Journal of Mathematical Systems, Estimation, and Control
Vol. 7, No. 2, 1997, pp. 1–12


# On Smoothness of Sub-Riemannian Minimizers*

S. Nikitin

**Abstract**

This paper presents new conditions under which sub-Riemannian distance can be measured by means of a $C^\infty$ sub-Riemannian geodesic.

**Key words**: nonlinear systems, geodesics, calculus of variations, optimal control, sub-Riemannian geometry

**AMS Subject Classifications**: 49K15, 53C22, 70F25

## 1 Introduction

The problems of sub-Riemannian geometry and Carnot-Caratheodory spaces are of great importance. For example, methods developed in the field of sub-Riemannian geometry find their numerous applications in the theory of geometric phases [17], [22] and in nonholonomic motion planning [8], [13]. On the other hand, analysis of sub-Riemannian minimizers is a challengeable problem for the modern geometrical control theory [1], [2], [10].

The state of the art in the field of sub-Riemannian geometry until 1985 is outlined in the paper [23]. More recent information about this subject can be found in [7], [12], [17], [25].

Although many interesting and important results on sub-Riemannian geodesics are already obtained [15], [16], [17], [18], [24], [25], the following fundamental question is still unanswered. Is it always true that sub-Riemannian distance can be measured by means of infinitely smooth sub-Riemannian geodesics? This basic question arises from the fact that sometimes sub-Riemannian distance is measured by means of so-called abnormal extremals [11], [15], [16], [18]. For a class of homogeneous systems

*Received July 10, 1995; received in final form December 8, 1995. Summary appeared in Volume 7, Number 2, 1997.

whose state spaces are compact Lie groups [19] and for distributions with the strong bracket-generating condition [24] it was shown that all sub-Riemannian minimizers are smooth.

In this paper we consider a sub-Riemannian structure generated by $C^\infty-$ vector fields

$$B(x) = \{b_1(x), b_2(x), \ldots .b_m(x)\} \quad x \in \mathbf{R}^n.$$

The goal of this paper is to present new conditions under which the sub-Riemannian distance between any two points can be always measured by a $C^\infty$ sub-Riemannian geodesic.

Though our main result is proved for the sub-Riemannian structures defined on $\mathbf{R}^n$, it can be easily rephrased in the language of differential geometry where $\mathbf{R}^n$ is replaced with a smooth paracompact manifold.

## 2 Sub-Riemannian Geodesics and a Variational Problem with Fixed Boundaries

It is well-known [2], that sub-Riemannian geodesics are the extremals of the following optimal control problem.

$$\int_0^1 \mid u(\tau) \mid^2 d\tau \to inf, \tag{2.1}$$

where

$$\mid u(\tau) \mid^2 = \sum_{i=1}^m (u_i(\tau))^2$$

and $u(\cdot) : [0,1] \to \mathbf{R}^m$ is subjected to the additional constraining relations introduced by

$$\begin{aligned} \dot{x} &= B(x)u(t), \\ x(0) &= \bar{x}, \;\; x(1) = \hat{x}, \end{aligned} \tag{2.2}$$

with $x \in \mathbf{R}^n, \quad B(x) = \{b_1(x), b_2(x), \ldots .b_m(x)\}$.

The points $\bar{x}, \hat{x} \in \mathrm{R}^n$ are assumed to be fixed beforehand. The minimum of (2.1) is said to be the sub-Riemannian distance between $\bar{x}$ and $\hat{x}$.

The vector fields

$$B(x) = \{b_1(x), b_2(x), \ldots .b_m(x)\}$$

are assumed to be complete $C^\infty$-vector-fields such that the Lie algebra generated by $\{b_j(x)\}_{j=1}^m$ has the full rank at any point $x \in \mathbf{R}^n$. Sometimes such family of vector fields is said to be bracket generating [24] and/or controllable [20]. Under this condition, for any two points $\bar{x}, \hat{x} \in \mathbf{R}^n$, one can find a bounded piecewise-smooth control which steers the system

$$\dot{x} = B(x)u(t)$$

from $\bar{x} \in \mathbf{R}^n$ to $\hat{x} \in \mathbf{R}^n$ in finite time. For a detailed justification of this fact see [20].

Thus the controllable family $B(x)$ generates a metric on $\mathbf{R}^n$, since for any two $\bar{x}, \hat{x} \in \mathbf{R}^n$ the sub-Riemannian distance between $\bar{x}$ and $\hat{x}$ is well defined by (2.1), (2.2). $\mathbf{R}^n$ endowed with this sub-Riemannian metric is called the sub-Riemannian structure (on $\mathbf{R}^n$).

We call a curve $\gamma \subset \mathbf{R}^n$ a sub-Riemannian geodesic (corresponding to the sub-Riemannian structure) if there exists a parametrization $x_\gamma(\tau)$ of $\gamma$ such that $x_\gamma(\tau)$ is the $x$-component of a solution for the following hamiltonian system.

$$\frac{d}{d\tau}x(\tau) = \frac{\partial}{\partial p}H(x(\tau), p(\tau)),$$
$$\frac{d}{d\tau}p(\tau) = -\frac{1}{2}\frac{\partial}{\partial x}H(x(\tau), p(\tau)),$$

where

$$H(x,p) = \frac{1}{2}\left|B^T(x)p\right|^2.$$

Recall the following standard notations:

- $C^k[0,1]$ - the set of $k-$ times continuously differentiable on [0,1] functions.
- $\|\cdot\|_k$ denotes the uniform norm on $C^k[0,1]$, i.e.,

$$\|u\|_k = \max_{t\in[0,1], 0\le i\le k} \mid (\frac{d}{dt})^i u(t) \mid$$

  for $u(t) \in C^k[0,1]$. $(\frac{d}{dt})^0 u(t)$ is another notation for $u(t)$.
- $C^\infty[0,1]$ denotes $\cap_k C^k[0,1]$, where the intersection is taken over all non-negative integers $k$.

The goal of this paper is to prove that under certain conditions sub-Riemannian distance is measured by means of a $C^\infty$ sub-Riemannian geodesic.

In order to accomplish this task let us replace (2.1), (2.2) by the following variational problem with fixed boundaries

$$\frac{1}{2}\int_0^1 \mid \dot{x}(\tau) - B(x(\tau))\dot{y}(\tau) \mid^2 d\tau + \frac{\varepsilon}{2}\cdot\int_0^1 \mid \dot{y}(\tau) \mid^2 d\tau \to inf, \qquad (2.3)$$

where $\varepsilon$ is a positive real number and

$$x(0) = \bar{x}, \quad x(1) = \hat{x}$$
$$\varepsilon \dot{y}(t) = (B(x(t)))^T(\dot{x}(t) - B(x(t))\dot{y}(t)) \qquad \text{for } t = 0 \text{ and } t = 1. \tag{2.4}$$

Let $(x_\varepsilon(t), y_\varepsilon(t))$ denote a minimizer, i.e., a solution for problem (2.3), (2.4). After imposing some mild constraints on $B(x)$, it is possible to show that a minimizer $(x_\varepsilon(t), y_\varepsilon(t))$ always exists for sufficiently small values of $\varepsilon$. Indeed, assume that $B(x)$ satisfies the following condition which is called C-condition in the sequel.

(C) *For any $\bar{x} \in \mathbf{R}^n$ and any positive constant $Q$ there exist a positive real number $\delta$ and a constant $A > 0$ such that*

$$\|x_{u,\xi}(t, \bar{x})\|_0 \leq A,$$

*where $x_{u,\xi}(t, \bar{x})$ is the solution for the following initial value problem*

$$\dot{x} = B(x)u(t) + \xi(t),$$
$$x(0) = \bar{x},$$

*with arbitrary continuous functions $u(t) : [0,1] \to \mathbf{R}^m$, $\xi(t) : [0,1] \to \mathbf{R}^n$ satisfying the inequalities*

$$\int_0^1 |u(t)|\, dt \leq Q \quad \text{and} \quad \int_0^1 |\xi(t)|\, dt \leq \delta.$$

If $B(x)$ satisfies C-condition, then the Hilbert's direct methods (see [3] pp.420-443 and [5], Chapter 7) allow us to claim that a minimizer $(x_\varepsilon(t), y_\varepsilon(t))$ always exists for sufficiently small values of $\varepsilon$. The proof of this fact is fairly standard. Nonetheless we sketch the main steps of this proof.

**Lemma 2.1** *Let $B(x)$ be a controllable family of $C^\infty$ vector fields and let C-condition hold. Then there exists a real number $\varepsilon_0 > 0$ such that for any $0 < \varepsilon \leq \varepsilon_0$ a minimizer $(x_\varepsilon(t), y_\varepsilon(t)) \in C^\infty[0,1]$, i.e., a solution for problem (2.3), (2.4), exists.*

**Proof:** It is well-known [2], [4], that problems (2.3), (2.4) and (2.5), (2.6) have the same family of extremals. Therefore instead of the variational problem (2.3), (2.4) we can consider

$$J_\varepsilon(\gamma) = \int_0^1 \sqrt{|\,\dot{x}(\tau) - B(x(\tau))\dot{y}(\tau)\,|^2 + \varepsilon \cdot |\,\dot{y}(\tau)\,|^2}\, d\tau \to inf, \tag{2.5}$$

where $\gamma = \{(x(t), y(t)); t \in [0,1]\}$, $\varepsilon$ is a positive real number and

$$x(0) = \bar{x}, \quad x(1) = \hat{x}$$
$$\varepsilon \dot{y}(t) = (B(x(t)))^T (\dot{x}(t) - B(x(t))\dot{y}(t)) \qquad \text{for } t = 0 \text{ and } t = 1. \tag{2.6}$$

Set $\rho_\varepsilon = \inf J_\varepsilon(\gamma)$. Then there exists a sequence of $C^\infty$-curves $\gamma_\varepsilon^n = \{(x_\varepsilon^n(t), y_\varepsilon^n(t)); t \in [0,1]\}$, $n = 1, 2, \ldots$, such that $\lim_{n\to\infty} J(\gamma_\varepsilon^n) = \rho_\varepsilon$. Since $B(x)$ is controllable, one can find a bounded piecewise smooth control $u(t) : [0,1] \to \mathbf{R}^m$ which steers the system

$$\dot{x} = B(x)u$$

from $\bar{x}$ to $\hat{x}$ and for which

$$\forall \varepsilon > 0 \quad \exists \ N(\varepsilon) > 0 \quad \text{such that} \quad \forall\, n \geq N(\varepsilon) \quad J(\gamma_\varepsilon^n) \leq \sqrt{\varepsilon} \cdot \int_0^1 |u(t)|\, dt.$$

Hence

$$\int_0^1 |\ \dot{x}_\varepsilon^n(t) - B(x_\varepsilon^n(t))\dot{y}_\varepsilon^n(t)\ |\, dt \leq \sqrt{\varepsilon} \cdot \int_0^1 |u(t)|\, dt,$$
$$\int_0^1 |\dot{y}_\varepsilon^n(t)|\, dt \leq \int_0^1 |u(t)|\, dt,$$

and therefore, C-condition implies the existence of $\varepsilon_0 > 0$, $N(\varepsilon)$ and a compact set $K$ such that

$$\forall\, n \geq N(\varepsilon),\ \varepsilon \leq \varepsilon_0 \quad \gamma_\varepsilon^n \subset K.$$

Thus we can apply the Hilbert's direct methods (see [5], Theorem 7.17, pp. 193-195). That yields the existence of a continuous curve

$$\gamma_\varepsilon = \{(x_\varepsilon(t), y_\varepsilon(t)); t \in [0,1]\} \subset K$$

for which $J(\gamma_\varepsilon) = \rho_\varepsilon$. Moreover, $(x_\varepsilon(t), y_\varepsilon(t))$ is of bounded variation, and therefore almost everywhere differentiable on [0,1]. Thus it follows from Weierstrass-Erdmann corner conditions and Euler first-order necessary conditions of extremum (see, e.g. [6]) that $x_\varepsilon(t)$ and $y_\varepsilon(t)$ are $C^\infty$ functions. Indeed, Weierstrass-Erdmann corner conditions imply that

$$\dot{x}_\varepsilon(t) - B(x_\varepsilon(t))\dot{y}_\varepsilon(t) \in C^0[0,1],$$
$$\dot{y}_\varepsilon(t) \in C^0[0,1].$$

Therefore $(\dot{x}_\varepsilon(t),\ \dot{y}_\varepsilon(t)) \in C^0[0,1]$. On the other hand, $(x_\varepsilon(t),\ y_\varepsilon(t))$ is a solution for the Euler's equations and that implies $(\ddot{x}_\varepsilon(t),\ \ddot{y}_\varepsilon(t)) \in C^0[0,1]$.

By differentiating the Euler's equations and carrying out the steps of mathematical induction with respect to the order of differentiating we conclude that for all positive integers $n$

$$(\left(\frac{d}{dt}\right)^n x_\varepsilon(t),\ \left(\frac{d}{dt}\right)^n y_\varepsilon(t)) \in C^0\,[0,1]\,.$$

Q.E.D.

It was shown in [9] for some special sub-Riemannian metrics, that as $\varepsilon \to 0$, $(x_\varepsilon(t), y_\varepsilon(t))$ converges uniformly on $[0,1]$ to a sub-Riemannian geodesic. We now impose some conditions which strengthen this result. One of them is D-condition which is stated as follows.

(D) *We say that a system*

$$\dot{x} = B(x)u(t)$$

*satisfies D-condition (at the points* $\bar{x},\ \hat{x} \in \mathbf{R}^n$*), if there exist real numbers* $\delta > 0$, $Q > 0$ *and* $P > 0$ *such that*

$$\forall \quad 0 < \varepsilon \leq \delta \quad \mid \sqrt{\varepsilon} \cdot p_\varepsilon(0) \mid \leq Q$$

*implies*

$$\forall \quad 0 < \varepsilon \leq \delta \quad \mid p_\varepsilon(0) \mid \leq P.$$

*Here* $p_\varepsilon(0)$ *has to be chosen so that* $x_\varepsilon(1) = \hat{x}$, *where* $x_\varepsilon(t)$ *is the* $x$*-component of a solution of the following hamiltonian system,*

$$\begin{aligned} \dot{x} &= \frac{\partial}{\partial p} H_\varepsilon(x,p), \\ \dot{p} &= -\frac{\partial}{\partial x} H_\varepsilon(x,p), \\ x(0) &= \bar{x}, \end{aligned} \tag{2.7}$$

*where*

$$H_\varepsilon(x,p) = \frac{1}{2}\left|B^T(x)p\right|^2 + \frac{\varepsilon}{2} \mid p \mid^2 . \tag{2.8}$$

The following theorem presents the main result of this paper.

**Theorem 2.1** *Let* $B(x) = \{b_1(x), b_2(x), \ldots b_m(x)\}$ *be a controllable family of* $C^\infty$ *vector fields for which C- and D-conditions hold. Then for any*

*$\bar{x}, \hat{x} \in \mathbf{R}^n$ one can find a $C^\infty$ sub-Riemannian geodesic $x(t)$ which measures the sub-Riemannian distance between $\bar{x}$ and $\hat{x}$. Thus $x(t)$ is the $x$-component of a solution for the following hamiltonian system*

$$\begin{aligned} \dot{x} &= \frac{\partial}{\partial p} H(x,p), \\ \dot{p} &= -\frac{\partial}{\partial x} H(x,p), \end{aligned} \tag{2.9}$$

*where*

$$H(x,p) = \frac{1}{2} \left| B^T(x) p \right|^2 . \tag{2.10}$$

**Proof:** Consider the variational problem (2.3), (2.4). The functions $(x_\varepsilon(t), y_\varepsilon(t))$ representing a solution for (2.3), (2.4) necessarily satisfy the Euler's equations

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}} L_\varepsilon(x,y,\dot{x},\dot{y})\right) - \frac{\partial}{\partial x} L_\varepsilon(x,y,\dot{x},\dot{y}) = 0, \\ \frac{d}{dt}\left(\frac{\partial}{\partial \dot{y}} L_\varepsilon(x,y,\dot{x},\dot{y})\right) - \frac{\partial}{\partial y} L_\varepsilon(x,y,\dot{x},\dot{y}) = 0, \end{aligned} \tag{2.11}$$

where

$$L_\varepsilon(x,y,\dot{x},\dot{y}) = \frac{1}{2} \mid \dot{x} - B(x)\dot{y} \mid^2 + \frac{\varepsilon}{2} \cdot \mid \dot{y} \mid^2 .$$

We define $p_\varepsilon$ by setting

$$p_\varepsilon = \frac{1}{\varepsilon} \frac{\partial}{\partial \dot{x}} L_\varepsilon(x,y,\dot{x},\dot{y}). \tag{2.12}$$

We will use the following form of (2.11)

$$\begin{aligned} \dot{x}_\varepsilon &= \frac{\partial}{\partial p} H_\varepsilon(x_\varepsilon, p_\varepsilon), \\ \dot{p}_\varepsilon &= -\frac{\partial}{\partial x} H_\varepsilon(x_\varepsilon, p_\varepsilon), \\ \dot{y}_\varepsilon &= B^T(x_\varepsilon) p_\varepsilon, \end{aligned} \tag{2.13}$$

where

$$H_\varepsilon(x,p) = \frac{1}{2} \mid B^T(x) p \mid^2 + \frac{\varepsilon}{2} \mid p \mid^2 . \tag{2.14}$$

Due to the boundary conditions (2.4)

$$x_\varepsilon(0) = \bar{x}$$

and the initial condition for $p_\varepsilon(t)$ has to be chosen so that $x_\varepsilon(1) = \hat{x}$. It follows from Weierstrass-Erdmann corner conditions (see the proof of Lemma 1 presented in this paper or , e.g. the book [6]) that $x_\varepsilon(t)$ and $p_\varepsilon(t)$ are $C^\infty$ functions. On the other hand, since the family of vector fields $B(x)$ is controllable, there exists a bounded piecewise smooth control

$$v : [0,1] \to \mathbf{R}^m$$

which steers the system

$$\dot{x} = B(x)v(t)$$

from $\bar{x}$ to $\hat{x}$.

If $(x_\varepsilon(t), y_\varepsilon(t))$ is a minimizer for (2.3), then

$$\int_0^1 \frac{1}{2} \mid \dot{x}_\varepsilon(t) - B(x_\varepsilon(t))\dot{y}_\varepsilon(t) \mid^2 + \frac{\varepsilon}{2} \cdot \mid \dot{y}_\varepsilon(t) \mid^2 dt \le \frac{\varepsilon}{2} \cdot \int_0^1 |v(t)|^2 \, dt.$$

Setting

$$D = \frac{1}{2} \int_0^1 |v(t)|^2 \, dt$$

and using (2.12), (2.13) we obtain

$$H_\varepsilon(x_\varepsilon(t), p_\varepsilon(t)) \le D \quad \forall \varepsilon \ge 0, \quad t \in [0,1]. \tag{2.15}$$

Hence $\|\varepsilon \cdot p_\varepsilon\| \to 0$ as $\varepsilon \to 0$ and, therefore, in accordance with $C-$condition, there exists a positive constant $\delta$ such that for $\|\varepsilon \cdot p_\varepsilon\| \le \delta$ the function $x_\varepsilon(t)$ is uniformly bounded on $[0,1]$, i.e., $\|x_\varepsilon\| \le A$ for some positive real number $A$.

Thus $\|\frac{\partial}{\partial p} H_\varepsilon(x_\varepsilon(t), p_\varepsilon(t))\|$ is bounded uniformly with respect to small enough $\varepsilon \ge 0$ and so is $\|x_\varepsilon\|_1$, i.e.,

$$\|x_\varepsilon\|_1 \le G_1, \tag{2.16}$$

where $G_1$ is some positive constant which does not depend on $\varepsilon$. Then in accordance with Ascoli's theorem [14] there exist a sequence $\{\varepsilon_j\}_{j=1}^\infty$ and a continuous function $\breve{x}(t)$ such that

$$\varepsilon_j \to 0 \qquad \text{as } j \to \infty$$

and

$$\|x_{\varepsilon_j} - \breve{x}\| \to 0 \qquad \text{as } j \to \infty.$$

On the other hand, for any control $v(t)$ which steers the system

$$\dot{x} = B(x)v(t)$$

from $\bar{x}$ to $\hat{x}$ we have, for any $\varepsilon > 0$,

$$\int_0^1 \mid \dot{y}_\varepsilon(t) \mid^2 dt \leq \int_0^1 \mid v(t) \mid^2 dt.$$

Therefore $\breve{x}(t)$ is a minimizer which measures the sub-Riemannian distance between $\bar{x}$ and $\hat{x}$. Our goal is to prove that $\breve{x}(t) \in C^\infty[0,1]$. Notice, that D-condition implies the existence of a subsequence $\{\varepsilon_{j'}\}_{j'=1}^\infty$ of the sequence $\{\varepsilon_j\}_{j=1}^\infty$ such that $\mid p_{\varepsilon_{j'}}(0) \mid$ is bounded uniformly with respect to $j'$, then for any non-negative integer $k$,

$$\left\| \left(\frac{d}{dt}\right)^k \left(B^T(x_{\varepsilon_{j'}}(t))p_{\varepsilon_{j'}}(t)\right) \right\|_0 \leq P_k,$$

where the real number $P_k$ does not depend on $j'$.

It follows from (2.15) that for any $t \in [0,1]$

$$\begin{aligned} |\langle b_k(x_\varepsilon(t)), p_\varepsilon(t)\rangle| &\leq \sqrt{2 \cdot D} \quad k = 1, 2, \ldots m, \\ \sqrt{\varepsilon} \cdot |p_\varepsilon(t)| &\leq \sqrt{2 \cdot D}. \end{aligned}$$

That yields that

$$\forall j',\ k \qquad \left\| x_{\varepsilon_{j'}} \right\|_k \leq G_k,$$

where $G_k$ does not depend on $j'$.

Thus the Ascoli's theorem (see, e.g., [14]) implies the existence of a subsequence $\{\varepsilon_{j''}\}_{j''=1}^\infty$ of $\{\varepsilon_{j'}\}_{j'=1}^\infty$ such that

$$\varepsilon_{j''} \to 0 \quad \text{as} \quad j'' \to \infty \tag{2.17}$$

and one can find $C^\infty$ functions $\breve{x}(t),\ \breve{p}(t)$ such that

$$\forall k \quad \|x_{\varepsilon_{j''}} - \breve{x}\|_k \to 0 \ \text{ and } \ \left\|p_{\varepsilon_{j''}} - \breve{p}\right\|_k \to 0 \ \text{ as } \quad j'' \to \infty. \tag{2.18}$$

$x_{\varepsilon_{j''}}(t),\ p_{\varepsilon_{j''}}(t)$ satisfy the differential equations

$$\begin{aligned} \dot{x}_{\varepsilon_{j''}} &= \frac{\partial}{\partial p} H_{\varepsilon_{j''}}(x_{\varepsilon_{j''}}, p_{\varepsilon_{j''}}), \\ \dot{p}_{\varepsilon_{j''}} &= -\frac{\partial}{\partial x} H_{\varepsilon_{j''}}(x_{\varepsilon_{j''}}, p_{\varepsilon_{j''}}) \end{aligned} \tag{2.19}$$

for any non-negative integer $j''$. Therefore taking the limit of (2.19) as $j'' \to \infty$ we conclude that $(\breve{x}(t),\ \breve{p}(t))$ is a solution for (2.9). Q.E.D.

## 3 Conclusion

In this paper new conditions were established under which sub-Riemannian distance can be measured by means of a $C^\infty$ sub-Riemannian geodesic. The abnormal minimizers appear if

$$\lim_{\varepsilon\to 0} p_\varepsilon(0) = \infty,$$

where $p_\varepsilon$ is defined in (2.12). Consider the function $q_\varepsilon(t) = p_\varepsilon(t)/p_\varepsilon(0)$. It is easy to see that there exists a sequence $\{\varepsilon_j\}_{j=1}^\infty$ such that

$$\lim_{j\to\infty} \varepsilon_j = 0$$

and $q_{\varepsilon_j}(t)$ converges to the $q$-component of a solution for

$$\begin{aligned}\dot{x} &= \frac{\partial}{\partial q} H(x, q, u(t)),\\ \dot{q} &= -\frac{\partial}{\partial x} H(x, q, u(t)),\end{aligned}$$

where

$$H(x, q, u) =< u, B^T(x)q >$$

and

$$\lim_{j\to\infty} B^T(x_{\varepsilon_j}(t))p_{\varepsilon_j}(t) = u(t).$$

On the other hand,

$$B^T(x(t))q(t) = 0$$

for $x(t)$ being the limit of $x_{\varepsilon_j}(t)$ as $j \to \infty$.

The approach presented in this paper may lead to new results on abnormal minimizers. The further development of this approach and its applications to the analysis of abnormal minimizers will be published elsewhere.

## 4 Acknowledgment

The author is happy to express his gratitude to Prof. Matthias Kawski for interesting and useful discussions on the topics of sub-Riemannian geometry. The author would like also to thank Prof. Gianna Stefani, Prof. Pierluigi Zezza and Prof. Richard Montgomery for helpful comments.

## References

[1] R. Brockett. Control theory and singular Riemannian geometry, in *New Directions in Applied Mathematics,* (P. Hilton and G. Young, ed.). New York: Springer-Verlag, 1990, 11-27.

[2] R. Brockett and L. Dai. Non-holonomic kinematics and the role of elliptic functions in constructive controllability, in *Nonholonomic Motion Planning,* (Z.X.Li and J.Canny, ed.). Amsterdam: Kluwer, 1992.

[3] O. Bolza. *Vorlesungen über Variationsrechnung.* New York: Chelsia Publ. Co., 1909.

[4] B.A. Dubrovin, A.T. Fomenko, S.P. Novikov. *Modern Geometry—Methods and Applications, Part I: Graduate Texts in Mathematics 93.* New York: Springer-Verlag, 1984. *Part II: Graduate Texts in Mathematics 104.* New York: Springer-Verlag, 1985.

[5] G.M. Ewing. *Calculus of Variations with Applications.* New York: W.W. Norton and Co., 1969.

[6] I.M. Gelfand and S.V. Fomin. *Calculus of Variations.* New Jersey: Prentice-Hall, 1963.

[7] M. Gromov. Carnot-Caratheodory spaces seen from within, preprint of Institute des Hautes Etudes Scientifiques, 1994.

[8] C. Fernandes, L. Gurvits, and Z. Li. Near-optimal nonholonomic motion planning for a system of coupled rigid bodies, *IEEE Trans. on Aut. Contr.,* **39** (1994), 450-463.

[9] A. Koranyi. Geometric properties of Heisenberg-type groups, *Advances in Math.*, **56** (1985), 28-38.

[10] A. Krener and S. Nikitin. Dido's problem with a fixed center of mass, in *Proc. of Symposium on Nonlinear Control System Design,* 26-28 June,1995, Tahoe City, California, USA.

[11] I. Kupka. Abnormal extremals, preprint,1992.

[12] J. Laumond. Singularities and topological aspects in nonholonomic motion planning, in *Nonholonomic Motion Planning,* (Z.X. Li and J. Canny, ed.). Amsterdam: Kluwer, 1992.

[13] Z. Li and J. Canny. Motion of two rigid bodies with rolling constraint, *IEEE Trans. Robot. and Automat.*, **RA2-06** (1990), 62-72.

[14] S. Lang. *Analysis II.* New Jersey: Addison-Wesley Pub. Co., 1969.

[15] W. Liu and H.J. Sussmann. Abnormal sub-Riemannian minimizers, preprint, 1992.

[16] R. Montgomery. Geodesics which do not satisfy the geodesic equations, preprint, 1991.

[17] R. Montgomery. Isoholonomic problems and some applications, *Commun. in Math. Phys.*, **128** (1990), 565-592.

[18] R. Montgomery. Abnormal minimizers, *SIAM J. Control and Optimization*, **32** (1994), 1-15.

[19] R. Montgomery. Singular extremals on Lie groups, *Math. Control Signals Systems,* **7** (1994), 217-234.

[20] S. Nikitin. *Global Controllability and Stabilization of Nonlinear Systems.* Hong Kong: World Scientific, 1994.

[21] L.S. Pontryagin, V.G. Boltyanskii, R.V. Gamkrelidze and E.F. Mischenko. *The Mathematical Theory of Optimal Processes.* New York: Wiley and Sons, 1962.

[22] A. Shapere and F. Wilczek. *Geometric Phases in Physics.* Hong Kong: World Scientific, 1989.

[23] R. Strichartz. Sub-Riemannian Geometry, *J. Differential Geometry,* **24** (1986), 221-263.

[24] R. Strichartz. Correction to 'Sub-Riemannian Geometry', *J. Diff. Geom.*, **30** (1989), 595-596.

[25] A.M. Vershik and V.Ya. Gershkovich. Nonholonomic dynamical systems, geometry of distributions and variational problems, in *Dynamical Systems VII*, Vol. I, (V.I. Arnol'd and S.P. Novikov, eds.). New York: Springer-Verlag, 1994, 4-79.

x

DEPARTMENT OF MATHEMATICS, ARIZONA STATE UNIVERSITY, TEMPE, AZ-85287-1804

Communicated by Clyde F. Martin