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Asymptotic Null Controllability of Nonlinear Neutral Volterra Integrodifferential Systems*

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Abstract

In this paper, sufficient conditions are obtained for asymptotic null controllability of nonlinear neutral Volterra integrodifferential systems. The results are obtained by using the Leray–Schauder fixed point theorem.

Key words: controllability, Volterra integrodifferential systems

1 Introduction

Compartmental models are frequently used in, e.g., theoretical epidemiology, physiology, population dynamics, the analysis of ecosystems, and chemical reaction kinetics. These models are used to describe the evolution of systems which can be divided into separate compartments, marking the pathways of material flow between compartments and the possible outflow to and inflow from the environment of the system (see [1, 7–11] and the references particularly in [1, 7, 8]). For some such models the time required for the material flow between compartments cannot be neglected, i.e., is not instantaneous. A paradigm for such systems can be visualized as one in which compartments are connected by pipes through which material passes in definite time. Because of the time lags caused by the passage though the pipes, the model equations for such systems are differential equations with deviating arguments; this is contrary to the classical case where model equations have transport time that can be considered negligible and thus can be modelled satisfactorily using ordinary differential equations. For

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more details, we refer to the work of Anderson [1], Gyori [7, 8] and Gyori and Wu [9].

A concrete example of a compartmental model is the radiocardiogram, where the two compartments correspond to the left and right ventricles of the heart and the pipes between these compartments represent the pulmonary and systematic circulation. Pipes coming out from and returning into the same compartment may represent shunts and the coronary circulation (see [7]). Other applications arise in tracer kinetics, in modeling the uptake of potassium by red blood cells, as well as in such environmentally oriented applications as modeling the kinetics of lead in a body (see [1]). A system for representing such models is obtained using nonlinear neutral Volterra integrodifferential equations of the form (1.1) given below. The aim of this paper is to study the controllability problem for such systems and to significantly extend the application of controllability to the above class of models (see [1, Section 21]).

Consider the nonlinear neutral Volterra integrodifferential systems of the form

$$\frac{d}{dt}\left[x(t) - \int_0^t C(t-s)x(s)ds - g(t)\right] = Ax(t) + \int_0^t G(t-s)x(s)ds + B(t)u(t) + f(t, x(t), u(t)), \qquad x(0) = x_0 \tag{1.1}$$

where $x \in E^n$, $u \in E^m$, $t \in J = [0, \infty)$, $C(t)$ and $G(t)$ are continuous $n \times n$ matrix valued functions, $B(t)$ is a continuous $n \times m$ matrix valued function, A is a constant $n \times n$ matrix and $f: J \times E^n \times E^m \to E^n$ and $g: J \to E^n$ are, respectively, continuous and absolutely continuous vector valued functions.

In this paper we develop conditions for the system (1.1) to be asymptotically null controllable. Let C^k denote the space of bounded continuous k-vector valued functions defined on J with the usual sup norm.

Definition 1.1 *The system (1.1) is said to be asymptotically null controllable if for every $x_0 \in E^n$ there exists a control u defined on J such that $x(0) = x_0$ and $\lim_{t\to\infty} x(t) = 0$.*

Balachandran and Dauer [2] introduced the concept of , -controllability by using an idea of Russell [14]. A similar concept was developed by Chukwu [4] to study controllability to affine manifolds, and Eke [5, 6] extended the concept to perturbed nonlinear systems. Recently Balachandran and Balasubramaniam [3] studied the null controllability of nonlinear systems. The work reported in [2–6] are based on the work of Kartsatos [12, 13] on generalized boundary conditions in ordinary differential equations.

2 Preliminary Results

Consider the linear system

$$\begin{aligned}\frac{d}{dt}\left[x(t)-\int_0^t C(t-s)x(s)ds-g(t)\right]&\\ = Ax(t)+\int_0^t G(t-s)x(s)ds+B(t)u(t).&\qquad (2.1)\end{aligned}$$

The solution of (2.1) can be written as (see [15])

$$x(t)=Z(t)[x(0)-g(0)]+g(t)+\int_0^t \dot{Z}(t-s)g(s)ds+\int_0^t Z(t-s)B(s)u(s)ds$$

where $Z(t)$ is an $n\times n$ continuously differentiable matrix satisfying

$$\frac{d}{dt}\left[Z(t)-\int_0^t C(t-s)Z(s)ds\right]=AZ(t)+\int_0^t G(t-s)Z(s)ds \qquad (2.2)$$

with $Z(0)=I$, the identity matrix. Assume the following limits exist

$$\lim_{t\to\infty} Z(t)=Z\neq 0, \qquad \lim_{t\to\infty}\dot{Z}(t)=\bar{Z},$$

$$\lim_{t\to\infty} g(t)=k \text{ (constant)}, \qquad \lim_{t\to\infty} W(t)=W$$

where

$$W(t)=\int_0^t ZB(s)(ZB(s))^* ds, \qquad (2.3)$$

the asterisk denotes the matrix transpose.

Further, (see Wu [15]) the solution of the system (1.1) is given by

$$\begin{aligned}x(t) \quad = \quad & Z(t)[x(0)-g(0)]+g(t)+\int_0^t \dot{Z}(t-s)g(s)ds\\ &+\int_0^t Z(t-s)[B(s)u(s)+f(s,x(s),u(s))]ds.\end{aligned}$$

Theorem 2.1 *The system* (2.1) *is asymptotically null controllable if and only if* W *is nonsingular.*

Proof: Assume that W is nonsingular, then for each $x_0\in E^n$, $x_0\neq 0$, define the control function u on J as

$$u(t)=-(Z(t)B(t))^*W^{-1}\left[Z[x(0)-g(0)]+k+\int_0^\infty \bar{Z}g(s)ds\right].$$

Clearly $x(0) = x_0$ and $\lim_{t\to\infty} x(t) = 0$, and so system (2.1) is asymptotically null controllable.

Conversely, assume that W is singular. Then there exists a vector $y \neq 0$ such that $y^*Wy = 0$. It follows that

$$\int_0^\infty y^*ZB(s)(y^*ZB(s))^*ds = 0.$$

Therefore,

$$y^*ZB(s) = 0 \quad \text{for } s \in J.$$

Since the solution is asymptotically null controllable, there exists a control $u(\cdot)$ such that

$$\begin{aligned} \lim_{t\to\infty} x(t) &= \lim_{t\to\infty} \Big\{ Z(t)[x(0) - g(0)] + g(t) + \int_0^t \dot{Z}(t-s)g(s)ds \\ &\quad + \int_0^t Z(t-s)B(s)u(s)ds \Big\} = 0. \end{aligned}$$

Letting $g = 0$, we have

$$Zx_0 + \int_0^\infty ZB(s)u(s)ds = 0.$$

So that

$$y^*Zx_0 + \int_0^\infty y^*ZB(s)u(s)ds = 0,$$

which implies that $y^*Zx_0 = 0$. So $y^* = 0$, which is a contradiction to the fact that $y \neq 0$. Hence W must be nonsingular. □

Example 2.1 Consider the linear system (2.1) with

$$\begin{gathered} G(t-s) = C(t-s) = -\exp(t-s), \quad g(t) = \exp(-t) \\ A = -1, \quad B(t) = \tfrac{1}{2}\exp(-t). \end{gathered} \tag{2.4}$$

Therefore $Z(t) = 2\exp(-t) - 1$ satisfies (2.2), so that

$$Z(0) = 1, \quad \lim_{t\to\infty} Z(t) = -1.$$

Hence from (2.3) we have $W = \frac{1}{4}\int_0^\infty \exp(-2s)ds = \frac{1}{8}$ and $|W| = \frac{1}{8}$ is nonsingular. Therefore the system is asymptotically null controllable.

To approach the asymptotic null controllability for the system (1.1), we define an operator $S: C^n \times C^m \to C^n \times C^m$, $S(x,u) = (z,v)$, for which any fixed point $(x,u) \in C^n \times C^m$ will satisfy (1.1) with $x(0) = x_0$ and $\lim_{t\to\infty} x(t) = 0$. Now, for each $(x,u) \in C^n \times C^m$, we define

$$\begin{aligned} z(t) &= Z(t)[x(0)-g(0)] + g(t) + \int_0^t \dot{Z}(t-s)g(s)ds \\ &\quad + \int_0^t Z(t-s)[B(s)u(s) + f(s,x(s),u(s))]ds \end{aligned}$$

$$\begin{aligned} v(t) &= (Z(t)B(t))^* W^{-1}\Big\{-Z[x(0)-g(0)] - k \\ &\quad - \int_0^\infty \bar{Z}g(s)ds - \int_0^\infty Zf(s,x(s),u(s))ds\Big\}. \end{aligned}$$

Note that if $(x,u) \in C^n \times C^m$ is a fixed point of S, we have

$$\begin{aligned} x(t) &= Z(t)[x(0)-g(0)] + g(t) + \int_0^t \dot{Z}(t-s)g(s)ds \\ &\quad + \int_0^t Z(t-s)[B(s)u(s) + f(x(s),u(s))]ds \end{aligned}$$

$$\begin{aligned} u(t) &= (Z(t)B(t))^* W^{-1}\Big\{-Z[x(0)-g(0)] - k \\ &\quad - \int_0^\infty \bar{Z}g(s)ds - \int_0^\infty Zf(s,x(s),u(s))ds\Big\}. \end{aligned}$$

Thus $x(t)$ is the solution of (1.1) corresponding to the control $u(t)$ with $x(0) = x_0$ and $\lim_{t\to\infty} x(t) = 0$. To find such a fixed point, we introduce a parameter $\mu \in [0,1]$ into the problem (1.1) as follows,

$$\begin{aligned} &\frac{d}{dt}\left[x(t) - \int_0^t C(t-s)x(s)ds - g(t)\right] = Ax(t) \\ &+ \int_0^t G(t-s)x(s)ds + \mu\{B(t)u(t) + f(t,x(t),u(t))\}. \end{aligned}$$

Consider the operator

$$S(x,u,\mu) = (z,v)$$

where

$$\begin{aligned} z(t) &= \mu\Big\{Z(t)[x(0)-g(0)] + g(t) + \int_0^t \dot{Z}(t-s)g(s)ds \\ &\quad + \int_0^t Z(t-s)[B(s)u(s) + f(s,x(s),u(s))]ds\Big\} \end{aligned}$$

$$v(t) = \mu\Big\{(Z(t)B(t))^* W^{-1}[-Z[x(0) - g(0)] - k - \int_0^\infty \bar{Z}g(s)ds - \int_0^\infty Zf(s, x(s), u(s))ds]\Big\}.$$

We want to show that, in an appropriate Banach space D, there exists a function pair $(x, u) \in D$ with $S(x, u, 1) = (x, u)$. For that we need the following Leray–Schauder fixed point theorem [6]. Note that $S(x, u, \mu)$ is completely continuous in (x, u) if, for each $\mu \in [0, 1]$, $S(x, u, \mu)$ is continuous in (x, u) and maps every bounded subset of D into a relatively compact set.

Theorem 2.2 *Let D be a Banach space. For the equation*

$$S(x, u, \mu) - (x, u) = 0 \tag{2.5}$$

assume the following:

(i) *$S(x, u, \mu)$ is defined on $D \times [0, 1]$, with values in D and is completely continuous in (x, u). Moreover, if K is a bounded subset of D, $S(x, u, \mu)$ is continuous in μ uniformly with respect to $(x, u) \in K$.*

(ii) *$S(x, u, \mu_0) = 0$, for some $\mu_0 \in [0, 1]$ and for every $(x, u) \in D$.*

(iii) *If there are any solutions of (2.5), then they belong to some closed ball $\bar{B}$ of D, independently of μ.*

Then there exists a continuum of solutions of (2.5), corresponding to all values of $\mu \in [0, 1]$, and all of these solutions lie in $\bar{B}$.

3 Main Result

We now state and prove sufficient conditions for the system (1.1) to be asymptotically null controllable. For this result, we denote $\sigma = \|W^{-1}\|$ and for any bounded interval $\hat{J}$, we let $C(\hat{J}, E^d)$ $(d = n+m)$ be the Banach space of all continuous E^d-valued functions on $\hat{J}$ with the sup norm.

Theorem 3.1 *Assume the following to hold for system* (1.1):

(i) *The fundamental matrix solution $Z(t)$ is such that*

$$\|Z(t)\| \le k_1 \text{ and } \|g(t)\| \le k_2, \quad \text{for all } t \in J,$$

where k_1 and k_2 are some positive constants.

(ii) *The $n \times m$ continuous matrix $B(t)$ is bounded on J, with*

$$\max_{t \in J} \|B(t)\| \le \beta, \quad \text{for some } \beta > 0.$$

(iii) *There exists constants H, P, R, N such that for all $s \in J$ we have*

$$\limsup_{t\to\infty}\left\|\int_0^t \dot{Z}(t-s)g(s)ds\right\| \le P,$$

$$\limsup_{t\to\infty}\left\|\int_0^t Z(t-s)B(s)ds\right\| \le H,$$

$$\limsup_{t\to\infty}\left\|\int_0^t Z(t-s)f(s,x(s),u(s))ds\right\| \le R(\|x\|+\|u\|)+N.$$

(iv) *The matrix W is nonsingular.*

If we can choose the constants satisfying $[(H+R)+\beta k_1\sigma R]<1$, then the system (1.1) is asymptotically null controllable.

Proof: Let $J_1=[0,1]$, $d=n+m$ and let the sup norm of $C(J_1,E^d)$ be $\|\cdot\|_1$. Assume that $w=(x,u)\in C(J_1,E^d)$ and consider the function $\bar{w}(\cdot)$ defined by

$$\begin{aligned}\bar{w}(t) &= w(t), \quad t\in J_1\\ \bar{w}(t) &= w(1), \quad t\in[1,\infty).\end{aligned}$$

Clearly, the set of all such $\bar{w}$ is a Banach space, which we designate by C_1 with the norm $\|\bar{w}\|_2=\|w\|_1$. Note that $C_1=D_1\times H_1$, where D_1 is defined with elements $x\in C(J_1,E^n)$ and H_1 is defined with elements $u\in C(J_1,E^m)$. So

$$\|\bar{w}\|_2=\|\bar{x}\|_{D_1}+\|\bar{u}\|_{H_1}, \quad \text{where} \quad \bar{w}=(\bar{x},\bar{u})\in C_1=D_1\times H_1.$$

Consider the operator $S\colon C_1\to C_1$ defined by

$$S(\bar{x},\bar{u},\mu)=(\bar{z},\bar{v})$$

where

$$\begin{aligned}\bar{z}(t) &= \mu\Big\{Z(t)[x(0)-g(0)]+g(t)+\int_0^t \dot{Z}(t-s)g(s)ds\\ &\quad +\int_0^t Z(t-s)[B(s)\bar{u}(s)+f(s,\bar{x}(s),\bar{u}(s))]ds\Big\}\end{aligned}$$

$$\begin{aligned}\bar{v}(t) &= \mu\Big\{(Z(t)B(t))^*W^{-1}[-Z[x(0)-g(0)]-k\\ &\quad -\int_0^\infty \bar{Z}g(s)ds-\int_0^\infty Zf(s,\bar{x}(s),\bar{u}(s))ds\Big\}.\end{aligned}$$

Our aim is to show that $S(\bar{x},\bar{u},1)$ has a fixed point. First, we shall prove that $S(\bar{x},\bar{u},\mu)$ is continuous in μ. To see this, let $\mu_1,\mu_2 \in [0,1]$ and $(\bar{x},\bar{u}) \in C_1$. Then we have

$$\begin{aligned}
&|S(\bar{x},\bar{u},\mu_1)(t) - S(\bar{x},\bar{u},\mu_2)(t)| \\
&\quad \le |\mu_1-\mu_2|\Big\{|Z(t)[x_0-g(0)]| + |g(t)| + \Big|\int_0^t \dot{Z}(t-s)g(s)ds\Big| \\
&\qquad + \Big|\int_0^t Z(t-s)[B(s)\bar{u}(s) + f(\bar{x}(s),\bar{u}(s))]ds\Big|\Big\} \\
&\qquad + |\mu_1-\mu_2|\Big|(Z(t)B(t))^* W^{-1}[-Z[x(0)-g(0)] - k \\
&\qquad - \int_0^\infty \bar{Z}g(s)ds - \int_0^\infty Zf(s,\bar{x}(s),\bar{u}(s))ds]\Big| \\
&\quad \le |\mu_1-\mu_2|[k_1|x_0-g(0)| + k_2 + P + H\|\bar{u}\| + R(\|\bar{u}\| + \|\bar{x}\|) + N] \\
&\qquad + |\mu_1-\mu_2|[\beta k_1\sigma\{k_1|x_0-g(0)| + k + P + R(\|\bar{u}\| + \|\bar{x}\|) + N\}].
\end{aligned}$$

Because

$$\|S(\bar{x},\bar{u},\mu_1) - S(\bar{x},\bar{u},\mu_2)\|_2 = \sup_{t\in J_1} |S(\bar{x},\bar{u},\mu_1)(t) - S(\bar{x},\bar{u},\mu_2)(t)|$$

it follows that the operator $S(\bar{x},\bar{u},\mu)$ is continuous in μ, uniformly on any bounded subset of C_1. Let

$$\bar{w} = (\bar{x},\bar{u}), \quad \bar{w}_n = (\bar{x}_n,\bar{u}_n), \quad \bar{w},\bar{w}_n \in C_1, \quad n = 1,2,\dots,$$

$$\bar{y}_n = (\bar{z}_n,\bar{v}_n) = S(\bar{x}_n,\bar{u}_n,\mu), \quad \bar{y} = (\bar{z},\bar{v}) = S(\bar{x},\bar{u},\mu).$$

Suppose that

$$\lim_{n\to\infty} \|\bar{w}_n - \bar{w}\|_2 = \lim_{n\to\infty} \|w_n - w\|_1 = 0,$$

that is

$$\lim_{n\to\infty} \|\bar{x}_n - \bar{x}\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|\bar{u}_n - \bar{u}\| = 0.$$

Then we have

$$\|\bar{y}_n - \bar{y}\| = \sup_{t\in J_1} |y_n(t) - y(t)| = \sup_{t\in J_1} [|z_n(t) - z(t)| + |v_n(t) - v(t)|].$$

Now,

$$\begin{aligned}
\sup_{t\in J_1} |z_n(t) - z(t)| \le \int_0^\infty &\|Z(s)[B(s)(\bar{u}_n(s) - \bar{u}(s)) \\
&+ f(s,\bar{w}_n(s)) - f(s,\bar{w}(s))]\|ds
\end{aligned}$$

and the integrand tends to zero as $n \to \infty$. Thus, it follows from the Lebesgue dominated convergence theorem that $\lim_{n\to\infty} \|\bar{z}_n - \bar{z}\| = 0$. Similarly,

$$\lim_{n\to\infty} \|\bar{v}_n - \bar{v}\| = 0.$$

Therefore,

$$\lim_{n\to\infty} \|\bar{y}_n - \bar{y}\| = 0.$$

This proves the continuity of $S(\bar{w}, \mu)$ with respect to $\bar{w}$. Let K be a bounded subset of C_1 with bound b_k. We now show that the family of functions $\bar{y} = S(\bar{w}, \mu)$, $\bar{w} \in K$, $\mu \in [0, 1]$, are equicontinuous. Let $t_1, t_2 \in [0, 1]$. Then,

$$|\bar{y}(t_1) - \bar{y}(t_2)| = |\bar{z}(t_1) - \bar{z}(t_2)| + |\bar{v}(t_1) - \bar{v}(t_2)|.$$

Now,

$$\begin{aligned} |\bar{z}(t_1) - \bar{z}(t_2)| &\le \|Z(t_1) - Z(t_2)\| \, |x_0 - g(0)| + |g(t_1) - g(t_2)| \\ &+ \left\| \int_0^{t_1} \dot{Z}(t_1 - s) g(s) ds - \int_0^{t_2} \dot{Z}(t_2 - s) g(s) ds \right\| \\ &+ \left\| \int_0^{t_1} Z(t_1 - s)[B(s)\bar{u}(s) + f(s, \bar{x}(s), \bar{u}(s))] ds \right. \\ &- \left. \int_0^{t_2} Z(t_2 - s)[B(s)\bar{u}(s) + f(s, \bar{x}(s), \bar{u}(s))] ds \right\| \\ &\le \|Z(t_1) - Z(t_2)\| \, |x_0 - g(0)| + \|g(t_1) - g(t_2)\| \\ &+ \int_0^{t_1} \|\dot{Z}(t_1 - s) - \dot{Z}(t_2 - s)\| \, \|g(s)\| ds \\ &- \int_{t_1}^{t_2} \|\dot{Z}(t_2 - s)\| \, \|g(s)\| ds \\ &+ \int_0^{t_1} \|Z(t_1 - s) - Z(t_2 - s)\| \, \|B(s)\bar{u}(s) + f(s, \bar{x}(s), \bar{u}(s))]\| ds \\ &+ \int_{t_1}^{t_2} \|Z(t_2 - s)[B(s)\bar{u}(s) + f(s, \bar{x}(s), \bar{u}(s))]\| ds \end{aligned}$$

and

$$\begin{aligned} |\bar{v}(t_1) - \bar{v}(t_2)| \quad &\le \quad \|(Z(t_1)B(t_1))^* - (Z(t_2)B(t_2))^*\| \sigma [k_1 |x_0 - g(0)| \\ &\quad + k + P + R\|\bar{w}\| + N]. \end{aligned}$$

These estimates show that the given family of functions is equicontinuous. That the family is uniformly bounded follows from the following argument

$$\|\bar{y}\|_2 = \|\bar{z}\|_{D_1} + \|\bar{v}\|_{H_1},$$

where

$$\|\bar{z}\|_{D_1} \leq k_1|x_0 - g(0)| + k_2 + P + H(\|\bar{u}\|_{H_1} + \|\bar{x}\|_{D_1}) + N$$
$$\|\bar{v}\|_{H_1} \leq \beta k_1 \sigma[k_1|x_0 - g(0)| + k + P + R(\|\bar{u}\|_{H_1} + \|\bar{x}\|_{D_1}) + N].$$

Thus, the given family of functions is uniformly bounded and equicontinuous. Hence $S(k,\mu)$ is relatively compact in C_1 for each $\mu \in [0,1]$. Now, assume that the equation

$$S(\bar{w},\mu) - \bar{w} = 0 \tag{3.1}$$

has a solution in C_1 and that $\bar{w}$ is such a solution which corresponds to fixed $\mu \in [0,1]$. Then

$$\begin{aligned} |\bar{w}(t)| &= |\bar{x}(t)| + |\bar{u}(t)| \leq k_1|x_0 - g(0)| + k_2 + P + H\|\bar{u}\| + R\|\bar{w}\| \\ &\quad + N + \beta k_1\sigma[k_1|x_0 - g(0)| + k + P + R\|\bar{w}\| + N] \\ &\leq \delta_1 + \delta_2\|\bar{w}\|, \quad t \in [0,1] \end{aligned}$$

where

$$\begin{aligned} \delta_1 &= (1 + \beta k_1\sigma)[k_1|x_0 - g(0)| + P + N] + k_2 + \beta k_1 \sigma k \\ \delta_2 &= (H + R) + \beta k_1 \sigma R \end{aligned}$$

which implies

$$\|\bar{w}\|_{C_1} = \|w\| \leq \delta_1(1 - \delta_2)^{-1} \equiv \delta. \tag{3.2}$$

By assumption $[(H+R) + \beta k_1\sigma R] < 1$. Thus the solutions of (3.1) are bounded uniformly with respect to $\mu \in [0,1]$. It is clear that all the assumptions of Theorem 2.2 are satisfied, so that $S(\bar{w},1)$ has a fixed point $\bar{y} = S(\bar{w},1) = \bar{w} \in C_1$.

We now inductively repeat the process as follows. Let $J_m = [0,m]$ and let C_m be the Banach space of all functions $\bar{w}$ which are obtained from the functions $w \in C([0,m], E^d)$ as follows:

$$\begin{aligned} \bar{w}(t) &= w(t), \quad t \in [0,m], \\ \bar{w}(t) &= w(m), \quad t \in [m,\infty), \end{aligned}$$

with the sup norm

$$\|\bar{w}\|_{C_m} = \|w\|_m = \sup_{t \in J_m} |w(t)|.$$

Then, there is a sequence $\{x_m\}$, $m = 1, 2, \ldots$ and a corresponding sequence $\{u_m\}$ such that x_m is a solution of

$$\frac{d}{dt}\left[x_m(t) - \int_0^t C(t-s)x_m(s)ds - g(t)\right] = Ax_m(t)$$
$$+ \int_0^t G(t-s)x_m(s)ds + B(t)u_m(t) + f(t, x_m(t), u_m(t))$$

with the property that

$$\bar{y}_m = (\bar{x}_m, \bar{u}_m) \in C_m, \quad \|y_m\|_m = \|y_m\| \leq \delta,$$

where δ is identified in (3.2),

$$\begin{aligned} x_m(t) &= Z(t)[x(0) - g(0)] + g(t) + \int_0^t \dot{Z}(t-s)g(s)ds \\ &\quad + \int_0^t Z(t-s)[B(s)u_m(s) + f(s, x_m(s), u_m(s))]ds \end{aligned}$$

$$\begin{aligned} u_m(t) &= (Z(t)B(t))^* W^{-1}[-Z(x(0) - g(0)) - k \\ &\quad - \int_0^\infty \bar{Z}g(s)ds - \int_0^\infty Zf(s, x_m(s), u_m(s))ds]. \end{aligned}$$

Since, obviously, the sequence $\{y_m(t)\} = \{(x_m(t), u_m(t))\}$, $m = 1, 2, \ldots$, is uniformly bounded and equicontinuous on $[0,1]$, there exists a subsequence $\{y^1_m(t)\}$, such that

$$\lim_{m\to\infty} y^1_m(t) = y(t) = (x(t), u(t)) \in C([0,1], E^d).$$

In the same way, there exists a subsequence $\{y^2_m(t)\}$ of $\{y^1_m(t)\}$ which is uniformly convergent to a function $p(t)$, $t \in [0,2]$, such that

$$p(t) = y(t), \quad t \in [0,1].$$

Using the diagonal process, we see that there exists a subsequence $\{y_{k_m}(t)\}$ of the original sequence $\{y_m(t)\}$ such that

$$\lim_{m\to\infty} |y_{k_m}(t) - y(t)| = 0 \tag{3.3}$$

uniformly on every finite subinterval of J, where $y \in C(J, E^d)$ and $\|y\| \leq \delta$. Clearly, given an arbitrary finite interval $I = [0, c] \subset J$, there exists some m_1 such that $k_{m_1} \geq c$ for $m \geq m_1$. Hence, $\{y_m(t)\}$ is defined for $m \geq m_1$, and the limit of (3.3) is well defined. It is evident that

$$\lim_{m\to\infty} |y_{k_m}(t) - y(t)| = 0$$

uniformly on any finite subinterval of J. Now, fix $c > 0$ and let $m(t) = (x(t), u(t))$, where

$$\begin{aligned} x(t) &= Z(t)[x(0) - g(0)] + g(t) + \int_0^t \dot{Z}(t-s)g(s)ds \\ &\quad + \int_0^t Z(t-s)[B(s)u(s) + f(s, y)]ds \end{aligned}$$

$$\begin{aligned} u(t) &= (Z(t)B(t))^* W^{-1}\Big[-Z(x_0 - g(0)) - k \\ &\quad - \int_0^\infty \bar{Z} g(s)ds - \int_0^\infty Z f(s, y(s))ds\Big]. \end{aligned}$$

Then, by the Lebesgue dominated convergence theorem,

$$\lim_{m\to\infty} |y_{k_m}(t) - m(t)| = 0, \quad t \in [0, c].$$

Since c was chosen arbitrarily, $m(t) = y(t)$ for $t \in [0, \infty)$. Thus

$$\begin{aligned} x(t) &= Z(t)[x(0) - g(0)] + g(t) + \int_0^t \dot{Z}(t-s)g(s)ds \\ &\quad + \int_0^t Z(t-s)[B(s)u(s) + f(s, x(s), u(s))]ds \end{aligned}$$

$$\begin{aligned} u(t) &= (Z(t)B(t))^* W^{-1}\Big[-Z(x(0) - g(0)) - k \\ &\quad - \int_0^\infty \bar{Z} g(s)ds - \int_0^\infty Z f(s, x(s), u(s))ds\Big]. \end{aligned}$$

Hence the proof is completed. □

Example 3.1 Consider the system (1.1) with G, C, g, A, B are as in (2.4) and

$$f(t, x, u) = \varepsilon \exp(-t)[\log(1 + |x|) + u] + (\sin x)/(1 + t^2).$$

Now, since $Z(t) = 2\exp(-t) - 1$, we have $\|Z(t)\| \le 3$, $\|Z^*(t)\| \le 3$, $\|g(t)\| \le 1$ for all $t \in J$ and $\max_t \|B(t)\| \le 1$. Also, we have

$$\limsup_{t\to\infty} \left\| \int_0^t \dot{Z}(t-s)g(s)ds \right\| \le P = 1,$$

$$\limsup_{t\to\infty} \left\| \int_0^t Z(t-s)B(s)ds \right\| \le H = \frac{1}{2},$$

$$\limsup_{t\to\infty} \left\| \int_0^t Z(t-s)f(s, x(s), u(s))ds \right\| \le R(\|x\| + \|u\|) + N,$$

where $R = \varepsilon$, $N = \pi$ and W is nonsingular.

It follows from Theorem 3.1 that, for sufficiently small $\varepsilon > 0$, there is at least one solution to the problem (1.1) and the system is asymptotically null controllable.

References

[1] D.H. Anderson. *Compartmental Modeling and Tracer Kinetics*, in Lecture Notes in Biomathematics. New York: Springer-Verlag, 1983.

[2] K. Balachandran and J.P. Dauer. Controllability of nonlinear systems to affine manifolds, *Journal of Optimization Theory and Applications*, **64** (1990), 15–27.

[3] K. Balachandran and P. Balasubramaniam. Null controllability of nonlinear perturbations of linear systems, *Dynamic Systems and Applications*, **2** (1993), 47–60.

[4] E.N. Chukwu. Total controllability to affine manifolds of control systems, *Journal of Optimization Theory and Applications*, **42** (1984), 181–199.

[5] A.N. Eke. Total controllability for linear control systems, *Institute of Mathematical and Computer Science*, **3** (1990), 149–154.

[6] A.N. Eke. Total controllability for nonlinear perturbed systems, *Institute of Mathematical and Computer Science*, **3** (1990), 335–340.

[7] I. Gyori. Delay differential and integrodifferential equations in biological compartmental models, *Systems Science*, (Wroclaw) Poland, **8** (1982), 167–187.

[8] I. Gyori. Connections between compartmental systems with pipes and integrodifferential equations, *Mathematical Modelling*, **7** (1986), 1215–1238.

[9] I. Gyori and J. Wu. A neutral equation arising from compartmental systems with pipes, *Journal of Dynamics and Differential Equations*, **3** (1991), 289–311.

[10] J.A. Jacquez. *Compartmental Analysis in Biology and Medicine.* New York: Elsevier, 1972.

[11] J.A. Jacquez. Compartmental models of biological systems: linear and nonlinear, in *Applied Nonlinear Analysis*, (V. Lakshmikantham, ed.). New York, 1979, pp. 185–205.

[12] A.G. Kartsatos. The Leray–Schauder theorem and the existence of solutions to boundary-value problem on an infinite interval, *Indiana University Mathematics Journal*, **23** (1974), 1021–1029.

[13] A.G. Kartsatos. A boundary-value problem on an infinite interval, *Proceedings of the Edinburg Mathematical Society*, **19** (1974–75), 245–252.

[14] D.L. Russell. *Mathematics of Finite-Dimensional Control Systems.* New York: Marcel Dekker, 1979.

[15] J. Wu. Globally stable periodic solution of linear neutral Volterra integrodifferential equations, *Journal of Mathematical Analysis and Applications*, **130** (1988), 474–483.

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