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Generalized Isoperimetric Problem*

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Abstract

In this paper the differential equations describing the minimal length curves satisfying the integral constraining relations of a general type are obtained. Moreover, an additional necessary condition supplementing Pontryagin maximum principle for the generalized isoperimetric problem is established. All results are illustrated by the analysis of generalized Dido's problem.

Key words: nonlinear systems, calculus of variations, sub-Riemannian geometry, isoperimetric problem

AMS Subject Classifications: 49B10, 49B27, 49K15, 53C22, 70F25

1 Introduction

The problems of sub-Riemannian geometry and/or Carnot - Caratheodory metrics have attracted much attention. It appears to be that both geometric phases in physics [16] and nonholonomic motion planning in robotics [3], [10] can be treated from the unified point of view of sub-Riemannian geometry. On the other hand, the classical isoperimetric problems [1] and their generalizations [2], [6] not only occupy a very important place in the sub-Riemannian geometry but also, under certain conditions, admit complete mathematical characterization of all their extrema. At the same time, calculation of sub-Riemannian geodesics arisen in the generalized isoperimetric problems is a challengeable problem for the modern geometrical control theory, since even for "simple examples" the calculation of length minimizers is not a trivial exercise [1], [2], [6].

The state of the art in the field of sub-Riemannian geometry until 1985 is outlined in the paper [17]. More recent information about this subject can be found in [9], [22].

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Many interesting and important results on the sub-Riemannian geodesics are already proved [12], [13], [18], [22]. In this paper we will derive necessary conditions for a geodesic to be the length minimizer in the generalized isoperimetric problem.

Though all our results are proved for isoperimetric problems defined in $\mathbf{R}^n$, they can be easily reformulated for the case where $\mathbf{R}^n$ is replaced by a smooth paracompact manifold.

The main theorems presented in this paper are illustrated with the analysis of generalized Dido's problem.

2 Necessary Conditions of Extremum

The generalized isoperimetric problem is stated as follows.

$$\int_0^1 \mid u(\tau) \mid^2 d\tau \rightarrow inf, \tag{2.1}$$

i.e., find the minima of the functional

$$\int_0^1 \mid u(\tau) \mid^2 d\tau,$$

where $u(\cdot) : [0,1] \rightarrow \mathbf{R}^m$ is subject to the additional constraining relations, i.e.,

$$\begin{aligned} \dot{x} &= u(t), \\ \dot{y} &= B(x)u(t), \\ x(0) &= \bar{x}, x(1) = \hat{x}, \\ y(0) &= \bar{y}, y(1) = \hat{y} \end{aligned} \tag{2.2}$$

with $x \in \mathbf{R}^m$, $y \in \mathbf{R}^n$, $\quad B(x) = \{b_1(x), \ldots, b_m(x)\}$.

The points $(\bar{x}, \bar{y}), (\hat{x}, \hat{y}) \in \mathbf{R}^{m+n}$ are assumed to be fixed beforehand. The minimum of (2.1) is said to be a sub-Riemannian distance between $(\bar{x}, \bar{y})$ and $(\hat{x}, \hat{y})$. We address it also as a sub-Riemannian length.

The vector fields

$$B(x) = \{b_1(x), b_2(x), \ldots, b_m(x)\}$$

are assumed to be C^∞-vector-fields such that the Lie algebra generated by

$$\{\frac{\partial}{\partial x_j} + \sum_i b_{ji}(x)\frac{\partial}{\partial y_i}\}_{j=1}^m$$

has the full rank at any point $x \in \mathbf{R}^m$. We will call such family of vector fields controllable [14] and also refer to $B(x)$ as controllable family of vector fields.

Introduce the matrix

$$G(x(t))p = \left\{ \left\langle \frac{\partial}{\partial x_i} b_j(x) - \frac{\partial}{\partial x_j} b_i\,(x)\,,p \right\rangle \right\}_{j,i=1}^{m}, \tag{2.3}$$

where j and i enumerate rows and columns, respectively.

Theorem 1 *Let $B(x)$ be a controllable family of vector fields. Then for any $(\bar{x},\bar{y}),(\hat{x}.\hat{y}) \in \mathbf{R}^{m+n}$ one can find a sub-Riemannian length minimizer $(x(t),y(t))$ which measures the sub-Riemannian distance between $(\bar{x},\bar{y})$ and $(\hat{x},\hat{y})$. Moreover, $(x(t),y(t))$ is a solution of the following boundary value problem*

$$\begin{cases} p_0 \cdot \ddot{x} = (G(x(t))p)\dot{x}, \\ \dot{y} = B(x)\dot{x}, \end{cases} \tag{2.4}$$

$$x(0) = \bar{x}, x(1) = \hat{x},$$

$$y(0) = \bar{y}, y(1) = \hat{y}$$

where $(p_0,p) \in \mathbf{R}^{1+n} \setminus 0$ is a real vector.

Proof: The Pontryagin maximum principle [15] yields that for all $t \in [0,1]$ $u(t)$ satisfies the equation

$$\frac{\partial}{\partial u} H(x,y,u,q,p) = 0 \tag{2.5}$$

where

$$H(x,y,u,q,p) = -\frac{p_0}{2} \cdot |u|^2 + \langle q,u\rangle + \langle p, B(x)u\rangle$$

and $(p_0,p) \in \mathbf{R}^{1+n} \setminus 0$;

$$\dot{q} = -\frac{\partial}{\partial x} H(x,y,u,q,p).$$

It follows from (2.5) that

$$p_0 \cdot \dot{x} = q + B^T(x)p, \tag{2.6}$$

where $B^T(x)$ is transpose of $B(x)$. Differentiation of (2.6) implies the first line in (2.4) and the assertion of Theorem 1 follows. Q.E.D

The real constant p_0 in (2.4) can not be omitted by taking it simply equal to 1, since it might happen [7], [12], [13] that the sub-Riemannian distance can be measured only by an abnormal geodesic with $p_0 = 0$. In

[18] it was established that p_0 can be taken equal to 1 if $B(x)$ satisfies strong bracket-generating condition.

Theorem 1 gives us a family of extremals. It might happen that there are several geodesics satisfying the same boundary value problem (2.4). In order to sort out local strong minimizers of the sub-Riemannian length we can use results from [24], [25] which imply that a regular sub-Riemannian geodesic without conjugate points is a local strong minimizer.

However, in general, the complete characterization of the global extremum seems to be a hopeless problem. Therefore any additional necessary condition which supplements (2.4) is important. We propose a necessary condition of extremum when $B(x)$ being Λ-uniform, i.e., defined as follows.

Definition *The family of vector fields $B(x)$ is said to be Λ-uniform if there exists a linear operator$\Lambda : \mathbf{R}^n \to \mathbf{R}^n$ such that*

$$\sum_\nu x_\nu \cdot \frac{\partial}{\partial x_\nu}\left(\frac{\partial}{\partial x_i} b_j(x) - \frac{\partial}{\partial x_j} b_i(x)\right) = \Lambda\left(\frac{\partial}{\partial x_i} b_j(x) - \frac{\partial}{\partial x_j} b_i(x)\right)$$

$$\forall x \in \mathbf{R}^m \quad \textit{and} \quad \forall\ 1 \le i, j \le m.$$

Q.E.D.

Notice that a family of vector fields $B(x)$ is $\Lambda--$uniform if, for each $1 \le i \le n, 1 \le j \le m$, $b_{ij}(x)$ is a uniform polynomial of degree $\deg(b_{ij})$ and this degree does not depend on $1 \le j \le m$. It is easy to see that under these assumptions Λ has a diagonal matrix.

Theorem 2 *Let $B(x)$ be a controllable family of vector fields. Suppose further that $B(x)$ is Λ -uniform. Then for any $(\bar{x}, \bar{y}), (\hat{x}.\hat{y}) \in \mathbf{R}^{m+n}$ one can find a sub-Riemannian length minimizer $(x(t), y(t))$ which measures the sub-Riemannian distance between $(\bar{x}, \bar{y})$ and $(\hat{x}, \hat{y})$. Moreover, $(x(t), y(t))$ is a solution of the boundary value problem (2.4), where for $(p_0, p) \in \mathbf{R}^{1+n}\setminus 0$ the following condition holds*

$$\begin{aligned} p_0 \cdot |\dot{x}(t)|^2 &= p_0 \cdot (\langle \hat{x}, \dot{x}(1)\rangle - \langle \bar{x}, \dot{x}(0)\rangle) \\ &+ \left\langle (I + \frac{1}{2}\cdot\Lambda)^T p, \bar{y} - \hat{y} - \int_0^1 B(\hat{x} + (\bar{x} - \hat{x})\tau)(\bar{x} - \hat{x})d\tau \right\rangle \\ &+ \int_0^1 \langle (G(\hat{x} + (\bar{x} - \hat{x})\tau)p)(\bar{x} - \hat{x}), \hat{x} + (\bar{x} - \hat{x})\tau\rangle\, d\tau,\ \forall t \in [0, 1], \end{aligned}$$

where $G(x)p$ is defined in (2.3).

Proof: Integration by parts implies

$$p_0 \cdot \int_0^1 |\dot{x}(\tau)|^2 \, d\tau = p_0 \cdot (\langle \hat{x}, \dot{x}(1) \rangle - \langle \bar{x}, \dot{x}(0) \rangle) - \\ \oint_\gamma \langle x, (G(x)p)\, dx \rangle + \\ \int_0^1 \langle (G(\hat{x} + (\bar{x} - \hat{x})\tau)p)(\bar{x} - \hat{x}), \hat{x} + (\bar{x} - \hat{x})\tau \rangle \, d\tau, \tag{2.7}$$

where the closed curve $\gamma \subset \mathbf{R}_x^m$ is composed out of the extremal

$$\{(x(t), y(t))\,; 0 \le t \le 1\}$$

and the segment of the straight line connecting the points $\bar{x}$ and $\hat{x}$. On the other hand,

$$\hat{y} - \bar{y} = \oint_\gamma B(x) dx - \int_0^1 B(\hat{x} + (\bar{x} - \hat{x})\tau)(\bar{x} - \hat{x}) d\tau = \\ \sum_{i,j} \int\int_\wp \left(\frac{\partial}{\partial x_i} b_j(x) - \frac{\partial}{\partial x_j} b_i(x) \right) dx_i \wedge dx_j - \\ \int_0^1 B(\hat{x} + (\bar{x} - \hat{x})\tau)(\bar{x} - \hat{x}) d\tau, \tag{2.8}$$

where $dx_i \wedge dx_j$ is the wedge or exterior product of dx_i and dx_j; $\wp$ is the set enclosed by the curve γ. Making use of Stokes' theorem we obtain

$$\oint_\gamma \langle x, (G(x)p)\, dx \rangle = \sum_{i,j} \int\int_\wp \left\langle \frac{\partial}{\partial x_i} b_j(x) - \frac{\partial}{\partial x_j} b_i(x), p \right\rangle dx_j \wedge dx_i + \\ \frac{1}{2} \int\int_\wp \sum_{i,j,\nu} x_j \cdot \frac{\partial}{\partial x_j} \left\langle \frac{\partial}{\partial x_\nu} b_i(x) - \frac{\partial}{\partial x_i} b_\nu(x), p \right\rangle dx_i \wedge dx_\nu. \tag{2.9}$$

The family of vector fields $B(x)$ is Λ-uniform. Therefore Eq. (2.9) can be represented as follows

$$\oint_\gamma \langle x, (G(x)p)\, dx \rangle = \\ \sum_{i,j} \int\int_\wp \left\langle \frac{\partial}{\partial x_i} b_j(x) - \frac{\partial}{\partial x_j} b_i(x), (I + \frac{1}{2} \cdot \Lambda)^T p \right\rangle dx_j \wedge dx_i \tag{2.10}$$

where I is the identity matrix and $(I + \frac{1}{2} \cdot \Lambda)^T$ denotes the transpose of $(I + \frac{1}{2} \cdot \Lambda)$.

It follows from (2.8) and (2.10) that

$$\oint_\gamma \langle x, (G(x)p)\, dx\rangle = \left\langle (I + \frac{1}{2}\cdot\Lambda)^T p, \hat{y} - \bar{y} + \int_0^1 B(\hat{x} + (\bar{x} - \hat{x})\tau)(\bar{x} - \hat{x})d\tau \right\rangle. \quad (2.11)$$

Combining (2.7) with (2.11) yields the assertion of Theorem 2. Q.E.D.

Suppose $B(x)$ is a controllable $\Lambda - -$uniform family of vector fields. Then it immediately follows from Theorems 1 and 2 that all regular geodesics can be represented in the form

$$(\mu \cdot x(\frac{t}{\mu}), \mu \cdot y(\frac{t}{\mu})),$$

where $\mu \in \mathbf{R} \setminus 0$ and $(x(t), y(t))$ is a solution of the following initial value problem

$$\begin{aligned} \dot{x} &= e^{\Omega(t)}\xi, \\ \dot{\Omega} &= G(x(t))p, \\ \dot{y} &= B(x)\dot{x}, \end{aligned}$$

$$x(0) = \bar{x},$$
$$y(0) = \bar{y},$$

with $\xi \in \mathbf{R}^m$ satisfying the following condition

$$\begin{aligned} |\xi|^2 = {} & (\langle \hat{x}, \dot{x}(1)\rangle - \langle \bar{x}, \dot{x}(0)\rangle) + \\ & \left\langle (I + \frac{1}{2}\cdot\Lambda)^T p, \bar{y} - \hat{y} - \int_0^1 B(\hat{x} + (\bar{x} - \hat{x})\tau)(\bar{x} - \hat{x})d\tau \right\rangle + \\ & \int_0^1 \langle (G(\hat{x} + (\bar{x} - \hat{x})\tau)p)(\bar{x} - \hat{x}), \hat{x} + (\bar{x} - \hat{x})\tau\rangle\, d\tau. \end{aligned}$$

Notice that $\Omega(t) = -\Omega^T(t)$.

3 Generalized Dido's Problem

Let us begin with the plot of the legend as it is narrated in "The Aeneid" by the Roman poet P. Virgilius Maro:

> *Mercatigue solum, facti de nomine Byrsam Taurino quantum possent circumdare tergo...*
>
> "They bought a space of ground, which (Byrsa called, from the bull's hide) they first enclosed..." translated by J. Dryden (p.144, [23]).

> The Phoenician princess Dido fled from her brother, the tyrant Pygmalion. Dido and her companions chose a good place on the north coast of Africa (at present the shore of the Gulf of Tunis) and wanted to found a settlement there. Among the natives there was not much enthusiasm for this idea. However, Dido managed to persuade their chieftain Hiarabas to give her as much land as she could enclose with the hide of a bull. Only later did the simple hearted Hiarabas understand how cunning and artful Dido was: she then cut the bull's hide into thin strips, tied them together to form an extremely long thong, and surrounded with it a large extent of territory and founded the city of Carthage there. In commemoration of this event the citadel of Carthage was called Byrsa. According to the legend, all these events occurred in 825 (or 814) B.C.[1]

The situation described in the legend can be stated as the following optimization problem:

- find the optimal form of a lot of land of the maximum area S for a given perimeter L.

Clearly, its solution is circle. Several other possibilities of stating optimization Dido's problems are described in [20].

3.1 N-th order generalized Dido's problem

In this section we will illustrate the application of Theorems 1 and 2 by the analysis of generalized Dido's problem of the N-th order. This problem is formulated as follows:

$$\int_0^1 \mid \dot{x}(\tau) \mid^2 d\tau \rightarrow inf, \tag{3.12}$$

where $x(\cdot) : [0,1] \rightarrow \mathbf{R}^2$ is subject to the additional constraining relations, i.e.,

$$\begin{aligned} \dot{y}_i &= a_i \cdot x^i \cdot (x_1 \cdot \dot{x}_2 - x_2 \cdot \dot{x}_1), \;\; 0 \le |i| \le N, \\ x(0) &= \bar{x}, \;\; x(1) = \hat{x}, \;\; y(0) = \bar{y}, \;\; y(1) = \hat{y}, \end{aligned} \tag{3.13}$$

where $a_i \in \mathbf{R} \setminus 0$, $i = (i_1, i_2)$ is multi-index, $|i| = i_1 + i_2$ and $x^i = x_1^{i_1} \cdot x_2^{i_2}$. $(\bar{x}, \bar{y})$, $(\hat{x}, \hat{y})$ are chosen beforehand. It is easy to verify that the family

[1]The description of the legend is taken from [20].

of vector fields

$$\frac{\partial}{\partial x_1} - \sum_i a_i \cdot x^i \cdot x_2 \frac{\partial}{\partial y_i},$$
$$\frac{\partial}{\partial x_2} + \sum_i a_i \cdot x^i \cdot x_1 \frac{\partial}{\partial y_i}$$

is controllable and is Λ-uniform with Λ being such that

$$(\Lambda y)_i = |i| \cdot y_i \qquad \forall\, 0 \le |i| \le N.$$

Therefore we can use Theorem 2 established in the previous section.

By Theorem 2 the extremals of the generalized Dido's problem are the solutions of the following boundary value problem

$$\begin{aligned} p_0 \ddot{x}_1 &= -\left(\sum_i (2+|i|) \cdot a_i \cdot p_i \cdot x^i\right) \dot{x}_2, \\ p_0 \ddot{x}_2 &= \left(\sum_i (2+|i|) \cdot a_i \cdot p_i \cdot x^i\right) \dot{x}_1, \\ \dot{y}_i &= a_i \cdot x^i \cdot (x_1 \cdot \dot{x}_2 - x_2 \cdot \dot{x}_1), \;\; 0 \le |i| \le N, \\ x(0) &= \bar{x}, \;\; x(1) = \hat{x}, \;\; y(0) = \bar{y}, \;\; y(1) = \hat{y}, \end{aligned} \tag{3.14}$$

where $(p_0, p) \in \mathbf{R}^{1+n} \setminus 0$ is a constant vector satisfying

$$\begin{aligned} & p_0 \cdot \int_0^1 |\dot{x}(\tau)|^2 \, d\tau = p_0 \cdot (\langle \hat{x}, \dot{x}(1)\rangle - \langle \bar{x}, \dot{x}(0)\rangle) + \\ & \sum_i (2+|i|) \cdot p_i \cdot (\bar{y}_i - \hat{y}_i) - \\ & 2 \cdot \sum_i (2+|i|) \cdot a_i \cdot p_i \cdot \int_0^1 (\hat{x} + (\bar{x} - \hat{x})\tau)^i \cdot (\hat{x}_1 \cdot \bar{x}_2 - \hat{x}_2 \cdot \bar{x}_1) d\tau. \end{aligned} \tag{3.15}$$

Considering the regular geodesics one can always put $p_0 = 1$. Assume that $\hat{x} = 0, \bar{x} = 0$. Then due to (3.14), (3.15)

$$|\dot{x}(t)|^2 = \sum_j (2+|j|) \cdot p_j \cdot (\bar{y}_j - \hat{y}_j) \qquad \forall \;\; t \in [0,1]$$

and

$$\frac{d}{dt}(x_1 + i \cdot x_2) \;\; = \;\; \sqrt{\sum_j (2+|j|) \cdot p_j \cdot (\bar{y}_j - \hat{y}_j)} \cdot e^{i \cdot \varpi},$$

$$\begin{aligned} \dot{\varpi} &= \sum_j (2+|j|)\cdot a_j \cdot p_j \cdot x^j, \\ \dot{y}_j &= a_j \cdot x^j \cdot (x_1\cdot \dot{x}_2 - x_2 \cdot \dot{x}_1),\ \ 0\le |j| \le N, \\ x(0) &= 0,\ \ x(1)=0,\ \ y(0)=\bar{y},\ \ y(1)=\hat{y}, \end{aligned} \tag{3.16}$$

where $i^2 = -1$. Thus all regular geodesics of generalized Dido's problem are solutions of (3.16).

If $p_0 = 0$, then we are dealing with abnormal geodesics which are either points or curves satisfying

$$\begin{aligned} &\sum_j (2+|j|)\cdot a_j \cdot p_j \cdot x^j = 0, \\ \dot{y}_j &= a_j \cdot x^j \cdot (x_1\cdot \dot{x}_2 - x_2 \cdot \dot{x}_1),\ \ 0\le |j| \le N, \\ x(0) &= 0,\ \ x(1)=0,\ \ y(0)=\bar{y},\ \ y(1)=\hat{y}. \end{aligned} \tag{3.17}$$

3.2 Generalized Dido's problem of the first order

For $N = 0$ generalized Dido's problem coincides with the problem solved by princess Dido, according to the legend, in 825 (or 814) B.C.

Consider generalized Dido's problem of the first order with $a_j = 1 \ \ \forall\, 0 \le |j| \le 1$. Let us begin with $x(0) = 0,\ \ x(1) = 0, y(0) = 0$. Then it follows from (3.17) that all abnormal geodesics are either points or straight lines corresponding to $y = 0$. Therefore all abnormal geodesics are extrema of generalized Dido's problem of the first order.

Now turn our attention to regular geodesics. Without loss of generality we can assume that $p_0 = 1$. Therefore all regular geodesics are solutions of the following problem

$$\begin{aligned} \frac{d}{dt}(x_1 + i\cdot x_2) &= \sqrt{2\cdot p_{00}\cdot \hat{y}_{00} + 3\cdot p_{10}\cdot \hat{y}_{10} + 3\cdot p_{01}\cdot \hat{y}_{01}}\cdot e^{i\cdot\varpi}, \\ \dot{\varpi} &= -(2\cdot p_{00} + 3\cdot p_{10}\cdot x_1 + 3\cdot p_{01}\cdot x_2), \\ \dot{y}_{00} &= x_1\cdot \dot{x}_2 - x_2\cdot \dot{x}_1, \\ \dot{y}_{10} &= x_1\cdot (x_1\cdot \dot{x}_2 - x_2\cdot \dot{x}_1), \\ \dot{y}_{01} &= x_2\cdot (x_1\cdot \dot{x}_2 - x_2\cdot \dot{x}_1), \\ x(0) &= 0,\ \ x(1) = 0,\ \ y(0) = 0,\ \ y(1) = \hat{y}. \end{aligned} \tag{3.18}$$

Those regular geodesics for which the function

$$\varphi(p) = 2\cdot p_{00}\cdot \hat{y}_{00} + 3\cdot p_{10}\cdot \hat{y}_{10} + 3\cdot p_{01}\cdot \hat{y}_{01}$$

attains its minimal value under the constraints given by (3.18) are extrema of generalized Dido's problem of the first order.

If

$$(x_1(t) + i \cdot x_2(t), y_{00}(t), y_{10}(t) + i \cdot y_{01}(t))$$

is a regular geodesic, then for any $\lambda \in \mathbf{C}$-the field of complex numbers-

$$(\lambda \cdot (x_1(t) + i \cdot x_2(t)), |\lambda|^2 y_{00}(t), |\lambda|^2 \cdot \lambda \cdot (y_{10}(t) + i \cdot y_{01}(t)))$$

is also a regular geodesic. Therefore the set of regular geodesics consists of orbits of the group isomorphic to $\mathbf{C}$-complex numbers. Taking this fact into account we can considerably simplify the analysis of (3.18) by putting $\hat{y}_{10} = 0,\ \hat{y}_{00} = 1$ and $\varpi(0) = 0$. This simplification and the fact that the differential equation defined by the first two lines in (3.18) is completely integrable in elliptic functions allow us to calculate numerically (see [6]) those $p \in \mathbf{R}^3$ for which the function

$$\varphi(p) = 2 \cdot p_{00} + 3 \cdot p_{01} \cdot \hat{y}_{01}$$

attains its minimal values under the constraints given by (3.18) with $\hat{y}_{10} = 0,\ \hat{y}_{00} = 1$ and $\varpi(0) = 0$. That gives us all the closed curves which are length minimizers of generalized Dido's problem of the first order. The relationship between $C = (\hat{y}_{01})^2$ and the shape of the length minimizer is depicted in Fig.1.

Now we consider the nonholonomic unit wave front corresponding to the sub-Riemannian structure generated by generalized Dido's problem of the first order. More precisely we are interested in the set of endpoints of length 1 geodesics, $\varphi(p) = 1$. Moreover the geodesics corresponding to $\varphi(p) = 1$ are invariant with respect to the actions defined by

$$(\lambda \cdot \left(x_1 \left(\frac{t}{|\lambda|} \right) + i \cdot x_2 \left(\frac{t}{|\lambda|} \right) \right), |\lambda|^2 y_{00}(\frac{t}{|\lambda|}), |\lambda|^2 \cdot \lambda \cdot (y_{10}(\frac{t}{|\lambda|}) + i \cdot y_{01}(\frac{t}{|\lambda|})))$$

for any $\lambda \in \mathbf{C}$ and

$$(x_1(t) - i \cdot x_2(t), -y_{00}(t), -y_{10}(t) + i \cdot y_{01}(t)). \tag{3.19}$$

Therefore we need only to solve the following system of differential equations

$$\begin{aligned}
&\frac{d}{dt}(x_1 + i \cdot x_2) = e^{i \cdot \varpi}, \\
&\dot{\varpi} = -(r + x_1 \cdot \cos(\psi) + x_2 \cdot \sin(\psi)), \\
&\dot{y}_{00} = x_1 \cdot \dot{x}_2 - x_2 \cdot \dot{x}_1, \\
&\dot{y}_{10} = x_1 \cdot (x_1 \cdot \dot{x}_2 - x_2 \cdot \dot{x}_1), \\
&\dot{y}_{01} = x_2 \cdot (x_1 \cdot \dot{x}_2 - x_2 \cdot \dot{x}_1), \\
&x(0) = 0, \quad y(0) = 0,\ \varpi(0) = 0,
\end{aligned} \tag{3.20}$$

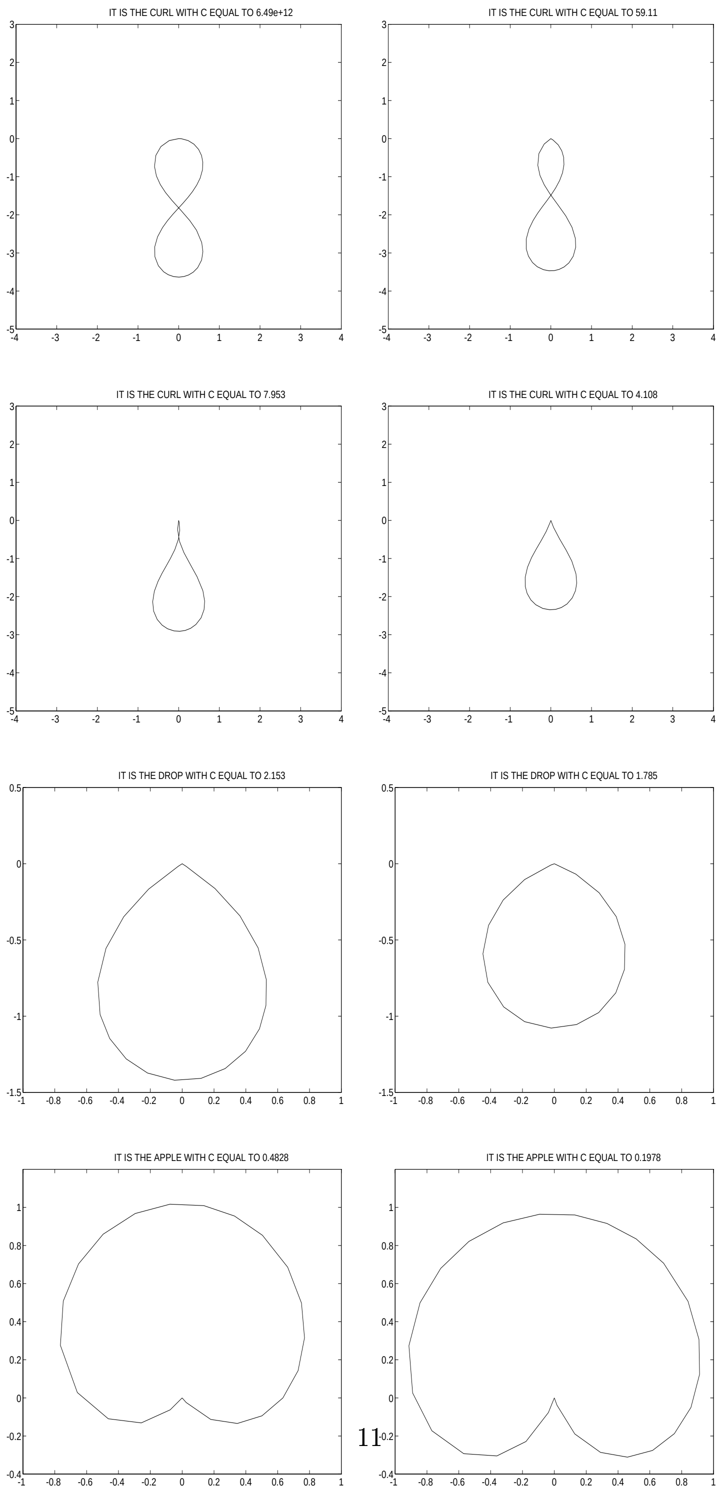

Figure 1: Length minimizers for different values of C.

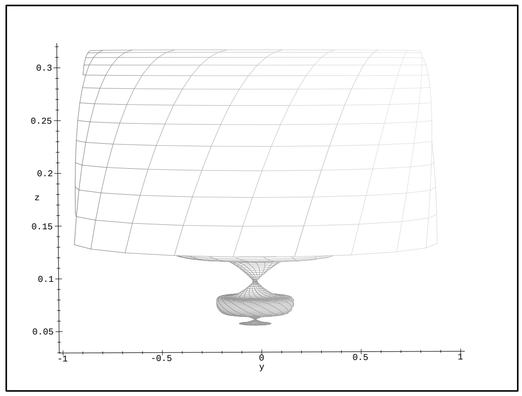

Figure 2: The projection of $\Im_0^+(1)$ in (x_1, x_2, y_{00})-space.

where $r \geq 0$ and $0 \leq \psi < 2\pi$ are real numbers.

The unit wave front $\Im$ can be represented as follows

$$\Im = \cup_{\mu > 0} \cup_{\psi=0}^{2\pi} (\Im_\psi^-(\mu) \cup \Im_\psi^+(\mu)),$$

with

$$\begin{aligned}\Im_\psi^\pm(\mu) \quad = \quad & \{(\mu \cdot e^{i\cdot\theta} \cdot (x_1(\frac{1}{\mu}, r, \psi) \pm i \cdot x_2(\frac{1}{\mu}, r, \psi)), \pm\mu^2 y_{00}(\frac{1}{\mu}, r, \psi), \\ & \mu^3 \cdot e^{i\cdot\theta} \cdot (\pm y_{10}(\frac{1}{\mu}, r, \psi) + i \cdot y_{01}(\frac{1}{\mu}, r, \psi))); r \geq 0, 0 \leq \theta < 2 \cdot \pi\},\end{aligned}$$

where $(x(t, r, \psi), y(t, r, \psi))$ is the solution of the initial value problem (3.20). A part of the projection of $\Im_0^+(1)$ in (x_1, x_2, y_{00})-space is depicted in Figure 2 and Figure 3. The set $\Im_0^-(1)$ is easy to visualize due to the fact that $\Im_0^-(1)$ is obtained from $\Im_0^+(1)$ by the reflection (3.19). The singularities of $\Im_0^+(1)$ occur in the points of y_{00}-axes corresponding to the closed curves. Some of these closed curves are shown in Figure 1. It is easy to see that there are infinitely many such singular points and they are all located on $y_{00}-$axes and they accumulate towards the origin. The origin corresponds to "figure eight" minimizer shown in Figure 1.

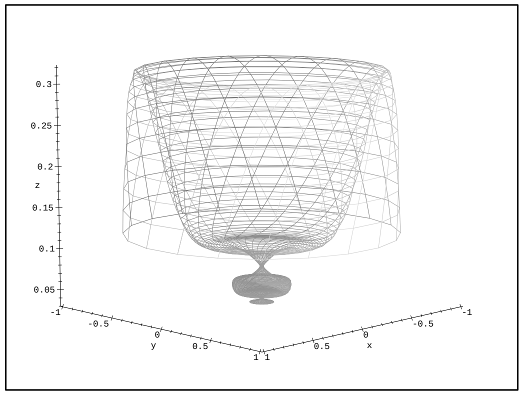

Figure 3: Another view of the projection of the nonholonomic unit wave front.

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