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SUMMARY
Robust and Risk-Sensitive Output Feedback Control for Finite State Machines and Hidden Markov Models*

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Abstract

The purpose of this paper is to develop a framework for designing controllers for finite state systems which are robust with respect to uncertainties. A deterministic model for uncertainties is introduced, leading to a dynamic game formulation of the robust control problem. This problem is solved using an appropriate information state. A risk-sensitive stochastic control problem is formulated and solved for Hidden Markov Models, corresponding to situations where the model for the uncertainties is stochastic. The two problems are related using small noise limits.

A *finite state machine* (FSM) is a discrete–time system defined by the model

$$\begin{cases} x_{k+1} &= f(x_k, u_k), \\ y_{k+1} &= g(x_k), \qquad k = 0, 1, \ldots, M, \end{cases}$$

where the state x_k evolves in a finite set $\mathbf{X}$, and the control u_k and output y_k take values in finite sets $\mathbf{U}$ and $\mathbf{Y}$, respectively. These sets have n, m, and p elements, respectively. The behavior of the FSM is described by a state transition map $f : \mathbf{X} \times \mathbf{U} \to \mathbf{X}$ and an output map $g : \mathbf{X} \to \mathbf{Y}$.

FSM models, together with accompanying optimal control problems, have been used widely in applications. However, it is typically the case that

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deterministic treatments of such problems do not specifically deal with *disturbances*, e.g., as arising from modelling errors, sensor noise, etc. In this paper we propose and solve a general *robust* control problem for FSMs, paralleling the framework that has been developed for linear systems. The approach we adopt is motivated by the information state methods developed by James-Baras-Elliott (1994), James-Baras (1995). We thus develop a general framework for robust output feedback control of FSMs which specifically incorporates a deterministic model for disturbances and their effects.

Hidden Markov Models (HMM) are a different but closely related class of models, and numerous filtering, estimation, and control problems for them have been proposed and employed in applications. These models use a *probabilistic* description of disturbances. However, the majority of applications to date use a *risk-neutral* stochastic optimal control formulation. It is clear from the work of Jacobson (1973), Whittle (1981) and others that a controller more conservative than the risk-neutral one can be very useful. Indeed, it is well known that risk-sensitive controllers are very closely related to robust controllers. Here, we formulate and solve such a *risk-sensitive* stochastic optimal control problem for HMMs. Our solution, which is interesting in itself, leads us to the solution of the robust control problem for FSMs mentioned above. This is achieved by using a HMM which is designed to be a small random perturbation of the FSM, and employing large deviation limits. It is possible to solve the robust control problem directly using an appropriate information state, once it is known, as in James-Baras (1995) (the large deviation limit identifies an information state).

The FSM model predicts that if the current state is x and a control input u is applied, the next state will be $x' = f(x,u)$. However, a disturbance may affect the actual system and result in a transfer to a state $x'' \neq x'$ instead. Similarly, the model predicts the next output to be $y' = g(x)$, whereas a disturbance may cause an output $y'' \neq y'$ to be observed. Additionally, the initial state x_0 may be unknown, and consequently we shall regard it as a disturbance.

We model the influence of disturbances as follows. Consider the following FSM model with two additional (disturbance) inputs w and v:

$$\begin{cases} x_{k+1} &= b(x_k, u_k, w_k), \\ y_{k+1} &= h(x_k, v_k), \qquad k = 0, 1, \ldots, M, \end{cases}$$

where, w_k and v_k take values in finite sets $\mathbf{W}$ and $\mathbf{V}$ respectively, $x_k \in \mathbf{X}$, $y_k \in \mathbf{Y}$, $u_k \in \mathbf{U}$. Thus $x'' = b(x,u,w)$ for some $w \in \mathbf{W}$, and $y'' = h(x,v)$ for some $v \in \mathbf{V}$. The functions $b : \mathbf{X} \times \mathbf{U} \times \mathbf{W} \to \mathbf{X}$ and $h : \mathbf{X} \times \mathbf{V} \to \mathbf{Y}$ are required to satisfy consistency conditions involving null inputs $w_\emptyset$ and

$v_\emptyset$, so that when no disturbances are present (i.e. $w_k \equiv w_\emptyset$, and $v_k \equiv v_\emptyset$), the behavior of the FSM with null disturbance inputs is the same as the original FSM. Cost functions $\phi_w : \mathbf{W} \times \mathbf{X} \times \mathbf{U} \to \mathbf{R}$, and $\phi_v : \mathbf{V} \times \mathbf{X} \to \mathbf{R}$, $\beta : \mathbf{X} \to \mathbf{R}$, are defined to quantify the effect of the disturbances. As part of the problem specification, one defines an additional output quantity

$$z_{k+1} \;=\; \ell(x_k, u_k), \tag{1}$$

where z_k takes values in a finite set $\mathbf{Z}$, and $\ell : \mathbf{X} \times \mathbf{U} \to \mathbf{Z}$. A cost function ϕ_z for this output is also specified.

The state variable x_k is not measured directly, and so the controller must make use of information available in the output signal $y_{0,k}$; i.e., the controller must be an *output feedback* controller. We denote by $\mathcal{O}_{k,l}$ the set of non–anticipating control policies defined on the interval $[k, l]$; i.e., those controls for which there exist functions $\bar{u}_j : \mathbf{Y}^{j-k+1} \to \mathbf{U}$ such that $u_j = \bar{u}_j(y_{k+1,j})$ for each $j \in [k, l]$.

The *output feedback robust control problem* we wish to solve is the following: given $\gamma > 0$ and a finite time interval $[0, M]$ find an output feedback controller $u \in \mathcal{O}_{0,M-1}$ such that

$$\sum_{k=0}^{M-1} \phi_z(z_{k+1}) \;\le\; \beta(x_0) + \gamma \sum_{k=0}^{M-1} \left(\phi_w(w_k; x_k, u_k) + \phi_v(v_k; x_k)\right)$$

for all $(w, v) \in \mathbf{W}^M \times \mathbf{V}^M$, $x_0 \in \mathbf{X}$.

This problem is solved by converting it into an output feedback dynamic game problem, which in turn is solved by reducing it to an equivalent state feedback problem in terms of an *information state*. The information state is defined by the recursion

$$\begin{cases} p_k^\gamma &= \Lambda^{\gamma\,*}(u_{k-1}, y_k) p_{k-1}^\gamma \\ p_0^\gamma &= -\beta, \end{cases}$$

where

$$\Lambda^{\gamma\,*}(u_{k-1}, y_k) p(x'') = \max_{x \in \mathbf{X}} \left\{ \Lambda^\gamma(u, y'')_{x,x''} + p(x) \right\}.$$

The value function satisfies the dynamic programming equation

$$\begin{cases} W^\gamma(p, k) &= \min_{u \in \mathbf{U}} \max_{y \in \mathbf{Y}} \left\{ W^\gamma(\Lambda^{\gamma\,*}(u, y) p, k+1) \right\} \\ W^\gamma(p, M) &= (p, 0), \end{cases}$$

where $(p,\, q) \;\stackrel{\triangle}{=}\; \max_{x \in \mathbf{X}} \{p(x) + q(x)\}$ is called the "sup-pairing," and is analogous to the familiar L_2 inner product.

The information state together with the dynamic programming equation characterize the solution to the robust control problem, as the following theorem shows.

Theorem (Necessity) *Assume that there exists an output feedback controller* $u^o \in \mathcal{O}_{0,M-1}$ *solving the output feedback robust control problem. Then there exists a solution* $W^\gamma(p,k)$ *of the dynamic programming equation such that* $W^\gamma(-\beta,0)=0$. (Sufficiency) *Assume that there exits a solution* $W^\gamma(p,k)$ *of the dynamic programming equation such that* $W^\gamma(-\beta,0)=0$, *and let* $\bar{u}^*_k(p)$ *be a control value achieving the minimum in the dynamic programming equation. Then* $\bar{u}^*_k(p^\gamma_k(\cdot\,;y_{1,k}))$ *is an output feedback controller which solves the output feedback robust control problem.*

It is important to note that the information state is related to a modification of the conditional distribution for a risk-sensitive stochastic control problem. Indeed, we obtain the solution to the robust control problem just described via the risk-sensitive control problem. To explain this, a HMM is constructed from the FSM by a random perturbation with a perturbation parameter $\varepsilon > 0$. A modified conditional distribution $\sigma^{\gamma,\varepsilon}_k$ is an information state for a risk-sensitive problem with cost

$$J^{\gamma,\varepsilon}(u) \;=\; \mathbf{E}^u\left[\exp\frac{1}{\gamma\varepsilon}\sum_{l=0}^{M-1}\phi_z\left(\ell(x^\varepsilon_l,u_l)\right)\right].$$

The value function is a function $S^{\gamma,\varepsilon}(\sigma,k)$. The stochastic control and deterministic game problems are related by the small noise limits

$$\lim_{\varepsilon\to 0}\gamma\varepsilon\log\sigma^{\gamma,\varepsilon}_k(x) = \; p^\gamma_k(x),$$

and

$$W^\gamma(p,k) \;=\; \lim_{\varepsilon\to 0}\gamma\varepsilon\log S^{\gamma,\varepsilon}(e^{\frac{1}{\gamma\varepsilon}p},k).$$

The results for the risk-sensitive stochastic control problem are of independent interest.

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