}_i$ $(i = 1, 2, \cdots, N^*)$ are nonzero matrices of appropriate dimensions.

Based on the structure of the matrices (4.2), the system equations (2.1) and (3.1) can equivalently be written as

$$\dot{x}_i(t) = \tilde{A}_i x_i(t) + \tilde{B}_i u_i^*(t) + w_i(t) \tag{4.3a}$$

$$y_i^*(t) = \tilde{C}_i x_i(t) + v_i(t), \qquad i = 1, 2, \cdots, N^* \tag{4.3b}$$

where $x_i(t)$, $w_i(t)$ and $v_i(t)$ are vectors determined correspondingly by (4.2) and the system equations (2.1) and (3.1).

From equations (4.3), it is clear that the case where the system (2.1) is controlled by a decentralized controller is equivalent to the case where each subsystem in (4.3) is controlled by an individual decentralized controller. Also, it can be seen from (4.2) that, for a certain ith subsystem, $w_i(t)$ and $v_i(t)$ will not contain information about $x_i(t)$, regardless of the structure and type of the decentralized controller. In other words, $w_i(t)$ and $v_i(t)$ can be considered as exogenic disturbances to their respective subsystems. Therefore, to analyze the decentralized stabilizability of the global system, it suffices to analyze individually the decentralized stabilizability of each subsystem in (4.3), in the presence of the disturbances. Furthermore, the system (2.1) is stable or stable with a DES δ if and only if the state of each subsystem (4.3) is stable or stable with DES $\delta_i \geq \delta$, respectively.

It is easy to show that the subsystems described by the triples ($\tilde{C}_i$, $\tilde{A}_i$, $\tilde{B}_i$), $i = 1, 2, \cdots, N^*$, are strongly connected, with respect to the control stations in their respective strongly connected components. These subsystems are therefore called strongly connected subsystems of the system (2.1) [5].

Let Λ_i^c denote the set of centralized uncontrollable and/or unobservable modes of the strongly connected subsystem $(\tilde{C}_i, \tilde{A}_i, \tilde{B}_i)$. Then, the following holds [21, Lemma 1].

$$\Lambda_i^c = \bigcap_{K_i^* \in R^{m_i \times r_i}} \sigma\left(\tilde{A}_i + \tilde{B}_i K_i^* \tilde{C}_i\right), \qquad i = 1, 2, \cdots, N^*. \tag{4.4}$$

Using (3.2) and (4.4), it is not difficult to show that

$$\Lambda^* = \bigcup_{i=1}^{N^*} \Lambda_i^c. \tag{4.5}$$

Equation (4.5) states that the set of quotient fixed modes of the system (2.1) is just the union of the centralized uncontrollable and/or unobservable modes of all of the strongly connected subsystems of this system. Now, we may apply the well-known Kalman decomposition [8] to the strongly connected subsystems $(\tilde{C}_i, \tilde{A}_i, \tilde{B}_i)$ and present straightforward proof of Theorem 1.

Proof of the necessity part of Theorem 1:

If the system (2.1) has a quotient fixed mode, λ, then by (4.5), there is at least one strongly connected subsystem of the system (2.1), say, the subsystem $(\tilde{C}_p, \tilde{A}_p, \tilde{B}_p)$, which has a centralized uncontrollable and/or unobservable mode λ. Consider now the pth subsystem (4.3) corresponding to the strongly connected subsystem $(\tilde{C}_p, \tilde{A}_p, \tilde{B}_p)$. When a decentralized controller (even a centralized controller) for this subsystem is applied, the state response of the resulting closed-loop system will contain terms proportional to $e^{\lambda t}$ when the initial state corresponding to the mode λ is nonzero. According to Kalman [8], this statement is true regardless of the structure and type of the applied controller, i.e., regardless of whether the controller is linear or nonlinear, dynamic or static, time-varying or time-invariant, or any other kind of controller. This means that the state of the pth subsystem (4.3) can not be stabilized with an arbitrarily prescribed DES. By the same reason, if such a λ exists and is unstable ($\lambda \notin C^-$), then, that subsystem can not be stabilized by any controller. This completes the proof of the necessity part of both statements (i) and (ii) in Theorem 1.

Proof of the sufficiency part of Theorem 1:

For any prescribed positive real number δ, let δ_i, $i = 1, 2, \cdots, N^*$, be numbers such that $\delta \leq \delta_1 \leq \delta_2 \leq \cdots \leq \delta_{N^*}$. In the following, it is shown that if $\Lambda^* = \emptyset$, then there exists individual decentralized controllers for each subsystem (4.3) so that the states of the resulting closed-loop systems are stable with DES $\delta_1, \delta_2, \cdots, \delta_{N^*}$, respectively.

If $\Lambda^* = \emptyset$, then, by (4.5), each strongly connected subsystem of the system (2.1) is centralized controllable and observable. Consider first the N^*-th subsystem in (4.3) where $w_{N^*}(t) = v_{N^*}(t) = 0$. According to Lemma 1, there exists a decentralized controller for this subsystem so that the state $x_{N^*}(t)$ of the resulting closed-loop system is stable with a DES δ_{N^*}. Then, by noticing the matrix structure in (4.2) and using the properties of stable time-invariant vectors, it follows that the disturbances $w_{N^*-1}(t)$ and the $v_{N*-1}(t)$ to $(N^* - 1)$-th subsystem in (4.3) are stable with a DES δ_{N^*}. Using Lemma 1 again, it follows that there exists a decentralized controller for the $(N^* - 1)$-th subsystem in (4.3) so that the state of the resulting closed-loop system is stable with a DES δ_{N^*-1}.

Repeating the above arguments for the rest of the subsystems in (4.3), i.e. the subsystems in (4.3) with $i = N^*-2, N^*-3, \cdots, 2, 1$, it is concluded that there exists N^* decentralized controllers for each subsystem in (4.3), so that the states of the resulting closed-loop systems are stable with DES $\delta_{N^*}, \delta_{N^*-1}, \delta_{N^*-2}, \cdots, \delta_2, \delta_1$, respectively. This implies that there exists a decentralized controller (2.3) for system (2.1) so that the resulting closed-loop system is stable with the prescribed DES δ. This completes the proof of the sufficiency part of statement (i) in Theorem 1.

If the system (2.1) has no unstable quotient fixed modes, then, by (4.5), each strongly connected subsystem of the system (2.1) has no unstable centralized uncontrollable and/or unobservable modes. Then, by using Kalman's decomposition methods [8] and arguments similar to the above, it is not difficult to show that there exists a decentralized controller for the system (2.1) so that the resulting closed-loop system is stable. This completes the proof of the sufficiency part of statement (ii) in Theorem 1, and consequently the proof of Theorem 1.

Khargonekar and Ozgular [11] showed that a linear time-invariant system can be stabilized by a decentralized periodically time-varying controller if and only if there exist no unstable "incompleting zero" in the complementary subsystems having zero transfer function matrices. It is shown in the following that these "incompleting zero" are exactly the unstable quotient fixed modes defined in this paper.

Consider the decentralized control system (2.1) with the matrices A, B and C in the form given by (4.2). Then, according to the definition of strongly connected digraphs, all the complementary subsystems with zero transfer function matrices have the form $(\hat{C}_j, A, \hat{B}_j)$, $j \in \{1, 2, \cdots, N^*\}$, with

$$\hat{B}_j = \left[B_1^* \; B_2^* \; \cdots, \; B_j^*\right] \qquad \text{and} \qquad \hat{C}_j = \begin{bmatrix} C_{j+1}^* \\ C_{j+2}^* \\ \vdots, \\ C_{N^*}^* \end{bmatrix} \tag{4.6}$$

where B_i^* $(i = 1, 2, \cdots, j)$ and C_i^* $(i = j+1, j+2, \cdots, N^*)$ are matrices given in (4.2).

Consider now the system matrix

$$S_j = \begin{bmatrix} \lambda I - A & \hat{B}_j \\ -\hat{C}_j & 0 \end{bmatrix}. \tag{4.7}$$

By the definition given in [11], a number λ is an incompleting zero of the complementary subsystems $(\hat{C}_j, A, \hat{B}_j)$ iff the rank of the matrix S_j is less than n, the dimension of the matrix A. Due to the triangular structure of the matrices A, B and C as shown in (4.2), we have

$$\text{Rank} S_j = \text{Rank} \begin{bmatrix} M_1 & \times & \cdots & \times \\ 0 & M_2 & \cdots & \times \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & M_{N^*} \end{bmatrix} \tag{4.8}$$

where

$$M_i = \begin{bmatrix} \lambda I - \tilde{A}_i & \tilde{B}_i \end{bmatrix}, \qquad i = 1, 2, \cdots, j$$

$$M_i = \begin{bmatrix} \lambda I - \tilde{A}_i \\ -\tilde{C}_i \end{bmatrix}, \qquad i = j+1, j+2, \cdots, N^*.$$

Using a result by Gong and Aldeen [5, Lemma A1], it can be shown easily that the rank of the matrix S_j is less than n if and only if there exists a matrix M_i ($i \in \{1, 2, \cdots, N^*\}$) which doesn't have full rank. Then, by using the well-known rank tests for controllability and observability of the system $(\tilde{C}_i, \tilde{A}_i, \tilde{B}_i)$ and the result of (4.5), it is concluded that the set of the incompleting zeros of the complementary subsystems with zero transfer function matrices are exactly the set of the quotient fixed modes of the system (2.1).

5 Conclusion

The problem of stabilization of linear time-invariant systems under general decentralized feedback schemes is investigated in this paper. By using the notion of quotient fixed modes, the paper presented a necessary and sufficient condition for the existence of a decentralized controller to stabilize a given linear time-invariant system (Theorem 1). This result gives a complete answer to the question on what kind of fixed modes can actually be eliminated under the constraint of decentralized control. The argument by Willems [22] is now extended to the following: "Structure fixed modes" caused by the fact that the system is not strongly connected are also fixed modes with respect to decentralized time-varying output feedback controllers; and in fact they are fixed modes with respect to all kind of decentralized controllers, including nonlinear ones.

Appendix

Proof of Lemma 1.

In this appendix, a proof of Lemma 1 is provided. Assume that the system (2.1) is centralized controllable and observable, and strongly connected. According to Anderson and Moore [2], there exists a decentralized time-varying feedback controller, described by

$$u_i(t) = \hat{K}_i(t)y_i(t) + \hat{u}_i(t), \qquad i = 1, 2, \cdots, N \tag{A.1}$$

where $\hat{K}$ are bounded matrices, for system (2.1) so that the resulting time-varying closed-loop system, described by

$$\dot{x}(t) = \hat{A}x(t) + \sum_{i=1}^{N} B_i\hat{u}_i(t) \tag{A.2a}$$

$$y_i(t) = C_i x(t), \qquad i = 1, 2, \cdots, N \tag{A.2b}$$

where $\hat{A} = A + \sum_{i=1}^{N} B_i\hat{K}_i(t)C_i$, is uniformly completely controllable and observable by an arbitrarily predetermined control station of the system, say by control station 1.

Then, according to Ikeda et al. [7], there exists a controller at control station 1 for the system (A.2) so that the closed-loop system is uniformly stable with any prescribed DES. Such a controller may be described by

$$\hat{u}_1(t) = F(t)z(t) \tag{A.3a}$$

$$\hat{u}_i(t) = 0, \qquad i = 2, 3, \cdots, N \tag{A.3b}$$

where $F(t)$ is a bounded feedback gain matrix and $z(t)$ is the estimate of the state vector $x(t)$ obtained by the observer

$$\dot{z}(t) = (\hat{A}(t) - H(t)C_1)z(t) + B_1\hat{u}_1(t) + H(t)y_1(t)$$

where $H(t)$ is a bounded observer gain matrix. Then the closed-loop system is given by

$$\dot{x}_c(t) = A_c(t)x_c(t) \tag{A.4}$$

where

$$x_c(t) = \begin{bmatrix} x(t) \\ z(t) \end{bmatrix}, \quad A_c(t) = \begin{bmatrix} \hat{A}(t) & B_1F(t) \\ H(t)C_1 & \hat{A}(t) - H(t)C_1 + B_1F(t) \end{bmatrix}.$$

Now, apply the decentralized controller as described by (A.1) and (A.3) to the system (4.1). This leads to the following closed-loop system.

$$\dot{x}_c(t) = A_c(t)x_c(t) + w_c(t) \tag{A.5}$$

where

$$w_c(t) = \begin{bmatrix} d(t) + \sum_{i=1}^{N} B_i\hat{K}_i(t)g_i(t) \\ H(t)g_1(t) \end{bmatrix}.$$

As $d(t)$ and $g_i(t)$ $(i = 1, 2, \cdots, N)$ are all stable with a DES α, then, by noticing the property of stable time-varying vectors, it follows that $w_c(t)$ is stable with a DES α.

For a given number $\delta \leq \alpha$, let β be a number such that $\beta \neq \alpha$ and $\delta = \min\{\alpha, \beta\}$. Design the controller (A.3) so that the system (A.4) is uniformly stable with a DES β. Let $\Phi(\cdot,\cdot)$ denote the transition matrix associated with the system (A.4). Since the system (A.4) is uniformly stable with a DES β, it is easy to show that

$$\|\Phi(t,t_0)\| \leq a e^{-\beta(t-t_0)}, \qquad \forall t_0, t \geq t_0 \qquad (A.6)$$

where a is a constant. The state response of the system (A.5) is given by

$$x_c(t) = \Phi(t,t_0)x_c(t_0) + \int_{t_0}^{t} \Phi(t,\tau)w_c(\tau)d\tau, \qquad \forall t_0, t \geq t_0. \qquad (A.7)$$

As $w_c(t)$ is stable with a DES α, i.e.

$$\|w_c(t)\| \leq f_1(t_0)e^{-\alpha(t-t_0)}, \qquad \forall t_0, t \geq t_0 \qquad (A.8)$$

where $f_1(\cdot)$ is a bounded function, it follows that

$$\begin{aligned}\|x_c(t)\| &\leq a\|x_c(t_0)\|e^{-\beta(t-t_0)} + f_1(t_0)\int_{t_0}^{t} e^{-\beta(t-\tau)}e^{-\alpha(\tau-t_0)}d\tau \\ &= a\|x_c(t_0)\|e^{-\beta(t-t_0)} + \frac{af_1(t_0)}{\beta-\alpha}\left\{e^{-\alpha(t-t_0)} - e^{-\beta(t-t_0)}\right\} \\ &\leq f_2(t_0)e^{-\delta(t-t_0)} \end{aligned} \qquad (A.9)$$

where $f_2(\cdot)$ is a bounded function given by

$$f_2(t_0) = a\|x_c(t_0)\| + \frac{af_1(t_0)}{|\alpha-\beta|}.$$

Equation (A.9) implies that the system (A.5), and therefore the system (4.1), is stable with the given DES δ. This completes the proof of part (i) of Lemma 1.

For the case where the system (2.1) is not centralized controllable and observable, assume that the system has no unstable centralized uncontrollable and/or unobservable modes and strongly connected. Then, by using Kalman's canonical system decomposition form [8] and the similar arguments to the above, it is not difficult to show that there exists a decentralized controller which stabilizes the system (4.1). This completes the proof of Lemma 1.

References

[1] B.D.O. Anderson and D.J. Clements. Algebraic characterization of fixed modes in decentralized control, *Automatia* **17** (1981), 703-712.

[2] B.D.O. Anderson and J.B. Moore. Time-varying feedback laws for decentralized control, *IEEE Trans. Automatic Control* **26** (1981), 1133-1138.

[3] J.P. Corfmat and A.S. Morse. Decentralized control of linear multivariable systems, *Automatica* **12** (1976), 479-495.

[4] E.J. Davison and T.N. Chang. Decentralized stabilization and pole assignment for general proper systems, *IEEE Trans. Automat. Contr.* **35** (1990), 652-664.

[5] Z. Gong and M. Aldeen. On the characterization of fixed modes in decentralized control, *IEEE Trans. Automatic Control* **37** (1992), 1046-1050.

[6] A.N. Gundes and C.A. Desoer. *Algebraic Theory of Linear Feedback Systems with Full and Decentralized Compensators.* Berlin: Springer-Verlag, 1990.

[7] M. Ikeda, H. Maeda and S. Kodama. Estimation and feedback in linear time-varying systems: a deterministic theory, *SIAM J. Control* **13** (1975), 304-326.

[8] R.E. Kalman. Mathematical description of linear dynamical systems, *SIAM J. Control* **1** (1963), 152-192.

[9] H. Kobayashi and T. Yoshikawa. Graph-theoretic approach to controllability and localizability of decentralized control systems, *IEEE Trans. Automatic Control* **27** (1982), 1096-1108.

[10] H. Kobayashi, H. Hanafusa and T. Yoshikawa. Controllability under decentralized information structure, *IEEE Trans. Automatic Control* **23** (1978), 182-188.

[11] P.P. Khargonekar and A.B. Ozguler. Decentralized control and periodic feedback, *IEEE Trans. Automat. Contr.* **39** (1994), 877-882.

[12] S. Lee, M. Meerkov and T. Runolfsson. Vibrational Feedback Control: Zero placement capabilities, *IEEE Trans. Automat. Contr.* **32** (1987), 604-611.

[13] A.B. Ozguler. Decentralized control: A stable proper fractional approach, *IEEE Trans. Automat. Contr.* **35** (1990), 1109-1117.

[14] U. Ozguner and E.J. Davison. Sampling and decentralized fixed modes, *Proc. 1985 American Control Conference* (1985), 257-262.

[15] K. Reinschke. Graph-theoretic characterization of fixed modes in centralized and decentralized control, *Int. J. Control* **39** (1984), 715-729.

[16] M.E. Sezer and D.D. Siljak. Structurally fixed modes, *Systems and Control Letters* **1** (1981), 60-64.

[17] L. Trave, A. Tarras and A. Titli. An application of vibrational control to cancel unstable decentralized fixed modes, *IEEE Trans. Automat. Contr.* **30** (1985), 283-286.

[18] L. Trave, A. Titli and A. Tarras. *Large Scale Systems: Decentralization, Structure Constraints and Fixed Modes.* Berlin: Springer-Verlag, 1989.

[19] K.A. Unyelioglu, A.B. Ozguler and P.P. Khargonekar. Decentralized simultaneous stablisation and reliable control using periodic feedback, *Systems and Control Letters* **18** (1992), 23-31.

[20] S.H. Wang. An example in decentralized control systems, *IEEE Trans. Automat. Contr.* **23** (1989), 938-938.

[21] S.H. Wang and E.J. Davison. On the stabilization of decentralized control systems, *IEEE Trans. Automatic Control* **18** (1973), 473-478.

[22] J.L. Willems. Time-varying feedback for the stabilization of fixed modes in decentralization control systems, *Automatica* **25** (1989), 127-131.

School of Electrical and Electronic Engineering, Nanyang Technological University, Nanyang Avenue, Singapore 639798

Department of Electrical and Electronic Engineering, The University of Melbourne, Parkville, Victoria, Australia 3052

Communicated by Clyde F. Martin