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The Value Function of a Slow Growth Control Problem with State Constraints*

M. Motta F. Rampazzo

Dedicated to Aldo Bressan on his seventieth birthday

Abstract

The formal extension of the conventional theory of dynamic programming to control problems with slow growth and a state constraint $x \in \overline{\Theta}$ encounters two major drawbacks: on one hand the formal Hamiltonian may happen to be discontinuous; on the other hand, just as in the case of bounded controls, the imposition of a state constraint possibly gives rise to a discontinuous value function.

On the contrary we provide conditions on the vectogram (at the points of $\partial\Theta$) which, whenever the L^1 norms of the controls are bounded, guarantee the continuity of the value function. Subsequently, we are able to establish a dynamic programming differential equation involving a continuous Hamiltonian and enjoying uniqueness properties.

Key words: unbounded controls, slow growth, state constraints, dynamic programming

AMS Subject Classifications: 34A37, 49N25, 49L20, 49L25

1 Introduction

We shall be concerned with an optimal control problem of the form

$$\begin{cases} minimize \quad \Psi\big(x(T), u(T)\big) \\ \dot{x} = f(t, x, u, v, \dot{u}), \qquad (x, u)(\bar{t}) = (\bar{x}, \bar{u}), \\ x(t) \in \overline{\Theta}, \qquad \int_{\bar{t}}^{T} |\dot{u}(t)|\, dt \leq K - \bar{k}, \end{cases}$$

where the triple $(u, v, \dot{u})$ represents the control exerted on the system; $\overline{\Theta}$ is the closure of an open set; K is the maximal bound for the integral of

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$|\dot{u}|$ (the total variation of u); and $\bar{k}$ is an initial datum less than or equal to K. More precisely, v is an ordinary control which takes values in a compact subset $V \subset \boldsymbol{R}^q$, u is an absolutely continuous map, bounded in the $W^{1,1}([\bar{t},T],\boldsymbol{R}^m)$ norm and taking values in a closed set $U \subset \boldsymbol{R}^m$, and the derivative $\dot{u}$ is constrained in a closed cone $C \subset \boldsymbol{R}^m$. It is clear that an equivalent control system governed by the control pair $(v,w) \in V \times C$ can be obtained by adding the state variables $\xi^i = u^i$, $i = 1,\ldots,m$ and the equations $\dot{\xi}^i = w^i$. The vector field f is *sublinear* in the variable $\dot{u}$, which can be considered as the unbounded component of the control policy. We refer to [3, 4, 5, 6, 7], [17], [19], [22], [24, 25] [28, 29, 30, 31], [33, 34, 35], [37], [39, 40, 41, 42, 43], [45, 46], [48, 49] for some examples where this sublinearity condition is fulfilled.

We recall that the theory of dynamic programming for optimal control problems of the form

$$\begin{array}{ll} minimize & \Psi(T,x(T)) \\ \dot{x} = h(t,x,a) & x(\bar{t}) = \bar{x} \end{array}$$

is usually formulated within one of the following conditions: i) for every (t,x) the map $a \to h(t,x,a)$ takes values in a bounded set as the control a ranges on its domain A; ii) a coercivity assumption, commonly stated in terms of a superlinearity hypothesis on the maps $a \to h(t,x,a)$, allows one to reduce–(a posteriori)–the problem to an equivalent one satisfying i). For example the so-called linear quadratic problems agree with condition ii). In fact, under either i) or ii) the Hamiltonian

$$K(t,x,p) \doteq \min_{a \in A} \left\{ p \cdot h(t,x,a) \right\}$$

turns out to be continuous at each (t,x,p). On the contrary, because of the unboundedness of $\dot{u}$ and of the sublinearity of f, the problem above disagrees with both hypothesis i) and hypothesis ii). Hence dynamic programming cannot be developed as a formal extension of the classical theory, in that this would yield a Hamiltonian function which possibly takes the value $-\infty$ at some point.

Referring to a reparametrization technique introduced in [35] (see also [31]) we overcome these drawbacks by embedding our problem into a new problem whose definition involves the *recession function of* f (see (2.3)). This auxiliary problem enjoys two main properties: first, it has the same value function as the original one; secondly, the associated Hamiltonian is continuous.

The difficulty related to the slow growth of f overlaps the one caused by the imposition of the state constraint $x(t) \in \overline{\Theta}$. Indeed, just as in the case of bounded controls, such a state constraint possibly gives rise to a discontinuous value function. We recall that, starting with H.M. Soner

[47], conditions concerning the directions of the vectogram on the boundary $\partial\Theta$ were established which guarantee the continuity of the value function. However the boundedness of the vectogram plays an essential role in Soner's result. On the contrary we neither have this boundedness hypothesis nor can we deduce it by a superlinearity condition.

We prove that under conditions $(H1)$ and $(H2)$ below the value function $V(\bar{t},\bar{x},\bar{u},\bar{k})$ of the considered control problem is continuous. These conditions still concern the directions of the vectogram at the boundary of the constraint set. Hence they are of the same nature as Soner's. Nevertheless the latter are no longer sufficient as soon as one allows unbounded controls $\dot{u}$ (see Examples 3.1, 3.2). We also remark that $(H1)$ and $(H2)$ allow for a *dense* embedding of the trajectories graphs of the original problem into the set of trajectories of the auxiliary problem (see [35]). As a consequence they guarantee that the value function $V(\bar{t},\bar{x},\bar{u},\bar{k})$ of the original problem coincides with the value function $V(\bar{t},\bar{x},\bar{u},\bar{k})$ of the extended problem. Finally, let us observe that the problem of the interaction between state constraints and unbounded controls already arises—in a simplified form—in any (state-constrained) problem of the Calculus of Variations with slow growth. For instance, this is the case for the classical problem of the minimal surface of revolution.

The last part of the paper is devoted to proving that the value function is the unique continuous (viscosity) solution of a boundary value problem involving the Hamilton–Jacobi–Bellman equation defined by the (continuous) Hamiltonian H. This extends a result of [34], which concerned the case with no constraint and a vector field affine in the derivative $\dot{u}$.

We conclude by remarking that several applications motivate the study of slow growth control problems. For instance, let us mention [8, 9] [10] [38], as applications in Lagrangian mechanics, [15] [18] [44], which concern subjects from economics, and [21] [27] [36] in the field of space–navigation. Furthermore, a numerical approach to an advertising problem modeled by a slow growth control system has been proposed in [12], while in [11] approximations are studied for the general problem with impulses.

2 The Control Problem and Its Space–Time Extension

We consider the control system

$$\begin{array}{c} \dot{x} = f(t,x(t),u(t),v(t),\dot{u}(t)) \\ (x,u)(\bar{t}) = (\bar{x},\bar{u}), \end{array} \tag{2.1}$$

on the time interval $[\bar{t},T]$. Here the control v is a Borel measurable map from $[\bar{t},T]$ into a compact subset $V \subset \boldsymbol{R}^q$, the control u is an absolutely continuous map from $[\bar{t},T]$ into a closed subset $U \subseteq \boldsymbol{R}^m$, and the derivative

$\dot{u}$ is constrained in a closed cone $C \subseteq \boldsymbol{R}^m$. We posit an integral bound of the form

$$\int_{\bar{t}}^{T} |\dot{u}(t)|\, dt \leq a,$$

$a \geq 0$, while no restrictions are assumed on the magnitude of $\dot{u}$. Hence $\dot{u}$ has to be regarded as the unbounded component of the control policy.

We subject the state variable to the relation

$$x(t) \in \overline{\Theta},$$

where Θ is an open subset of $\boldsymbol{R}^n$ and $\overline{\Theta}$ denotes its closure. Observe that the dependence of f on u could be neglected by simply adding the equations $\dot{x}_{n+\alpha} = \dot{u}_\alpha$ (and the initial conditions $x_{n+\alpha}(\bar{t}) = \bar{u}_\alpha$). Hence the variable u can be thought both as a state parameter and as a control parameter, and the condition $u \in U$ can be regarded as a further (trivial) state constraint. In particular the actual control policy is represented by the pair $(v, \dot{u})$, and there is not loss of generality in considering the derivative $\dot{u}$ in place of an unbounded control w taking values in C.

Hypothesis $(H0)$ below, which concerns the Lipschitz continuity and the sublinear growth of the map f, will be assumed throughout the paper.

$(H0)$ i) The function f is continuous on $\mathcal{D} \doteq [0,T] \times \boldsymbol{R}^n \times U \times V \times C$. Moreover, there exists a constant $L > 0$ such that

$$\begin{array}{l} |f(t',x',u',v,w) - f(t,x,u,v,w)| \leq L(1+|w|)|(t',x',u') - (t,x,u)| \\ \forall (t',x',u',v,w),\ (t,x,u,v,w) \in \mathcal{D}. \end{array} \tag{2.2}$$

ii) There is a continuous map f^∞, called the *recession function of* f, such that

$$\lim_{r \to +\infty} r^{-1} f(t,x,u,v,rw) = f^\infty(t,x,u,v,w) \tag{2.3}$$

uniformly on compact subsets of $\mathcal{D}$. Observe that the recession function satisfies the homogeneity condition, $f^\infty(t,x,u,v,rw) = r f^\infty(t,x,u,v,w)$ $\forall r \geq 0$.

Let us fix $a \geq 0$ and let us consider the set of controls

$$W_a(\bar{t},\bar{u}) \doteq \left\{ \begin{array}{c} (v,u) \in \mathcal{B}([\bar{t},T],V) \times W^{1,1}([\bar{t},T],U) :\ u(\bar{t}) = \bar{u}, \\ \int_{\bar{t}}^{T} |\dot{u}|\, dt \leq a, \quad \dot{u} \in C \ \text{ a.e. in } \ [\bar{t},T] \end{array} \right\},$$

where $\mathcal{B}([\bar{t},T],V)$ denotes the set of Borel measurable maps from $[\bar{t},T]$ into V and $W^{1,1}([\bar{t},T],U)$ is the (Sobolev) space of absolutely continuous maps from $[\bar{t},T]$ into U. Under condition i) in $(H0)$ (which, of course, can be weakened), for every control $(v,u) \in W_a(\bar{t},\bar{u})$ there exists a unique global solution to (2.1), which will be denoted by $x[\bar{t},\bar{x},\bar{u};v,u](\cdot)$. Whenever the initial condition is understood from the context we adopt the notation $x[v,u](\cdot)$ instead of $x[\bar{t},\bar{x},\bar{u};v,u](\cdot)$.

Given an initial condition $(\bar{t}, \bar{x}, \bar{u})$ a control $(v, u) \in W_a(\bar{t}, \bar{u})$ will be called *admissible* if the corresponding solution of (2.1) satisfies the state–constraint $x[\bar{t}, \bar{x}, \bar{u}; v, u](t) \in \overline{\Theta} \quad \forall t \in [\bar{t}, T]$. The set of admissible controls corresponding to the initial condition $(\bar{t}, \bar{x}, \bar{u})$ will be denoted by $W_a^c(\bar{t}, \bar{x}, \bar{u})$.

Remark 2.1 It is obvious that under hypothesis $(H1)$ below the set $W^c_{K-\bar{k}}(\bar{t}, \bar{x}, \bar{u})$ is not empty for any initial condition $(\bar{t}, \bar{x}, \bar{u}) \in [0, T) \times \overline{\Theta} \times U$ and any $\bar{k} \in [0, K]$.

Let $\Psi : \boldsymbol{R}^n \times U \to \boldsymbol{R}$ be a bounded continuous map. For any $(\bar{t}, \bar{x}, \bar{u}, \bar{k}) \in [0, T) \times \overline{\Theta} \times U \times [0, K]$ we consider the following optimal control problem of Mayer type:

$$\text{minimize} \left\{ \Psi\big(x[\bar{t}, \bar{x}, \bar{u}; v, u](T), u(T)\big) : \ (v, u) \in W^c_{K-\bar{k}}(\bar{t}, \bar{x}, \bar{u}) \right\} \qquad (\mathcal{P}).$$

Remark 2.2 Actually the boundedness assumption on Ψ is not restrictive. Indeed a problem equivalent to $(\mathcal{P})$ and satisfying this hypothesis is promptly obtained by replacing Ψ with the map $\tilde{\Psi} \doteq \arctan \circ \Psi$.

Correspondingly, let us define the value function $V : [0, T) \times \overline{\Theta} \times U \times [0, K] \to \boldsymbol{R}$ by setting

$$V(\bar{t}, \bar{x}, \bar{u}, \bar{k}) \doteq \inf_{(v,u) \in W^c_{K-\bar{k}}(\bar{t}, \bar{x}, \bar{u})} \Psi\big(x[\bar{t}, \bar{x}, \bar{u}; v, u](T), u(T)\big).$$

By Remark 2.1, the function V turns out to be well defined as soon as one assumes hypothesis $(H1)$ below.

We shall not pursue a formal extension of the dynamic programming approach, for, as pointed out in the introduction, the lack of any coercivity hypothesis and the unboundedness of the control $\dot{u}$ would yield a discontinuous Hamiltonian. Instead, aiming to avoid this drawback, we embed (see [31] [35]) the problem into a new one which has the same value function and involves only *bounded* controls (see also [2], where an idea of reparametrization is also involved).

Definition 2.1 *Let f be a vector field satisfying hypothesis $(H0)$. For every $(t, x, u, v, w_0, w) \in \mathcal{D}_e \doteq [0, T] \times \boldsymbol{R}^n \times U \times V \times [0, +\infty) \times C$ we set*

$$\overline{f}(t, x, u, v, w_0, w) \doteq \begin{cases} f(t, x, u, v, w/w_0) w_0 & \text{if } w_0 \neq 0 \\ f^\infty(t, x, u, v, w) & \text{if } w_0 = 0, \end{cases}$$

where f^∞ is the recession function defined by (2.3). *The vector field $\overline{f}$ will be called the* space-time extension of f.

Let us introduce the *space-time system* relative to (2.1),

$$\begin{cases} t' = w_0(s) \\ x' = \overline{f}(t(s), x(s), u(s), v(s), w_0(s), w(s)) \\ u' = w(s) \\ (t, x, u)(0) = (\bar{t}, \bar{x}, \bar{u}) \end{cases} \qquad s \in [0, 1], \tag{2.4}$$

where the state and the control are identified with the the triples (t, x, u) and (v, w_0, w), respectively, and the prime denotes differentiation with respect to s.

For instance, in [34] we considered the dynamics

$$f(t, x, u, v, \dot{u}) = g_0(t, x, u, v) + \sum_{i=1}^{m} g_i(t, x, u, v)\dot{u}_i,$$

whose corresponding space–time system has the form

$$\begin{cases} t' = w_0(s) \\ x' = g_0(t, x, u, v)(s)w_0(s) + \sum_{i=1}^{m} g_i(t, x, u, v)(s)w_i(s) \\ u' = w(s) \\ (t, x, u)(0) = (\bar{t}, \bar{x}, \bar{u}) \end{cases} \qquad s \in [0, 1].$$

For $z \in \boldsymbol{R}^p$ and $r \geq 0$ let us denote by $z + B_p[r] \doteq \{\zeta \in \boldsymbol{R}^p : \ |\zeta - z| \leq r\}$ the closed ball of center z and radius r.

Definition 2.2 *A space–time control for the initial condition* $(\bar{t}, \bar{x}, \bar{u}, \bar{k}) \in [0, T] \times \boldsymbol{R}^n \times U \times [0, K]$ *is a Borel measurable map* $(v, w_0, w) : [0, 1] \to V \times [0, K + T] \times \big(C \cap B_m[K + T]\big)$ *which satisfies*

$$\bar{t} + \int_0^1 w_0(\sigma)\, d\sigma = T,$$

$$\bar{u} + \int_0^s w(\sigma)\, d\sigma \in U \quad \forall s \in [0, 1],$$

$$\int_0^1 |w(\sigma)|\, d\sigma \leq K - \bar{k}.$$

The set of space–time controls for the initial condition $(\bar{t}, \bar{x}, \bar{u}, \bar{k})$ *will be denoted by* $,_{K-\bar{k}}(\bar{t}, \bar{u})$.

It is well known that for every initial condition $(\bar{t}, \bar{x}, \bar{u}, \bar{k}) \in [0, T] \times \boldsymbol{R}^n \times U \times [0, K]$ and every space–time control $(v, w_0, w) \in\ ,_{K-\bar{k}}(\bar{t}, \bar{u})$ there exists a unique solution $(t, x, u)[\bar{t}, \bar{x}, \bar{u};\ v, w_0, w]$ of the Cauchy problem (2.4) defined on the whole interval $[0, 1]$. Such solutions will be called *space–time trajectories.*

Definition 2.3 *Given an initial condition* $(\bar{t},\bar{x},\bar{u},\bar{k})$*, a space–time control is called* admissible *if the corresponding solution of* (2.4) *satisfies the relation*

$$x[\bar{t},\bar{x},\bar{u};v,w_0,w](s)\in\overline{\Theta}$$

for every $s\in[0,1]$*. If this is the case, also the corresponding space–time trajectory* $x[\bar{t},\bar{x},\bar{u};v,w_0,w]$ *is called* admissible*. The subset of admissible space–time controls of* $,\ _{K-\bar{k}}(\bar{t},\bar{u})$ *will be denoted by* $,\ ^{c}_{K-\bar{k}}(\bar{t},\bar{x},\bar{u})$*.*

Correspondingly to the space–time control system (2.4) we consider the optimal control problem

$$\text{minimize}\left\{\Psi\big(x[\bar{t},\bar{x},\bar{u};v,w_0,w](1),u(1)\big):\ (v,w_0,w)\in,\ ^{c}_{K-\bar{k}}(\bar{t},\bar{x},\bar{u})\right\}\qquad(\mathcal{P}_e)$$

which can be considered as an *extension* of problem $(\mathcal{P})$. The value function associated with $(\mathcal{P}_e)$ is defined by

$$\mathcal{V}(\bar{t},\bar{x},\bar{u},\bar{k})\doteq\inf_{(v,w_0,w)\in,\ ^{c}_{K-\bar{k}}(\bar{t},\bar{x},\bar{u})}\Psi\big(x[\bar{t},\bar{x},\bar{u};v,w_0,w](1),u(1)\big).$$

Note that the extended problem above is an ordinary control problem (with state constraints), i.e. it involves only controls taking values in a given compact set. As a particular case of Corollary 2.1 below, in the case with no state constraints, the value function of the original problem coincides with the value function of the extended problem. Hence the dynamic programming for the unbounded control problem $(\mathcal{P})$ can be studied by means of the equivalent *ordinary* problem $(\mathcal{P}_e)$. Actually the utility of $(\mathcal{P}_e)$ goes beyond this goal, for (in the unconstrained case) the (graphs of the) trajectories of $(\mathcal{P})$ are dense in the set of trajectories of $(\mathcal{P}_e)$ (see Theorem 2.1 below). In other words, $(\mathcal{P}_e)$ is a *proper extension* of $(\mathcal{P})$.

However the situation becomes much more involved as soon as one tackles the problem involving an actual constraint of the form $x\in\overline{\Theta}$, $\Theta\neq\boldsymbol{R}^n$, for which in general the above density result is false (see [35]). On the other hand, in [35] we established conditions $(H1)$, $(H2)$ below—involving the directions of the field $\overline{f}$ at the points of $\partial(\Theta\times U)$—which ensure that $(\mathcal{P}_e)$ is a proper extension of $(\mathcal{P})$. To state this density result let us recall the notion of *regular space–time trajectory,* which loosely speaking, is a reparametrization of the graph of a trajectory of the original system.

Definition 2.4 *Given an initial condition* $(\bar{t},\bar{x},\bar{u},\bar{k})\in[0,T)\times\overline{\Theta}\times U\times[0,K]$ *a space–time trajectory* $(t,x,u)(\cdot)$ *starting at* $(\bar{t},\bar{x},\bar{u})\in[0,T)\times\overline{\Theta}\times U$ *is called* regular *if there is a control* $(\tilde{v},\tilde{u})\in W_{K-\bar{k}}(\bar{t},\bar{u})$ *so that*

$$(t,x,u)(s)=(t(s),\tilde{x}\circ t(s),\tilde{u}\circ t(s))$$

for every $s\in[0,1]$*, where* $\tilde{x}\equiv x[\bar{t},\bar{x},\bar{u};\tilde{v},\tilde{u}]$*.*

We now recall conditions $(H1)$, $(H2)$ from [35]:

$(H1)$ *there exist a function* $\nu_1 \in UC\big([0,T) \times \overline{\Theta} \times U; V\big)$ *and positive constants* q_1*,* r_1 *such that for any* $\bar{y} \doteq (\bar{t}, \bar{x}, \bar{u}) \in [0,T) \times \overline{\Theta} \times U$ *one has*

$$\bar{x} + hf(\bar{y}, \nu_1(\bar{y}), 0) + B_n[hr_1] \subset \Theta \qquad \forall h \in (0, q_1],$$

where $UC(E,F)$ *denotes the space of uniformly continuous maps from a metric space* E *to a metric space* F*;*

$(H2)$ *there exist a function* $(\nu_2, \omega) \in UC\big([0,T] \times \overline{\Theta} \times U; V \times (B_m[1] \cap C)\big)$ *and positive constants* q_2*,* r_2 *such that for any* $\bar{y} \doteq (\bar{t}, \bar{x}, \bar{u}) \in [0,T] \times \overline{\Theta} \times U$ *and any* $\bar{w}_0 \in [0,1]$ *one has*

$$\big(\bar{x} + h\overline{f}(\bar{y}, \nu_2(\bar{y}), \bar{w}_0, \omega(\bar{y})), \bar{u} + h\omega(\bar{y})\big) + B_{n+m}[hr_2] \subset \Theta \times \overset{\circ}{U} \qquad \forall h \in (0, q_2],$$

where $\overset{\circ}{U}$ *denotes the interior of* U*.*

Remark 2.3 If we consider the conventional control problem obtained by taking $\dot{u} \equiv 0$, condition $(H1)$ is nothing but a slight generalization of Soner's one [47]. Actually, the generalization consists in the consideration of time dependent fields f and in weakening (see also [20]) the assumptions on $\partial\Theta$ (so that, in particular, condition $(H1)$ could not be formulated in terms of the normal vector to $\partial\Theta$).

Theorem 2.1 *[35] Assume* $(H1)$*,* $(H2)$*, and let* $(\bar{t}, \bar{x}, \bar{u}, \bar{k}) \in [0,T) \times \overline{\Theta} \times U \times [0,K]$*. Then for any* $\varepsilon > 0$ *and any admissible space–time trajectory* (t,x,u) *there is an admissible regular trajectory* $(\check{t}, \check{x}, \check{u})$ *such that*

$$\|(\check{t}, \check{x}, \check{u})(\cdot) - (t,x,u)(\cdot)\|_\infty < \varepsilon, \quad \int_0^1 |\dot{\check{u}}(s)|\, ds \le \int_0^1 |\dot{u}|\, ds. \tag{2.5}$$

As an easy consequence of Theorem 2.1 we have

Corollary 2.1 *Under hypotheses* $(H1)$ *and* $(H2)$ *one has*

$$\mathcal{V}(\bar{t}, \bar{x}, \bar{u}, \bar{k}) = V(\bar{t}, \bar{x}, \bar{u}, \bar{k})$$

for every $(\bar{t}, \bar{x}, \bar{u}, \bar{k}) \in [0,T) \times \overline{\Theta} \times U \times [0,K]$*.*

3 The Continuity of the Value Function

As soon as hypotheses $(H1)$ and $(H2)$ are satisfied, Corollary 2.1 allows us to replace V with the value function $\mathcal{V}$ of the bounded control problem $(\mathcal{P}_e)$. Actually we will demonstrate that, under the same hypotheses, $\mathcal{V}$ is continuous. Hence $\mathcal{V}$ is nothing but the continuous extension of V to the closed domain $[0,T] \times \overline{\Theta} \times U \times [0,K]$.

Theorem 3.1 *Under hypotheses* $(H1)$, $(H2)$ *the value function* $\mathcal{V}$ *is continuous.*

Before proving Theorem 3.1 let us make some comments upon hypotheses $(H1)$ and $(H2)$. The continuity of the value function of a problem with state constraints and bounded controls has been originally studied by H.M.Soner [47] and successively by other authors —see e.g. [13] [14] [20] [26] [32]. In particular, Soner has shown that the value function of an infinite horizon problem is continuous provided the vectogram of the boundary points of $\overline{\Theta}$ contains a vector uniformly pointing inwards. We begin by observing that Soner's original hypothesis is formulated in connection with an infinite horizon problem. Hence, as a minor byproduct, Theorem 3.1 provides an analogue of Soner's result for a (possibly time–dependent) finite horizon problem. Moreover we wish to get rid of the idea that by simply stating Soner's condition for the (x, u)–variables in the extended problem $(\mathcal{P}_e)$—which is a problem with bounded controls—one could obtain a sufficient condition for the continuity of $\mathcal{V}$. In fact, on one hand, this is false as shown by the two examples below. On the other hand, if one takes the variable k into consideration as well, Soner's condition will prescribe that the k–component of the vectogram points towards the negative direction of the k–variable. This is always untrue at the boundary points where $k = K$, for we have

$$\dot{k} = |w|.$$

The following two examples show that it is hard to weaken hypotheses $(H1)$, $(H2)$. Actually, in each of these examples, a (x, u)–component of the dynamics at a boundary point can be selected which points inwards. In particular in the first example —which enlightens the special role of the t–variable in connection with the possible occurrence of impulses— condition $(H2)$ holds true, while $(H1)$ is fulfilled except for the requirement of uniform continuity. In the second example, where the particular role of the variable k is put in evidence, the opposite situation occurs, $(H1)$ being valid and $(H2)$ being violated at only one point.

Example 3.1 Let us set

$$\Theta \doteq \boldsymbol{R}^2 \setminus \Big(\big\{ (x_1, x_2) \in \boldsymbol{R}^2 : \; x_1^2 + x_2^2 \le 4, \; x_2 \ge 0 \big\} \cup] - \infty, 0] \times [0, 2] \Big),$$
$$T \doteq \pi, \quad U \doteq \boldsymbol{R}^2, \quad K = 1,$$

and consider the problem of minimizing $\Psi(x(1)) = |2 - x_2(1)|$ over all terminal points of the constrained control system

$$\begin{cases} \dot{x} = g_0(v) + g_1(x)\dot{u}_1 + g_2(x)\dot{u}_2, \quad \forall t \in [\bar{t}, \pi], \\ (x, u)(\bar{t}) = (\bar{x}, \bar{u}), \quad \int_{\bar{t}}^{\pi} |\dot{u}(t)| \, dt \le 1 - \bar{k}, \quad x(t) \in \overline{\Theta}, \end{cases}$$

where

$$g_0(v) \doteq \begin{pmatrix} -\sin v \\ \cos v \end{pmatrix}, \quad v \in [0, 3\pi/4], \quad g_1(x) \doteq \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \quad g_2(x) \doteq \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

No directional constraints are imposed on $\dot{u}$. Observe that hypothesis $(H2)$ is easily satisfied. Indeed it is sufficient to set $v = \nu_2(x) = 3\pi/4$, $(w_1, w_2)(x) = (\sqrt{3}/2, -1/2)$ at each point $(x_1, x_2) \in \partial\Theta$ with $x_2 = 0$, and to extend continuously the map (ν_2, w_0, w_1) over the closed set $\overline{\Theta}$. On the contrary, $(H1)$, does not hold true, for each ν_1 satisfying the relation involved in $(H1)$ would be discontinuous at $x = (2, 0)$. Actually if one starts at $(\bar{t}, \bar{x}, \bar{u}, \bar{k}) = (0, 0, 0, 0, 0, 0)$, the space–time control

$$(v, w_0, w_1, w_2)(s) \doteq \begin{cases} (0, 0, 2, 0) & \forall s \in [0, 1/2] \\ (\pi(s - 1/2), 2\pi, 0, 0) & \forall s \in]1/2, 1] \end{cases}$$

turns out to be optimal for the corresponding space–time trajectory (t, x_1, x_2, u_1, u_2)—which coincides with $(0, 4s, 0, 2s, 0)$ if $s \in [0, 1/2]$ and with $(2\pi(s - 1/2), 2\cos\pi(s - 1/2), 2\sin\pi(s - 1/2), 1, 0)$ if $s \in]1/2, 1]$—reaches the point $(x_1(1), x_2(1)) = (0, 2)$ where Ψ has a minimum. Actually, in order to steer the state (x_1, x_2) from $(0, 0)$ to $(2, 0)$ one spends the whole variation $K = 1$ available for u. Hence, since the vector field g_0 has a non-positive x_1–component, as soon as the initial value $\bar{k}$ is greater than 0 —i.e. as soon as one has the constant $1 - \bar{k} < 1$—, for any choice of the control (v, w_0, w_1, w_2) the terminal value $x_2(1)$ turns out to be nonpositive. Since the value function has a strict minimum at $x_2 = 2$ this implies that the map $k \mapsto \mathcal{V}(0, 0, 0, 0, 0, k)$ is discontinuous at $k = 0$.

Example 3.2 Let $K = \pi$ and Ψ, Θ, U, and T be as in the previous example and let us consider the constrained trajectories of the control system

$$\dot{x} = g_0(v_1) + g_1(v_2)\dot{u}_1 + g_2(x)\dot{u}_2 \quad t \in [\bar{t}, \pi], \quad (x, u)(\bar{t}) = (\bar{x}, \bar{u})$$

where

$$\{ \; g_0(v_1) \doteq \begin{pmatrix} \frac{2}{\pi}\cos v_1 \\ \frac{2}{\pi}\sin v_1 \end{pmatrix}, \quad g_1(v_2) \doteq \begin{pmatrix} -\sin v_2 \\ \cos v_2 \end{pmatrix},$$

$$g_2(x) \doteq \begin{pmatrix} -(2 - x_1) \\ 0 \end{pmatrix},$$

$$v_1 \in [0, 2\pi], \quad v_2 \in [0, 3\pi/4], \quad C = [0, +\infty[\times[0, +\infty[.$$

In this case hypothesis $(H1)$ is trivially satisfied, while $(H2)$ does not hold, in that the involved map ν_2 cannot be continuous at $(2, 0)$. The space–time control

$$(v_1, v_2, w_0, w_1, w_2)(s) \doteq \begin{cases} (0, 0, 2\pi, 0, 0) & \forall s \in [0, 1/2] \\ (0, \pi(s - 1/2), 0, 2\pi, 0) & \forall s \in]1/2, 1] \end{cases}$$

turns out to be optimal for the initial data $(\bar{t}, \bar{x}, \bar{u}, \bar{k}) = (0,0,0,0,0,0)$. Indeed the corresponding trajectory

$$(t,x,u)(s) = \begin{cases} (2\pi s, 4s, 0, 0, 0) & \forall s \in [0, 1/2] \\ (\pi, 2\cos\pi(s-1/2), 2\sin\pi(s-1/2), 2\pi(s-1/2), 0) & \forall s \in]1/2, 1] \end{cases}$$

steers (0,0) to (0,2). However, as soon as the initial time $\bar{t}$ is greater than 0, x_1 cannot reach the value 2 and the terminal value $x_2(1)$ is non positive. It follows that the value function $\mathcal{V}$ is discontinuous at $(\bar{t}, \bar{x}_1, \bar{x}_2, \bar{u}_1, \bar{u}_2, \bar{k}) = (0,0,0,0,0,0)$.

We shall exploit the following result concerning reparametrizations of space–time controls and trajectories.

Proposition 3.1 *[35] Let (v, w_0, w) be any space–time control and define the map σ by setting*

$$\sigma(s) \doteq \frac{\int_0^s |(w_0,w)|\, ds'}{\int_0^1 |(w_0,w)|\, ds'}.$$

Then there exists, uniquely defined almost everywhere, a space–time control (v^c, w_0^c, w^c), called the canonical representative of (v, w_0, w), *such that*

i) $|(w_0^c, w^c)|(s) = \int_0^1 |(w_0,w)|(\xi)\, d\xi'$ *for a.e.* $s \in [0,1]$;

ii) for any $s \in [0,1]$ *one has*

$$(v^c \circ \sigma(s), (w_0^c \circ \sigma(s))\frac{d\sigma}{ds}(s), (w^c \circ \sigma(s))\frac{d\sigma}{ds}(s)) = (v(s), w_0(s), w(s));$$

iii) for any $\xi \in [0,1]$ *one has*

$$(t,x,u)[\bar{t}, \bar{x}, \bar{u}; v, w_0, w](\sigma^{-1}(\xi)) = (t,x,u)[\bar{t}, \bar{x}, \bar{u}; v^c, w_0^c, w^c](\xi).$$

Proof of Theorem 3.1: Let $(\bar{t}, \bar{x}, \bar{u}, \bar{k})$ be a point in $[0,T] \times \overline{\Theta} \times U \times [0,K]$ and fix $\varepsilon, \delta' > 0$. For any pair (t_1, x_1, u_1, k_1), $(t_2, x_2, u_2, k_2) \in [0,T] \times \overline{\Theta} \times U \times [0,K]$ satisfying

$$\begin{cases} |(t_1,x_1,u_1,k_1) - (\bar{t}, \bar{x}, \bar{u}, \bar{k})| \leq \delta', \quad |(t_2,x_2,u_2,k_2) - (\bar{t}, \bar{x}, \bar{u}, \bar{k})| \leq \delta', \\ \mathcal{V}(t_2,x_2,u_2,k_2) - \mathcal{V}(t_1,x_1,u_1,k_1) \geq 0, \end{cases}$$

choose an admissible space–time control $(v, w_0, w) \in \,,^{c}_{K-k_1}(t_1, x_1, u_1)$ whose corresponding trajectory (t, x, u) satisfies the inequality

$$\mathcal{V}(t_1, x_1, u_1, k_1) \geq \Psi[x(1), u(1)] - \varepsilon. \tag{3.6}$$

Thanks to Proposition 3.1 it is not restrictive to assume that (v, w_0, w) coincides with its canonical representative. Let us consider the space–time control $(v, \tilde{w}_0, \tilde{w})$ defined as follows.

If $t_1 < t_2$, set

$$\tilde{w}_0(s) \doteq \begin{cases} 0 & s \in [0, \bar{s}] \\ w_0(s) & s \in (\bar{s}, 1] \end{cases},$$

$$\text{where } \bar{s} \doteq \min \left\{ s \in [0,1] : \ t_1 + \int_0^s w_0 \, d\sigma = t_2 \right\};$$

if $t_1 \geq t_2$, set

$$\tilde{w}_0(s) \doteq w_0(s) + (t_1 - t_2).$$

Now set

$$\tilde{w}(s) \doteq \begin{cases} w(s) & s \in [0, \bar{\bar{s}}] \\ 0 & s \in (\bar{\bar{s}}, 1] \end{cases},$$

$$\text{where } \bar{\bar{s}} \doteq \max \left\{ s \in [0,1] : \ \int_0^s |w| \, d\sigma \leq K - k_2 \right\},$$

which in the event $k_2 \leq k_1$, implies $\tilde{w} = w$ on $[0,1]$. Setting $(\tilde{t}, \tilde{x}, \tilde{u}) \doteq (t, x, u) \ [t_2, x_2, u_2; v, \tilde{w}_0, \tilde{w}]$ one has

$$\tilde{t}(1) = T, \quad |\tilde{t}(s) - t(s)| \leq |t_2 - t_1|, \quad |\tilde{u}(s) - u(s)| \leq |u_2 - u_1| + |k_2 - k_1| \quad (3.7)$$

and the space–time control $(v, \tilde{w}_0, \tilde{w})$ belongs to $\Gamma_{K-k_2}(t_2, u_2)$. In order to estimate the quantity $\tilde{x} - x$ we need Lemma 3.1 below. Let us observe that (see Corollary 2.1 in [35]) all space–time trajectories whose initial points belong to $(\bar{t}, \bar{x}, \bar{u}) + B_{1+n+m}[\delta']$ remain inside a compact subset $Q' \subset [0,T] \times \boldsymbol{R}^n \times U$, and let us consider the maps $\mu_0 : [0, K+T] \to [0, +\infty[$, $\mu : B_m(C) \doteq B_m[K+T] \cap C \to [0, +\infty[$ defined by

$$\mu_0(w_0) \doteq \max\left\{ |\overline{f}(y, v, w_0, w) - \overline{f}(y, v, 0, w)| : \ (y, v, w) \in Q' \times V \times B_m(C) \right\},$$

$$\mu(w) \doteq \max\left\{ |\overline{f}(y, v, w_0, w) - \overline{f}(y, v, w_0, 0)| : \ (y, v, w_0) \in Q' \times V \times [0, K+T] \right\}.$$

Lemma 3.1 *If* $w_0 \in L^1([0,1], [0, K+T])$, $w \in L^1([0,1], B_m[K+T] \cap C)$ *then* $\mu_0 \circ w_0$, $\mu \circ w \in L^1([0,1], \boldsymbol{R})$. *Moreover for any* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that*

$$i) \ \int_0^1 \mu_0(w_0(s)) \, ds \leq \varepsilon \quad \text{if} \quad \int_0^1 w_0(s) \, ds \leq \delta,$$
$$ii) \ \int_0^1 \mu(w(s)) \, ds \leq \varepsilon \quad \text{if} \quad \int_0^1 |w(s)| \, ds \leq \delta.$$

The proof of ii) can be found in [35], and with the same arguments one can prove i) as well.

Let us write y and $\tilde{y}$ in place of (t,x,u), $(\tilde{t},\tilde{x},\tilde{u})$, respectively. In the event $t_1 < t_2$ by exploiting the homogeneity and the Lipschitzianity of $\overline{f}$ one obtains:

$$\begin{aligned}
&|\tilde{x}(s) - x(s)| \le |x_2 - x_1| + \Big|\int_0^{\min\{\bar{s},\bar{\bar{s}}\}} \big[\overline{f}(\tilde{y},v,0,w)(s') - \\
&- \overline{f}(y,v,w_0,w)(s')\big]\, ds' + \\
&+ \int_{\min\{\bar{s},\bar{\bar{s}}\}}^{\max\{\bar{s},\bar{\bar{s}}\}} \big[\overline{f}(\tilde{y},v,\tilde{w}_0,\tilde{w})(s') - \overline{f}(y,v,w_0,w)(s')\big]\, ds' + \\
&+ \int_{\max\{\bar{s},\bar{\bar{s}}\}}^{s} \big[\overline{f}(\tilde{y},v,w_0,0)(s') - \overline{f}(y,v,w_0,w)(s')\big]\, ds'\Big| \le \\
&\le |x_2 - x_1| + \Big|\int_0^{\min\{\bar{s},\bar{\bar{s}}\}} \big[\overline{f}(\tilde{y},v,0,w)(s') - \overline{f}(\tilde{y},v,w_0,w)(s')\big]\, ds' + \\
&+ \int_{\min\{\bar{s},\bar{\bar{s}}\}}^{\max\{\bar{s},\bar{\bar{s}}\}} \big[\overline{f}(\tilde{y},v,\tilde{w}_0,\tilde{w})(s') - \overline{f}(\tilde{y},v,w_0,w)(s')\big]\, ds' + \\
&+ \int_{\max\{\bar{s},\bar{\bar{s}}\}}^{s} \big[\overline{f}(\tilde{y},v,w_0,0)(s') - \overline{f}(\tilde{y},v,w_0,w)(s')\big]\, ds' + \\
&+ \int_0^s \big[\overline{f}(\tilde{y},v,w_0,w)(s') - \overline{f}(y,v,w_0,w)(s')\big]\, ds'\Big| \le \\
&\le |x_2 - x_1| + \int_0^{\bar{s}} \mu_0(w_0(s'))\, ds' + \max\Big\{0, M\int_{\bar{\bar{s}}}^{\bar{s}} (\tilde{w}_0(s') + |w(s')|)\, ds\Big\} + \\
&+ \int_{\bar{\bar{s}}}^1 \mu(w(s'))\, ds' + L\int_0^s (w_0(s') + |w(s')|)|\tilde{y}(s') - y(s')|\, ds',
\end{aligned} \tag{3.8}$$

where $M \doteq \max_{Q' \times V \times [0,1] \times (B_m[1] \cap C)} |\overline{f}|$. If $t_1 \ge t_2$, we have

$$\begin{aligned}
&|\tilde{x}(s) - x(s)| \le |x_2 - x_1| + \Big|\int_0^{\bar{\bar{s}}} \big[\overline{f}(\tilde{y},v,\tilde{w}_0,w)(s) - \overline{f}(y,v,w_0,w)(s)\big]\, ds + \\
&+ \int_{\bar{\bar{s}}}^{s} \big[\overline{f}(\tilde{y},v,\tilde{w}_0,0)(s) - \overline{f}(y,v,w_0,w)(s)\big]\, ds\Big| \le |x_2 - x_1| + \\
&+ L\int_0^s (\tilde{w}_0(s) + |w(s)|)|\tilde{y}(s) - y(s)|\, ds + \int_0^s \big|\overline{f}(y,v,\tilde{w}_0,w)(s) \\
&- \overline{f}(y,v,w_0,w)(s)\big|\, ds + \int_{\bar{\bar{s}}}^1 \mu(w(s))\, ds.
\end{aligned} \tag{3.9}$$

Observe that from the definitions of $\bar{s}$ and $\bar{\bar{s}}$ one has $\int_0^{\bar{s}} w_0(s)\, ds \le |t_2 - t_1|$ and $\int_{\bar{\bar{s}}}^1 |w(s)|\, ds \le |k_2 - k_1|$. In view of (3.2)–(3.4) and Lemma 3.1, Gronwall's Lemma implies that for any $\eta > 0$ there exists some $\delta < \delta'$ such that

$$d((\tilde{x}(s),\tilde{u}(s)), \theta \times U) \le |(\tilde{t},\tilde{x},\tilde{u})(s) - (t,x,u)(s)| \le \eta \qquad \forall s \in [0,1]$$

provided (t_1,x_1,u_1,k_1), $(t_2,x_2,u_2,k_2) \in (\bar{t},\bar{x},\bar{u},\bar{k}) + B(\delta)$.

By Theorem 2.1 there is an admissible control

$$(\check{v},\check{w}_0,\check{w}) \in \Gamma^c_{K-k_2}(t_2,x_2,u_2)$$

whose corresponding trajectory $\check{y}$ satisfies

$$|\check{y}(s) - y(s)| \le |\check{y}(s) - \tilde{y}(s)| + |\tilde{y}(s) - y(s)| \le \rho(\eta) + \eta,$$

where ρ is infinitesimal as η tends to 0. By

$$0 \le \mathcal{V}(t_2,x_2,u_2,k_2) - \mathcal{V}(t_1,x_1,u_1,k_1) \le \Psi(\check{y}(1)) - \Psi(y(1)) + \varepsilon \tag{3.10}$$

and by the uniform continuity of Ψ on the compact set Q', it follows that

$$0 \le \mathcal{V}(t_2,x_2,u_2,k_2) - \mathcal{V}(t_1,x_1,u_1,k_1) \le 2\varepsilon$$

provided δ is chosen sufficiently small. □

4 Dynamic Programming Principle and Dynamic Programming Equation

In this section we show that the value function is the unique solution of a boundary value problem involving a suitable Hamilton–Jacobi–Bellman equation. The Hamiltonian we shall consider does not enjoy the usual nondegeneracy condition in the coefficient of the t–partial derivative, which yields the uniqueness of the solution in standard problems. However the uniqueness of the solution is recovered by a more subtle nondegeneracy property which involves both the t–partial derivative coefficient and the k–partial derivative's one.

Let us define the Hamiltonian $H : [0,T]\times\boldsymbol{R}^n \times U\times\boldsymbol{R}^{1+n+m+1} \to\boldsymbol{R}$ by setting

$$H(t,x,u;p_t,p_x,p_u,p_k) \doteq \min_{\substack{v\in V\\(w_0,w)\in S^m_+}} \mathcal{H}(t,x,u;p_t,p_x,p_u,p_k,v,w_0,w), \quad (4.11)$$

where $\mathcal{H}$ denotes the unminimized Hamiltonian

$$\mathcal{H}(t,x,u;p_t,p_x,p_u,p_k,v,w_0,w) \doteq \left\{ \left(p_t w_0 + p_x\cdot \overline{f}(t,x,u,v,w_0,w) + p_u\cdot w + p_k|w|\right\}, \quad (4.12)$$

while $S^m = \big\{(w_0,w)\boldsymbol{R}^{1+m} : \ |(w_0,w)| = 1\big\}$, $S^m_+ \doteq S^m \cap \big([0,+\infty[\times C\big)$. Furthermore, let us introduce the domain

$$\Omega \doteq [0,T)\times\Theta\times \overset{\circ}{U} \times[0,K),$$

and the boundary's subsets

$$\begin{aligned} \partial_T\Omega &\doteq \{T\}\times\overline{\Theta}\times U\times[0,K], \\ \partial'\Omega &\doteq \partial\Omega\setminus\partial_T\Omega, \end{aligned} \quad (4.13)$$

where $\partial\Omega$ denotes the boundary of Ω. Let us recall the definitions of viscosity subsolution and supersolution (see e.g. [23], [16]).

Definition 4.1 *Let E be any subset of $\boldsymbol{R}^N$. A function $\mathcal{V}\in C^0(E)$ is a* viscosity subsolution [supersolution] *at $(t,x,u,k)\in E$ of the Dynamic Programming Equation*

$$-H\big(t,x,u,\nabla\mathcal{V}\big) = 0,$$

if for any $\lambda\in C^\infty(\boldsymbol{R}^N)$ such that (t,x,u,k) is a local maximum [minimum] point of $\mathcal{V}-\lambda$ on E one has

$$-H\big(t,x,u,\nabla\lambda(t,x,u,k)\big)\le 0 \qquad [-H\big(t,x,u,\nabla\lambda(t,x,u,k)\big)\ge 0],$$

where $\nabla\lambda$ denotes the gradient of λ. $\mathcal{V} \in C^0(E)$ is a viscosity solution *of (DPE) at (t,x,u,k) if it is both a viscosity subsolution and a viscosity supersolution.*

Theorem 4.1 (Dynamic Programming Equation and Boundary Conditions) *Assume hypotheses $(H1)$ and $(H2)$. Then*

a) $\mathcal{V}$ is a viscosity solution on Ω of the dynamic programming equation (DPE);

b) $\mathcal{V}$ satisfies

$$\mathcal{V}(T,x,u,k) \leq \Psi(x,u) \qquad \forall (T,x,u,k) \in \partial_T\Omega; \tag{4.14}$$

c) $\mathcal{V}$ is a viscosity supersolution of (DPE) on $\partial'\Omega$ and at any point

$$(T,x,u,k) \in \partial_T\Omega$$

where

$$\mathcal{V}(T,x,u,k) < \Psi(x,u).$$

The proof of Theorem 4.1 is based on the following dynamic programming principle.

Proposition 4.1 (Dynamic Programming Principle) *The value function $\mathcal{V}$ enjoys the following properties:*

i) for an initial condition

$$(\bar{t},\bar{x},\bar{u},\bar{k}) \in [0,T] \times \overline{\Theta} \times U \times [0,K]$$

and an admissible control

$$(v,w_0,w) \in \;,\;{}^{c}_{K-\bar{k}}(\bar{t},\bar{x},\bar{u}),$$

let

$$(t,x,u) \doteq (t,x,u)[\bar{t},\bar{x},\bar{u};v,w_0,w]$$

be the corresponding trajectory of the extended system (2.4). Then the map

$$s \mapsto \mathcal{V}\big(t(s),x(s),u(s),\bar{k} + \int_0^s |w(\sigma)|\,d\sigma\big) \tag{4.15}$$

is nondecreasing;

ii) if in i) the control (v,w_0,w) is optimal, then the map in (4.5) is constant.

Proof: Since for each $(\bar{t}, \bar{x}, \bar{u}, \bar{k}) \in \overline{\Omega}$ the set $, {}^{c}_{K-\bar{k}}(\bar{t}, \bar{x}, \bar{u})$ is not empty, the proof of Proposition 4.1 can be straightforwardly obtained from the proof of Proposition 4.1 in [34]. □

Minor changes to the proof of Theorem 4.1 in [34] are sufficient to obtain the proof of Theorem 4.1. These modifications are related to the presence of the state constraint $x \in \overline{\Theta}$ and to the augmented generality of the dynamics, which here is no longer linear in $\dot{u}$.

Proof of Theorem 4.1: Let us begin by replacing the set Ω in [34] with the set Ω introduced above and let us recall that by Corollary 2.1 in [35] for any $(\bar{t}, \bar{x}, \bar{u}, \bar{k}) \in \overline{\Omega}$ the set of admissible space–time trajectories is not empty and locally bounded. Then the proof that $\mathcal{V}$ is a viscosity subsolution of (DPE) on Ω can be deduced by obvious modifications of the first part of the proof of Theorem 4.1 in [34]. Secondly, in order to prove that $\mathcal{V}$ is a viscosity supersolution of (DPE) on $\Omega \cup \partial'\Omega$ and at any point $(T, \bar{x}, \bar{u}, \bar{k}) \in \partial_T\Omega$ where $\mathcal{V}(T, \bar{x}, \bar{u}, \bar{k}) < \Psi(\bar{x}, \bar{u})$, it is sufficient to replace the trajectories $y^n(\cdot)$ and the functions $\mathcal{H}$ and H in the second part of the proof of Theorem 4.1 in [34] with admissible trajectories of (2.4) and the functions $\mathcal{H}$ and H defined here, respectively. What makes the proof work after these changes is the homogeneity property

$$\mathcal{H}(t, x, u; p, v, w_0, w) = \mathcal{H}(t, x, u; p, v, \frac{w_0}{|(w_0, w)|}, \frac{w}{|(w_0, w)|})|(w_0, w)|,$$

which in turn is a consequence of the positive homogeneity of $\overline{f}$ in (w_0, w). □

For the reader's convenience we state below an uniqueness theorem and a verification theorem which are corollaries of the subsequent comparison theorem. These theorems are straighforward generalizations of analogous results (see [34]) concerning the unconstrained case with $f = f_0(t, x, u, v) + \sum_{i=1}^{m} f_i(t, x, u, v)\dot{u}_i$.

Theorem 4.2 (Uniqueness) *Assume hypotheses $(H1)$ and $(H2)$. Then the value function $\mathcal{V}$ is the unique bounded continuous viscosity solution of (DPE) on Ω which satisfies the boundary conditions b) and c) of Theorem 4.1.*

Theorem 4.3 (Verification) *Let $Z \in C(\overline{\Omega})$ be a bounded viscosity subsolution of (DPE) on Ω which satisfies the condition $Z \leq \Psi$ on $\partial_T\Omega$. Then*

$$Z \leq \mathcal{V} \qquad on\ \overline{\Omega}.$$

Moreover, if for a given $(\bar{t}, \bar{x}, \bar{u}, \bar{k}) \in \overline{\Omega}$ there exists a space–time control $(v, w_0, w) \in , {}^{c}_{K-\bar{k}}(\bar{t}, \bar{x}, \bar{u})$ such that

$$\Psi\big(x[\bar{t}, \bar{x}, \bar{u}; v, w_0, w](1), u(1)\big) \leq Z(\bar{t}, \bar{x}, \bar{u}, \bar{k}),$$

then the space–time control (v, w_0, w) *is optimal and*

$$Z(\bar{t}, \bar{x}, \bar{u}, \bar{k}) = \mathcal{V}(\bar{t}, \bar{x}, \bar{u}, \bar{k}).$$

Theorem 4.4 (Comparison) *Assume hypotheses* $(H1)$ *and* $(H2)$. *Let* $\mathcal{V}_1$ *be a bounded continuous viscosity subsolution of (DPE) on* Ω *which satisfies*

$$\mathcal{V}_1(T, x, u, k) \leq \Psi(x, u) \qquad \forall (T, x, u, k) \in \partial_T \Omega.$$

Let $\mathcal{V}_2$ *be a bounded continuous viscosity supersolution of (DPE) on* $\Omega \cup \partial' \Omega$ *such that for any* $(T, x, u, k) \in \partial_T \Omega$ *either* $\mathcal{V}_2$ *satisfies the inequality*

$$\mathcal{V}_2(t, x, u, k) \geq \Psi(x, u)$$

or it is a viscosity supersolution of (DPE).

Then

$$\mathcal{V}_1 \leq \mathcal{V}_2 \qquad \text{on } \overline{\Omega}.$$

Remark 4.1 In the particular case investigated in [34]—where

$$f = g_0(t, x, u, v) + \sum_{i=1}^{m} g_i(t, x, u, v) \dot{u}_i$$

and no space constraints acted on x—hypothesis $(H1)$ is trivially satisfied. We recall that the value function corresponding to this case was proved to be continuous under one of the two following hypotheses:

$(H2)_C$ The set U coincides with the whole $\boldsymbol{R}^m$.

$(H2)_U$ the cone C coincides with the whole $\boldsymbol{R}^m$; moreover for any $\varepsilon > 0$ and $u_1 \in U$ there exists a $\delta > 0$ such that for each $u_2 \in U \cap B(u_1, \delta)$, there is a path $\gamma_{13} \in W^{1,1}([0,1], U)$ satisfying $\gamma_{13}(0) = u_1$, $\gamma_{13}(1) = u_2$, and

$$\int_0^1 |\gamma_{13}'(s)| \, ds \leq \varepsilon.$$

It is not difficult to check that if $C = \boldsymbol{R}^n$ hypothesis $(H2)$, assumed here in Theorems 3.1 and 4.1, is stronger than $(H2)_U$. However, on one hand, $(H2)$ yields the continuity of the value function in the constrained sublinear case and in the presence of an actual cone constraint on the directions of $\dot{u}$; and on the other hand $(H2)$ coincides with condition $(H3)$ of [34] which was essential to prove the uniqueness of the solution of the dynamic programming equation. As a consequence we have that the uniqueness and verification results which in [34] had been obtained under assumptions $(H2)_C$, $(H3)$ or $(H2)_U$, $(H3)$ remain still valid if one assumes the sole hypothesis $(H3)$.

Remark 4.2 In [34] we extended the Pontryagin maximum principle to optimal space–time trajectories in the special case where $f = g_0 + \sum_{i=1}^m g_i \dot{u}_i$, $\Theta = \boldsymbol{R}^n$ and $U = \boldsymbol{R}^m$. Moreover we showed that for those kind of problems a version of the known relationship between value function and adjoint variables is still valid. All these results can be generalized by simply replacing $g_0 w_0 + \sum_{i=1}^m g_i w_i$ with the more general vector field $\overline{f}$ considered in this paper.

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Dipartimento di Matematica Pura ed Applicata, Via Belzoni 7 - 35131 Padova, Italy

Dipartimento di Matematica Pura ed Applicata, Via Belzoni 7 - 35131 Padova, Italy

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