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A Sliding Horizon Feedback Control Problem with Feedforward and Disturbance*

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Introduction

We consider a technique producing a near optimal tracking controller that can be implemented on line. A quadratic performance measure is adopted with a finite horizon. The variational form of the state-costate optimality conditions is approximated using the time finite element method, obtaining thereby an open loop solution. The approximate solution then is used as a control input to the system over the length of time spanned by the subinterval associated with the first time finite element. The state of the controlled system at the end of this subinterval is in turn treated as a new initial condition and the suboptimal control is recomputed for the a finite horizon problem over an interval of the same length but translated by the length of the above subinterval. Using updated trajectory and disturbance information a new open loop solution is computed and employed as described above. This process repeated again and again is referred to as a sliding horizon solution. The method utilizes a fixed gain discrete feedback/feedforward control design in which the feedforward component produces control inputs reflecting the known future of the desired trajectory.

The approximation of the quadratic regulator problem via the time finite element method has been discussed in the literature [1,3,5]. In these cases it appears that the Q-matrix is taken to by symmetric and positive definite. We consider Q only to be semidefinite. An appropriate approximation theory is obtained by perturbing to the positive definite case. We present a rate for the resulting approximations based on the perturbation

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variable. In this way we may justify the consideration of the positive definite case. Engineering applications of the time finite element method have been suggested in [6,12-15].

The literature includes many articles describing tracking control designs. The standard LQR tracking problem is flawed by a required knowledge of the reference trajectory for all future time. Recent research has been aimed at developing methods that provide tracking when the desired trajectory is known only over a finite future interval of time. For example, Lee, et al. [10] describe the use of an instantaneous optimal controller producing a terrain tracking design for an aircraft, without requiring the solution of a two point boundary value problem. Kwon, et al. [8,9] have examined the use of an optimal receding horizon control design for tracking and disturbance rejection. However, the design requires the solution of a Ricatti type equation online. The adaptation of the sliding horizon was first discussed in [13,14]. In [15] the technique of sliding horizon control was extended to systems with previewable disturbances.

In the remainder of this section we pose the basic optimal control problem considered in this study and specify the regularity assumptions we require. In Section 1 we discuss the time finite element approximation for Q positive definite. In Section 2 we develop an approximation theory for Q only semidefinite based on perturbations to the positive definite case. We obtain results that establish convergence and a rate of convergence as the perturbation is allowed to approach zero. In Section 3 estimates are studied for the sliding horizon problem. Finally, in Section 4 we discuss the numerical implementation of the method and present an application to the variational approximation of the terrain tracking problem for an aircraft.

We denote by E^n the vector space $\mathbb{R}^n$ with Euclidean norm $|u| = (\sum_{i=1}^n u_i^2)^{1/2}$. The space of linear operators on E^n we denote by $\mathcal{L}(E^n)$ and the space of linear operators from E^m into E^n by $\mathcal{L}(E^m, E^n)$. Consider control of the system on a sample interval (t_0, T_0)

$$\begin{aligned} x'(t) &= A(t)x(t) + b(t)u(t) + d(t) \text{ in } (t_0, T_0) \\ x(t_0) &= x^0 \,. \end{aligned} \tag{0.1}$$

Although (0.1) is solved on intervals of length $T = T_0 - t_0$, we make the following assumptions:

$$\begin{aligned} &t \mapsto A(t) \text{ belongs to } H^1(0,\infty,\mathcal{L}(E^n)) \cap L^\infty(0,\infty,\mathcal{L}(E^n)) \\ &t \mapsto b(t) \text{ belongs to } H^1(0,\infty,\mathcal{L}(E^m,E^n)) \cap L^\infty(0,\infty,\mathcal{L}(E^m,E^n)) \\ &t \mapsto d(t) \text{ belongs to } H^1(0,\infty;E^n) \cap L^\infty(0,\infty;E^n)\,. \end{aligned} \tag{R.1}$$

The real-valued function $t \mapsto u(t)$ serves as a control with

$$u \in L^2(t_0, T_0; E^m)\,.$$

The solution $t \mapsto x(t) = x(t; u) \in E^n$ associated with the control u has meaning as the unique solution of the integral equation

$$x(t) = x^0 + \int_{t_0}^{t} [A(s)x(s) = b(s)u(s) = d(s)]\, \mathrm{ds}\,. \tag{0.2}$$

Hence, x belongs to $H^1(t_0, T_0; E^n)$. The control criterion considered in this study is

$$J(u) = \int_{t_0}^{T_0} \{(x(t;u) - z(t))^* Q(t) x(t;u) - z(t)) \\ + u(t)^* R(t) u(t)\}\, \mathrm{dt}, \tag{0.3}$$

where the superscript "*" denotes the transpose of a vector or matrix. In general, for each t, $Q(t)$ is assumed to be a positive semidefinite symmetric $n \times n$ matrix unless otherwise specified, and $R(t)$ is a positive definite symmetric $m \times m$ matrix. Hence, there exists a positive real number ρ such that for any $t \in [0, \infty]$ and any $v \in E^m$

$$v^* R(t) v \geq \rho |v|^2\,. \tag{0.4}$$

We assume that

$$\begin{array}{l} t \mapsto Q(t) \text{ is in } H^1(0,\infty;\mathcal{L}(E^n)) \cap L^\infty(0,\infty;\mathcal{L}(E^n)) \\ t \mapsto R(t) \text{ is in } H^1(0,\infty;\mathcal{L}(E^m)) \cap L^\infty(0,\infty;\mathcal{L}(E^m)) \\ t \mapsto z(t) \text{ is in } H^1(0,\infty;E^n). \end{array} \tag{R.2}$$

Define the following quantities:

$$\mu_A = \|A\|_{L^\infty(0,\infty;\mathcal{L}(E^n))}\,,$$
$$\mu_Q = \|Q\|_{L^\infty(0,\infty;\mathcal{L}(E^n))}\,,$$
$$\mu_b = \|b\|_{L^2(0,\infty;\mathcal{L}(E^m,E^n))}\,,$$

and

$$\mu_d = \|d\|_{L^2(0,\infty;E^n)}\,.$$

Given $0 \leq t_0 < T_0 < +\infty$ and the interval (t_0, T_0), we use the notation

$$H = L^2(t_0, T_0; E^n)\,, \quad U = L^2(t_0, T_0; E^m)\,, \quad \text{and} \quad V = H^2(t_0, T_0; E^n)$$

to denote the Hilbert spaces without explicit reference to the particular interval unless it is necessary to avoid ambiguities.

Finally, the linear quadratic regulator problem is posed as the minimization problem given by

$$\begin{array}{l} \text{Find } u_0 \in L^2(t_0, T_0; E^m) \text{ such that} \\ J(u_0) = \text{infimum}\{J(y) : u \in L^2(t_0, T_0; E^m)\}\,. \end{array} \tag{0.5}$$

Furthermore, it is well-known that this problem has a unique solution under the assumptions expressed above [11].

1 Preliminaries

In this section we consider without loss of generality the time finite element approximation of optimal controls on the interval $(0,T)$. We assume that the matrix-valued function $t \mapsto Q(t)$ is symmetric and positive definite on $[0,T]$. The finite element solution in this case is investigated by Borsarge and Johnson [1]. The approach here is similar to that in [1] in that we also focus on a weak formulation of the optimality conditions, (1.16). In [1] this formulation is used to obtain error estimates for time finite element approximations of the optimal control. We include this discussion since it provides the framework of elliptic boundary value problems that motivates our analysis in the next section where Q is assumed only to be semidefinite. This viewpoint enables us to provide useful estimates to improve the error estimates in [1] and to analyze the sliding horizon approach. Accordingly, we assume there exists a positive number ϵ_1 such that for each $t \in [0,T]$ and any $u \in E^n$

$$u^* Q(t) u \geq \epsilon_1 |u|^2 \,. \tag{1.1}$$

Our starting point is the well-known optimality conditions for the optimal control problem (0.5).

$$\begin{aligned} x' = A(t)x + b(t)u + d(t) \text{ in } (0,T) \\ x(0) = x^0 \end{aligned} \tag{1.2}$$

$$\begin{aligned} -p' = A(t)^* p + Q(t)(x-z) \text{ in } (0,T) \\ p(T) = 0 \end{aligned} \tag{1.3}$$

$$u = -R^{-1}(t) b(t)^* p \text{ in } (0,T). \tag{1.4}$$

Hence, the state-costate system is given by

$$x' = Ax - Bp + d \text{ in } (0,T) \tag{1.5}$$

$$-p' = A^* p + Q(x-z) \text{ in } (0,T) \tag{1.6}$$

with initial conditions and final conditions

$$x(0) = x^0 \qquad p(T) = 0 \tag{1.7}$$

where

$$B = bR^{-1}b^* \,,$$

and where we have suppressed the dependence on t. Since these equations are obtained from (1.2)–(1.4), there exists a solution to (1.5)–(1.7). We note that B defined in (1.7) is symmetric and positive semidefinite. We give for reference in the following sections estimates for x and p based on Gronwall inequality arguments.

Lemma 1.1 *The solutions x and p of the initial value problems (1.2) and (1.3) satisfy the inequalities*

$$|x(t)| \le (|x^0| + \mu_b \|u\|_U + \mu_d) \exp(\mu_A t) \tag{1.8}$$

and

$$|p(t)| \le (\mu_Q \|x\|_H + \|z\|_H) \exp(\mu_A (T-t)) , \tag{1.9}$$

respectively for any $t \in [0,T]$.

Since Q is positive definite, we may solve (1.6) for x

$$x = -Q^{-1}(p' + A^* p) + z$$

and substitute into (1.5) to obtain the second-order equation

$$-(Q^{-1}(p' + A^* p))' + AQ^{-1}(p' + A^* p) + Bp = \tilde{d} \tag{1.10}$$

where $\tilde{d} = -z' + Az + d$. We supplement equation (1.10) with boundary conditions

$$\begin{aligned} p'(0) + A^* p(0) &= Q(z(0) - x^0) \\ p(T) &= 0 \,. \end{aligned} \tag{1.11}$$

Define the spaces

$$V_0 = V_{0R} = \{\psi \in V : \psi(T) = 0\}$$

and

$$V_{0L}\{\psi \in V : \psi(0) = 0\}$$

with the norm

$$\|\psi\|_{V_0}^2 = \int_0^T |\psi'(t)|^2 \, \mathrm{dt} \,.$$

and the norm for V_{0L} defined similarly. We note, for example, for any $\psi \in V_{0L}$ and for any $t \in [0,T]$

$$|\psi(t)| \le \frac{1}{2} T^{1/2} \|\psi\|_{V_{0L}} \tag{1.12}$$

and

$$\|\psi\|_H \le \frac{2}{\pi} T \|\psi\|_{V_{0L}} \tag{1.13}$$

with similar inequalities for V_0. Further, the symmetric bilinear form on V_0 given by

$$[\psi,\varphi]=\int_0^T\{(\psi'(t)+A(t)^*\psi(t))^*Q^{-1}(t)(\varphi'(t)+A(t)^*\varphi(t)) + \psi^*(t)B(t)\varphi(t)\}\,\mathrm{dt} \tag{1.14}$$

is continuous. Certainly, it is clear that there is a positive number κ_1 such that

$$[\varphi,\psi]\le\kappa_1\|\varphi\|_{V_0}\|\psi\|_{V_0}\,. \tag{1.15}$$

We may now give the weak formulation of (1.10)–(1.11) as the variational problem:

Find $p\in V_0$ such that

$$[p,\psi]=\int_0^T\tilde{d}^*\psi\,\mathrm{dt}+(z(0)-x_0)^*\psi(0) \tag{1.16}$$

for any $\psi\in V_0$.

We note that

$$\begin{aligned}[p,p]&=\int_0^T\{(x(t)-z(t))^*Q(t)(x(t)-z(t))+p(t)^*B(t)p(t)\}\,\mathrm{dt}\\ &\ge\epsilon_1\|x-z\|_H^2\end{aligned}$$

and

$$[p,p]\ge\epsilon_1\|x-z\|_H^2\,. \tag{1.17}$$

On the other hand, from Lemma 1.1 we may obtain the estimate that

$$|p(t)|\le\mu_Q\|x-z\|_H\exp(\mu_AT)\,,$$

and it follows that

$$\|p\|_H\le\mu_QT^{1/2}\exp(\mu_AT)\|x-z\|_H\,. \tag{1.18}$$

Finally, it follows that

$$\begin{aligned}\int_0^T|p'(t)|^2\,\mathrm{dt}&=\int_0^T(A^*p+Q(x-z))^*(A^*p+Q(x-z))\,\mathrm{dt}\\ &\le\int_0^T\{p^*AA^*p+(x-z)^*Q^2(x-z)\\ &\quad+p^*AQ(x-z)+(x-z)^*QA^(p\}\,\mathrm{dt}.\end{aligned}$$

Applying Cauchy's inequality, there is positive constant κ such that

$$\int_0^T |p'(t)|^2 \,\mathrm{dt} \leq \kappa \int_0^T (|p(t)|^2 + |x-z|^2)\,\mathrm{dt}.$$

Applying (1.18), we have then

$$\int_0^T |p'(t)|^2 \,\mathrm{dt} \leq \kappa(\mu_Q^2 T \exp(2\mu_A T)_1)\|x-z\|_H^2 \,,$$

and from (1.17) it follows that

$$\int_0^T |p'(t)|^2 \,\mathrm{dt} \leq \kappa_0[p,p] \tag{1.19}$$

with $\kappa_0 = \kappa(\mu_Q^2 T \exp(2\mu_A T) + 1)/\epsilon_1$.

The following estimates hold for the solution of (1.16).

Proposition 1.1 *Let p be the solution of (1.16). Then*

$$\|p\|_V \leq \kappa_0 C(\tilde{d}, T, x^0, z) \tag{1.20}$$

where

$$C(\tilde{d}, T, x^0, z) = \frac{2}{\pi} T\|\tilde{d}\|_H + \frac{1}{2} T^{1/2}|x^0 - z(0)|\,.$$

From (1.10), (1.20), and the differentiability assumptions on Q, A and B, we have the following result from (1.6).

Proposition 1.2 *Let p be the solution of (1.16) and let (R.1) and (R.2) hold. Then*

$$\|p''\|_H \leq C_0(\tilde{d}, T, x^0, z) \tag{1.21}$$

where $C_0(\tilde{d}, T, x^0, z)$ may also depend on the A, Q^{-1}, B, and their derivatives.

Finally, it is useful to estimate the dependence of p upon the initial condition. This may be accomplished by an argument similar to that above. Hence, for $i = 1, 2$ we have

$$\begin{aligned} -(Q^{-1}(p_i' + A^* p_i))' + AQ^{-1}(p_i' + A^* p_i) + Bp_i &= \tilde{d} \\ p_i(T) &= 0 \\ p_i'(0) + A^* p_i(0) &= Q(z(0) - x_i^0)\,. \end{aligned} \tag{1.22}$$

Proposition 1.3 *Let p_1 and p_2 be solutions of (1.22) for x_1^0 and x_2^0, respectively. Then*

$$\|p_1 - p_2\|_{V_0} \leq \frac{1}{2}\kappa_0 T^{1/2}|x_1^0 - x_2^0|\,. \tag{1.23}$$

Proof: Setting $\rho = p_1 - p_2$ and $\xi = x_1^0 - x_2^0$, we obtain the boundary value problem

$$-(Q^{-1}(\rho' + A^*\rho))' + Aq^{-1}(\rho' + A^*\rho) + B\rho = 0$$
$$\rho(T) = 0$$
$$\rho'(0) + A^*\rho(0) = Q\xi\,.$$

Using the same argument as that used to obtain (1.20) we obtain the result. □

Corollary 1.4 *The solution to (1.5)–(1.7) is unique.*

For the purpose of approximation, we introduce a finite dimensional subspace ${V_0}^N$ of V_0, and we look for a solution p^N such that

$$[p^N, \psi] = \int_{t_0}^{T_0} \tilde{d}^*\psi \,\mathrm{dt} + ((x^0 - z(0))^*\psi(0)$$

for all $\psi \in {V_0}^N$. By Cea's lemma [4] we obtain

$$\|p - p^N\|_{V_0} \leq (\kappa_1^2/\kappa_0)^{1/2} \inf\{\|\psi - p\|_{V_0} : \psi \in {V_0}^N\}. \tag{1.24}$$

We make the following assumption on the approximation space ${V_0}^N$.

The subspace ${V_0}^N$ has the property that there is an operator $I^N : V_0 \mapsto {V_0}^N$ such that if $\varphi \in V_0$ and $\|\varphi''\|_H < +\infty$ cf. [16], then

$$\|I^N\varphi - \varphi\|_{V_0} \leq C(N)\|\varphi''\|_H \tag{A.1}$$

where $C(N) \to 0$ as $N \to +\infty$. We may now obtain the following estimate.

Proposition 1.4 *Let p be the solution of (1.16) and suppose that (R.1), (R.2) and (A.1) hold. Then*

$$\|p - p^N\|_{V_0} \leq (\kappa_1^2/\kappa_0)^{1/2}\, C(N)C_0(d, T, x^0, z)\,. \tag{1.25}$$

Proof: The result follows from (1.24) under the assumption (A.1) and Proposition 1.5. □

We now use

$$u^N = -R^{-1}b^*p^N \tag{1.26}$$

as an approximation of the optimal control. The resulting state x^N is obtained as $x^N = x(\cdot, u^N)$. That is, by solving the initial boundary value problem

$$\begin{aligned} x^{N'}(t) &= A(t)x^N(t) + b(t)u^N(t) + d(t) \\ x^N(0) &= x^0\,. \end{aligned} \tag{1.27}$$

Thus, we may determine bounds on the error in the control and the resulting state from (1.26) and Gronwall's inequality.

Lemma 1.5 *Under the assumptions of Proposition 1.4, we have the estimates*

$$\|u^N - u\|_{V_0} \leq \underline{\rho}\mu_b(\kappa_1^2/\kappa_0)^{1/2}\, C(N)C_0(d,T,x^0,z)\,. \tag{1.28}$$

$$|x^N(t) - x(t)| \leq \mu_a\mu_b \exp(\mu_a T)\int \vee 0|u^N(s) - u(s)|\,\mathrm{ds} \tag{1.29}$$
$$+ \mu_b|u^N(t) - u(t)|$$

and

$$|x^{N'}(t) - x'(t)| \leq \mu_a|x^N(t) - x(t)| + \mu_b|u^N(t) - u(t)|\,. \tag{1.30}$$

Proposition 1.6 *Under the assumption of Proposition 1.4 there exists a positive number $\widetilde{K}$ that depends on x^0, z, d, T, A, Q, b such that*

$$\|u^N - u\|_{V_0} \leq \widetilde{K}\, C(N)$$

$$\|x^N - x\|_{W^1(0,T;E)} \leq \widetilde{K}\, C(N)\,.$$

2 Tracking with Disturbance

In this section we consider the case in which the matrix Q is only symmetric positive semidefinite. Our approach is to perturb the system so that the resulting Q is positive definite. The results from the previous section may now be applied to obtain numerical solutions. We then show that the original problem is the limiting case of the perturbed problems. Further, we obtain an estimate of the perturbation error.

Our starting point is the system of state-costate equations (1.5)–(1.7) that we restate as

$$\begin{aligned} x' - Ax + Bp &= d \text{ in } (0,T) \\ p' + A^*p + Qx &= Qz \text{ in } (0,T) \\ x(0) &= x^0 \\ p(T) &= 0\,. \end{aligned} \tag{2.1}$$

We note that for each $t \in [0,T]$, the matrix $B(t) = b(t)R^{-1}b(t)^*$ is an $n \times n$ positive semidefinite matrix.

Remark 2.1 As noted, there is a unique solution to (0.5). It is characterized by the optimality conditions of the form (1.2)–(1.4). This does not depend on the positive definiteness of Q. Thus, we conclude there exists a solution to (2.1).

To solve the system (2.1) numerically, we look for a solution of the weak formulation of (2.1). First, we change variables to obtain a homogeneous boundary condition by setting $x = y + x_0$. In this way we obtain the system

$$\begin{aligned} y' - Ay + Bp &= d + Ax^0 \\ p' + A^*p + Qy &= -Qx^0 \\ y(0) &= 0 \\ p(T) &= 0. \end{aligned} \tag{2.2}$$

Define the space

$$\mathbb{W} = V_{0L} \times V_{0R}\,.$$

To obtain the weak formulation of (2.1), let $\psi \in V_{0R}$ and $\varphi \in V_{0L}$ and consider

$$\begin{aligned} \int_0^T \psi^*(y' - Ay + Bp)\,\mathrm{dt} + \int_0^T \varphi^*(p' + A^*p + Qy)\,\mathrm{dt} \\ = \int_0^T \{\psi^*(d + Ax^0) - \varphi^* Qx^0\}\,\mathrm{dt}\,. \end{aligned} \tag{2.3}$$

We may rewrite (2.3) in terms of a bilinear form on $\mathbb{W}$. Setting $\Phi = \mathrm{co}\ell(\varphi, \psi)$ and $Y = \mathrm{co}\ell(y, p)$ in $\mathbb{W}$, the bilinear form is given by

$$\mathcal{A}(\Phi, Y) = \int_0^T [\varphi^* \psi^*] \begin{bmatrix} q & \frac{d}{\mathrm{dt}} + A^* \\ \frac{d}{\mathrm{dt}} - A & B \end{bmatrix} \begin{bmatrix} y \\ p \end{bmatrix} \mathrm{dt}\,. \tag{2.4}$$

The weak form of the problem may be written more compactly as

Find $Y \in \mathbb{W}$ such that for any $\Phi \in \mathbb{W}$

$$\mathcal{A}(\Phi, Y) = \langle \Phi, f \rangle \tag{2.5}$$

where

$$f = \mathrm{co}\ell(-Qx^0, d + Ax^0)\,.$$

The solvability of (2.5) depends on whether $\mathcal{A}(\cdot,\cdot)$ is positive definite. For the present case, we have

$$\mathcal{A}(Y, Y) = \int_0^T \{p^*(y' - Ay + Bp) + y^*(p' + A^*p + Qy)\}\,\mathrm{dt}\,.$$

Integrating the first term by parts, we see that

$$\mathcal{A}(Y,Y) = \int_0^T \{y^* Q y + p^* B p\}\,\mathrm{dt}\,.$$

We note that the matrices Q and B in (2.6) are symmetric but only positive semidefinite. Hence, the uniqueness and well-posedness of solutions to (2.5) cannot be guaranteed. (We already know that there exists a solution.)

As an alternative, we consider the regularization of the optimality conditions (2.2) by introducing the matrix-valued function

$$Q_\epsilon(t) = Q(t) + \epsilon I$$

in place of Q. Thus, we obtain

$$\begin{aligned} x' - Ax + Bp &= d \text{ in } (0,T) \\ p' + A^* p + Q_\epsilon x &= Q_\epsilon z \text{ in } (0,T) \\ x(0) &= x^0 \\ p(T) &= 0\,. \end{aligned} \tag{2.7}$$

Of course these equations are the state-costate equations of the optimal control problem

Find $u_\epsilon \in U$ such that

$$J_\epsilon(u_\epsilon) = \text{infimum}\{J_\epsilon(u) : u \in U\} \tag{2.8}$$

where

$$J_\epsilon(u) = \int_0^T \{(x(u) - z)^* Q_\epsilon (x(u) - z) + u^* R u\}\,\mathrm{dt}\,. \tag{2.9}$$

Note that the perturbation is only in the functional and does not involve the equation (0.1). Hence, it follows that

$$J_\epsilon(u) = J(u) + \epsilon \|x(u) - z\|_H^2\,. \tag{2.10}$$

The optimality conditions are given by

$$\begin{aligned} x_\epsilon' - Ax_\epsilon - bu_\epsilon &= d \text{ in } (0,T) \\ p_\epsilon' + A^* p_\epsilon + Q_\epsilon x_\epsilon &= Q_\epsilon z \text{ in } (0,T) \\ u_\epsilon &= -R^{-1} b^* p_\epsilon \text{ in } (0,T) \\ x(0) &= x^0 \\ p(T) &= 0\,, \end{aligned} \tag{2.11}$$

and (2.8)–(2.9) has a unique solution.

For $\epsilon > 0$, the existence of a unique solution to (2.11) follows from the results of Section 1.

Lemma 2.1 *For any $\epsilon > 0$ there exist a unique solution (x_ϵ, p_ϵ) to the state-costate equations (2.11). Furthermore, defining the symmetric bilinear form*

$$[\psi, \varphi]_\epsilon = \int_0^T \{(\psi'(t) + A(t)^*\psi(t))^* Q_\epsilon^{-1}(t)(\varphi'(t) + A(t)^*\varphi(t))$$
$$\psi(t)^* B(t)\varphi(t)\}\,\mathrm{dt}\,,$$

we obtain the estimate

$$\|p'_\epsilon\|_H^2 \leq \frac{\kappa}{\epsilon} C(d, T, x^0, z)\,. \tag{2.12}$$

We now investigate the convergence of the solutions of (2.7) as $\epsilon \to 0$. Our approach is to use the result that for fixed $\epsilon \geq 0$, u_ϵ is the unique solution of (2.8).

Proposition 2.2 *Let $\epsilon \to 0$. Then $x(u_\epsilon) \to x(u_0)$ and $p_\epsilon \to p_0$ in E^n uniformly for $t \in [0,T]$ and $u_\epsilon \to u_0$ in E^m uniformly in $[0,T]$.*

Proof: In the case of null control in which $u =$ the zero function θ, we note

$$J_\epsilon(\theta) = J(\theta) + \epsilon\|x(\theta) - z\|_H^2$$
$$\geq \rho \int_0^T |u_\epsilon(t)|^2\,\mathrm{dt}\,.$$

Hence, it follows that there is a sequence $\{u_{\epsilon_i}\}_{i=1}^\infty$ such that $u_{\epsilon_i} \to \tilde{u}_0$ weakly in H. Further, it follows from Gronwall's inequality that the sequence $\{x(u_{\epsilon_i}\}_{i=1}^\infty$ converges strongly in H to $x(\tilde{u}_0)$. From (2.10) it follows that for any $u \in H$

$$J_\epsilon(u) \geq J_\epsilon(u_\epsilon) = J(u_\epsilon) + \epsilon\|x(u_\epsilon) - z\|_H^2\,.$$

The weak lower semicontinuity of the Hilbert space norm implies that for any $u \in H$

$$J(u) \geq \underline{\lim} J_\epsilon(u_\epsilon) \geq J(\tilde{u}_0)\,.$$

We conclude that $\tilde{u}_0$ is the unique optimal control for the unperturbed problem, and in fact $u_\epsilon \to u_0$ weakly as $\epsilon \to 0$.

On the other hand, the solution of the adjoint equation in (2.11) satisfies

$$p_\epsilon(t) = -\int_t^T \{A^*(s)p_\epsilon(s) + Q_\epsilon(x(s; u_\epsilon) - z(s))\}\,\mathrm{ds}\,.$$

Thus, $x(u_\epsilon) \to x(u_0)$ in H implies again from Gronwall's inequality that $p_\epsilon \to p_0$ uniformly in $t \in [0,T]$ as $\epsilon \to 0$. Now appealing to equation (2.11) we see that, in fact, it must be true that

$$u_\epsilon \to u_0$$

uniformly in $[0,T]$ as $\epsilon \to 0$. Continuing it follows that $x_\epsilon \to x_0$ uniformly in $[0,T]$. □

The above theorem yields the convergence of u_ϵ to u_0 as ϵ approaches zero. In fact, we can determine a rate of convergence. Since we know from the previous section that the state-costate system of the perturbed problem has a unique solution, our approach is to examine the terms in a formal ϵ-series expansion of x_ϵ and p_ϵ. Specifically, consider the expansions

$$x_\epsilon = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots + \epsilon^\ell x_\ell + \cdots$$

and

$$p_\epsilon = p_0 + \epsilon p_1 + \epsilon^2 p_2 + \cdots + \epsilon^\ell p_\ell + \cdots .$$

We obtain conditions such that the above series converge in H. Hence, substituting these expansions into the equations of (2.7) and collecting like terms, we formally obtain the following systems:

For $\ell = 0$

$$\begin{aligned} x_0' - Ax_0 + Bp_0 &= d \\ p_0' + A^* p_0 + Qx_0 &= Qz \\ x_0(0) &= x^0 \\ p_0(T) &= 0\,, \end{aligned} \tag{2.14}$$

for $\ell = 1$

$$\begin{aligned} x_1' - Ax_1 + Bp_1 &= 0 \\ p_1' + A^* p_1 + Qx_1 + x_0 &= z \\ x_1(0) &= 0 \\ p_1(T) &= 0\,, \end{aligned} \tag{2.15}$$

and for $\ell \geq 2$

$$\begin{aligned} x_\ell' - Ax_\ell + Bp_\ell &= 0 \\ p_\ell' + A^* p_\ell + Qx_\ell + x_{\ell-1} &= 0 \\ x_\ell(0) &= 0 \\ p_\ell(T) &= 0\,. \end{aligned} \tag{2.16}$$

It will be useful also to state (2.14)–(2.16) in terms of the corresponding systems of integral equations. For $\ell = 0$

$$\begin{aligned} x_0(t) &= x^0 + \int_0^t [A(s)x_0(s) - B(s)p_0(s) + d(s)]\,\mathrm{ds} \\ p_0(t) &= -\int_t^T [A(s)^* p_0(s) + Q(s)(x_0(s) - z(s))]\,\mathrm{ds} \end{aligned} \tag{2.17}$$

for $\ell = 1$

$$\begin{aligned} x_1(t) &= \int_0^t [A(s)x_1(s) - B(s)p_1(s)]\,\mathrm{ds} \\ p_1(t) &= -\int_t^T [A(s)^* p_1(s) + Q(s)x_1(s) + (x_0(s) - z(s))]\,\mathrm{ds} \end{aligned} \tag{2.18}$$

for $\ell \geq 2$

$$\begin{aligned} x_\ell(t) &= \int_0^t [A(s)x_\ell(s) - B(s)p_\ell(s)]\,\mathrm{ds} \\ p_\ell(t) &= -\int_t^T [A(s)^* p_\ell(s) + Q(s)x_\ell(s) + x_{\ell-1}(s))]\,\mathrm{ds}\,. \end{aligned} \tag{2.19}$$

Theorem 2.4 *Let K_0 be such that*

$$\begin{aligned} K_0 \geq \max\{&\|x_0\|_H, \|x_1\|_H, \|x_1\|_{L^\infty(0,T;\mathbb{R}^n)}, \|x_1\|_{L^\infty(0,T;\mathbb{R}^n)}, \\ &\|p_0\|_H, \|p_1\|_H, \|p_0\|_{L^\infty(0,T;\mathbb{R}^n)}, \|p_1\|_{L^\infty(0,T;\mathbb{R}^n)}\}, \\ K &= \exp(\mu_A T)\,, \\ \mu &= \max\{1, \mu_Q\}\,, \end{aligned} \tag{2.20}$$

and without loss of generality let

$$\alpha = 2\mu_b K^2 T/\rho > 1\,.$$

Then for any $\epsilon \in [0, \frac{1}{\alpha})$

$$\|x_\epsilon - x_0\|_H \leq \epsilon K_0$$

and

$$\|p_\epsilon - p_0\|_H \leq \epsilon K_0(1 + 2\mu K T)\,.$$

We present the proof of this theorem through several lemmas.

Lemma 2.5 *Let $\ell \geq 2$ then*

$$\|x_\ell\|_H \leq 2\mu_b^2 K^2 \rho^{-1} T \|x_{\ell-1}\|_H \tag{2.21}$$

and

$$\|p_\ell\|_H \leq \mu K T^{1/2} (\|x_\ell\|_H + \|x_{\ell-1}\|_H)\,. \tag{2.22}$$

Proof: Consider the ℓth system given in (2.16) and observe that this system constitutes the state-costate system for the following control problem. The underlying control system is given by

$$\begin{aligned} x' - Ax &= bu \\ x(0) &= 0 \end{aligned} \tag{2.23}$$

with the optimization problem given by:

$$\text{Find } u_\ell \in U \text{ such that } J_\ell(u_\ell) = \inf\{J_\ell(u) : u \in U\} \tag{2.24}$$

where $J_\ell(\cdot)$ is given by

$$J_\ell(u) = \int_0^T \{x^*(Qx - 2x_{\ell-1}) + u^* R u\}\,\mathrm{dt}\,. \tag{2.25}$$

Clearly, there exists a unique solution u_ℓ to the problem (2.23)–(2.25) with the optimality system in which $x_\ell = x(u_\ell)$ on $(0,T)$ is given by

$$\begin{aligned} x_\ell' - Ax_\ell &= bu_\ell \\ p_\ell' + A^* p_\ell + Qx_\ell &= -x_{\ell-1} \\ x_\ell(0) &= 0 \\ p_\ell(T) &= 0 \\ u_\ell &= -R^{-1} b^* p_\ell\,. \end{aligned} \tag{2.26}$$

Applying inequality (1.8) with $x^0 = 0$ and $d = 0$ and inequality (1.9) with $z = x_{\ell-1}$, we see that

$$\|x_\ell\|_H \leq \mu_b K T^{1/2} \|u_\ell\|_U \tag{2.27}$$

and

$$\|p_\ell\|_H \leq \mu K T^{1/2} (\|x_\ell\|_H + \|x_{\ell-1}\|_H)$$

which gives (2.22). Now from the optimality of u_ℓ, it follows that with $\theta =$ the zero control

$$0 = J_\ell(\theta) \geq J_\ell(u_\ell)\,.$$

Hence, we find that

$$0 \geq -2\|x_{\ell-1}\|_H \|x_\ell\|_H + \rho \|u_\ell\|_U^2 \,.$$

From the inequality (2.27) it follows

$$\|u_\ell\|_U \leq 2\rho^{-1}\mu_b K T^{1/2} \|x_{\ell-1}\|_H \,.$$

Substituting back into (2.27), we obtain the recursion estimate (2.21). □

From the recursion relation above and the assumptions of Theorem 2.4, we have the following result.

Lemma 2.6 *Under the hypotheses of Theorem 2.4 and with*

$$\alpha = 2\mu_b^2 K^2 \rho^{-1} T \tag{2.28}$$

the following hold

$$\begin{aligned} \|x_0\|_H &\leq K_0 \\ \|x_1\|_H &\leq K_0 \\ \|x_2\|_H &\leq \alpha K_0 \\ &\vdots \\ \|x_\ell\|_H &\leq \alpha^{\ell-1} K_0 \end{aligned}$$

and

$$\begin{aligned} \|p_0\|H &\leq K_0 \\ \|p_1\|_H &\leq K_0 \\ \|p_2\|_H &\leq \mu K T \alpha K_0 \\ &\vdots \\ \|p_\ell\|_H &\leq 2\mu K T \alpha^{\ell-1} K_0 \,. \end{aligned}$$

Let us define the nth partial sums in H

$$\mathcal{X}^n = x_0 + \epsilon x_1 + \cdots + \epsilon^n x_n \tag{2.29}$$

and

$$\mathcal{P}^n = p_0 + \epsilon p_1 + \cdots + \epsilon^n p_n \,. \tag{2.30}$$

Lemma 2.7 *Let $\epsilon \in (0, 1/\alpha)$ where α satisfies (2.28) and let the hypotheses of Theorem 2.4 hold. Then the series given in (2.29) and (2.30) converge in H to x_ϵ and p_ϵ, respectively as $n \to \infty$.*

Proof: Since $\epsilon\alpha \in (0,1)$, we see that the series (2.29) and (2.30) are absolutely convergent. From the completeness of H of H, it follows that the series converge strongly in H. Thus,

$$\mathcal{X}^n \to \mathcal{X} \text{ and } \mathcal{P}^n \to \mathcal{P} \text{ in } H\,.$$

By summing the integral equations $\ell = 0,\, 1, \dots, n$, we see that

$$\begin{aligned} \mathcal{X}^n(t) &= x^0 + \int_0^t [A(s)\mathcal{X}^n(s) - B(s)\mathcal{P}^n(s) + d(s)]\,\mathrm{ds} \\ \mathcal{P}^n(t) &= -\int_t^T [A(s)^*\mathcal{P}^n(s) + Q(s)(\mathcal{X}^n(s) - z(s))]\,\mathrm{ds}\,. \end{aligned} \tag{2.31}$$

It follows that in the limit (2.27) yields

$$\begin{aligned} \mathcal{X}(t) &= x^0 + \int_0^t [A(s)\mathcal{X}(s) - B(s)\mathcal{P}(s) + d(s)]\,\mathrm{ds} \\ \mathcal{P}(t) &= -\int_t^T [A(s)^*\mathcal{P}(s) + Q(s)(\mathcal{X}(s) - z(s))]\,\mathrm{ds}\,. \end{aligned} \tag{2.32}$$

In fact (2.32) implies that $\mathcal{X}$ and $\mathcal{P}$ are solutions of (2.7). Hence, from uniqueness we see that $x_\epsilon = \mathcal{X}$ and $p_\epsilon = \mathcal{P}$. □

We now give the proof of Theorem 2.4.

Proof of Theorem 2.4: Consider the expansions

$$x_\epsilon - x_0 = \epsilon x_1 + \epsilon^2 x_2 + \cdots$$

and

$$p_\epsilon - p_0 = \epsilon p_1 + \epsilon^2 + \cdots\,.$$

From Lemma 2.6, we see that

$$\begin{aligned} \|x_\epsilon - x_0\|_H &\le \epsilon K_0 + \epsilon^2 \epsilon K_0 + \cdots + \epsilon^\ell \alpha^{\ell-1} K_0 + \cdots \\ &\le \epsilon K_0 \{1 + \epsilon\alpha + \cdots + (\epsilon\alpha)^{\ell-1} + \cdots\} \\ &\le \epsilon K_0 \frac{1}{1-\epsilon\alpha}\,. \end{aligned}$$

For p_ϵ we have similarly

$$\begin{aligned} \|p_\epsilon - p_0\|_H &\le \epsilon K_0 + \epsilon^2 (2\mu K T \alpha K_0) + \cdots + \\ &\quad + \epsilon^\ell (2\mu K T \alpha^{\ell-1} K_0) + \cdots \\ &\quad \le \epsilon K_0 \{1 + 2\mu K T \frac{\epsilon\alpha}{1-\epsilon\alpha}\}\,, \end{aligned}$$

and the result follows. ◻

Finally, we obtain the estimates assumed in Theorem 2.4.

Lemma 2.8 *There exists a positive number K_0 satisfying (2.20) that depends only on T, μ_A, μ_Q, μ_b, μ_d, x^0, z, and ρ.*

Proof: Consider the optimality conditions

$$\begin{aligned} x_0' - Ax_0 - bu_0 &= d \\ x_0(0) &= x^0 \\ p_0' + A^* p_0 + Qx_0 &= Qz \\ p_0(T) &= 0 \\ u_0 &= -R^{-1} b^* p_0 \,. \end{aligned}$$

The control problem associated with this system is given as follows. The underlying system is

$$\begin{aligned} x' - Ax - bu &= d \\ x(0) &= x^0 \end{aligned}$$

with optimization problem

$$\text{Find } u_0 \in U \text{ such that } J_0(u_0) = \inf\{J_0(u) : u \in U\}$$

where $J_0(\cdot)$ is given by

$$J_0(u) = \int_0^T [(x-z)^* Q (x-z) + u^* R u] \, \mathrm{dt} \,.$$

From the optimality of u_0 we see that, with θ the zero element in U,

$$\widehat{K}_0 = J_0(\theta) \geq J_0(u_0) \geq \rho \|u_0\|_U^2 \,.$$

From Gronwall's inequality Lemma 1.1 with $x_0 = x(u_0)$

$$|x_0(t) \leq (|x^0| + \mu_b (\widehat{K}_0/\rho)^{1/2} + \mu_d) \exp(\mu_A T) = K_{01}$$

and

$$|p_0(t)| \leq T^{1/2} \mu_Q (\|x_0\|_H + \|z\|_H) \exp(\mu_A T) = K_{02}$$

for any $t \in [0,T]$.

By a similar argument, we obtain positive constants K_{11} and K_{12} such that

$$|x_1(t)| \leq K_{11}$$

and

$$|p_1(t)| \le K_{12}$$

for any $t \in [0, T]$. Finally, choose K_0 such that

$$K_0 = \max\{K_{01}, K_{02}, T^{1/2}K_{01}, T^{1/2}K_{02},\\ K_{11}, K_{12}, T^{1/2}K_{11}, T^{1/2}K_{12}\}$$

to obtain the result. ◻

Corollary 2.9 *Let the assumptions of Theorem 2.4 hold. Then there is a positive constant K_1 such that*

$$\|x'_\epsilon - x'_0\|_H \le \epsilon K_1 \,,$$
$$\|p'_\epsilon - p'_0\|_H \le \epsilon K_1 \,,$$

and

$$\|p''_\epsilon - p''_0\|_H \le \epsilon K_2$$

where K_1 and K_2 depend only on the problem parameters and $\epsilon \in (0, 1/\alpha)$ where α satisfies (2.28).

Proof: This result follows immediately by applying Theorem 2.4 to (2.11) and (2.1). ◻

Remark 2.10 Note that this improves the estimate on p_ϵ obtained above from elliptic estimates. Thus, we have

$$\|p'_\epsilon\|_H \le \widetilde{K}_1$$

and

$$\|p''_\epsilon\|_H \le \widetilde{K}_2$$

where $\widetilde{K}_1$ and $\widetilde{K}_2$ depend on problem parameters and are independent of ϵ.

To approximate numerically the solution of (2.1), we approximate the associated perturbed problem for $\epsilon \in (0, 1/\alpha)$. This allows us to utilize the theory from Section 1 along with the estimates of Theorem 2.4 and Corollary 2.9.

Theorem 2.11 *Let $\epsilon \in (0, 1/\alpha)$ and suppose that (R.1), (R.2), and (A.1) hold. Then*

$$\|p_0 - p^N_\epsilon\|_{V_0} \le \epsilon K_0 + C(N)\widetilde{K}_2 \,.$$

Proof: From the above estimates and (A.1), we have

$$\begin{aligned}\|p_0 - p^N_\epsilon\|_{V_0} &\le \|p_0 - p_\epsilon\|_{V_0} + \|p_\epsilon - p^N_\epsilon\|_{V_0}\\ &\le \epsilon K_0 + C(N)\|p''_\epsilon\|_H \,.\end{aligned}$$

From Remark 2.10 the result follows. □

Finally, if we set $\epsilon = C(N)$, we immediately have the approximation result.

Corollary 2.12 *Let N be sufficiently large that $C(N) \in (0, 1/\alpha)$ and suppose that (R.1), (R.2), and (A.1) hold. Then with $\epsilon = C(N)$*

$$\|p_0 - p_\epsilon^N\|_{V_0} \leq C(N)K_3$$

where $K_3 = K_0 + \widetilde{K}_2$.

3 Approximate Sliding Horizon Control

Estimate on the Approximate Sliding Horizon Control

The estimates of the previous section under the assumptions (R.1) and (R.2) depend only on the length of the time interval (as well as d, x^0, and z) and not on its location. In this section we consider the relation between the optimal controls on subintervals (t_0, T_0) and (t_1, T_1) of $(0, +\infty)$ that are of the same length

$$T = T_0 - t_0 = T_1 - t_1$$

but overlap so that

$$t_0 \leq t_1 \leq T_0 \leq T_1$$

and are such that the initial condition of the problem on (t_1, T_1) is the state of the problem on (t_0, T_0) evaluated at $t = t_1$. Denote by $\tau = T_1 - T_0 = t_1 - t_0$. We assume that

$$Q = Q(t) \text{ satisfies (1.1) for each } t \in [0, +\infty)$$

and will indicate the differences in results when Q is only positive semidefinite.

On (t_0, T_0) we consider the control system

$$\begin{aligned} x_0' &= Ax_0 + bu + d_0 \\ x_0(t_0) &= x^0 \end{aligned} \tag{3.1}$$

under the criterion

$$J_0(u) = \int_{t_0}^{T_0} \{(x_0(t:u) - z(t))^* Q(t)(x_0(t;u) - z(t)) + u(t)^* R(t) u(t)\} \, \mathrm{dt} \, .$$

While on the interval (t_1, T_1), we have the problem

$$\begin{aligned} x_1' &= Ax_1 + bu + d_1 \\ x_1(t_1) &= x_0(t_1) \end{aligned} \tag{3.2}$$

with criterion

$$\begin{aligned} J_1(u) = \int_{t_1}^{T_1} &\{(x_1(t;u) - z(t))^* Q(t)(x_1(t;u) - z(t)) \\ &+ u(t)^* R(t) u(t)\} \, \mathrm{dt}. \end{aligned}$$

Introducing the adjoint variables p_0 and p_1 for the problems, we obtain state-costate equations

$$x_0' = Ax_0 - Bp_0 + d_0 \text{ in } (t_0, T_0) \tag{3.3}$$
$$x_0(t_0) = x^0$$
$$-p_0' = A^* p_0 + Q(x_0 - z) \text{ in } (t_0, T_0) \tag{3.4}$$
$$p_0(T_0) = 0$$

and

$$x_1' = Ax_1 - Bp_1 + d_1 \text{ in } (t_{1,1}) \tag{3.5}$$
$$x_1(t_1) = x_0(t_1)$$
$$-p_1' = A^* p_1 + Q(x_1 - z) \text{ in } (t_1, T_1)$$
$$p_1(T_1) = 0\,,$$

respectively. Correspondingly, we obtain the boundary value problems

$$\begin{aligned} -(Q^{-1}(p_0' + A^* p_0))' + AQ^{-1}(p_0' + A^* p_0) + Bp_0 &= \tilde{d}_0 \text{ in } \ (t_0, T_0) \\ p_0(T_0) &= 0 \\ p_0'(t_0) + A^* p_0(t_0) &= Q(z(t_0) - x^0), \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} -(Q^{-1}(p_1' + A^* p_1))' + AQ^{-1}(p_1' + A^* p_1) + Bp_1 &= \tilde{d}_1 \text{ in } (t_1, T_1) \\ p_1(T_1) &= 0 \\ p_1'(t_1) + A^* p_1(t_1) &= Q(z(t_1) - x^0(t_1)) \end{aligned} \tag{3.8}$$

respectively, similar to (1.6) in Section 1. Here

$$\tilde{d}_0 = d_0 - z' + Az \text{ in } (t_0, T_0)$$

and

$$\tilde{d}_1 = d_1 - z' + Az \text{ in } (t_1, T_1)\,.$$

On the intersection (t_1, T_0) of the intervals (t_0, T_0) and (t_1, T_1), set

$$x = x_1 - x_0, p = p_1 - p_0, \text{ and } d = \tilde{d}_0 - \tilde{d}_1 .$$

Thus, subtracting the state-costate equations (3.3), (3.4) from (3.5), (3.6), we obtain the system

$$\begin{gathered} x' - Ax - Bp + d \text{ in } (t_1, T_0) \\ x(t_1) = 0 \end{gathered} \tag{3.9}$$

$$\begin{gathered} -p' = A^* p + Qx \text{ in } (t_1, T_0) \\ p(T_0) = p_1(T_0) \end{gathered} \tag{3.10}$$

and the boundary value problem in (t_1, T_0)

$$\begin{aligned} 1(Q^{-1}(p' + A^* p))' + AQ^{-1}(p' + A^* p) + Bp = d \text{ in } (t_1, T_0) \\ p(T_0) = p_1(T_0) \\ p'(t_1) + A^* p(t_1) = 0. \end{aligned} \tag{3.11}$$

It is convenient to introduce $\eta = p - p_1(T_0)$ so that we may consider a problem with homogeneous essential boundary conditions in (t_1, T_0)

$$\begin{gathered} -Q^{-1}(\eta' + A^*\eta)' + AQ^{-1}(\eta' + A^*\eta) + B\eta = \delta \\ \eta(T_0) = 0 \\ \eta'(t_1) + A^*\eta(t_1) = -A^*(t_1)p_1(T_0) \end{gathered} \tag{3.12}$$

where

$$\delta = d + [(Q^{-1}A^*)' - (AQ^{-1}A^* + B)]p_1(T_0) .$$

Define the bilinear form

$$[\eta, \eta] = \int_{t_1}^{T_0} \{(\eta' + A^*\eta)^* Q^{-1}(\eta' + A^*\eta) + \eta^* B\eta\} \, \mathrm{dt} .$$

By arguments analogous to those of Section 1, we may establish the existence of a positive number $\tilde{\kappa}$ that depends on problem parameters ϵ_1, μ_A, μ_Q, μ_b, μ_d, and T such that

$$[\eta, \eta] \geq \tilde{\kappa} \int_{t_1}^{T_0} \eta(t)^* \delta(t) \, \mathrm{dt}.$$

It follows that there exist positive constants K_1 and K_2 depending on problem parameters such that

$$[\eta, \eta] \leq (K_1|p_1(T_0)| + K_2\|d\|_H)(\int_{t_1}^{T_0} |\eta'(t)|^2 \, \mathrm{dt})^{1/2} ,$$

and thus, from (3.13)

$$\tilde{\kappa}(\int_{t_1}^{T_0} |\eta'(t)|^2 \,\mathrm{dt})^{1/2} \le K_1|p_1(T_0)| + K_2\|d\|_H \,.$$

Noting that

$$\int_{t_1}^{T_0} |\eta'(t)|^2 \,\mathrm{dt} = \int_{t_1}^{T_0} |p'(t)|^2 \,\mathrm{dt}\,,$$

we have the result.

Proposition 3.1 *Let Q satisfy (1.1). Then*

$$\|p_1 - p_0\|_V \le (1/\tilde{\kappa})\{K_1|p_1(T_0)| + K_2\|d\|_H\}\,.$$

To complete the estimate, we bound $|p_1(T_0)|$.

Lemma 3.2 *There is a positive constant that $\mathcal{C}$ depending on T, x^0, and problem parameters such that for any $t \in (t_0, T_0)$*

$$|x_0(t)| \le \mathcal{C}\,.$$

Proof: From the optimality of u_0, we have for the zero function θ

$$J_0(\theta) \ge J_0(u_0)\,.$$

Accordingly, it follows that

$$K_0 = J_0(\theta) \ge \rho\|u_0\|_{U_0}^2\,. \tag{3.15}$$

We note that the value of $J_0(\theta)$ depends only on T, x^0, z and d.

Applying Lemma 1.1, we have

$$|x_0(t)| \le (|x^0| + \mu_b\|u_0\|_{U_0} + \mu_d)\exp(\mu_A t) \tag{3.16}$$

and for any $t \in (t_0, T_0)$

$$|x_0(t)| \le (|x^0| + \mu_b(K_0/\rho)^{1/2} + \mu_d)\exp(\mu_A T)\,. \quad \square \tag{3.17}$$

Lemma 3.3 *There exist a positive constant $\mathcal{C}$ depending on T and problem parameters such that*

$$|p_1(T_0)| \le \mathcal{C}\tau\,.$$

Proof: We note that

$$p_1(t) = \int_t^{T_1} [A(s)^* p_1(s) + Q(s)(x_1(s) - z(s))] \, \mathrm{ds}$$

and

$$|p_1(t)| \leq \int_t^{T_1} [\mu_A |p_1(s)| + \mu_Q |x_1(s) - z(s)|] \, \mathrm{ds} \,. \tag{3.20}$$

By elliptic estimates for the solution of (3.8), we may obtain as in Proposition 1.1

$$\int_{t_1}^{T_1} |p_1'(t)|^2 \, \mathrm{dt} \leq \kappa_0 C(\tilde{d}_1, T, x_1(t_1), z) \,. \tag{3.21}$$

Since $x_1(t_1) = x_0(t_1)$, we see from Lemma 3.2 that $|x_1(t_1)| \leq \mathcal{C}$. It follows from (1.8) that for any $t \in [t_1, T_1]$

$$|x_1(t)| \leq (|x_0(t_1)| + \mu_b (K_0/\rho)^{1/2} + \mu_d) \exp(\mu_A T)$$

and from Lemma 3.2

$$|x_1(t)| \leq (\mathcal{C} + \mu_b (K_0/\rho)^{1/2} + \mu_d) \exp(\mu_A T) \,. \tag{3.22}$$

Estimates (3.21) and (3.22) in (3.20) yield the result. □

The estimate comparing the costates on the interval of intersection now follows from Proposition 3.1 and Lemma 3.3.

Theorem 3.4 *There exist a constant $\mathcal{K}$ depending only on problem parameters such that*

$$\|p_1 - p_0\|_V = \mathcal{K}(\tau + \|d\|_H) \,. \tag{3.23}$$

Remark 3.5 If the disturbances d_0 and d_1 are the same over (t_1, T_0), then $\|d\|_H = 0$ and

$$\|p_1 - p_0\|_V \leq \mathcal{K}\tau \,.$$

The estimates above involve Q^{-1}. If Q is only semidefinite, then the estimate for the perturbed case with $Q_\epsilon = Q + \epsilon I$

$$\|p_1 - p_0\|_V \leq \frac{\mathcal{K}}{\epsilon}(\tau + \|d\|_H) \,.$$

Now we turn to bounding the difference between the costate and the Galerkin approximation of the computed costate on the interval (t_1, T_1). To this end, we introduce the elliptic problem on the interval (t_1, T_1) with boundary condition involving $x_0^N(t_1)$ where u_0^N and x_0^N is determined by equations as in (1.26) and (1.27)

$$\begin{aligned} -Q^{-1}(\tilde{p}_1' + A^*\tilde{p}_1)' + AQ^{-1}(\tilde{p}_1' + A^*\tilde{p}_1) + B\tilde{p}_1 = d_1 \text{ in } (t_1, T_1) \\ \tilde{p}_1(T_1) = 0 \\ \tilde{p}_1'(t_1) + A^*\tilde{p}_1(t)1) = Q(z(t_1) - x_0^N(t_1)) \,. \end{aligned} \tag{3.24}$$

We refer to $\tilde{p}_1$ as the compute costate. We estimate the error between p_1 and $\tilde{p}_1$ using Proposition 1.4 by

$$\|p_1 - \tilde{p}_1\|_{V_1} \le \frac{1}{2}\kappa_0 T^{1/2} |x_0^N(t_1) - x_0(t_1)| \,.$$

We now apply Proposition 1.6 to obtain the following.

Lemma 3.6 *Under the assumptions of Proposition 1.4 the estimate*

$$\|p_1 - \tilde{p}_1\|_{V_1} \le \frac{1}{2}\kappa_0 T^{1/2} \widetilde{K}\, C(N) \tag{3.25}$$

holds.

The error of the Galerkin approximation now is obtained in the following.

Theorem 3.7 *Under the assumptions of Proposition 1.4, there exits a positive constant $\widetilde{\mathcal{K}}$ depending on problem parameters such that*

$$\|p_1 - \tilde{p}_1^N\|_{V_1} \le \widetilde{\mathcal{K}}\, C(N) \,.$$

Proof: The result follows from the triangle inequality

$$\|p_1 - \tilde{p}_1^N\|_{V_1} \le \|p_1 - \tilde{p}_1\|_{V_1} + \|\tilde{p}_1 - p_1^N\|_{V_1} \,,$$

Lemma 3.6, and the Galerkin estimate Proposition 1.4. □

We may also compare the Galerkin approximation of the computed costate on (t_1, T_1) with the costate on (t_0, T_0) over the interval of intersection (t_1, T_0) to measure the effect of the disturbance.

Theorem 3.8 *There exists positive numbers $\mathcal{K}_0$ and $\mathcal{K}_1$ such that*

$$\|p_0 - \tilde{p}_1^N\|_V \le \mathcal{K}_0 \triangle + \mathcal{K}_1\, C(N) \,.$$

Proof: Estimating the quantity $\|p_0 - \tilde{p}_1^N\|_V$ is accomplished by bounding the terms on the right side of the inequality

$$\|p_0 - \tilde{p}_1^N\|_V \le \|p_0 - p_1\|_V + \|p_1 - \tilde{p}_1\|_{V_1} + \|\tilde{p}_1 - \tilde{p}_1^N\|_{V_1} \,. \tag{3.26}$$

The first term is bounded by (3.23). The second term is estimated by (3.25). Finally, the last term is dominated means of an argument similar to (1.25). □

From relations similar to (1.4) and (1.26) and the assumptions (0.4) and (R.2), we have the following.

Corollary 3.9 *Under the assumptions of Proposition 1.4 there exists* $\widehat{\mathcal{K}}_0$*,* $\widehat{\mathcal{K}}_1$ *and* $\widehat{\mathcal{K}}_2$ *such that*

$$\|u_1 - \tilde{u}_1^N\|_{V_1} \leq \widehat{\mathcal{K}}_0\, C(N)$$

and

$$\|u_0 = \tilde{u}_1^N\|_V \leq \widetilde{\mathcal{K}}_1 \triangle + \widehat{\mathcal{K}}_2\, C(N)\,.$$

This section determines two bounds for sliding horizon control. The first isolates the approximation of the computed costate with the costate of the problem on (t_1, T_1). This measures error introduced by using an approximation of the state function of the problem on (t_0, T_0) at t_1 as an initial condition for the problem on (t_1, T_1). This error is combined with the error from the Galerkin approximation in Theorem 3.7. Hence, this estimate bounds the error between the computed sliding horizon controller and the sliding horizon controller for a single step and therefore represents a local error. On the other hand, the estimate contained in Theorem 3.8 compares the approximation of the computed adjoint with the original adjoint over the interval (t_1, T_0). It includes the effects that may arise if the disturbance varies over sampling intervals.

Finally, we mention that if the state of the plant is sampled at t_1 and this measurement gives enough information to determine the whole state, then these observations may be used in place of the computed $x_0^N(t_1)$. The error from this method is determined by how well the plant is modeled by the state equations.

4 A Numerical Feedback Control Algorithm

In this section we present an algorithm based on the sliding horizon concept for feedback with disturbance and feedforward. It is based on the finite element solution of an elliptic boundary value problem such as that in (1.10), (1.11) coupled with equation (1.25). We also apply this algorithm to the problem of aircraft terrain tracking.

Our starting point is the boundary value system on a sample interval $(\hat{t}, \widehat{T})$ where $\hat{x}$ is the initial value of the state equation.

$$-(Q^{-1}(p' + A^*p))' + AQ^{-1}(p' + A^*p) + Bp = \tilde{d} \text{ in } (\hat{t}, \widehat{T})$$

$$\begin{aligned} p'(\hat{t}) + A^* p(\hat{t}) &= Q(\hat{t})(z(\hat{t}) - \hat{x}) \\ p(\widehat{T}) &= 0\,. \end{aligned} \tag{4.1}$$

The variational form of (4.1) is given in terms of the bilinear form $[\cdot,\cdot]$ defined in (1.14) over the space V_0 given in Section 1. Thus, we have the statement

$$\begin{aligned} &\text{Find } p \in V_0 \text{ such that for any } \psi \in V_0 \\ &[p,\psi] = (\tilde{d},\psi)_H + (\hat{x} - z(\hat{t}))\psi(\hat{t})\,. \end{aligned} \tag{4.2}$$

Let $\{\phi_j\}_{j=1}^M$ be a basis for the finite dimensional subspace V_0^M and V_0 and let $p^M = \sum_{i=1}^M c_i\phi_i$. The Galerkin approximation p^M to p over V_0^M satisfies the equation

$$[p^M, \phi_j] = (\tilde{d}, \phi_j)_H + (\hat{x} - z(\hat{t}))\phi_j(\hat{t}) \tag{4.3}$$

for $j = 1, \ldots, M$. Define the symmetric, positive definite $M \times M$ matrix G by

$$G_{ij} = [\phi_i, \phi_j]\,,$$

the $n \times M$ matrix-valued function Φ by

$$\Phi(t) = [\phi_1(t), \ldots, \phi_M(t)]$$

with

$$\widehat{\Phi}^* = \begin{bmatrix} \phi_1(\hat{t})^* \\ \vdots \\ \phi_M(\hat{t})^* \end{bmatrix}.$$

We also define the M-vectors $\widetilde{D}$ by

$$\widetilde{D}_i = (\tilde{d}, \phi_i)_H \text{ for } i = 1, \ldots, M$$

and

$$D_z = \widetilde{D} - \widehat{\Phi}^* z(\hat{t})\,.$$

We rewrite (4.3) as

$$Gc = D_z + \widehat{\Phi}^*\hat{x} \tag{4.4}$$

and represent the function $p^N(t)$ by

$$p^N(t) = \Phi(t)c\,, \tag{4.5}$$

where $c = \mathrm{co}\ell(c_1, \ldots, c_M)$.

Using (1.26), (1.27) and (4.5), we obtain the initial-value problem

$$x^{N'}(t) = A(t)x^N(t) - B(t)\Phi(t)c + d(t)$$
$$x^N(\hat{t}) = \hat{x}. \tag{4.6}$$

Now given the value $\hat{x} = x_0$ at $\hat{t} = t_0$, a one step method for obtaining an approximating value at $t_1 = t_0 + h$, $x_1 \cong x^N(t_1)$ may be expressed by the equation

$$x_1 = ,\ x_0 - \Lambda c + \delta \tag{4.7}$$

where

, is an $n \times n$ invertible matrix that may depend on the matrix A, t_0, and h

Λ is an $n \times M$ matrix that may depend on the matrices A, B, Φ, t_0, and h

d is an n-vector that may depend on the matrix A, the n-vector d, t_0, and h.

The algorithm is realized by the following procedure. Given t_0, x_0, and h with $j = 0$

(i) $\hat{t} = t_j$
$\hat{x} = x_j$

(ii) Calculate $\hat{c}: G\hat{c} = D_z + \widehat{\Phi}^* \hat{x}$
Set $c_j = \hat{c}$

(iii) Calculate $x_{j+1}: x_{j+1} = ,\ \hat{x} - \Lambda\hat{c} + \delta$

(iv) Update $j = j + 1$, $t_j = t_j + h$, and $x_j = x_{j+1}$. Return to (i).

We note that solving equation (4.4) for c and substituting into equation (4.7) leads to the recursion relation

$$x_{j+1} = (,\ - \Lambda G^{-1}\widehat{\Phi}^*)x_j + (\delta - G^{-1}D_z)$$

and then

$$Gc_{j+1} = \widehat{\Phi}^* x_{j+1} + D_z\,.$$

We now apply the above algorithm to the problem of aircraft terrain tracking control. A linear model of the longitudinal dynamics of the vehicle is assumed (fig. 1). The state is defined as

$$x = \begin{bmatrix} h \\ \theta \\ q \\ \alpha \end{bmatrix}$$

and the control

$$u = \delta_e$$

where $x_1 = h$ is the vertical height above the terrain, $x_2 = \theta$ is the pitch angle, $x_3 = q$ is the pitch rate, and $x_4 = \alpha$ is the angle of attack. The control $u = \delta_e$ is the elevator deflection angle. The state equation takes the form

$$x' = Ax + bu$$

where

$$A = \begin{bmatrix} 0 & V & 0 & -V \\ 0 & 0 & 1 & 0 \\ 0 & 0 & M_q & M_\alpha \\ 0 & 0 & 1 & Z_{\alpha/V} \end{bmatrix}$$

and

$$b = \begin{bmatrix} 0 \\ 0 \\ M_E \\ Z_{e/V} \end{bmatrix} .$$

The variable V is the forward velocity of the aircraft (200 m/sec). Stability derivative valves for the AFTI-16 are used [5]. They are $M_1 = -.4932$, $M_\alpha = 1.4168$, $M_E = -1.645$, $Z_{\alpha/V} = -.5164$, and $Z_{e/V} = -.0717$.

The following quadratic performance measure is employed [2]

$$J = \frac{1}{2} \int_{t_0}^{T_0} (Q_1 \cdot (h - (r_1 + c))^2 + Q_2\theta^2 + Q_3 q^2 + Q_4\alpha^2 + R\delta_e^2) \, \mathrm{dt}$$

where $c = 20$m corresponding to the desired elevation above the terrain. The significant components of the state vector are h and θ. Accordingly, the penalties Q_1 and Q_2 are set as

$$Q_1 = 10^2 \quad \text{and} \quad Q_2 = 10^3 .$$

The penalties Q_3 and Q_4 may be viewed as perturbations. Hence, Q_3 and Q_4 in the present example are taken as

$$Q_3 = Q_4 = 10^{-1} .$$

Computational experience indicates that Q_3 and Q_4 may be taken even smaller. The performance of the design is simulated using a finite forward preview of the horizon of 1200m. The craft is flying straight and level when a ramp-type terrain feature is detected. The trajectory of the center of mass of the plane is shown in Fig. 2. The approximate solution employs a 20 element discretization of the horizon interval. The "dip" down before the climb reflects a momentum trade off. The pitch, angle of attach and elevator time histories are shown in Fig. 3. Simulations with shorter horizons (300m and 600m) have also been conducted. While not shown, those results indicate that the design results within acceptable aerodynamics.

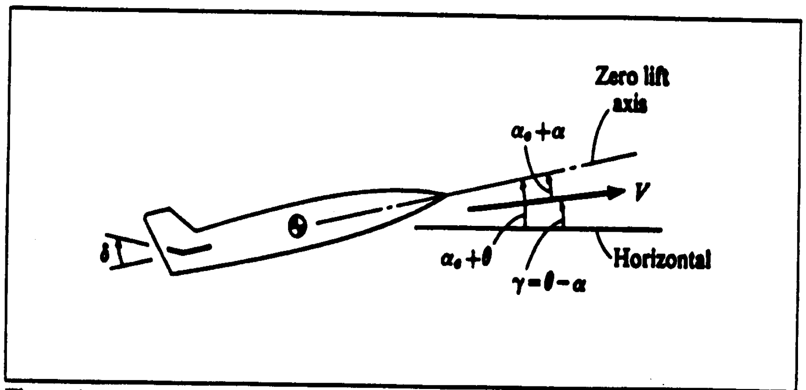

Figure 1 Nomenclature for Controlling Longitudinal Motions of an Aircraft.

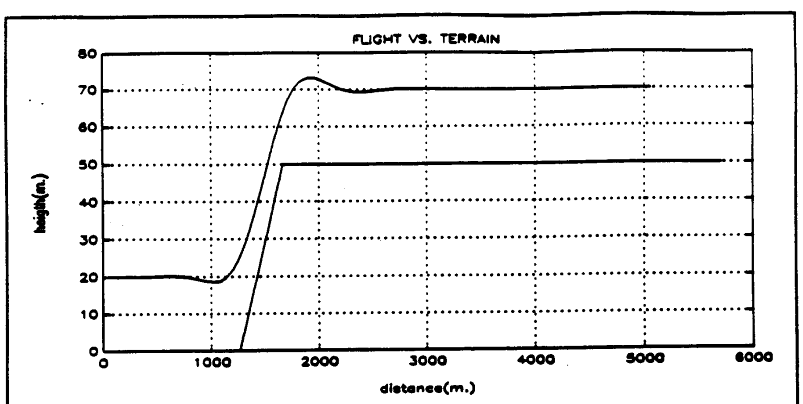

Figure 2 Terrain Tracking: Time History of The Elevation of The Aircraft: (V = 200 m/s, Preview Length = 1200m)

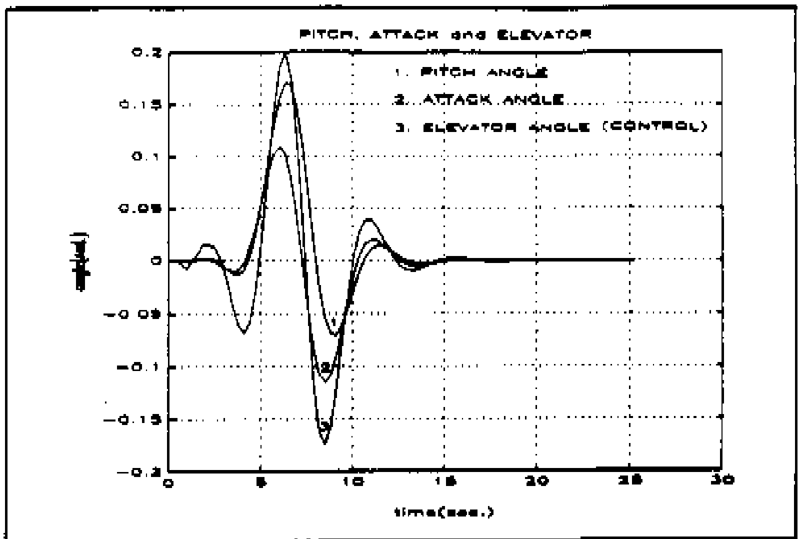

Figure 3 Terrain Tracking Time History of the Pitch Angle, Angle of Attach and Elevator Angle. (V = 200 m/s, Preview Length = 1200m)

5 Conclusions

A method is developed that employs a sliding horizon procedure with a time finite element approximate solution of the open loop quadratic tracking problem with a finite horizon. This method produces an online discrete feedback/feedforward controller that makes it possible to achieve optimal terrain tracking and/or command following. The time finite element procedure is extended to problems with semidefinite Q-matrices by means of a perturbation technique that produces errors no worse than those introduced by finite element approximations. Estimates of errors introduced in approximations for sliding horizon controllers are established using elliptic methods. A numerical example is presented for aircraft terrain tracking control.

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