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Control Lyapunov Functions, Input-To-State Stability and Applications to Global Feedback Stabilization for Composite Systems*

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Abstract

The concepts of control Lyapunov function and the feedback stabilization are extended for the case of parameterized control systems. These concepts are related with the input-to-state stability condition introduced by Sontag and the corresponding results consist generalizations of the Artstein's theorem on stabilization. Versions of the input–to–state stability properties are also discussed. We use these results in order to face the partial-state feedback global stabilizability problem for composite nonlinear systems specially those having triangular structure.

Key words: nonlinear parameterized systems, smooth feedback, global stabilizability, control Lyapunov function, input-to-state stability

1 Introduction

Our goal is to derive sufficient conditions for partial-state global stabilization by smooth (C^∞) static feedback for composite systems, specially those having triangular structure. The main tools we use in order to face this problem are certain properties and results relative to the *input-to-state-stability condition* (I.S.S.C.) introduced by Sontag in [14,15] and some extensions of the Artstein's theorem on stabilization (see [2,13,21]) concerning parameterized systems of the form

$$\begin{gathered} \dot{y} = F(y,u,x) \\ (y,u,x) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^k \end{gathered} \tag{1.1}$$

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where the map F is continuous on $\mathbb{R}^{n+m+k}$ (C^0) vanishing at zero, i.e. $F(0,0,0)=0$, y is the state, u is the input and x is a continuous time varying parameter.

In Section 2 we give the notion of the input–to–state–attractivity condition (I.S.A.C.) which is a weak version of the I.S.S.C. and was originally introduced in [24]. We also provide extensions of the concepts of the "control Lyapunov functions" and the "global feedback stabilization" for parameterized systems (1.1). For instance, it is reasonable to define a system (1.1) to be globally feedback stabilizable, if there exists a static feedback $u=u(y)$ such that the corresponding closed-loop system

$$\dot{y}=F(y,u(y),x) \quad \text{with } x \text{ as input} \tag{1.2}$$

satisfies the I.S.S.C. In Section 3 we give extensions of the Artstein's theorem for parameterized systems (1.1). Special emphasis is given for single-input systems which are affine in the control, namely systems of the form

$$\dot{y}=A(y,x)+uB(y), \tag{1.3}$$

where the mappings A and B are C^0 and A vanishes at zero. Theorem 3.5 in Section 3 consists one of our main results offering sufficient conditions for global feedback stabilization for the following affine in the control case.

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} G_1(x,y) \\ G_2(x,y) \end{pmatrix} + u \begin{pmatrix} 0 \\ \xi(y) \end{pmatrix} \tag{1.4}$$

$$y:=(y_1^{'},y_2)^{'} \quad (^{'} \text{ stands for transpose}),\ y_1\in\mathbb{R}^n,\ y_2\in\mathbb{R}, x\in\mathbb{R}^k,$$

where $G_1:\mathbb{R}^{n+k+1}\to\mathbb{R}^n$, $G_2:\mathbb{R}^{n+k+1}\to\mathbb{R}$ and $\xi:\mathbb{R}^{n+1}\to\mathbb{R}$ are C^1 mappings with G_1 and G_2 vanishing at zero and ξ being everywhere strictly positive. We use this theorem to study the output feedback stabilizability problem for composite systems (1.4) together with a given subsystem $\dot{x}=G_0(x,y)$, particularly for systems of the form

$$\begin{pmatrix} \dot{x} \\ \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} G_0(x,y) \\ G_1(x,y) \\ G_2(x,y) \end{pmatrix} + u \begin{pmatrix} 0 \\ 0 \\ \xi(y) \end{pmatrix} \tag{1.5}$$

where $G_0:\mathbb{R}^{n+k+1}\to\mathbb{R}^k$ is C^0 vanishing at zero, the subsystem

$$\dot{x}=G_0(x,y) \quad \text{with } y \text{ as input} \tag{1.6}$$

satisfies the I.S.A.C. and we assume that only the y-component of the state (x,y) is available (Corollary 3.9). The previous results are applied in Section 4 to derive sufficient conditions for the output feedback stabilizability problem for general triangular single-input systems of the form

$$\dot{x}=f(x,y_1)$$

$$\dot{y}_i = g_i(x, y_1, \ldots, y_i) + h_i(y_1, \ldots, y_{i+1}), \quad 1 \leq i \leq n; \tag{1.7a}$$

$$u := y_{n+1}$$

with

$$h_n(y_1, \ldots, y_n, u) \equiv u \tag{1.7b}$$

where $y := (y_1, \ldots, y_n)'$ is the output and $(x, y) \in \mathbb{R}^k \times \mathbb{R}^n$ is the state of the system and we assume that the subsystem

$$\dot{x} = f(x, y_1) \quad \text{with } y_1 \text{ as input} \tag{1.8}$$

satisfies the I.S.A.C. The corresponding result (Theorem 4.1) consists a considerable generalization of our recent work [25] also dealing with systems (1.7) where some more strict assumptions than those of the present work had imposed. It also consists an extension of earlier works, see for instance [6,7,10-12,18-20,22], and most notably of [4,24] dealing with the particular case (1.7) with

$$h_i(y_1, \ldots, y_{i+1}) = y_{i+1}, \quad \forall\, i = 1, \ldots, n \Leftrightarrow 1.$$

Our approach combines and simultaneously extends the methodology employed in [23-26].

2 Definitions

Throughout the paper we use the notation K for the class of all increasing C^0 functions $a : \mathbb{R}^+ \to \mathbb{R}^+$ with $a(0) = 0$. A function a is said to be of class K_∞, if $a \in K$ and $a(s) \to +\infty$ as $s \to +\infty$. A C^0 function $a : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is of class KL if for every fixed t $a(\cdot, t)$ is of class K and for each s the function $a(s, \cdot)$ is nonincreasing, tending to zero at infinity. We say that the system

$$\dot{y} = F(y, u), \quad (y, u) \in \mathbb{R}^n \times \mathbb{R}^m \quad \text{with } u \text{ as input} \tag{2.1}$$

F being C^1 vanishing at zero, satisfies the I.S.S.C., if it is complete and there exists a pair of functions $\alpha \in KL$ and $\beta \in K$ such that for every essentially bounded input u and for almost all $t \geq 0$ it holds that

$$|y(t, y_0, u)| \leq \alpha(|y_0|, t) + \beta(||u_t||) \tag{2.2}$$

where $y(t, y_0, u)$ denotes the trajectory of (2.1) starting from y_0 at time $t = 0$ with input u, u_t equals $u(t)$ for $0 \leq t \leq T$ and is zero otherwise and $|| \, ||$, $| \, |$ are the L^∞ and the usual Euclidean norms, respectively.

Remark 2.1 It is known from [14,15,8] that, if there exists a positive definite, uniformly unbounded (p.d.u.u.) C^1 function $V : \mathbb{R}^n \to \mathbb{R}^+$ and C^0 functions a, b, where a is of class K, b is positive definite and such that

$$DV(y)F(y,u) \leq \Leftrightarrow b(|y|)$$

$$\forall\ y, \quad a(|u|) \leq |y| \tag{2.3}$$

(DV means the derivative of V), then (2.1) satisfies the I.S.S.C. Furthermore, in the recent contributions [9,16] of Lin, Sontag and Wang it has been established that the I.S.S.C. and its Lyapunov description (2.3) are equivalent, provided that the map F is C^1.

A weaker version of the I.S.S.C. is the following property which originally proposed and studied in [24].

Definition 2.2 *We say that (2.1) satisfies the input-to-state-attractivity condition (I.S.A.C.) if there exists a function $\gamma \in K$ such that for any initial y_0, input u and time $T \leq +\infty$ for which the corresponding trajectory $y(t, y_0, u)$ of (1.1) exists on $[0, T)$ and satisfies*

$$|u(t)| \leq \gamma(|y(t, y_0, u)|), \quad \forall\ t \in [0, T) \tag{2.4}$$

it holds that

$$\overline{\lim_{t \to T}} |y(t, y_0, u)| < +\infty; \tag{2.5a}$$

particularly, for $T = +\infty$

$$\lim_{t \to +\infty} y(t, y_0, u) = 0. \tag{2.5b}$$

It can be shown that I.S.S.C. implies I.S.A.C.; particularly, if (2.2) is satisfied then I.S.A.C. holds with γ be any function of class K_∞ such that $\gamma(\beta(s)) < s$, $\forall\ s > 0$ (see [24]).

We now provide some extensions of the concepts of the control Lyapunov function (clf) and the feedback stabilization for the case (1.1).

Definition 2.3 *Let $a, b : \mathbb{R}^+ \to \mathbb{R}$ be a pair of C^0 functions, a being of class K and b being positive definite. We say that the function $V : \mathbb{R}^n \to \mathbb{R}^+$ is an (a, b)-clf (or simply clf) with respect to (1.1), if it is p.d.u.u. and C^1 on $\mathbb{R}^n$ and there exists a set $M \subset \mathbb{R}^n \times \mathbb{R}^m$ with $(y, u) \in M$ for every $y \in \mathbb{R}^n$ and for some u depending on y such that the following holds.*

$$DV(y)F(y,u,x) < \Leftrightarrow b(|y|)$$

$$\forall\ (y, u) \in M, \quad y \neq 0, \quad a(|x|) < |y|. \tag{2.6}$$

Remark 2.4 It can be easily established that for the affine in the control case (1.3) the existence of an (a,b)-clf V is equivalent to the following condition

$$\begin{aligned} DV(y)B(y) = 0, \quad y \neq 0 \Rightarrow \\ DV(y)A(y,x) < -b(|y|), \quad a(|x|) < |y|. \end{aligned} \tag{2.7}$$

Remark 2.5 The definition of the control Lyapunov function given above is a reasonable extension of the corresponding notion introduced in [13] by Sontag for systems without parameters, which according to Artstein's theorem is equivalent to global stabilization by means of a relaxed feedback controller. Specifically, for the affine in the control single-input systems

$$\dot{y} = A(y) + uB(y) \tag{2.8}$$

condition (2.7) is reduced to the following implication:

$$DV(y)B(y) = 0, \quad y \neq 0 \Rightarrow DV(y)A(y) < -b(|y|),$$

which according to Artstein's theorem is equivalent to the fact that (2.8) is *globally asymptotically stabilizable* at the origin ($G.A.S.$) by means of an *ordinary* static feedback being C^∞ on $\mathbb{R}^n \setminus \{0\}$.

Finally, we give a direct extension of the notion of stabilization for parameterized systems (1.1).

Definition 2.6 *We say that (1.1) is input-to-state stabilizable at the origin (I.S.S.), if there exists an ordinary static feedback $u = u(y)$ such that the closed-loop parameterized system (1.2) satisfies the I.S.S.C.*

As in the case of systems without parameters and taking into account the recent results [9,16] it can be established that the existence of an (a,b)-clf with respect to (1.3) where a is of class K_∞ implies that (1.3) is I.S.S. by means of an ordinary static feedback $u = u(y)$ being C^∞ on $\mathbb{R}^n \setminus \{0\}$ (see Proposition 3.1 in Section 3); the converse claim is also true provided that the mappings A and B as well as the feedback stabilizer are at least C^1.

3 Global Feedback Stabilization

The following result consists a direct extension of the Artstein's theorem for the affine in the control case. Version of this result is also obtained in the recent work [5] of Freeman-Kokotovic.

Proposition 3.1 *Consider the parameterized single-input system (1.3) and suppose that there exists an (a,b)-clf V with $a \in K_\infty$. Then there exists a function $u : \mathbb{R}^n \to \mathbb{R}$ which is C^∞ on $\mathbb{R}^n \setminus \{0\}$ such that*

$$DV(y)(A(y,x) + u(y)B(y)) < -b(|y|)$$

$$\forall\ y \neq 0, \quad |x| < a^{-1}(|y|), \tag{3.1}$$

hence (1.3) is I.S.S. If furthermore we assume that there exists a C^k $(0 \leq k \leq \infty)$ function u_s vanishing at zero such that (3.1) holds for y near zero with u_s instead of u, then the corresponding stabilizer u can be constructed in such a way that (3.1) is satisfied and further $u(y) = u_s(y)$ for y near zero.

Proof: (Outline). Condition (3.1) follows by taking into account Remark 2.4 and applying standard partition of unity based arguments. From (3.1) and Remark 2.1 it follows that (1.3) is I.S.S. by means of the ordinary static feedback $u = u(y)$. The rest part of the proof follows easily by applying the same discussion with that given in [25] or [26]. $\square$

Next we derive sufficient conditions for the existence of a feedback stabilizer being linear near zero and C^∞ on the whole state space. First, we need the following elementary result.

Lemma 3.2 *Consider a general parameterized system (1.1). Assume that the mapping $F : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}^n$ is C^1 and there exist C^0 functions $a,\ b : \mathbb{R}^+ \to \mathbb{R}^+$ a being linear (without any loss of generality we may assume that $a \not\equiv 0$) and b quadratic near zero and a C^2 positive definite function $V : \mathbb{R}^n \to \mathbb{R}^+$ such that the matrix $D^2V(0)$ is positive definite, i.e.,*

$$D^2V(0) > 0 \tag{3.2}$$

and the following implication holds:

$$y'D^2V(0)\frac{\vartheta F}{\vartheta u}(0,0,0) = 0,\ y \neq 0 \ \text{ near zero}$$
$$\Rightarrow y'D^2V(0)F(y,0,x) < \Leftrightarrow b(|y|);\ \ \forall\,|x| < a^{-1}(|y|). \tag{3.3}$$

Then there exists a constant c_0 such that for any $c \geq c_0$ it holds that

$$y'D^2V(y)F(y,\phi_c(y),x) < \Leftrightarrow b(|y|), \forall |x| < a^{-1}(|y|), y \neq 0 \ \text{ near zero} \tag{3.4a}$$

with

$$\phi_c(y) := cy'D^2V(0)\frac{\vartheta F}{\vartheta u}(0,0,0). \tag{3.4b}$$

Proof: (Outline). We take into account conditions (3.2) and (3.3), the fact that locally around zero the functions a and $b^{1/2}$ are linear and apply similar procedure with that in [1] for the linearization of (1.1) at the origin in order to determine a constant c_0 such that

$$y'D^2V(0)\Big\{\frac{\vartheta F}{\vartheta y}(0,0,0)y + \frac{\vartheta F}{\vartheta u}(0,0,0)\phi_c(y) + \frac{\vartheta F}{\vartheta x}(0,0,0)x\Big\} < \Leftrightarrow b(|y|)$$

$$\forall\ |x| < a^{-1}(|y|),\ y \neq 0 \ \text{ near zero}, c \geq c_0$$

with ϕ_c as defined in (3.4b). The desired conclusion is a direct implication of the previous inequality. □

An immediate consequence of Proposition 3.1 and Lemma 3.2 is the following result. Its proof is omitted, since it is analogous with that given in [25, Lemma 2.1].

Corollary 3.3 *Consider the system (1.3) and in addition to the hypothesis of Proposition 3.1 assume A and B are C^1, a and $b^{1/2}$ are linear near zero, V is C^2 and satisfies (3.2) and the following holds*

$$\begin{aligned} & y' D^2 V(0) B(0) = 0,\ y \neq 0 \ \textit{ near zero} \\ & \Rightarrow D^2 V(y) A(x,y) < \Leftrightarrow b(|y|),\ \forall\, |x| < a^{-1}(|y|). \end{aligned} \tag{3.5}$$

Then (1.3) is I.S.S. by means of a C^∞ static feedback. In particular, there exist a constant c_0 and a C^∞ function $u_c := u(y,c)$, $(y,u) \in \mathbb{R}^n \times [c_0, +\infty)$ which for $c \geq c_0$ and for y near zero coincides with ϕ_c (as the latter defined in (3.4b)) and further condition (3.1) is satisfied with u_c instead of u.

Next we deal with the stabilization problem for parameterized single–input systems of the form (1.4). We assume that the mappings G_1 and G_2 are C^1 vanishing at zero and ξ is a C^0 being everywhere strictly positive. Furthermore the following assumption is imposed.

Assumption 3.4. Suppose that there exist a closed subset $M \in \mathbb{R}^{n+1}$, a pair of disjoint open subsets U^+, $U^- \in \mathbb{R}^{n+1}$, a p.d.u.u. C^∞ function $V : \mathbb{R}^n \to \mathbb{R}^+$ satisfying (3.2), functions a, a_1, $a_2 \in K_\infty$ being linear near zero, a positive definite C^0 function $b : \mathbb{R}^+ \to \mathbb{R}^+$ being quadratic near zero and a linear map $\phi : \mathbb{R}^n \to \mathbb{R}$ with $\phi(0) = 0$ such that

(i) $\mathbb{R}^{n+1} = U^+ \bigcup U^- \bigcup M$, $\pi(M) = \mathbb{R}^n$ ($\pi(M)$ means the projection of M on $\mathbb{R}^n$ along the y_2 axis); for each compact set $Q \in \mathbb{R}$ the set $\{y := (y_1', y_2)' \in M,\ y_1 \in Q\}$ is bounded, the graph of the mapping $y_2 = \phi(y_1)$ coincides with the restriction of the set M near zero, i.e.,

$$\{y \in \mathbb{R}^{n+1} : y_2 = \phi(y_1),\ |y_1| \leq \delta\} = M \bigcap \{y \in \mathbb{R}^{n+1} \ :\ |y_1| \leq \delta\} \tag{3.6}$$

for some positive constant δ;

(ii)

$$\begin{aligned} a_1^2(|y|) < \hat{V}(y) := V(y_1) + C(y_2 \Leftrightarrow \phi(y_1))^2 < a_2^2(|y|) \\ \forall\, 0 < |y| \leq \delta \end{aligned} \tag{3.7}$$

for some positive constant C;

$$\max\{a_1(2s), a_1(k(s))\} < \frac{1}{2}a_2(s), \ \forall\, s > 0 \tag{3.8a}$$

for some $k \in K_\infty$ for which

$$\max\{|y_2|, \ y \in M\} < \frac{1}{2}k(|y_1|), \ \forall\, y_1 \neq 0 \tag{3.8b}$$

(iii)

$$DV(y_1)G_1(x,y) < -b(|y|), \ \forall |x| < a^{-1}(|y|), \ y \in M \Big\backslash \{0\}, \tag{3.9}$$

which in turns implies that V is an (a,b) - clf with respect to $\dot{y}_1 = G_1(x, y_1, y_2)$ with y_2 as input.

Theorem 3.5 *Consider the system (1.4) and suppose that Assumption 3.4 is fulfilled. Then there exists a C^∞ p.d.u.u. function $W(y)$, a constant c_0 and a C^∞ map $u_c := u(y_1; y_2; c), (y_1, y_2; c) \in \mathbb{R}^{n+1} \times [c_0, +\infty)$ such that*

$$W \ \textit{ coincides with } \hat{V} \textit{ for } y \textit{ near zero} \tag{3.10}$$

$$u_c \ \textit{ coincides with } \hat{\phi} := -2cC(y_2 - \phi(y_1))\xi(0,0) \textit{ near zero} \tag{3.11}$$

$$a_1^2(|y|) < W(y) < a_2^2(|y|), \ \forall\, y \in \mathbb{R}^{n+1} \Big\backslash \{0\} \tag{3.12}$$

and furthermore, if we call $G := (G_1', G_2)'$ then

$$DW(y)G(x,y) + (u_c\xi)(y)\frac{\vartheta W}{\vartheta y_2}(y) < -b(|y|)$$

$$\forall\, |x| < a^{-1}(|y|), \ y \neq 0, \ c \geq c_0, \tag{3.13}$$

which in turns implies that W is an (a,b)-clf with respect to (1.4) and (1.4) is I.S.S. by means of the static feedback $u = u_c$.

Remark 3.6.
• It should be noted that weaker versions of the previous theorem have been obtained in [3, 17, 21, 23, 26] for systems without parameters ("back-stepping design"). However, because of the additional requirements of the present work we need to apply a different and more technical procedure than these proposed in the previously mentioned works. To be more precise, repeating step–by–step the approach in [23] or [26] we can construct a pair of C^∞ functions W and u_c satisfying (3.10), (3.11) and (3.13) provided only that conditions (i) and (iii) of Assumption 3.4 hold. For the purposes of our present work the function W should be constructed in such a way

that the additional condition (3.12) is satisfied for given a_1, $a_2 \in K_\infty$. This is feasible by following a different approach from that in [23, 26] and assuming that the functions a_1, a_2 satisfy condition (ii) of Assumption 3.4.

• Conditions (i) and (iii) of Assumption 3.4 are fulfilled, if for instance we assume that the subsystem $\dot{y}_1 = G_1(x, y_1, y_2)$ with y_2 as input is I.S.S. by means of static C^∞ feedback $y_2 = \phi(y_1)$ being linear near zero. Indeed, in that case $M := \{y \in \mathbb{R}^{n+1} : y_2 = \phi(y_1)\}$ and (3.9) follows by applying the converse stability theorem in [16].

• The result of Theorem 3.5 can be used to obtain a general sufficient conditions for global **exponential** stabilizability for systems (1.4) without parameters, namely systems of the form $\dot{y}_1 = G_1(y_1, y_2)$, $\dot{y}_2 = G_2(y_1, y_2) + u\xi(y)$, where G_1, G_2, ξ are $\mathbf{C}, G_1, G_2$ vanishing at zero and ξ is everywhere strictly positive. Assume that condition (i) and (ii) of Assumption 3.4 are satisfied for some appropriate sets M, U^+, U^- and functions V, ϕ and b, (i.e. it holds that $DV(y_1)G_1(y) < \Leftrightarrow b(|y|)$, $\forall\, y \in M \setminus \{0\}$). Furthermore assume that the function b is quadratic and (3.8a) holds with k being globally Lipschitzian, i.e. $|k(y_1)| \le K|y_1|$, $\forall\, y_1$ for certain positive constant K. Then the overall system is globally exponentially smoothly stabilizable. Indeed, since V satisfies (3.2) and k is globally Lipschitzian a pair of linear functions $a_1, a_2 : \mathbb{R}^+ \to \mathbb{R}^+$ can be determined satisfying (3.7) and (3.8a). The desired conclusion is then a direct consequence of (3.12) and (3.13) of the statement of Theorem 3.5 together with the fact that a_1^2, a_2^2 and b are quadratic.

Proof of Theorem 3.5: First we determine a locally finite partition of $\mathbb{R}^n \setminus \{0\}$ consisting of open bounded subsets D_i associated with nonnegative C^∞ functions ψ_i with $D_i := \text{supp}\ \psi_i$, $\Sigma\psi_i(y_1) = 1$ and in such a way that if we define

$$k_i := \sup\ \{a_1^2(2|y_1|) \Leftrightarrow V(y_1),\ y_1 \in D_i\} \tag{3.14a}$$

$$\ell_i := \inf\ \{\frac{1}{2}a_2^2(|y_1|) \Leftrightarrow V(y_1),\ y_1 \in D_i\}, \tag{3.14b}$$

then from (3.8a) the following property is fulfilled.

Property a.

$$k_i + \sup_{y_1 \in D_i} a_1^2(k(|y_1|)) < \ell_1. \tag{3.15}$$

In addition to the previous requirement we can determine for each i an open subset $\Delta_i \in \mathbb{R}$ such that if we define $\hat{M} := \bigcup D_i \times \Delta_i$ then the following property holds.

Property b. The set M is contained in the closure of $\hat{M}$, specifically, $\hat{M}$ is a neighborhood of $M \setminus \{0\}$ with $\hat{M} \bigcap \{y \in \mathbb{R}^{n+1} : y_1 = 0\} = \phi$ and properties (i) - (iii) of Assumption 3.4 are satisfied with $c\ell\hat{M}$ instead of M.

Particularly, instead of (3.9) and (3.8b) we may assume that

$$\begin{aligned} &DV(y_1)G_1(x,y) < \Leftrightarrow \hat{b}(|y|), \\ &\forall\, |x| < a^{-1}(|y|),\ \ y \in c\ell\hat{M}\backslash\{0\} \end{aligned} \tag{3.16a}$$

for some $\hat{b} : \mathbb{R}^+ \to \mathbb{R}^+$ being C^0 and positive definite with

$$\hat{b}(s) > b(s),\ \ \forall\, s > 0 \tag{3.16b}$$

and

$$|y_2| < \frac{1}{2}k(|y_1|),\ \ \forall\, y \in c\ell\hat{M}\backslash\{0\}. \tag{3.17}$$

Moreover, by (3.7) our partition can be constructed in such a way that the following additional local property is satisfied.

Property c. There is a positive constant $\delta^* < \delta$ such that for every y_1 belonging to the sphere $S(0,\delta^*)$ of radius δ^* centered at zero there exists an index i such that

$$\sup\ \{a_1^2(|y|) \Leftrightarrow V(y_1),\ y \in D_i \times \Delta_i\} < 0, \tag{3.18a}$$

$$0 < \inf\ \{a_2^2(|y|) \Leftrightarrow V(y_1),\ y \in D_i \times \Delta_i\}; \tag{3.18b}$$

furthermore we can assume that for every i for which $D_i \bigcap S^c(0,\delta) \neq \emptyset (S^c$ means the complement of S) we have $D_i \bigcap S(0,\delta^*) = \emptyset$. (Note that (3.18a) is a consequence of (3.6) and (3.7); indeed the latter imply $a_1(||y_1, y_2||) < V(y_1)$, for $y_1 \neq 0$ near zero and y_2 near $\phi_1(y_1)$).

Consider finally a C^0 function $\epsilon : \mathbb{R} \to \mathbb{R}^+$ vanishing at zero (whose existence is guaranteed by condition (i) of Assumption 3.4 and (3.16b)) such that for every $y_1 \neq 0$ it holds that

$$0 < \epsilon(y_1) \leq\ \max\left\{\frac{\hat{b}(|y|) \Leftrightarrow b(|y|)}{1 + (\Sigma|D\psi_i(y_1)|)|G_1(x,y)|},\ a(|x|) < |y|,\ y \in c\ell\hat{M}\right\}. \tag{3.19}$$

We now proceed to the construction of the desired clf as follows. First, by taking into account Properties a and b of the previous partition and recalling condition (i) of Assumption 3.4 we can construct for each i a C^∞ real function $\phi_i(y_2)$ with $D\phi_i(y_2) = 0$ for $y_2 \in c\ell\Delta_i$, $D\phi_i(y_2) > 0$ for $(y_1', y_2)' \in U^+\backslash D_i \times \Delta_i$, $D\phi_i(y_2) < 0$ for $(y_1', y_2)' \in U^-\backslash D_i \times \Delta_i$ and such that for every i and j for which $D_i \cap D_j \neq \emptyset$ the following hold:

$$|\phi_i(y_2) \Leftrightarrow \phi_j(y_2)| < \epsilon(y_1),\ \forall\ (y_1^{'}, y_2^{'}) \in (D_i \cap D_j) \times \mathbb{R}. \tag{3.20}$$

In particular, by (3.15) for every i for which

$$D_i \bigcap S^c(0,\delta^*) \neq \emptyset, \tag{3.21}$$

the corresponding ϕ_i can be constructed such that the following inequality is satisfied

$$k_i + a_1^2(k(|y_1|)) < \phi_i(y_2) < \ell_i,\ \forall\, y_2 \in \Delta_i,\ y_1 \in D_i,$$

which by (3.17) implies

$$k_i + a_1^2(2|y_2|) < \phi_i(y_2) < \ell_i,\ \forall\, y_2 \in \Delta_i. \tag{3.22}$$

Moreover, by (3.8a) for every index i for which (3.21) is satisfied, the function ϕ_i can be constructed such that in addition to (3.22) the following holds.

$$k_i + a_1^2(2|y_2|) < \phi_i(y_2) < \ell_i + \frac{1}{2}a_2^2(|y_2|),\ \forall\, y_2 \in \Delta_i^c. \tag{3.23}$$

From (3.22), (3.23) and the definition (3.14) of the pair k_i, ℓ_i we get

$$a_1^2(|(y_1,y_2)|) < a_1^2(2|y_1|) + a_1^2(2|y_2|) < V(y_1) + \phi_i(y_2)$$

$$< \frac{1}{2}(a_2^2(|y_1|) + a_2^2(|y_2|)) < a_2^2(|(y_1,y_2)|),\ \forall\,(y_1,y_2) \in D_i \times \mathbb{R} \tag{3.24}$$

and for every i for which (3.21) holds. Finally by using Property c (conditions (3.18a) and (3.18b)) for every i for which

$$D_i \subset S(0,\delta^*) \tag{3.25}$$

the corresponding ϕ_i can be constructed in such a way that (3.24) is satisfied for $y \neq 0$ and $D\phi_i(y_2) \neq 0$ for $y_2 \in \Delta_i^c$, particularly $D\phi_i(y_2) > 0$ for $(y_1', y_2) \in U^+\backslash D_i \times \Delta_i$, $D\phi_i(y_2) < 0$ for $(y_1', y_2)' \in U^-\backslash D_i \times \Delta_i$, and in addition

$$\phi_i(y_2) = 0,\ \forall\, y_2 \in c\ell\Delta_i. \tag{3.26}$$

Next we define

$$L(y) := \begin{cases} \Sigma\psi_i(y_1)\phi_i(y_2), & \text{for } y_1 \neq 0 \\ \quad 0 & \text{for } y_1 = 0 \end{cases} \tag{3.27a}$$

$$\Phi(y) := V(y_1) + L(y). \tag{3.27b}$$

We can easily verify that Φ is p.d.u.u. on $\mathbb{R}^{n+1}$ and smooth on the region $\mathbb{R}^{n+1}\backslash\{(y_1,y_2) \in \mathbb{R}^{n+1} : y_1 \neq 0\}$. Furthermore, by taking into account the definition (3.27) and the properties of ϕ_i it follows that

$$DL(y) = 0(y_1 \neq 0) \Rightarrow y \in c\ell\hat{M} \setminus \{0\}, \tag{3.28}$$

$$\frac{\vartheta L}{\vartheta y_2}(y) > 0 \ \text{for } y \in U^+/\, c\ell\hat{M} \setminus \{0\}, \tag{3.29}$$

$$\frac{\vartheta L}{\vartheta y_2}(y) < 0 \ \text{ for } y \in U^- / \, c\ell\hat{M} \setminus \{0\}. \tag{3.30}$$

Since (3.24) holds for all i, we also get

$$\begin{aligned} & a_1^2(|y|) < \Sigma\psi_i(y_1)(V(y_1) + \phi_i(y_2)) \\ & = V(y_1) + \Sigma\psi_i(y_1)\phi_i(y_2) = \Phi(y) < a_2^2(|y|), \ \forall\, y \neq 0. \end{aligned} \tag{3.31}$$

Furthermore, from the additional property (3.26)(which holds for every i satisfying (3.25)) and taking into account the fact that $D_i \bigcap S(0,\delta^*) = \phi$ for all i for which $D_i \bigcap S^c(0,\delta) \neq \phi$, it follows that for y such that $0 < y_1 \leq \delta^*$ it holds

$$L(y) = 0 \Longleftrightarrow y \in c\ell\hat{M}, \ (0 < y_1 \leq \delta^*). \tag{3.32}$$

This property is quite necessary for the construction of the desired clf; this explains the reason that we have considered separately the cases (3.21) and (3.25).

Next we show that the following implication holds.

$$\frac{\vartheta \Phi}{\vartheta y_2}(y) = 0, \ (y_1 \neq 0) \Rightarrow \frac{\vartheta \Phi}{\vartheta y_1} G_1(x, y_1) < \Leftrightarrow \hat{b}(|y|), \ \ \forall\, |x| < a^{-1}(|y|). \tag{3.33}$$

Indeed, consider any vector $\bar{y} := (\bar{y}_1,', \bar{y}_2)'$ with $\bar{y}_1 \neq 0$ and $\vartheta\Phi/\vartheta y_2(\bar{y}) = 0$. Then, since the summation in the right hand side in (3.27a) is finite, there exist integers $i_1, \ldots, i_\nu$ depending on $\bar{y}$ such that

$$\Sigma\psi_i\phi_i = \sum_{k=1}^{\nu} \psi_{i_k}(\bar{y}_1)\phi_{i_k}(\bar{y}_2).$$

It follows that

$$\sum_{k=1}^{\nu} \psi_{i_k}(\bar{y}_1) D\phi_{i_k}(\bar{y}_2) = 0.$$

The latter in conjunction with the facts that each $D\phi_{i_k}(\bar{y}_2)$ is strictly positive (negative) for y belonging to $U^+ \setminus c\ell\hat{M}$ ($U^- \setminus c\ell\hat{M}$, resp.) and ψ_{i_k} are nonnegative with $\sum_{k=1}^{\nu}\psi_{i_k}(\bar{y}_1) = 1$, imply that

$$\bar{y} \in c\ell\hat{M} \setminus \{0\}. \tag{3.34}$$

From (3.16a), (3.27) and (3.34) we get

$$\frac{\vartheta \Phi}{\vartheta y_1} G_1(x, \bar{y}) < \Leftrightarrow \hat{b}(|\bar{y}|) + \left(\sum_{k=1}^{\nu} D\psi_{i_k}(\bar{y}_1)\phi_i(\bar{y}_2) \right) G_1(x, \bar{y})$$

$$\forall\, |x| < a^{-1}(|y|). \tag{3.35}$$

Since $\sum_{k=1}^{\nu} D\psi_{i_k}(\bar{y}_1) = 0$ and assuming without any loss at generality that $D\psi_{i_\nu}(\bar{y}_1) \neq 0$ it follows by (3.16b), (3.19), (3.20) and (3.35) that

$$\begin{aligned}
&\tfrac{\vartheta \Phi}{\vartheta y_1} G_1(x,\bar{y}) < -\hat{b}(|\bar{y}|)\\
&+\Big(\sum_{k=1}^{\nu-1} |\phi_{i_n}(\bar{y}_2) - \phi_{i_k}(\bar{y}_2)||D\psi_{i_k}(\bar{y}_1)|\Big)|G_1(x,\bar{y})|\\
&\leq -\hat{b}(|\bar{y}|) + \varepsilon(\bar{y}_1)\Big(\sum_{k=1}^{\nu-1}|D\psi_{i_k}(\bar{y}_1)|\Big)|G_1(x,\bar{y})|\\
&< -\hat{b}(|\bar{y}|),\ \ \forall\, |x| < a^{-1}(|\bar{y}|)
\end{aligned}$$

hence the implication (3.33) is established. We are now in a position to build a pair of appropriate smooth functions W and u satisfying (3.10), (3.11) and (3.12). We proceed as in the proof of Theorem 1.1 in [26]. Let $\Theta : \mathbb{R}^n \to \mathbb{R}$ be a C^∞ map taking values on the interval [0,1] and such that $\Theta(y_1) = 0$ for $|y_1| < 1/2$ and $\Theta(y_1) = 1$ for $|y_1| > 1$. Then using (3.28) - (3.33) we can show quite similar to [26] that for appropriate small $\sigma > 0$ the mapping

$$W(y) := V(y_1) + \Theta(\sigma y_1)L(y) + C(1 - \Theta(\sigma y_1))(y_2 - \phi(y_1))^2 \tag{3.36}$$

is C^∞ and p.d.u.u. on $\mathbb{R}^{n+1}$, (particularly (3.7) and (3.31) imply the desired inequality (3.12)), and satisfies the implication

$$\begin{aligned}
&DW(y)\begin{pmatrix} 0 \\ \xi(y)\end{pmatrix} = \xi(y)\frac{\vartheta W}{\vartheta y_2}(y) = 0,\ \ y \neq 0 \Rightarrow \frac{\vartheta W}{\vartheta y_2}(y) = 0,\ \ y \neq 0\\
&\Rightarrow DW(y)\begin{pmatrix} G_1(x,y) \\ G_2(x,y)\end{pmatrix} = \frac{\vartheta W}{\vartheta y_1}(y)G_1(x,y) < -b(|y|),\ \ \forall\, |x| < a^{-1}(|y|).
\end{aligned} \tag{3.37}$$

(Note at this point that, as we have mentioned before, condition (3.32) is quite necessary to establish (3.37); details are left to the reader.)

It follows that property (3.1) of Proposition 3.1 is satisfied with respect to the parameterized system (1.4), hence there exists a map $v = v(y)$ being C^∞ on $\mathbb{R}^{n+1}\backslash\{0\}$ such that (3.13) is satisfied with v instead of u_c. We now take into account that according to (3.36) W coincides with $\hat{V}(y) = V(y_1) + C(y_2 - \phi(y_1))^2$ for y near zero, $D^2V(0)$ is positive definite and for y_1, y_2 near zero the functions ϕ, a and $b^{1/2}$ are linear. It follows that the matrix

$D^2V(0)$ is also positive definite and by (3.6) the following implication holds in a neighborhood of zero:

$$y'D^2W(0)\binom{0}{\xi(0)} = DW(y)\binom{0}{\xi(y)} = 0 \Leftrightarrow y_2 = \phi(y_1), \text{ near zero}$$

$$\Rightarrow DW(y)\binom{G_1(x,y)}{G_2(x,y)} = \frac{\vartheta W}{\vartheta y_1}(y)G_1(x,y) < \Leftrightarrow b(|y|), \ \forall\, |x| < a^{-1}(|y|).$$

Hence by Lemma 3.2 there exists a constant c_0 such that for every $c \geq c_0$ the linear map

$$\hat{\phi}(y) = \Leftrightarrow cy'D^2W(0)\begin{pmatrix} 0 \\ \xi(0) \end{pmatrix} = \Leftrightarrow 2cC(y_2 \Leftrightarrow \phi(y_1))\xi(0)$$

satisfies (3.13) with $\hat{\phi}$ instead of u_c locally around zero. The desired conclusion, namely the existence of a C^∞ map $u = u_c$ satisfying both (3.11) and (3.13) follows directly from the previous discussion and Corollary 3.3. □

Next we deal with the output feedback stabilization problem for composite systems, specially those having the general form (1.5). The corresponding result (see Corollary 3.9 below) is used in Section 4 in order to derive sufficient conditions for stabilization for the triangular case (1.7). We first need the following lemma whose proof consists a generalization of our analysis in [24].

Lemma 3.7 *Consider the system*

$$\dot{x} = G_0(x,y) \tag{3.38a}$$

$$\dot{y} = G(x,y) \tag{3.38b}$$

$$(x,y) \in \mathbb{R}^k \times \mathbb{R}^n$$

where the mappings G_0 and G are C^0 vanishing at zero. Assume that there exist functions $a_1, a_2 \in K_\infty$, $b : \mathbb{R}^+ \to \mathbb{R}^+$ being C^0 and positive definite and a C^∞ p.d.u.u. map $W : \mathbb{R}^n \to \mathbb{R}^+$ such that

A1. *The subsystem (3.38a) with y as input satisfies the I.S.A.C., particularly the implication (2.4) ⇒ (2.5) holds with x, y instead of y and u, respectively and*

$$\gamma = a_1^{-1}. \tag{3.39}$$

A2. *Condition (3.12) holds and further*

$$DW(y)G(x,y) < \Leftrightarrow b(|y|), \ \forall\, |x| < a_2(|y|), \ y \neq 0. \tag{3.40}$$

A3. *The origin $O \in \mathbb{R}^{k+n}$ is locally asymptotically stable with respect to (3.38).*

A4.

$$2|x'G_0(x,y)| \leq b(|y|),\ \ \forall\, |x| < a_2(|y|). \tag{3.41}$$

Then the origin is globally asymptotically stable with respect to (3.38).

Proof: We define

$$R := \{(x,y) : |y| < a_2^{-1}(|x|)\},$$

$$S := \{(x,y) : |y| < a_1^{-1}(|x|)\},$$

$$L := \{(x,y) : W(y) < |x|^2\}.$$

Obviously, by (3.12) we have $R \subset L \subset S$. Let N be an invariant open neighborhood of zero which is contained in the region of attraction of the origin with respect to (3.38) and whose existence is guaranteed by Assumption A3. We show that the region $L\bigcup N$ is positively invariant with respect to (3.38). Indeed, if we evaluate the derivative $\dot{W}$ of W along the trajectories of (3.38) we get from (3.40) and (3.41) that

$$\dot{W}(y) < 2x'G_0(x,y),\ \ \forall\, |x| < a_2(|y|),\ y \neq 0;$$

therefore,

$$\begin{aligned} &W(y(t)) = W(y(0)) + \textstyle\int_0^t \dot{W}(y(p))dp \\ &\leq |x(0)|^2 + \textstyle\int_0^t \frac{d}{dt}|x(p)|^2 dp \leq |x(t)|^2 \end{aligned} \tag{3.42}$$

for any trajectory $(x(t), y(t))$ of (3.38) starting from $(x(0), y(0)) \in L$ at $t = 0$ and such that $(x(\tau), y(\tau)) \in R^c$ for $0 \leq \tau \leq t$. Since $R \subset L$ the inequality (3.42) in conjunction with the positively invariance of N imply that $L\bigcup N$ is positively invariant. To complete the proof it suffices to show that each trajectory of (3.38) is defined for all $t \geq 0$ entering N after some finite time. Indeed, from (3.40) we get

$$\dot{W}(y) \leq \Leftrightarrow b(|y(t)|) \tag{3.43}$$

for each trajectory $(x(t), y(t))$ of (3.40) for which $|x(t)| < a_2(|y(t)|)$ with initial value $(x(0), y(0)) \in (N\bigcup L)^c$. This trajectory enters $N\bigcup L$ after some finite time, for otherwise, because of the positively invariance of $N\bigcup L$, there exists a constant $\theta > 0$ such that $|(x(t), y(t))| \geq \theta$ and $a_2(|y(t)|) > |x(t)|$ for all $t \geq 0$. It turns out that $b(|y(t)|) \geq \hat{\theta}$, $\forall\, t \geq 0$ for some constant $\hat{\theta} > 0$, hence by (3.46) we get

$$\begin{aligned} W(y(t)) \quad &\leq W(y(0)) \Leftrightarrow \textstyle\int_0^t b(|y(p)|)dp \\ &\leq W(y(0)) \Leftrightarrow t\hat{\theta},\ \ \forall\, t \geq 0, \end{aligned}$$

which is a contradiction, because W is p.d.u.u.

Since $L \subset S$ the previous discussion asserts that each trajectory of (3.38) enters S after some finite time. This in conjunction with our Assumption A1 that (3.38a) with y as input satisfies the I.S.A.C. with $\gamma = a_1^{-1}$ and the fact that N is contained in the region of (3.38) imply as in the proof of Theorem 3.1 in [24] that each trajectory of (3.38) is defined for all $t \geq 0$ tending to zero as $t \to +\infty$. □

Remark 3.8. The same result at the previous lemma holds if A2 is satisfied and instead of A1, A3 and A4 we assume that the subsystem (3.38a) with y as input satisfies the I.S.S.C., namely (2.2) holds with x, y instead of y, u and $\beta := \beta_1$ being a function of class K satisfying

$$(1 + p_1)\beta_1(s) < a_1((1 \Leftrightarrow p_2)s), \;\; \forall\, s > 0 \tag{3.44}$$

for some strictly positive constants p_1 and p_2. Indeed, by the main result in [14] condition A2 implies that the subsystem (3.38a) with x as input satisfies the I.S.S.C., namely (2.2) holds with x instead of u and

$$\beta_2 := (a_1^2)^{-1} \circ a_2^2 \circ a_2^{-1} \tag{3.45}$$

instead of β. Then by (3.44) and (3.45) we get

$$\beta_2((1 + p_1)\beta_1(s)) < (1 \Leftrightarrow p_2)s, \;\; \forall\, s > 0$$

which according to the small gain theorem of Jiang–Teel–Praly in [4] implies that zero is globally asymptotically stable with respect to (3.38). Mainly because of the absence of Condition A3, the previous assumptions seem to be rather weaker than those of Lemm 3.7. However the analysis of Lemma 3.7 is simpler than this of the small gain theorem and quite useful to explore in the global feedback stabilization problem for composite systems.

To be more specific, for composite control systems of the form:

$$\dot{x} = G_0(x, y) \tag{3.46a}$$

$$\dot{y} = G(x, y, u) \tag{3.46b}$$

$$(x, y, u) \in \mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}$$

where the mappings G_0 and G are C^0 vanishing at zero and (3.46a) satisfies assumption A1, Lemma 3.7 indicates the following algorithm scheme for stabilization by output feedback:

- First define $a_1 = \gamma^{-1}$, where γ is the characteristic function of the I.S.A.C. imposed for (3.46a).

- Next, consider any p.d.u.u. C^∞ $W : \mathbb{R}^n \Leftrightarrow\!\!\rightarrow \mathbb{R}^+$ and a function $a_2 \in K_\infty$ such that

$$a_1^2(|y|) < W(y) < a_2^2(|y|), \quad \forall\ y \neq 0.$$

- Determine a positive definite function $b : \mathbb{R}^+ \Leftrightarrow\!\!\rightarrow \mathbb{R}^+$ satisfying (3.41).

- Finally, determine if possible a feedback law $u = u(y)$ satisfying (3.40) and such that zero is locally asymptotically stable with respect to

$$\begin{aligned} \dot{x} &= G_0(x, y) \\ \dot{y} &= G(x, y, u(y)). \end{aligned} \tag{3.47}$$

Then Lemma 3.7 asserts that zero is globally asymptotically stable with respect to the closed-loop system (3.47).

The previous algorithm consists of a powerful tool to face the feedback stabilization problem for systems (3.46) with $n = 1$, namely when (3.46b) operates on the real line. To illustrate the usefulness of the previous scheme we consider the following example.

Example Consider the planar case

$$\dot{x} = \Leftrightarrow x^3 + x^2 y \tag{3.47a}$$

$$\dot{y} = u \Leftrightarrow x^3 + x^2 y^2 \phi(x, y) \tag{3.47b}$$

where ϕ is C^0 vanishing at zero and y is the output of the system. First, notice that (3.47a) satisfies the I.S.S.C. with y as input. Particularly, if one takes $V(x) = \frac{1}{2}x^2$, then it holds that

$$DV(x)(\Leftrightarrow x^3 + x^2 y) < 0, \quad \forall\ |y| < \frac{3}{4}|x|.$$

The latter implies that (3.47a) satisfies the I.S.S.C. with $\beta(s) = \frac{4}{3}s$, which in turns implies that I.S.A.C. is satisfied with γ being any function of class K_∞ such that $\gamma(\beta(s)) < s$, $s \neq 0$. Let $\gamma(s) := \frac{1}{2}s$, $a_1(s) := \gamma^{-1}(s) := 2s$, $a_2(s) := 3s$, $b(s) := 2^6 s$ and $W(s) := 5s^2$. By applying Proposition 3.1 for the subsystem (3.47b), we can determine a $\mathbf{C}^\infty$ map $n = u(y)$ such that

$$u(y) = \Leftrightarrow 30y, \quad y \text{ near zero};$$

$$DW(y)(u(y) \Leftrightarrow x^3 + x^2\phi(x, y)) < \Leftrightarrow b(|y|), \quad \forall\ |x| < a_2(|y|).$$

Then it can be easily established that all the requirements of Lemma 3.7 are satisfied hence the previous feedback globally asymptotically stabilizes the system (3.47) at $0 \in \mathbb{R}^2$. For reasons of completeness we note that under the previous choice of the map $u(y)$ near zero and if we evaluate the derivative $\Phi(x,y)$ of $\Phi(x,y) := \frac{1}{2}(x^2+y^2)$ along the trajectories of the closed-loop system we get $\dot{\Phi}(x,y) = \Leftrightarrow x^4 \Leftrightarrow 30y^2 + x^2y^3\phi(x,y) < 0$, for $(x,y) \neq 0$ near zero which guarantees that zero is locally asymptotically stable with respect to the corresponding closed-loop system.

We conclude this section by the following corollary which is an immediate consequence of Theorem 3.5 and Lemma 3.7 for the case (1.5). We use this result in Section 4 to face the global feedback stabilization problem for triangular systems.

Corollary 3.9 *Consider the composite system (1.5) and, in addition to the hypothesis of Theorem 3.5, assume that* $a = a_2^{-1}$ *and the subsystem (1.6) satisfies Conditions A1 and A4 of Lemma 3.7. Moreover, assume that* $0 \in \mathbb{R}^{k+n}$ *is locally asymptotically stable with respect to the closed–loop system (1.5) with* $u = u_c(y)$*, where* $u_c(y)$ *is the smooth output feedback satisfying (3.11) and (3.13). Then the same feedback globally asymptotically stabilizes (1.5) at the origin.*

4 Partial-State Stabilizability for Triangular Systems

We now state and prove our result concerning the output feedback global stabilizability problem for the general triangular case (1.7), where we assume that $y = (y_1,\ldots,y_n)'$ is the output of the system.

Theorem 4.1 *Consider the system (1.7) where the mappings* f*,* g_i *and* h_i *are* C^∞ *vanishing at zero. Assume that:*

A1 *The subsystem (1.8) satisfies the I.S.A.C. (or its stronger version I.S.S.C).*

A2 *The matrix* $(\vartheta f/\vartheta x)(0,0)$ *is Hurwitz.*

A3 *For each* $i = 1,\ldots,n \Leftrightarrow 1$ *and every nonzero* $w_i := (y_1,\ldots,y_i)'$ *there exists an* **odd** *integer* $n_i := n_i(w_i)$ *and a neighborhood* $N_i := N_i(w_i)$ *at* w_i *(both depending on* w_i*) such that*

$$c\ell\left(\bigcup_{w_i\in\mathbb{R}} \frac{\vartheta^{n_i} h_i}{\vartheta y_{i+1}^{n_i}}(N_i\times\mathbb{R})\right) \subset \mathbb{R}^+ (\mathbb{R}^- \mathit{resp})\backslash\{0\} \tag{4.1a}$$

particularly, for $w_i = 0$

$$\frac{\vartheta h_i}{\vartheta y_{i+1}}(0;0) \neq 0; \tag{4.1b}$$

moreover we assume that

$$h_i(0;y_{i+1}) = 0 \quad \textit{iff}\ y_{i+1} = 0. \tag{4.1c}$$

(Without any loss of generality we may assume in the sequel that the sets in the left hand side in (4.1a) is contained in $(\mathbb{R}^+ \setminus \{0\})$*.*

Under the previous hypothesis the system (1.7) is G.A.S. by means of a C^∞ *output feedback* $u = u(y)$ *being linear zero.*

Remark 4.2 For instance (4.1a) is fulfilled if each $h_i(w_i, y_{i+1})$ is a polynomial with respect to y_{i+1} at odd degree. Particularly, Condition A3 holds if it is assumed that for each $1 \leq i \leq n \Leftrightarrow 1$ there exist an odd integer n_i and C^∞ functions $a_{ij}(y_1, \dots, y_i)$, $0 \leq j \leq n_i \Leftrightarrow 1$ with $a_{i1}(0, \dots, 0) \neq 0$ such that

$$h_i(y_1, \dots, y_i, y_{i+1}) = y_{i+1}^{n_i} + \sum_{j=0}^{n_i - 1} a_{ij}(y_1, \dots, y_i) y_{i+1}^j$$

whereas the polynomial $\pi(y_{i+1}) := y_{i+1}^{n_i - 1} + \sum_{j=1}^{n} a_{ij}(0) y_{i+1}^{j-1}$ has no ordinary roots. Indeed, in that case $\partial h_i / \partial {y_{i+1}}^{n_i}$ is everywhere strictly positive, $(\partial h_i / \partial y_{i+1})(0;0) = a_{i1}(0) \neq 0$ and $h_i(0; y_{i+1}) = y_{i+1}\pi(y_{i+1}) = 0$ if and only if $y_{i+1} = 0$. Finally, note that (4.1b) implies that (1.7) is locally asymptotically stabilizable by linear feedback (see Lemma 4.3 below).

As we have mentioned in the introduction, Theorem 4.1 consists a considerable extension of the main result in [25]. Specifically, in [25] it is assumed that (1.8) satisfies a strong version of the I.S.S.C., namely, there exist a p.d.u.u. C^1 map $\Phi_0(x)$ positive constants c_0, R_1 and R_2 such that $|D\Phi_0(x)| \leq R_1|x|$, $|f(x, y_1) \Leftrightarrow f(x, 0)| \leq R_2|y_1|$ and $D\Phi_0(x) f(x, 0) < \Leftrightarrow c_0|x|^2$, $\forall\, x, y_1$. The previous assumptions imply that there exist positive constants K and L such that

$$D\Phi_0(x) f(x, y_1) < \Leftrightarrow K|x|^2, \quad \forall\; |y_1| < L|x|$$

which obviously consists of a special case of the input-to-state stability property. Furthermore, in [25] the following additional inequality is imposed:

$$g_i(x, y_1, \dots, y_i)| \leq M|x|, \quad \forall\; x, y_1, \dots, y_i,\; 1 \leq i \leq n$$

for certain positive constant M. The previous assumptions together with $(A3)$ permits us to apply in [25] a rather simplified Lyapunov technique in order to obtain the desired output feedback. Unfortunately, this approach is not applicable in our case, under the general hypothesis of Theorem 4.1.

In order to establish Theorem 4.1 we need first the following elementary results (Lemma 4.3) dealing with the local stabilizability problem for systems (1.7). The proof of Lemma 4.3 follows by repeating similar discussion with that given in [25] and is omitted.

Lemma 4.3 *Consider the system (1.7) and assume that Property A2 of Theorem 4.1 holds and* $(\vartheta h_i/\vartheta y_{i+1})(0,\ldots,0) \neq 0$, $i = 1,\ldots,n \Leftrightarrow 1$. *Let* P *be the positive definite solution of the Lyapunov matrix equation*

$$P\frac{\vartheta f}{\vartheta y}(0,0) + (\frac{\vartheta f}{\vartheta y}(0,0))'P = \Leftrightarrow I \tag{4.2}$$

and let

$$V_1(y_1) := C_1 y_1^2$$

$$V_i(y_1,\ldots,y_i) := V_1(y_1) + \sum_{j=1}^{i-1} C_{j+1}(y_{j+1} \Leftrightarrow \phi_j(y_1,\ldots,y_j))^2, \quad 1 \leq i \leq n \Leftrightarrow 1 \tag{4.3a}$$

$$\Phi_i(x,y_1,\ldots,y_i) := x'Px + V_i(x,y_1,\ldots,y_i), \quad 1 \leq i \leq n \Leftrightarrow 1 \tag{4.3b}$$

$C_1,\ldots,C_n$ *being arbitrary positive constants and*

$$\phi_1(y_1) := \Leftrightarrow c_1 DV_1(y_1)\frac{\vartheta h_1}{\vartheta y_2}(0,0) = \Leftrightarrow 2c_1C_1\frac{\vartheta h_1}{\vartheta y_2}(0,0)y_1,$$

$$\phi_i(y_1,\ldots,y_i) := \Leftrightarrow c_i DV_i(y_1,\ldots,y_i)\begin{pmatrix} 0 \\ \vdots \\ 0 \\ (\vartheta h_i/\vartheta y_{i+1})(0,\ldots,0)\end{pmatrix}$$

$$= \Leftrightarrow 2c_iC_i\frac{\vartheta h_i}{\vartheta y_{i+1}}(0,\ldots,0)(y_i \Leftrightarrow \phi_{i-1}(y_i,\ldots,y_{i-1}))^2, \quad 1 \leq i \leq n. \tag{4.4}$$

Then there exist constants $c_{10},\ldots,c_{i0}$ *such that for every* $c_j \geq c_{j0}$, $1 \leq j \leq i$ *the derivative of the Lyapunov function* Φ_j *along the trajectories of the linearization of the system*

$$\begin{aligned}
\dot{x} &= f(x,y_1), \\
\dot{y_1} &= g_1(x,y_1) + h_1(y_1,y_2), \\
\dot{y_2} &= g_2(x,y_1,y_2) + h_2(y_1,y_2,y_3),\ \ldots, \\
\dot{y_j} &= g_j(x,y_1,\ldots,y_j) + h_j(y_1,\ldots,y_j,\phi_j(y_1,\ldots,y_j))
\end{aligned}$$

at the origin is negative definite locally around zero.

The following lemma is a direct consequence of Lemmas 3.2 and 4.3. Details are left to the reader.

Lemma 4.4 *Under the same hypothesis with those of Lemma 4.3, for each pair of positive constants* a *and* b *there exist constants* $c_{10},\ldots,c_{i0}$ *such that* *for every* $C_j \geq c_{j0}$, $1 \leq j \leq i$ *the conclusion of Lemma 4.3 holds and in addition the following inequality is satisfied in a neighborhood at zero.*

$$\begin{aligned} &DV_j(\bar{y})(g_1+h_1, g_2+h_2, \ldots, g_j+h_j)'(x, y_1, \ldots, y_j, \phi_j(y_1, \ldots, y_j)) \leq \\ &\quad \Leftrightarrow b|(y_1, \ldots, y_j)|^2 \qquad \forall \bar{y} := (y_1, \ldots, y_j) \neq 0 \ ; \qquad |x| < a|\bar{y}|, \end{aligned}$$

where ϕ_j *is defined in (4.4).*

Proof of Theorem 4.1. For reasons of simplicity we consider the case $n = 3$, namely we prove that the $(k+3)$-dimensional system

$$\begin{aligned} \dot{x} &= f(x, y_1) \\ \dot{y_1} &= g_1(x, y_1) + h_1(y_1, y_2) \\ \dot{y_2} &= g_2(x, y_1, y_2) + h_2(y_1, y_2, y_3) \\ \dot{y_3} &= g_3(x, y_1, y_2, y_3) + u \end{aligned} \tag{4.5}$$

is G.A.S. by smooth output feedback. The proof of the general case follows similarly by induction. We divide our procedure into two steps.

Step I Global stabilization of

$$\dot{x} = f(x, y_1), \ \dot{y_1} = g_1(x, y_1) + h_1(y_1, y_2), \ \dot{y_2} = g_2(x, y_1, y_2) + u. \tag{4.6}$$

First, we need some elementary facts which follow directly from our hypothesis. Without any loss of generality we may assume that because of our Assumption A2 the characteristic function γ of the I.S.A.C., imposed for the subsystem (1.8), is linear near zero (see [24] for details). We define

$$a_1 := \gamma^{-1}$$

and let $a_2, \ k \in K_\infty$ being linear near zero such that

$$\max\{(a_1 \circ k)(s), \ a_1(2s)\} < \frac{1}{2} a_2(s), \ \forall s > o. \tag{4.7}$$

We consider the map

$$J(y_1) := \max\{|xf(x, y_1)|, \ |y_1| \geq a_2^{-1}(|x|)\} \tag{4.8a}$$

and let $\zeta \in K_\infty$ being linear near zero and a pair of constants σ_0 and C such that

$$J(y_1) \leq \zeta^2(|y_1|), \ \forall y_1, \tag{4.8b}$$

$$a_1^2(s) \leq Cs^2 \leq a_2^2(s), \ s > 0 \text{ near zero}, \tag{4.9a}$$

$$\begin{aligned} &a_1^2(|(y_1, y_2)|) < Cy_1^2 + C(y_2 \Leftrightarrow \sigma y_1)^2 < a_2^2(|(y_1, y_2)|) \\ &\qquad \forall (y_1, y_2) \neq 0 \text{ near zero}, \ \sigma \leq \sigma_0 \end{aligned} \tag{4.9b}$$

whose existence is guaranteed by the fact that both a_1 and a_2 are linear near zero.

We now recall condition (4.1b) of our Assumption A3 from which it follows that there exists a constant $\delta_1 > 0$ such that

$$\frac{\vartheta h_1}{\vartheta y_2}(y_1, y_2) \neq 0, \ \ \forall |(y_1, y_2)| \leq \delta_1. \tag{4.10}$$

Without any loss of generality we may assume that the functions $a_1(s)$, $a_2(s)$, $k(s)$, $\zeta(s)$ are linear and (4.9) holds for $0 \leq s \leq \delta_1$. Furthermore, we may assume that (4.8) and (4.9) hold for every $0 \leq |(y_1, y_2)| \leq \delta_1$. We now invoke Lemmas 4.3 and Corollary 3.3 as well as our Assumption A2 and (4.10) in order to determine a linear map of the form $\phi(y_1) = c_1 y_1$, $|y_1| \leq \delta_2$ for some appropriate constants c_1 and $0 \leq \delta_2 \leq \delta_1$, a C^∞ map $u_1(y_1)$, $y_1 \in \mathbb{R}$ vanishing at zero and being linear for $|y_1| \leq \delta_2$, a positive definite C^0 map $c_1(s)$, $s \in \mathbb{R}^+$ being quadratic for $0 \leq s \leq \delta_1$ such that if we define

$$V_1(y_1) := C y_1^2 \tag{4.11}$$

(C being the constant defined in (4.9)) then the following properties are satisfied:

• **Property a1.** The origin $O \in \mathbb{R}^{n+1}$ is locally asymptotically stable with respect to

$$\dot{x} = f(x, y_1), \ \ \dot{y_1} = g_1(x, y_1) + h_1(y_1, \phi_1(y_1)) \tag{4.12}$$

specifically, if we denote by P the solution of the matrix equation (4.2), then the derivative of the positive definite function $x'Px + V_1(y_1)$ (V_1 is defined by (4.11)) along the trajectories of the linearization of (4.12) at zero is negative definite for $0 \leq |(x, y_1)| \leq \delta_2$.

• **Property b1.**

$$DV(y_1)(g_1(x, y_1) + h_1(y_1, \phi_1(y_1))) < \min\{\Leftrightarrow q_1(|y_1|), \Leftrightarrow 2\zeta^2(|y_1|)\}$$
$$\forall\, 0 < |y_1| \leq \delta_2, \ \ |x| < a_2(|y_1|). \tag{4.13}$$

• **Property c1.**

$$DV(y_1)(g_1(x, y_1) + u_1(y_1)) < \min\{\Leftrightarrow q_1(|y_1|), \Leftrightarrow 2\zeta^2(|y_1|)\}$$
$$\forall\, |x| < a_2(|y_1|), \ \ y_1 \neq 0. \tag{4.14}$$

Note that Properties a1, b1 follow directly from Lemma 4.3 respectively, whereas Property c1 is a consequence of Corollary 3.3 for the case $\dot{y}_1 = g_1(x, y_1) + u$.

We now establish the first part of our theorem by using the previous properties and Theorem 3.5. However, a direct application of Theorem 3.5 for the case (4.6) is in general impossible; we need first to apply an appropriate global change of coordinates. We proceed as follows. Consider a function $r(s)$, $s \in \mathbb{R}^+$ of class K_∞ which is linear for $|s| \leq \delta_3$, δ_3 being a positive constant with $\delta_3 \leq \delta_2$ such that

$$\max\{|y_2| : h_1(y_1, y_2) = u_1(y_1)\} \leq r(|y_1|), \ \forall\, |y_1| \geq \delta_3; \tag{4.15a}$$

$$|\phi_1(y_1)| \leq r(|y_1|), \ \forall\, |y_1| \geq \delta_3. \tag{4.15b}$$

Without any loss of generality assume in the sequel that $\delta_1 = \delta_2 = \delta_3$. It should be noted that the existence of the function r satisfying (4.15a) follows directly from our Assumption A3 (see [23,26] where analogous statements are established). We now consider a diffeomorphish $m_1 : \mathbb{R} \to \mathbb{R}$ with $Dm_1(s) \neq 0$, $\forall\, s$; $m_1(s) \to \pm\infty$ as $s \to \pm\infty$ being linear for $|s| \leq \delta_1$, specifically

$$m_1(s) = \gamma_1 s, \ \text{ for } |s| \leq \delta_1$$

for some constant $\gamma_1 > 0$ and such that

$$(r \circ k^{-1})(2|s|) \leq |m_1(s)|, \ \forall\, s \ ; \tag{4.16a}$$

$$\frac{1}{\sigma_0}\frac{|\phi_1(s)|}{|s|} = \frac{c_1}{\sigma_0} \leq \gamma_1, \ \ 0 < |s| \leq \delta_1 \tag{4.16b}$$

where k and σ_0 are defined in (4.7) and (4.9), respectively. We apply the transformation

$$(x, y_1, y_2) \to (x, y_1, m_1^{-1}(y_2)). \tag{4.17}$$

In the new coordinates the system (4.6) takes the form

$$\begin{aligned} \dot{x} &= f(x, y_1) \\ \dot{y_1} &= g_1(x, y_1) + h_1(y_1, m_1(y_2)) \\ \dot{y_2} &= m_1^*(y_2) g_2(x, y_1, m_1(y_2)) + u m_1^*(y) \end{aligned} \tag{4.18}$$

where $m_1^* := Dm_1^{-1} \circ m_1$.

Note that our Assumption A3 remains invariant under the change of coordinates (4.17). Particularly, from (4.10) we get

$$\frac{\vartheta h_1}{\vartheta y_2}(y_1, m_1(y_2)) \neq 0, \ \forall |(y_1, y_2)| \leq \widehat{\delta} \tag{4.19}$$

for certain $\widehat{\delta} > 0$. Moreover in the new coordinates we can determine by using (4.13) and (4.14) a positive definite C^0 function $q_2(s)$, $s \in \mathbb{R}^+$ being quadratic near zero such that

$$DV(y_1)(g_1(x, y_1) + h_1(y_1, \phi_1(y_1))) < \Leftrightarrow q_2(|(y_1, y_2)|)$$

$$\forall \;\; |x| < a_2(|(y_1, y_2)|), \;\; y_1, y_2 \;\; \text{near zero,} \tag{4.20}$$

$$DV(y_1)(g_1(x, y_1) + u_1(y_1)) < \Leftrightarrow q_2(|y_1, y_2|)$$

$$\forall \, |x| < a_2(|(y_1, y_2)|); \tag{4.21}$$

$$\Leftrightarrow q_2(|(y_1, y_2)|) \leq \min\{\Leftrightarrow q_1(|y_1|), \Leftrightarrow 2\zeta^2(|y_1|)\}, \; \forall \; y_1, y_2. \tag{4.22}$$

Notice that from (4.8) and (4.22) we get

$$2|xf(x, y_1)| < q_2(|(y_1, y_2)|), \;\; \forall \, |x| < a_2(|(y_1, y_2)|). \tag{4.23}$$

Let $0 < \delta \leq \delta^*$ and define

$$\alpha_1(y_1) := \max\{y_2 : h_1(y_1, m_1(y_2)) = u_1(y_1)\},$$

$$\beta_1(y_1) := \min\{y_2 : h_1(y_1, m_1(y_2)) = u_1(y_1)\},$$

$$M_1 := \{(y_1, y_2) \in \mathbb{R}^2 : m_1(y_2) = \phi_1(y_1), \;\; |y_1| < \delta\}$$

$$\bigcup\{(y_1, y_2) \in \mathbb{R}^2 : y_2 \in C\Big(\{m_1^{-1}(\phi_1(y_1)), \alpha_1(y_1), \beta_1(y_1)\}\,\Big|_{|y_1| \, = \, \delta}\Big)$$

$$\bigcup\{(y_1, y_2) \in \mathbb{R}^2 : h_1(y_1, m_1(y_2)) = u_1(y_1), \;\; |y_1| > \delta\},$$

where $C(S)$denotes the convex hull at a subset S of a given vector space,

$$U_1^+ := \{(y_1, y_2) \in \mathbb{R}^2 : m_1(y_2) > \phi_1(y_1), \;\; |y_1| < \delta\}$$

$$\bigcup\{(y_1, y_2) \in \mathbb{R}^2 : y_2 > \max C\Big(\{m_1^{-1}(\phi_1(y_1)), \alpha_1(y_1)\}\,\Big|_{|y_1| \, = \, \delta}\Big)$$

$$\bigcup\{(y_1, y_2) \in \mathbb{R}^2 : h_1(y_1, m_1(y_2)) > u_1(y_1), \;\; |y_1| > \delta\},$$

$$U_1^- := \{(y_1, y_2) \in \mathbb{R}^2 : m_1(y_2) < \phi_1(y_1), \;\; |y_1| < \delta\}$$

$$\bigcup\{(y_1, y_2) \in \mathbb{R}^2 : y_2 < \min C\Big(\{m_1^{-1}(\phi_1(y_1)), \beta_1(y_1)\}\,\Big|_{|y_1| \, = \, \delta}\Big)$$

$$\bigcup\{(y_1, y_2) \in \mathbb{R}^2 : h_1(y_1, m_1(y_2)) < u_1(y_1), \;\; |y_1| > \delta\}.$$

We are now in a position to show that for appropriate small δ all conditions of Assumption 3.4 are satisfied with respect to (1.4) with dynamics

$$G_1(x, y_1, y_2) := g_1(x, y_1) + h_1(y_1, m_1(y_2))$$

$$G_2(x, y_1, y_2) := m_1^*(y_2) g_2(x, y_1, m_1(y_2)) \tag{4.24}$$

$$\xi(y_1, y_2) := m_1^*(y_2)$$

namely with respect to

$$
\begin{aligned}
\dot{y_1} &= g_1(x, y_1) + h_1(y_1, m_1(y_2)) \\
\dot{y_2} &= m_1^*(y_2) g_2(x, y_1, m_1(y_2)) + u m_1^*(y_2)
\end{aligned}
\tag{4.25}
$$

where M_1,: U_1^+, U_1^-, V_1, q_2, a_2^{-1} and $\hat{\phi}_1 := \phi_1/\gamma_1$ play the role of M, U^+, U^-, V, b, a and ϕ, respectively. Indeed, using our Assumption A3 (for $i = 1$) and taking into account that this property remains invariant under the change of coordinates (4.17) we can easily establish similar to [26] that property (i) of Assumption 3.4 is satisfied with respect to (4.25). Particularly, the intersection of M_1 with the region $\{(y_1, y_2) : |y_1| < \delta\}$ coincides with the graph of the function $\hat{\phi}_1 = m_1^{-1}(\phi_1(y_1)) = \phi_1/\gamma_1$ for $|y_1| < \delta$, hence (3.6) is satisfied. The most crucial observation is that in the new coordinates the corresponding system (4.25) satisfies conditions (3.7) and (3.8) of Assumption 3.4. Indeed, if we define

$$
\hat{V}_1(y_1, y_2) := V_1(y_1) + C(y_2 \Leftrightarrow \hat{\phi}_1(y_1))^2 \tag{4.26}
$$

then (3.7) is a direct consequence of (4.9), (4.11) and (4.16). Furthermore, for each y_1 we have

$$
2\max\{|y_2| : (y_1, y_2) \in M_1\} \leq k(|y_1|). \tag{4.27}
$$

Indeed, for given y_1 let $w := \max\{|y_2| : (y_1, y_2) \in M_1\}$. Then from (4.15) and the definition of M_1 we get $|m_1(w)| \leq r(|y_1|)$ which by (4.16a) implies $2w \leq k(|y_1|)$, hence (4.27) is satisfied. The latter in conjunction with (4.7) implies condition (3.8) of Assumption 3.4. Finally, from (4.20), (4.21) and the definition of the set M_1 we can easily establish as in [26] that (3.9) is also satisfied with $b = q_2$. For reasons of completeness we note that for appropriate small δ and because of (4.20), (4.21) and (4.19) it follows

$$
\begin{aligned}
&DV_1(y_1)(g_1(x, y_1) + h_1(y_1, m_1(y_2))) < \Leftrightarrow q_2(|(y_1, y_2)|) \\
&\forall\, y_2 \in C\Big(\{m_1^{-1}(\phi_1(y_1)), \alpha_1(y_1), \beta_1(y_1)\}\,\Big|_{|y_1| = \delta}\Big).
\end{aligned}
$$

(The inequality above is quite necessary to establish (3.9).) Consequently, according to Theorem 3.5 there exists a p.d.u.u. C^∞ function $V_2(y_1, y_2)$ associated with a real constant c_{20} and a C^∞ map $u_2 := u_2(y_1, y_2, c_2) \in \mathbb{R}^2 \times [c_{20}, +\infty)$ such that V_2 coincides with $\hat{V}_1$ near zero (as the latter is defined by (4.26)) and satisfies

$$
a_1^2(|(y_1, y_2)|) < V_2(y_1, y_2) < a_2^2(|(y_1, y_2)|), \ \forall\, (y_1, y_2) \neq 0, \tag{4.28}
$$

u_2 is linear for y in a neighborhood of zero, particularly

$$
u_2(y) = \Leftrightarrow 2c_2 C(y_2 \Leftrightarrow \hat{\phi}_1(y_1)), \ y := (y_1, y_2)' \ \text{ near zero} \tag{4.29}
$$

and such that

$$DV_2(y)\begin{pmatrix} G_1(x,y_1,y_2) \\ G_2(x,y_1,y_2) \end{pmatrix} + (u_2\xi)(y)\frac{\vartheta V_2}{\vartheta y_2}(y_1,y_2) < \Leftrightarrow q_2(|y|)$$

$$\forall\, |x| < a_2(|y|),\ y = (y_1,y_2)^{'} \neq 0, \tag{4.30}$$

where G_1, G_2 and ξ are defined in (4.24). In addition to the previous properties we can select the constant c_2 sufficiently large so that $0 \in \mathbb{R}^{k+2}$ is locally asymptotically stable with respect to the closed-loop system (4.18) with (4.29). This claim is a direct consequence of Lemma 4.3 for the case $i = 2$ and Property a1. Particularly, for appropriate c_2 the derivative of the Lyapunov function $x'Px + \hat{V}_1(y_1,y_2) = x'Px + Cy_1^2 + C(y_2 \Leftrightarrow \hat{\phi}_1(y_1))^2$ along the trajectories of the closed-loop system (4.18) with (4.29) is negative definite in a neighborhood of zero. The previous discussion in conjunction with (4.8), (4.23) and (4.30) and the fact that (1.8) satisfies the I.S.A.C. with $\gamma = a_1^{-1}$ imply that all conditions of Corollary 3.9 are satisfied with respect to (1.5) with dynamics $G_0(x,y) = f(x,y); G_1, G_2$ and ξ as defined in (4.24) and $b = q_2$. Therefore zero is globally asymptotically stable with respect to the closed-loop system (4.18) with (4.29). This property remains invariant under the change of coordinates (4.17), hence we conclude that (4.6) is G.A.S. by means of a C^∞ output feedback.

Step II Global stabilization of (4.5), or equivalently of the system

$$\begin{aligned}
\dot{x} &= f(x,y_1) \\
\dot{y_1} &= g_1(x,y_1) + h_1(y_1, m_1(y_2)); \\
\dot{y_2} &= m_1^*(y_2)g_2(x,y_1,m_1(y_2)) + m_1^*(y_2)h_2(y_1,m_1(y_2),y_3); \\
\dot{y_3} &= g_3(x,y_1,m_1(y_2),y_3) + u.
\end{aligned} \tag{4.31}$$

We use the result that we have derived in Step I and apply the same procedure. For reasons of completeness we briefly present the most important part of the proof.

Taking into account our hypothesis A2 and A3 and using Lemmas 4.3 and 3.2 we can find a linear map $\phi_2(y_1,y_2)$ vanishing at zero such that the following properties are satisfied.

Property a2. The origin $O \in \mathbb{R}^{k+2}$ is locally asymptotically stable with respect to

$$\begin{aligned}
\dot{x} &= f(x, y_1);\\
\dot{y_1} &= g_1(x, y_1) + h_1(y_1, m_1(y_2));\\
\dot{y_2} &= m_1^*(y_2) g_2(x, y_1, m_1(y_2)) + m_1^*(y_2) h_2(y_1, m_1(y_2), \phi_2(y_1, y_2)).
\end{aligned} \tag{4.32}$$

Particularly, the derivative of $x'Px + V_2(y_1, y_2)$, along the trajectories of the linearization of (4.32) at the origin is negative definite locally around zero.

Property b2.

$$\mathrm{DV_2(y_1, y_2)} \begin{pmatrix} g_1(x, y_1) + h_1(y_1, m_1(y_2)) \\ m_1^*(y_2) g_2(x, y_1, m_1(y_2)) + m_1^*(y_2) h_2(y_1, m_1(y_2), \phi_2(y_1, y_2)) \end{pmatrix}$$

$$< \Leftrightarrow q_2(|y_1, y_2|), \ \ \forall\, |x| < a_2(|y_1, y_2|),\ (y_1, y_2) \neq 0 \ \text{ near zero.}$$

For reasons of completeness we note that Property b2 is a direct consequence at Property b1 and Lemma 3.2. Furthermore from Step I we recall the fact that the following property holds.

Property c2. There exists a C^∞ map $u_2(y_1, y_2)$ such that (4.30) is fulfilled.

Next we consider a diffeomorphish $m_2 : \mathbb{R} \to \mathbb{R}$ with $Dm_2(y_3) \neq 0$, $\forall\, y_3$ with $m_2 \to \pm\infty$ as $y_3 \to \pm\infty$ being linear near zero, i.e. $m_2(y_3) = \gamma_2 y_3$ for some constant γ_2, and apply in (4.31) the transformation

$$(x, y_1, y_2, y_3) \to (x, y_1, y_2, m_2^{-1}(y_3)).$$

The resulting system is

$$\begin{aligned}
\dot{x} &= f(x, y_1)\\
\dot{y_1} &= g_1(x, y_1) + h_1(y_1, m_1(y_2))\\
\dot{y_2} &= m_1^*(y_2) g_2(x, y_1, m_1(y_2)) + m_1^*(y_2) h_2(y_1, m_1(y_2), m_2(y_3))\\
\dot{y_3} &= m_2^*(y_3) g_3(x, y_1, m_1(y_2), m_2(y_3)) + u m_2^*(y_3)
\end{aligned} \tag{4.33}$$

where $m_2^* = Dm_2^{-1} \circ m_2$. We define as in Step I

$$\alpha_2(y_1, y_2) := \max\{y_3 : h_2(y_1, m_1(y_2), m_2(y_3)) = u_2(y_1, y_2)\},$$

$$\beta_2(y_1, y_2) := \min\{y_3 : h_2(y_1, m_1(y_2), m_2(y_3)) = u_2(y_1, y_2)\},$$

and for sufficiently small $\delta > 0$ let

$$M_2 := \{(y_1, y_2, y_3) \in \mathbb{R}^3 : m_2(y_3) = \phi_2(y_1, y_2),\ |(y_1, y_2)| < \delta\}$$

$$\bigcup\{(y_1, y_2, y_3) \quad \in \quad \mathbb{R}^3 : y_3$$

$$\alpha_2(y_1, y_2), \beta_2(y_1, y_2)\} \Big|_{|(y_1,y_2)| = \delta}\Big)$$

$$\bigcup\{(y_1, y_2, y_3) \in \mathbb{R}^3 : h_2(y_1, m_1(y_2), m_2(y_3)) = u_2(y_1, y_2),\ |(y_1, y_2)| > \delta\}.$$

Analogously with Step I we define the sets U_2^+ and U_2^- and consider the functions

$$\hat{V}_2(y_1, y_2, y_3) := \hat{V}_1(y_1, y_2) + C(y_3 \Leftrightarrow \hat{\phi}_2(y_1, y_2))^2\ ;$$

$$\hat{\phi}_2(y_1, y_2) := \phi_2(y_1, y_2)/\gamma_2$$

Then by using our hypothesis A3 and Properties b2 and c2 we can establish that for appropriate choice of m_2 all conditions of Theorem 3.5 are satisfied with respect to (1.4) with dynamics

$$\mathrm{G}_1(\mathrm{x}, \mathrm{y}_1, \mathrm{y}_2) := \begin{pmatrix} g_1(x, y_1) + h_1(y_1, m_1(y_2)) \\ m_1^*(y_2) g_2(x, y_1, m_1(y_2)) + m_1^*(y_2) h_2(y_1, m_1(y_2), m_2(y_3)) \end{pmatrix} \tag{4.34a}$$

$$G_2(x, y_1, y_2) := m_2^*(y_2) g_3(x, y_1, m_1(y_2), m_2(y_3)) \tag{4.34b}$$

$$\xi := m_2^*(y_2) \tag{4.34c}$$

and M_2,: U_2^+, U_2^-, V_2, a_2^{-1} and $\hat{\phi}_2 := \phi_2/\gamma_2$ instead of M, U^+, U^-, V, a and ϕ, respectively and $b := q_3$, q_3 being an appropriate positive definite C^0 function being quadratic near zero and such that $q_2(s) \le q_3(s)$, $\forall\, s \ge 0$. Therefore we can determine a p.d.u.u. C^∞ function $V_3(y_1, y_2, y_3)$ which coincides with $\widehat{V}_2(y_1, y_2, y_3) := Cy_1^2 + C_1(y_2 \Leftrightarrow \hat{\phi}_2(y_1, y_2))^2$ near zero and satisfies $a_1^2(|(y_1, y_2, y_3)|) < V_3(y_1, y_2, y_3) < a_2^2(|(y_1, y_2, y_3)|)$ for all $(y_1, y_2, y_3) \neq 0$, associated with a constant c_{30} and a C^∞ map $u_3 = u_{c_3}(y_1, y_2, y_3; c_3)$ which coincides with $\Leftrightarrow 2c_3 C_1(y_3 \Leftrightarrow \hat{\phi}_2(y_1, y_2))$ near zero and such that (3.13) holds with $W = V_3$, $b = q_3$, $a = a_2^{-1}$ and G_1, G_2 and ξ as defined in (4.34). Furthermore, by evaluating the derivative at the Lyapunov function $x'Px + \hat{V}_3(y_1, y_2, y_3)$ along the trajectories of the closed-loop system (4.33) with $u = u_3$, we can determine by taking into account property the constant c_3 sufficiently large so that zero is locally asymptotically stable with respect to (4.33) with $u = u_3$. Since (1.8) satisfies the I.S.A.C. with $\gamma = a_1^{-1}$, from Corollary 3.9 we conclude that zero is globally asymptotically stable with respect to the closed-loop system (4.33) with $u = u_3$. □

5 Robust Stabilization

The result of Theorem 4.1 can be generalized for the following more general class of triangular systems whose dynamics also depend on some unknown time-varying parameters

$$\begin{aligned} &\dot{x} = f(x, y_1, \theta) \\ &\dot{y}_i = g_i(x, y, \theta) + h_i(y_1, \ldots, y_{i+1}),\ 1 \le i \le n \Leftrightarrow 1 \\ &\dot{y}_n = g_n(x, y, \theta) + u \\ &(x, y) := (x; y_1, \ldots, y_n) \in \mathbb{R}^k \times \mathbb{R}^n \end{aligned} \tag{5.1}$$

where $\theta = \theta(t) \in \mathbb{R}^\mu$ is a time varying vector parameter and there exist positive definite C^∞ mappings $\xi_0 : \mathbb{R}^{k+1} \Leftrightarrow\!\to \mathbb{R}^+$, $\xi_i : \mathbb{R}^{k+1} \Leftrightarrow\!\to \mathbb{R}^+$, $i = 1, 2, \ldots, n$ such that

$$|f(x, y_1, \theta)| + \left| \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) (x, y_1, \theta) \right| \le \xi_0(x, y_1)$$

$$|g_i(x, y, \theta)| \le \xi_i(x, y_1, \ldots, y_i), \quad \forall\ x, y, \theta.$$

Also assume that $x' L f(x, 0, \theta) \le \Leftrightarrow \ell |x|^2$, $\forall\, \theta, x$ near zero for certain positive constant ℓ and positive definite matrix L and the parameterized subsystem $\dot{x} = f(x, y_1, \theta)$ with y_1 as input satisfies a stronger version of the I.S.S.C. Particularly, we assume that each trajectory $x(t) \doteqdot x(t, x_0, y_1, \theta)$ is defined for almost all $t \ge 0$, essentially bounded input y_1 and parameter θ and there exists a pair of functions $a \in KL$ and $\beta \in K$ such that

$$|x(t)| \le \alpha(|x_0|, t) + \beta(||(y_1)_t||)$$

for all $t \ge 0$ and $\theta = \theta(t)$. (See also [27] where a weaker version, being analogous with the I.S.A.C. property, has been imposed.)

If in addition Condition A3 of Theorem 4.1 are satisfied, then by using a slight modification of the approach used in Section 4 it can be shown that (5.1) is G.A.S. (uniformly on θ) by means of a C^∞ output feedback $u = u(y)$ being independent of θ and vanishing at zero. The previous extension has been already obtained in [27] for the particular case of systems (5.1) with $h_i \equiv y_{i+1}$, $1 \le i \le n \Leftrightarrow 1$; in the previous mentioned paper this result was used to explore the dynamic output feedback stabilization for triangular systems where only the y_1 component is available.

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