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Equivalent Conditions for the Solvability of Nonstandard LQ-Problems with Applications to Partial Differential Equations with Continuous Input-Output Solution Map*

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Abstract

We consider an optimal control problem with indefinite cost for an abstract model which covers in particular hyperbolic and hyperbolic-like systems in a general bounded domain. Necessary and sufficient conditions are given for the synthesis of the optimal control, which is given in terms of the Riccati operator arising from a nonstandard Riccati equation. The theory also extends a finite-dimensional frequency theorem to the infinite-dimensional setting. Applications include the damped wave equation with Dirichlet control, damped Euler-Bernoulli and Kirchoff equations with control in various boundary conditions, and the damped Schrödinger equation with Dirichlet control.

AMS Subject Classifications: primary 47A, secondary 35B
Key words: Riccati equations, boundary control problems

1 Introduction

1.1 Problem setting

Let X (state) and U (control) be separable Hilbert spaces. Consider the following abstract dynamical system

$$\dot{x} = Ax + Bu \in [D(A^*)]', \quad x(0) = \alpha \in X \tag{1.1.1}$$

where $u \in L_2(0,\infty;U)$ is a control function. The problem that we wish to consider is: find

$$\inf_{u \in L_2(0,\infty;U)} J(x,u) > -\infty \tag{1.1.2}$$

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where

$$J(x,u) = \int_0^\infty F(x,u)dt \tag{1.1.3a}$$

$$\begin{aligned} F(x,u) &= (F_1x,x)_X + 2Re(F_2x,u)_U + (F_3u,u)_U \\ &\geq \rho(\|x\|_X^2 + \|u\|_U^2), \quad \rho \in \mathbb{R}. \end{aligned} \tag{1.1.3b}$$

The dynamics (1.1.1) – (1.1.3) is subject to the following assumptions which shall be maintained throughout the paper:

(H.1) A is the generator of a s.c. stable semigroup on X with margin of stability

$$\|e^{At}\|_X \leq Me^{-\omega_0 t}, \quad M \geq 1, \quad \omega_0 > 0. \tag{1.1.4}$$

(H.2) B : continuous $U \longrightarrow [D(A^*)]'$, or equivalently, $A^{-1}B \in \mathcal{L}(U;X)$, where $[D(A^*)]$ denotes the dual of $D(A)$ with respect to the X-topology, and A^* is the X-adjoint of A (without loss of generality, we take $A^{-1} \in \mathcal{L}(X)$).

(H.3) The following abstract trace regularity holds (see [L-T.3, class (H.2)]): the (closable) operator $B^*e^{A^*t}$ admits a continuous extension, denoted by the same symbol, from $X \longrightarrow L^2(0,T;U)$:

$$\int_0^T \|B^*e^{A^*t}x\|_U^2 dt \leq c_T\|x\|_X^2 \quad \forall T < \infty; \quad x \in X \tag{1.1.5}$$

where B^* is the dual of B, and $B^* \in \mathcal{L}(D(A);U)$ after identifying $[D(A)]''$ with $D(A)$.

(H.4) $F_1 \in \mathcal{L}(X)$, $F_2 \in \mathcal{L}(X,U)$, and $F_3 \in \mathcal{L}(U)$.

The solution to the state equation (1.1.1) is given explicitly by

$$x(t) = x(t,0;\alpha) = e^{At}\alpha + (Lu)(t) \tag{1.1.6}$$

where, we have defined

$$(Lu)(t) \equiv \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \tag{1.1.7a}$$

$$: \ \textit{continuous } L^2(0,\infty;U) \longrightarrow C([0,\infty);X) \tag{1.1.7b}$$

(continuity of L follows from (H.3) [Theorem in section 3 and Remark 3.3. in L-T.3])

Remark 1.1.1 Assumption (H.3) is an abstract trace theory property. It has been shown to hold true for many classes of partial differential equations by purely PDE methods (energy methods in either differential or pseudo-differential form) including: second order hyperbolic equations; Euler-Bernoulli and Kirchoff equations; Schrödinger equations, and first order hyperbolic systems, all in arbitrary space dimensions and on explicitly identified spaces (see [L-T.3, class (H.2)]).

1.2 Statement of main results

We first define the space of admissible control functions and state variables by $\mathcal{M}_\alpha \equiv \{(x,u) \in L^2(0,\infty,X) \times L^2(0,\infty;U) \; : \; (x,u)$ satisfying $\dot{x} = Ax + Bu,\ x(0) = \alpha,\ u(0) \in U, \}$. (Clearly, $\mathcal{M}_\alpha$ is nonempty for A stable as $u \equiv 0 \longrightarrow x \in L^2(0,\infty;X)$ for any $\alpha \in X$.)

We now state the main results of the paper.

Theorem 1.2.1 *Assume hypotheses (H.1) – (H.4).*

1. *If $\rho > 0$, then there exists a unique optimal response $\{x^0, u^0\}$ of (1.1.2) which depends continuously on the initial state $\alpha \in X$. The optimal value of the minimum $V^0(\alpha) \equiv \inf_{x,u} J(x,u)$ is a quadratic form on X, i.e., there exists a positive self-adjoint operator, $P \in \mathcal{L}(X)$ such that*

$$V^0(\alpha) \equiv (P\alpha, \alpha)_X \tag{1.2.1}$$

   for $\alpha \in X$.

2. *Let $\rho > 0$. Then,*

   (a) *the optimal response guaranteed by (i) satisfies the following Hamiltonian system of equations:*

$$\begin{aligned} \frac{dx^0}{dt} &= Ax^0 + Bu^0 \in [D(A^*)]', \quad x^0(0) = \alpha \in X && (1.2.2)\\ \frac{d\psi^0}{dt} &= -A^*\psi^0 + F_1x^0 + F_2^*u^0 \in [D(A^*)]', \\ &\quad \phi^0(T) = 0, \quad T > 0 && (1.2.3)\\ &F_2x^0(t;\alpha) + F_3u^0(t;\alpha) + B^*\psi^0(t;\alpha) = 0 && (1.2.4) \end{aligned}$$

   $\forall \alpha \in X$, a.e. in $t > 0$. where ψ^0 is the solution to the adjoint problem (1.2.3), and the quantity $B^*\psi^0(t;\alpha)$ is well-defined in U (see Proposition 3.2.1);

   (b) *the operator P guaranteed by Eq. (1.2.1) is expressed by $P\alpha = \psi^0(0;\alpha)$ for all $\alpha \in X$;*

3. *If $\rho < 0$ then $\inf_{x,u} J(x,u) = -\infty$.* □

Theorem 1.2.2 Frequency Theorem for the Nonsingular Case *Assume hypotheses (H.1) – (H.4). Let $\rho > 0$. Then,*

1. *there exists a unique bounded, positive self-adjoint operator, $P = P^* \in \mathcal{L}(X)$, (see Eq. (3.1.6)) which satisfies the following Algebraic Riccati Equation, (ARE) for all $x, z \in D(A)$; or else for all $x, z \in D(A_F)$, A_F defined in (1.2.12) below:*

$$(PAx, z)_X + (Px, Az)_X - ([(PB + F_2^*)F_3^{-1}(B^*P + F_2) - F_1]x, z)_X = 0 \tag{1.2.5}$$

(see Theorem 3.1) with the property (see Lemmas 3.4.1 and 3.5.1 below)

$$B^*P \in \mathcal{L}(D(A);X) \cap \mathcal{L}(D(A_F);X); \tag{1.2.6}$$

2. *there exists a unique pair of operators* $P = P^* \in \mathcal{L}(X)$ *and* h*, defined in Eq. (1.2.9) below, such that*

$$2Re(Ax+Bu,Px) + F(x,u) = \|F_3^{1/2}(u-hx)\|_U^2, \quad (x,u) \in D(A) \times U \tag{1.2.7}$$

   with the properties

$$B^*P \in \mathcal{L}(D(A);X) \cap \mathcal{L}(D(A_F);X); \tag{1.2.8}$$

   where

$$h = -F_3^{-1}(B^*P + F_2) \tag{1.2.9}$$

   (see Theorem 4.1 below)

3. *there exists an operator* $P = P^* \in \mathcal{L}(X)$ *such that*

$$2Re(Ax+Bu,Px) + F(x,u) \geq \delta(\|u\|_U^2 + \|x\|_X^2), \quad (x,u) \in D(A) \times U \tag{1.2.10}$$

   for some $\delta > 0$ *(see Theorem 4.1 below);*

4. *the optimal control* $u^0(\cdot\,;\alpha)$ *is given by (see Eq. (3.3.7))*

$$u^0(t;\alpha) = -F_3^{-1}(B^*P + F_2)x^0(t;\alpha) \in L_2(0,\infty;U); \tag{1.2.11}$$

   where P *is the unique solution to the Algebraic Riccati Equation (1.2.5) (see Corollary 3.3.2 below);*

5. *define the operator (*F *stands for "feedback") (see Lemma 3.4.2 below)*

$$A_F \equiv A - BF_3^{-1}(B^*P + F_2) \tag{1.2.12}$$

   with maximal domain, where P *is the unique solution to the Algebraic Riccati Equation (1.2.5). Then,* A_F *is the generator of a s.c. semigroup on* X *and, in fact, for* $\alpha \in X$ *(see Eq. (3.4.8c) below):*

$$\begin{aligned} y^0(t;\alpha) &= e^{A_F t}\alpha = e^{(A-BF_3^{-1}(B^*P+F_2))t}\alpha \\ &\in L_2(0,\infty;X) \cap C([0,\infty];X), \quad t \geq 0. \end{aligned} \tag{1.2.13}$$

   Moreover, the semigroup, $e^{A_F t}$ *is uniformly (exponentially) stable on* X*: there exist constants* $C_F \geq 1$ *and* $\rho_F > 0$ *such that (see Corollary 3.4.1 below)*

$$\|e^{A_F t}\|_{\mathcal{L}(X)} \leq C_F e^{-\rho_F t}, \quad t \geq 0; \tag{1.2.14}$$

6. *for* $\alpha \in X$ *(see Lemma 3.1.1 below)*

$$(P\alpha, \alpha) = J^0(\alpha) = J(u^0(\cdot\,;\alpha), x^0(\cdot\,;\alpha)) = \inf_u J(u, x); \tag{1.2.15}$$

7. *for the existence of operators* P *and* h *satisfying Eq. (1.2.7), the condition* $\rho \geq 0$ *is necessary.* □

Theorem 1.2.3 Frequency Theorem for the singular case *Assume hypotheses (H.1) – (H.4).*

1. *Let* $J(x, u) > -\infty$*. Then, there exists a bounded operator* $P \in \mathcal{L}(X)$ *satisfying the Linear Operator Inequality (LOI)*

$$(Ax + Bu, Px)_X + F(x, u) \geq 0 \quad \forall (x, u) \in D(A) \times U \tag{1.2.16}$$

   in a suitably weak sense (see Theorem 5.2.1 below).

2. *Conversely, let* P *be a self-adjoint operator which satisfies the LOI (1.2.16) for* $(x, u) \in X \times U$*. Then* $J(x, u) > -\infty$*.* □

Remark 1.2.2 There is no claim that P is a strong solution of the LOI (1.2.16). The lack of (proof of) the regularity properties of B^*P, in particular, if it is densely defined, prevents us from justifying the formal steps needed to give the desired conclusion that P satisfies the LOI (1.2.16) even for, say, $x \in D(A)$. Thus, Theorem 1.2.3 gives the solution of (1.2.16) in a suitably weaker sense.

1.3 Literature

In this paper, we consider the existence of solutions to an optimal control problem with indefinite cost functional for an abstract PDE model in a general bounded domain, $\Omega \subset R^n$. The model covers, in particular, the use of boundary controls for hyperbolic and hyperbolic-like systems, e.g. the wave equation with Dirichlet control, Euler-Bernoulli and Kirchoff plate equations with control acting in the displacement or Neumann boundary conditions, or as a bending moment.

We show that there exists an optimal control u to the minimization problem if and only if the cost functional is nonnegative. Moreover, when there does exist an optimal control, we distinguish between two cases: in the first case, the cost functional is coercive in the control, u; in the second, we only assume that the cost functional is nonnegative. In the coercive case, necessary and sufficient conditions are given for the synthesis of an optimal control in terms of the solution to a nonstandard Riccati equation. However, when the cost is nonnegative (and not necessarily coercive in u), there is no explicit synthesis of the optimal control. Instead, the optimal cost is given in terms of a weak solution to an (operator) dissipativity inequality.

For a finite-dimensional history of this type of optimal control problem, we refer the reader to [19]. The minimization problem was also extended to infinite-dimensional systems in [19], wherein a general dynamical system on a bounded domain, with the control function acting as a forcing function in the interior of the domain was considered. However, to do this problem, strong assumptions on the regularity of the solutions to the dynamical system were made. These assumptions were later removed in [18], again for the case where the control function acts in the interior of the domain. Extensions to dynamical systems with control acting on the boundary of the domain were later made in [24], wherein the Pritchard-Salamon class of problems was considered. However, this latter class of problems does not include many PDEs of interest, as the control operator is essentially bounded on the space of interest. In particular, it does not include the examples which we are presently considering, as in these examples the control operator is allowed to be as badly unbounded as the dynamics operator A, but the input-output solution map is continuous. To consider these problems of interest, we follow closely the ideas of [18]. However, new added technical difficulties arise since the control operator is badly unbounded, and these regularity questions must be handled differently than was the case in [18].

2 Preliminary Results

Following [1], [5], we now define the space of admissible control operators $\mathcal{F} \equiv \{K \in \mathcal{L}(D(A);U) \ : \ A_K = A + BK$ generates an exponentially stable s.c. semigroup on X, $D(A_K) \subset D(K)$; $Ke^{A_K t} \in \mathcal{L}(X; L^2(0,\infty;U))\}$. The set $\mathcal{F}$ is nonempty as $K = -B^* P_H \in \mathcal{F}$, where P_H satisfies the standard algebraic Riccati equation

$$(Ax, P_H z) + (P_H x, Az) - (B^* P_H x, B^* P_H z) + (x, z) = 0, \tag{2.1}$$

$\forall x, z \in D(A)$ *or* $x, z \in D(A_K)$, with the property that the (unclosable) operator $B^* P_K \in \mathcal{L}(D(A);U) \cap \mathcal{L}(D(A_K);U)$ (see [5] for details).

Lemma 2.1 *Assume hypotheses (H.1) – (H.4). If*

$$\inf_u \{J(x,u) \ : \ (x,u) \in \mathcal{M}_\alpha\} > -\infty \quad \forall \alpha \in X, \tag{2.2}$$

then $F_3 \geq 0$ *and*

$$F(x,u) \geq 0 \ \forall (\omega, x, u) \in \mathbb{R} \times X \times U \quad \textit{with } i\omega x = Ax + Bu. \tag{2.3}$$

Proof: (by contradiction). We follow closely the ideas of [L-W.1], taking into consideration the lower regularity of the state variable. Let $K \in \mathcal{F}$. Then $A_K \equiv A + BK$ is exponentially stable on X. Suppose that there exists $(\tilde{\omega}, \tilde{a}, \tilde{u}) \in \mathbb{R} \times X \times U$ with

$$i\tilde{\omega}\tilde{a} = A\tilde{a} + B\tilde{u} \quad \textit{and } F(\tilde{a}, \tilde{u}) < 0. \tag{2.4}$$

For each $T>0$ define the control function $u_T(\cdot\ )$ by

$$u_T(t) \equiv \begin{cases} e^{i\tilde{\omega}t}\tilde{u} & if\ t\in[0,T] \\ e^{i\tilde{\omega}T}Ke^{A_K(t-T)}\tilde{a} & if\ t>T. \end{cases} \tag{2.5}$$

Then, the solution of the Cauchy problem

$$\dot{x} = Ax_T + Bu_T, \quad x_T(0)=\tilde{a} \tag{2.6}$$

on $\mathbb{R}^+$ is

$$x_T(t) \equiv \begin{cases} e^{i\tilde{\omega}t}\tilde{a} & if\ t\in[0,T] \\ e^{i\tilde{\omega}T}e^{A_K(t-T)}\tilde{a} & if\ t>T. \end{cases} \tag{2.7}$$

So, for each $T>0$,

$$\begin{aligned} J(x_T,u_T) &= \int_0^T F(x_T,u_T)dt + \int_T^\infty F(x_T,u_T)dt \\ &= \int_0^T F(\tilde{a},\tilde{u})dt + \int_T^\infty F(e^{i\omega T}e^{A_K(t-T)}\tilde{a}, e^{i\omega T}Ke^{A_K(t-T)}\tilde{a})dt \\ &= TF(\tilde{a},\tilde{u}) + \int_0^\infty F(e^{A_K\tau}\tilde{a}, Ke^{A_K\tau}\tilde{a})d\tau. \end{aligned}$$

Since, by assumption, $F(\tilde{a},\tilde{u})<0$, $J(x_T,u_T)\longrightarrow -\infty$ as $T\longrightarrow +\infty$, which is a contradiction. Hence $F(x,u)\geq 0\ \ \forall(\omega,x,u)\in\mathbb{R}\times X\times U$ satisfying $i\omega x = Ax+Bu$.

To show that $F_3\geq 0$, we first suppose that there exists a $\tilde{u}$ such that $F_3<0$, i.e., $(F_3\tilde{u},\tilde{u})<0$.

For such $\tilde{u}$, define

$$v(t) = \begin{cases} 0 & t\neq N \\ \tilde{u} & t=N \end{cases}$$

for some $N\in R$. Then, let $a(t)$ satisfy $\dot{a} = Aa+Bv,\quad a(0)=0$, i.e., $a(t) = \int_0^t e^{A(t-\tau)}Bv(\tau)d\tau$, so that its corresponding Laplace transform satisfies $i\omega\hat{a} = A\hat{a}+B\hat{v}$, where $(\hat{\cdot}\ )$ denotes the Laplace transform of $(\cdot\ )$. Then, for such v, $a(t)\equiv 0\ \ \forall t\in R$. But,

$$\begin{aligned} F(a(N),v(N)) &= (F_1a(N),a(N))_X + 2Re(F_2a(N),u(N))_U \\ &\qquad +(F_3u(N),u(N))_U \\ &= F(0,u) = F_3(u,u) < 0 \end{aligned}$$

which is a contradiction, as $F(a,v)\geq 0$ for all pairs (a,v) such that $i\omega a = Av+Bv$, or equivalently that their corresponding inverse Laplace transforms satisfy $\dot{\hat{a}} = A\hat{a}+B\hat{v}$.

Hence $F_3\geq 0$. □

Lemma 2.2 *Assume hypotheses (H.1) – (H.4) and let* $\rho > 0$. *Then, for each* $\alpha \in X$, *there exists a unique point* $(x^0(\cdot\,;\alpha), u^0(\cdot\,;\alpha)) \in L_2(0,\infty;X) \times L_2(0,\infty;U)$ *minimizing* J. *Moreover, there exists a positive constant* $\delta > 0$ *such that* $F_3 \geq \delta I$.

Proof: If $\rho > 0$, then the first claim follows by convexity of the cost functional $J(x,u)$. To prove that $F_3 > 0$, we repeat the proof of Lemma 2.1 with the Hermitian form replaced by

$$F_1(x,u) = F(x,u) - \frac{1}{2}\rho\|u\|_U > 0 \tag{2.8}$$

as $F(x,u) \geq \rho(\|x\|_X^2 + \|u\|_U^2)$ to get that $F_3 - 1/2\rho \geq 0$, and hence $F_3 > 0$. Details are omitted. □

3 Feedback Relationship Between the Optimal Control Function u^0 and the Corresponding State Variable x^0

The main result of this section is the following.

Theorem 3.1 *Assume that* $\rho > 0$ *so that the operator* F_3 *is coercive, Then,*

1. *the map* $\alpha \longrightarrow (x^0(\cdot\,;\alpha), u^0(\cdot\,;\alpha))$ *from* $X \longrightarrow L^2(0,\infty;X) \times L^2(0,\infty;U)$ *is linear and continuous. Thus, the optimal cost*

$$J^0(x^0(\cdot\,;\alpha), u^0(\cdot\,;\alpha)) \equiv V^0(\alpha) \tag{3.1}$$

   is a continuous Hermitian form on X *(see above Eq. (3.1.6);*

2. *the self-adjoint operator* $P \in \mathcal{L}(X)$ *defined by* $(P\alpha,\alpha) \equiv V^0(\alpha)$ *for all* $\alpha \in X$, *is a solution of the algebraic Riccati equation (ARE)*

$$(Ax, Py) + (Px, Ay) - ([(PB + F_2^*)F_3^{-1}(B^*P + F_2) - F_1]x, y) = 0 \tag{3.2}$$

   $\forall x, y \in D(A)$, *(see Theorem 3.5.1), and the s.c. semigroup generated by* $A_F \equiv A - BF_3^{-1}(B^*P + F_2)$ *is exponentially stable on* X *(see Corollary 3.4.1);*

3. *for each* $\alpha \in X$, *the optimal control* $u^0(\cdot\,;\alpha)$ *admits the feedback form*

$$u^0(\cdot\,;\alpha) = -F_3^{-1}(B^*P + F_2)x^0(\cdot\,;\alpha) \in L_2(0,\infty;U) \tag{3.3}$$

   where the optimal state $x^0(\cdot\,;\alpha)$ *satisfies (1.1.1) and so*

$$x^0(\cdot\,;\alpha) = e^{A_F t}\alpha \in L_2(0,\infty;X) \cap C(0,\infty;X) \tag{3.4}$$

   (see Corollary 3.3.2 and Lemma 3.4.2);

4. *the (unclosable) operator* $B^*P \in \mathcal{L}(D(A);X) \cap \mathcal{L}(D(A_F);X)$ *(see Lemma 3.5.1).*

3.1 Regularity of the value function

Lemma 3.1.1 *the map* $\alpha \longrightarrow (x^0(\cdot\,;\alpha), u^0(\cdot\,;\alpha))$ *from* $X \longrightarrow L^2(0,\infty;X) \times L^2(0,\infty;U)$ *is linear and continuous. Thus, the optimal cost*

$$J^0(x^0(\cdot\,;\alpha), u^0(\cdot\,;\alpha)) \equiv V^0(\alpha)$$

is a continuous Hermitian form on X *(see above Eq. (3.1.6). Thus, there exists a nonnegative self-adjoint operator* $P \in \mathcal{L}(X)$ *such that*

$$V^0(\alpha) = (P\alpha, \alpha)_X, \quad \forall \alpha \in X.$$

Proof: (See [9] for an analogous proof in a setting different from ours.) We return to the dynamics (1.1.1) – (1.1.3): Using the regularity of F, the maps

$$\alpha \longrightarrow F_1 x, \quad \alpha \longrightarrow F_2 x \tag{3.1.1}$$

are continuous $X \longrightarrow X$ and $X \longrightarrow U$ respectively. Thus, for each $\alpha \in X$, the functional $J(x_u, u)$ is quadratic, continuous : $L_2(0,\infty;U) \longrightarrow \mathbb{R}$. By assumption, for each $\alpha \in X$, there exists a unique optimal solution $u^0(\cdot\,,\alpha,0) \in L_2(0,\infty;U)$. By (1.1.6) and (1.1.7), the corresponding optimal trajectory

$$x^0(\cdot\,,\alpha,0) \in L^2(0,\infty;X) \cap C(0,\infty;X) \tag{3.1.2}$$

Now, for each $(\alpha, u) \in X \times U$, $J(x_\alpha, u)$ is well-defined and by assumption, J^0 achieves its minimum at the point $u^0(t)$. By (H.4), we infer that the map

$$\alpha \longrightarrow (F_1 x^0, F_2 x^0, F_3 u) \tag{3.1.3}$$

is linear and continuous

$$X \longrightarrow L^2(0,\infty;X) \times [L_2(0,\infty;U)]^2 \tag{3.1.4}$$

Hence, in particular, the map

$$\begin{aligned} \alpha \longrightarrow J^0(\alpha) \quad &\equiv J(x^0(\cdot\,,\alpha,0), u^0(\cdot\,,\alpha,0)) \\ &= \inf_{u \in L_2(0,\infty;U)} J(x(\cdot\,,\alpha,0), u(\cdot\,,\alpha,0)) \end{aligned} \tag{3.1.5}$$

is a continuous quadratic form on X. Therefore, there exists a self-adjoint operator, $P \in \mathcal{L}(X)$, $P \geq 0$ such that

$$J^0(\alpha) = (\alpha, P\alpha)_X, \quad \alpha \in X \tag{3.1.6}$$

$\square$

3.2 The adjoint problem

Lemma 3.2.1 *Assume hypotheses (H.1) – (H.4). Let $\rho > 0$. Then, for each $\alpha \in X$, there exists an optimal solution denoted by $\{x^0(\cdot\ ;\alpha); u^0(\cdot\ ;\alpha)\} \in L_2(0,\infty; X \times U)$ to the minimization problem (1.1.2). Moreover, the optimal control and trajectory are related by the following:*

$$u^0(t;\alpha) = -F_3^{-1}(F_2x^0(t;\alpha) + L^*(F_1x^0(\cdot\ ;\alpha) + F_2^*u^0(\cdot\ ;\alpha))) \in L_2(0,\infty;U) \tag{3.2.1}$$

Proof: Since $\rho > 0$, the cost functional is strictly convex and hence there exists a unique minimum. We introduce the following Lagrangian, with $(\lambda, x, u) \in L_2(0,\infty; X \times X \times U)$ as free parameters:

$$\mathcal{L}(\lambda, x, u) = 1/2\{(F_1x, x)_X + 2Re(F_2x, u)_U + (F_3u, u)_U\} + (\lambda, x - e^{A\cdot}\ \alpha - Lu)_X \tag{3.2.2}$$

Using the optimality conditions

$$\mathcal{L}_x = 0 \quad \Longrightarrow F_1x^0 + F_2^*u^0 + \lambda^0 = 0 \tag{3.2.3}$$
$$\mathcal{L}_u = 0 \quad \Longrightarrow F_2x^0 + F_3u^0 - L^*\lambda^0 = 0. \tag{3.2.4}$$

Combining (3.2.3) and (3.2.4) yields that

$$u^0(\cdot\ ;\alpha) = -F_3^{-1}[F_2(x^0(\cdot\ ;\alpha) + L^*(x^0(\cdot\ ;\alpha) + u^0(\cdot\ ;\alpha))] \tag{3.2.5}$$

x paper where the inverse in (3.2.5) is well-defined by Lemma 2.2. □

We now introduce the "adjoint" state equation as follows:

$$\frac{d\psi^0}{d\tau} = -A^*\psi^0 + F_1^*x^0 + F_2^*u^0, \quad \tau > 0 \tag{3.2.6}$$

Then,

$$\psi^0(t) = \int_t^\infty e^{A^*(\tau - t)}(F_1^*x^0(\tau) + F_2^*u^0(\tau))d\tau \tag{3.2.7a}$$

$$\Longleftrightarrow \psi^0(t) = \int_t^\infty e^{A^*(\tau - t)}(F_1^*x^0(\tau) + F_2^*u^0(\tau))d\tau. \tag{3.2.7b}$$

The following technical result is important. Its proof is based on the regularity of the output operator L in (1.1.7).

Proposition 3.2.1 *Assume (H.1) – (H.4). With reference to $\psi^0(t;\alpha)$ given by Eq. (3.2.6), we have for fixed $T > 0$*

$$B^*\psi^0(t;\alpha) \in L_2(0,\infty;U) \qquad \forall t > 0. \tag{3.2.8}$$

Proof: It is a corollary of the regularity of the operator L defined in (1.1.7), applied to $F_1x^0(t;\alpha)+F_2^*u^0(t;\alpha)$ and hypothesis (H.2). Recalling the definition (1.1.7) of L, we have that

$$\begin{aligned} B^*\psi^0(t;\alpha) &= B^*\int_t^\infty e^{A^*(\tau-t)}[F_1x^0(\tau;\alpha)+F_2u^0(\tau;\alpha)]d\tau \\ &= \{L^*[F_1x^0+F_2^*u^0]\}(t) \\ &= I. \end{aligned} \tag{3.2.9}$$

But $I \in L_2(0,\infty;U)$ by the continuity (1.1.7) of L applied to $(F_1x^0+F_2^*u^0) \in L_2(0,\infty;X)$, and the result is proved. □

Lemma 3.2.2 *For all $\alpha \in X$, we have the following identity:*

$$P\alpha \equiv \psi^0(0,\alpha) \tag{3.2.10}$$

where P is the nonnegative self-adjoint operator defined in Lemma 3.1.1.

Proof: Consider the cost functional

$$\begin{aligned} J(u^0,x^0) &= \int_0^\infty F(x^0(t),u^0(t))dt \qquad (3.2.11) \\ &= \int_0^\infty [(F_1x^0,x^0)+2Re(F_2x^0,u^0)+(F_3u^0,u^0)]dt \\ &= \int_0^\infty [(x^0,F_1^*x^0+F_2^*u^0)+(u^0,F_2x^0)+(F_3u^0,u^0)]dt) \\ &= \int_0^\infty [(e^{At}+Lu,F_1^*x^0+F_2^*u^0)+(u^0,F_2x^0+F_3^*u^0)]dt \\ &= \int_0^\infty [(e^{At}\alpha,F_1^*x^0+F_2^*u^0)+(u^0,L^*(F_1^*x^0+F_2^*u^0)) \\ &\qquad +(u^0,F_2x^0+F_3^*u^0)]dt \\ &= \int_0^\infty [(e^{At}\alpha,F_1^*x^0+F_2^*u^0)+(u^0,B^*(\psi^0(t)) \\ &\qquad +(u^0,F_2x^0+F_3^*u^0)]dt \\ &= (\alpha,\int_0^\infty e^{At}(F_1^*x^0+F_2^*u^0)dt) \\ &\qquad +\int_0^\infty (u^0,B^*(\psi^0(t)-F_2x^0+F_3^*u^0)dt) \\ &= (\alpha,\psi^0(0,\alpha))+\int_0^\infty (u^0,F_2x^0+F_3^*u^0+B^*\psi^0)dt \qquad (3.2.12) \end{aligned}$$

by (3.2.7). Now,

$$J^0(x^0,u^0)=(\alpha,P\alpha)$$

$$= (\alpha, \psi^0(0,\alpha)) + \int_0^\infty (u^0, F_2 x^0 + F_3^* u^0 - B^* \psi^0) dt$$
$$= (\alpha, \psi^0(0,\alpha)) + \int_0^\infty (u^0, F_2 x^0 + F_3^* u^0 + L^*(F_1 x^0 + F_2^* u^0)) dt, \quad (3.2.13)$$

$t > 0$. But,

$$F_2 x^0 + F_3^* u^0 + L^*(F_1 x^0 + F_2^* u^0) = 0 \quad \in L_2(0,\infty;U) \quad (3.2.14)$$
$$\Longrightarrow F_2 x^0 + F_3^* u^0 + B^* \psi^0 = 0 \quad \in L_2(0,\infty;U) \quad (3.2.15)$$
$$\Longrightarrow u^0(\cdot\,;\alpha) = -F_3^{-1}(B^* \psi(\cdot\,;\alpha) + F_2 x^0(\cdot\,;\alpha) \in L_2(0,\infty;U) \quad (3.2.16)$$

by Lemma 3.2.1, (3.2.7), and (1.1.7). Thus,

$$(P\alpha, \alpha)_X = J^0(x^0, u^0) = J^0(x^0(\cdot\,;\alpha), u^0(\cdot\,;\alpha) = (\alpha, \psi^0(0\,;\alpha))_X. \quad (3.2.17)$$

Hence, $P\alpha \equiv \psi^0(0,\alpha)$. □

3.3 Semigroup properties for the optimal solution $\{x^0(t;\alpha);\ u^0(t;\alpha)\}$

We first state the following lemma.

Lemma 3.3.1 *Assume hypotheses (H.1) – (H.4). Define the cost function*

$$J_{t_0}(x,u) \equiv \int_{t_0}^\infty F(x(t), u(t)) dt. \quad (3.3.1)$$

Then, the optimal control and the corresponding solution for the minimization of the cost functional (3.3.1) where $u = u(t) \in L^2(t_0,\infty;U)$ *and* $x(t)$ *is the solution of*

$$\dot{x} = Ax + Bu \in [D(A^*)]', \quad x(t_0) = \alpha \in X \quad (3.3.2)$$

can be expressed by

$$u^0(t - t_0;\alpha); \qquad x^0(t - t_0;\alpha). \quad (3.3.3)$$

Proof: The proof is analogous to that in [25, pg. 468]. Details are omitted.
□

We are now ready to state the main result of this subsection.

Lemma 3.3.2 *Assume hypotheses (H.1) – (H.4). Then, the optimal solution* $\{x^0(t;\alpha); u^0(t;\alpha)\}$ *satisfies the following semigroup property in* X *and* U *respectively:*

$$x^0(t+\sigma;\alpha) = x^0(t; x^0(\sigma;\alpha)); \qquad u^0(t+\sigma;\alpha) = u^0(t; x^0(\sigma;\alpha)), \quad \alpha \in X. \quad (3.3.4)$$

Proof: The proof is analogous to [25, proof of Lemma 6]. Details are omitted.
□

Corollary 3.3.1 *Assume hypotheses (H.1) – (H.4). Then, $\psi^0(t;\alpha)$, defined in Eq. (3.2.6) satisfies the following semigroup property.*

$$\psi^0(t+\sigma;\alpha) = \psi^0(t;x^0(\sigma;\alpha)), \quad \alpha \in X. \tag{3.3.5}$$

Proof: We use Eq. (3.2.7) for $\psi^0(t;\alpha)$, together with Lemma 3.3.2. Thus, we have that

$$\begin{aligned}
\psi^0(t+\sigma;\alpha) \quad &= \int_{t+\sigma}^{\infty} e^{A^*\tau}[F_1^* x^0(\tau;\alpha) + F_2^* u^0(\tau;\alpha)]d\tau \\
(r=\tau-\sigma) \quad &= \int_{t}^{\infty} e^{A^*r}[F_1^* x^0(r+\sigma;\alpha) + F_2^* u^0(r+\sigma;\alpha)]d\sigma \\
(\textit{by Lemma } 3.3.2) \quad &= \int_{t}^{\infty} e^{A^*r}[F_1^* x^0(r;x^0(\sigma;\alpha)) + F_2^* u^0(r;x^0(\sigma;\alpha))]d\sigma \\
&= \psi^0(t;x^0(\sigma;\alpha)).
\end{aligned} \tag{3.3.6}$$

□

Corollary 3.3.2 *Assume hypotheses (H.1) – (H.4). In addition, assume that the constant ρ (Eq. 1.1.3(b)) is positive, so that there exists a positive constant, δ, such that the operator, $F_3 > \delta I$. Then,*

$$u^0(t;\alpha) = -F_3^{-1}[B^*P + F_2^*]x^0(t;\alpha), \quad \textit{a.e. in } t. \tag{3.3.7}$$

Proof: We use Corollary 3.3.1 in Eq. (3.2.16) for $u^0(t;\alpha)$. □

3.4 Definition of A_F and Its Properties

Define the semigroup $\Phi(t)$ guaranteed by Lemma 3.3.2 as follows:

$$\Phi(t)\alpha = x^0(t;\alpha), \quad \forall\ \alpha \in X. \tag{3.4.1}$$

Then, we have the following results:

Lemma 3.4.1 *Assume (H.1) – (H.4) and that $u^0(0) \in U$. Moreover, assume that there exists a positive constant, δ such that $F_3 \geq \delta I$. Then,*

$$X \supset D(B^*P) \supset D(A_F). \tag{3.4.2}$$

Thus, $D(B^*P)$ is dense in X.

Proof: Following [5, proof of Lemma 4.5], we use the implicit representation for $\psi^0(t;\alpha)$, (3.2.10). Thus, for $\alpha \in D(A_F)$,

$$B^*P\alpha \quad = B^*\psi^0(0;\alpha) = B^*A^{*-1}\int_0^{t_0} A^* e^{A^*\tau}[F_1^* x^0(\tau;\alpha) + F_2^* u^0(\tau;\alpha)]d\tau$$

$$
\begin{aligned}
&+B^*e^{A^*t_0}Px^0(t_0;\alpha)\\
&= B^*A^{*-1}\{e^{A^*t_0}[F_1^*x^0(t_0;\alpha)+F_2^*u^0(t_0;\alpha)]\\
&\qquad -[F_1^*x^0(0;\alpha)+F_2^*u^0(0;\alpha)]\}\\
&\quad +\int_0^{t_0} e^{A^*\tau}\frac{d}{d\tau}[(F_1^*-F_2^*F_3^{-1}(B^*P+F_2))\Phi(\tau)\alpha]d\tau\}\\
&\quad +B^*e^{A^*t_0}Px^0(t_0;\alpha)\\
&= B^*A^{*-1}\{e^{A^*t_0}[F_1^*x^0(t_0;\alpha)+F_2^*u^0(t_0;\alpha)]\\
&\qquad -[F_1^*\alpha+F_2^*u^0(0;\alpha)]\\
&\quad +\int_0^{t_0} e^{A^*\tau}[F_1^*-F_2^*F_3^{-1}(B^*P+F_2)]\Phi(\tau)A_F\alpha d\tau\}\\
&\quad +B^*e^{A^*t_0}Px^0(t_0;\alpha) \qquad (3.4.3)\\
&\in U \qquad (3.4.4)
\end{aligned}
$$

where, by assumption (H.2), $B^*A^{*-1}\in \mathcal{L}(X;U)$, and where

$$B^*e^{A^*t}P\Phi(t)x\in L^2(0,T;U),\qquad x\in D(A_F)\ \ \forall T<\infty,\ \ t<T. \qquad (3.4.5)$$

by the trace assumption (H.3) and Proposition 3.2.1, so that t_0 in (3.4.3) can be chosen (depending on α) such that $u^0(t_0)$ and the last term in (3.4.3) are well-defined in U (this can be done as the measure of all such $t_0's$ contained in $[0,t_1]$ is equal to t_1). □

Define the operator $F_0\equiv -BF_3^{-1}(B^*P+F_2)$.

Lemma 3.4.2 *Assume (H.1) – (H.4) and that there exists a positive constant, δ, such that the operator $F_3\geq\delta I$. Then, for $x\in X$,*

$$\frac{d\Phi(t)x}{dt}=[A-F_0]\Phi(t)x\in[D(A^*)]'\ \ \textit{a.e. in } t\geq 0. \qquad (3.4.6)$$

Thus,

$$
\begin{aligned}
[A-F_0]\Phi(t)x = A_F\Phi(t)x = \Phi(t)A_Fx\in X,\ x\in D(A_F),\ t\geq 0 \quad &(3.4.7a)\\
[A-F_0]x = A_Fx=\Phi(t)A_Fx\in X,\ x\in D(A_F); \quad &(3.4.7b)\\
\Phi(t)x=e^{A_Ft}x\in X,\ x\in X. \quad &(3.4.7c)
\end{aligned}
$$

Proof: We follow closely [5, proof of Lemma 4.6]. Recalling (3.3.7) and (1.1.6) for the optimal dynamics, we get

$$\Phi(t)\alpha=x^0(t;\alpha)=e^{At}\alpha-\int_0^t e^{A(t-\tau)}BF_3^{-1}(B^*P+F_2^*)\Phi(\tau)\alpha d\tau,\quad \alpha\in X \qquad (3.4.8)$$

Thus, differentiating (3.4.8) with $\alpha \in X$ and $\beta \in D(A^*)$,

$$
\begin{aligned}
(\tfrac{\Phi(t)\alpha}{dt}, z) &= (e^{At}\alpha, A^*\beta) - (BF_3^{-1}[B^*P + F_2^*]\Phi(t)\alpha, \beta) \\
&\quad -(\int_0^t e^{A(t-\tau)}BF_3^{-1}(B^*P + F_2^*)\Phi(\tau)\alpha d\tau, A^*\beta) \\
&= (\Phi(t)\alpha, A^*\beta) - (BF_3^{-1}([B^*P + F_2^*]\Phi(t)\alpha, \beta) \qquad (3.4.9)
\end{aligned}
$$

where the last term in (3.4.9) is well-defined a.e. in t as $A^{-1}BF_3^{-1}([B^*P + F_2^*]\Phi(t)\alpha, A^*\beta)$ is well-defined a.e. in t by (H.2) and Lemma 3.4.1. (3.4.9) yields (3.4.6) □

Corollary 3.4.1 *Assume (H.1) – (H.4) and that there exists a positive constant, δ, such that the operator $F_3 \geq \delta I$. Then, the semigroup, $\Phi(t)$, defined by (3.4.1) is exponentially stable on X, i.e., there exists constants $M \geq 1$ and $\rho > 0$ such that*

$$\|\Phi(t)\|_{\mathcal{L}(X)} \leq Me^{-\rho t}, \quad t > 0. \qquad (3.4.10)$$

.

Proof: We use Lemma 3.4.2, (3.4.7c), combined with Datko's result [4]. □

We now give the counterparts of Lemmas 4.7 and 4.8 of [5].

Lemma 3.4.3 *Assume (H.1) – (H.4) and that there exists a positive constant, δ, such that the operator $F_3 \geq \delta I$. Then, for A_F and P defined by (3.4.7) and (3.1.6) respectively.*

$$A^*P \in \mathcal{L}(D(A_F); U), \qquad (3.4.11)$$

$$A_F^*P \in \mathcal{L}(D(A); U). \qquad (3.4.12)$$

Moreover,

$$-A^*Px = F_1x - F_3^{-1}F_2x - F_3^{-1}B^*Px + PA_Fx \in X, \quad x \in D(A_F), \qquad (3.4.13)$$

$$-A_F^*Px = F_1x - F_3^{-1}F_2x - F_3^{-1}B^*Px + PAx \in X, \quad x \in D(A). \qquad (3.4.14)$$

Proof: The proof follows closely that of Lemmas 4.7 and 4.8 in [5] with R replaced by $F_1 - F_3^{-1}[B^*P + F_2]$. The proof relies heavily on the identity (3.3.10) for $\psi(0;\alpha) = P\alpha$.

3.5 The operator P is a solution of the algebraic Riccati equation

As in Corollary 4.9 of [5], we begin with a corollary to Lemma 3.4.3.

Lemma 3.5.1 *Assume (H.1) – (H.4) and that there exists a positive constant, δ, such that the operator $F_3 \geq \delta I$. Then, for P defined in (3.1.6) we have*

$$\mathrm{(B^*Px, B^*Pz)_U \text{ is } \mathit{well-defined\ either\ for}\ x, z \in D(A);\ \mathit{or\ for}\ x, z \in D(A_F)} \tag{3.5.1}$$

$$|(B^*Px, B^*Pz)_U| \leq C \begin{cases} \|Ax\|_X \ \ \|Az||_X, \quad x, z \in D(A) & (3.5.2) \\ \|A_F x\|_X \ \ \|A_F z||_X, \quad x, z \in D(A_F) & (3.5.3) \end{cases}$$

Proof: We first let $x \in D(A)$ so that $A_F^* Px \in X$ by (3.4.12) Next, we take $z \in X$ for now. Then,

$$|(A_F^* Px, z)_X| \leq c\|Ax\|_X \ \ \|z\|_X. \tag{3.5.4}$$

We now compute, with $A_F^{**} = A_F$ since A_F is closed, still with $x \in D(A)$, $z \in X$

$$\begin{aligned} (A_F^* Px, z)_X &= (Px, A_F z)_X = \\ &(Px, [A + BF_3^{-1}(B^*P + F_2)]z)_X \\ &= (Px, [I + BF_3^{-1}(B^*PA^{-1} + F_2A^{-1})]Az)_X \\ &= ([I + BF_3^{-1}(B^*PA^{-1} + F_2A^{-1})]^* Px, Az)_X & (3.5.5) \\ &= \mathit{well-defined\ for}\ x \in D(A), \ \ z \in X. & (3.5.6) \end{aligned}$$

Now, we restrict z to $z \in D(A)$, so that Az fills all of X as z runs over $D(A)$. Then (3.5.6) says that the left term in the inner product in (3.5.5) is in X, i.e. that

$$D([I + BF_3^{-1}(B^*PA^{-1} + F_2A^{-1})]^* P) \supset D(A). \tag{3.5.7}$$

But, since P is bounded self-adjoint and F_2 is bounded, we have that

$$\begin{aligned} &D([I + BF_3^{-1}(B^*PA^{-1} + F_2A^{-1})]^* P) \\ &= D([I + A^{*-1}PBF_3^{*-1}B^* + A^{*-1}F_2^*F_3^{*-1}B^*]P) \\ &= \{x \in X : \ Px \in D([I + A^{*-1}PBF_3^{*-1}B^* + A^{*-1}F_2^*F_3^{*-1}B^*])\} \\ &= \{x \in X : \ Px \in D(A^{*-1}PBF_3^{*-1}B^*)\} \\ &= D(A^{*-1}PBF_3^{*-1}B^*P). & (3.5.8) \end{aligned}$$

Combining (3.5.8) with (3.5.7) we obtain

$$D(A^{*-1}PBF_3^{*-1}B^*P) \supset D(A). \tag{3.5.9}$$

Finally, with $x, z \in D(A)$ we obtain, as desired

$$\begin{aligned} &c\|Ax\|_X \ \ \|Az\|_X \\ &\quad \geq |(A^{*-1}PBF_3^{*-1}B^*Px, Az)_X| (\mathit{well-defined\ by}\ (3.5.9)) \\ &\quad \geq c_2|(B^*Px, B^*Pz)_U|, \ c_2\ (\mathit{constant}) & (3.5.10) \end{aligned}$$

and the first case of (3.5.1) as well as (3.5.2) follow. The result in (3.5.1) and (3.5.3) for $x, z \in D(A_F)$ is contained in (3.4.13) of Lemma 3.4.3. □

We finally obtain the ultimate goal of our analysis in this section.

Theorem 3.5.1 *Assume (H.1) – (H.4). Then,*

1. *The operator P defined in (3.1.6) satisfies the Algebraic Riccati Equation, in (1.2.5) i.e.*

$$(Ax, Py) + (Px, Ay) - ([(PB + F_2^*)F_3^{-1}(B^*P + F_2) - F_1]x, y) = 0 \quad (3.5.11)$$

for all $x, z \in D(A)$; or else for all $x, z \in D(A_F)$.

2. *Moreover, such P is the unique solution of (3.5.11) within the class of self-adjoint operators $P \in \mathcal{L}(X)$ such that $B^*P \in \mathcal{L}(A_F; X) \cap \mathcal{L}(D(A); X)$.*

Proof:

(i) We combine Lemma 3.4.3 and Lemma 3.5.1.

(ii) Uniqueness of the ARE:

Let P_1 and P_2 be two solutions to the algebraic Riccati equation such that the semigroups generated by $F_i = A_K - BB^*P_i = A - BF_3^{-1}(B^*P_i + F_2)$, $i = 1, 2$ are exponentially stable on X:

$$\begin{aligned}(Ax, Py) + (Px, Ay) \quad & -(F_2^*F_3^{-1}F_2x, y) + (F_1x, y) - (F_3^{-1}B^*Px, B^*Py) \\ & -(F_3^{-1}B^*Px, F_2y) - (F_3^{-1}F_2x, B^*Py) \qquad (3.5.12) \\ & = 0. \qquad (3.5.13)\end{aligned}$$

Let $Q = P_1 - P_2$. Then, we have

$$\begin{aligned} 0 &= (QA_Kx, y) + (Qx, A_Ky) \\ &\quad -(F_3^{-1}B^*P_1x, B^*P_1y) - (F_3^{-1}B^*P_2x, B^*P_2y) \\ &= (QA_Kx, y) + (Qx, A_Ky) - (BB^*P_1x, Qy) + (BB^*P_2x, P_2y) \\ &\quad -(BB^*P_1x, P_2y) + (BB^*P_1x, P_2y) \\ &= (QA_Kx, y) + (Qx, A_Ky) - (BB^*P_1x, Qy) + (BB^*P_2x, P_2y) \\ &\quad +(P_1x, BB^*P_2y) \\ &= (QA_Kx, y) + (Qx, A_Ky) - (BB^*P_1x, Qy) - (Qx, BB^*P_2y) \\ &= (Q(A_K - BB^*P_1)x, y) - (Qx, (A_K - BB^*P_2)y) \\ &= (QA_{F_1}x, y) - (Qx, A_{F_2}y) \end{aligned}$$

where A_{F_1} and A_{F_2} are the exponentially stable semigroups generated by F_i, $i = 1, 2$. But this is just a Lyapunov equation so $P_1 - P_2 = 0$. □

This completes the proof of Theorem 3.1. □

3.6 Proof of Theorem 1.2.1

(i) follows from Lemma 2.4;

(ii)(a) follows from Lemma 2.5;

$(ii)(b) - (c)$ follow from Eqs. (1.1.1), (3.2.3), Proposition 3.2.1, and Lemma 3.2.1;

To prove condition (iii), we use the following Lemma.

Lemma 3.6.1 *If* $\rho < 0$, *then* $\inf_{\mathcal{M}_\alpha} J(x,u) = -\infty$.

Proof: Let $\rho < 0$. Then, the functional $F(x,u)$ is not bounded below on the set $\mathcal{M}_\alpha$ for some $\alpha \in X$. Thus, there exists a triple $\{\omega; \tilde{a}; \tilde{u}\}$ in $R \times D(A) \times U$ such that $i\omega\tilde{a} = A\tilde{+}B\tilde{u}$ and $F(\tilde{a}; \tilde{u}) < 0$. Proceeding as in the proof of Lemma 2.3 we then obtain that

$$\inf J(x,u) = -\infty \qquad \Box \tag{3.6.1}$$

This completes the proof of Theorem 1.2.1. □

4 Proof of the Frequency Theorem for the Non-singular Case

(i), $(iv) - (vii)$ follow from Theorem 3.1.
(ii), (iii):
By using the ARE for $x = y \in D(A)$, it is a simple exercise to show that

$$2Re(Ax + Bu, Px)_X + F(x,u) = \|F_3^{1/2}(u + F_3^{-1}(B^*P + F_2)x)\|_U^2, \tag{4.1}$$

$(x,u) \in D(A) \times U$, where all terms are well-defined by the regularity of B^*P. Thus,

$$2Re(Ax + Bu, Px)_X + F(x,u) \geq 0 \quad (x,u) \in D(A) \times U. \tag{4.2}$$

Now, consider the Hermitian form $F_1(x,u) = F(x,u) - \delta(\|x\|_X^2 + \|u\|_U^2)$, $0 < \delta < \alpha_2$. Then, Theorem 1.2.1 still applies to the form $F_1(x,u)$, and so the representation

$$2Re(Ax + Bu, P_1x)_X + F_1(x,u) \geq 0 \quad (x,u) \in D(A) \times U \tag{4.3}$$

for some $P_1 \in \mathcal{L}(X)$ holds true, and all the properties of Theorem 1.2.1 hold true for this P_1. But, this is equivalent to (4.1) and condition (iii) is proved. □

This completes the proof of Theorem 1.2.2. □

5 Proof of the Frequency Theorem for the Singular Case

In the remainder of this paper, we shall study the frequency theorem for the singular problem, i.e., when the cost functional J is only nonnegative on the state space X. We first prove the following result:

Lemma 5.1 *Let $P \in \mathcal{L}(X)$ be a self-adjoint solution of the LOI (1.2.16) for $(x,u) \in X \times U$. Then, for each $\alpha \in X$ and $(x,u) \in \mathcal{M}_\alpha$,*

$$J^0(x,u) \geq (P\alpha, \alpha) = V(\alpha). \tag{5.1}$$

Moreover, J^0 is bounded below in each $\mathcal{M}_\alpha$ for each α.

Proof: We choose an arbitrary constant $T > 0$ and a sequence $u_n(t) \in L_2([0,T];U) \cap C^1([0,T];U)$ such that $u_n(t) \longrightarrow u(t)$ in $L_2(0,\infty;U)$. Since $(Ax + Bu, Px)$ is well-defined for $(x,u) \in X \times U$, then the following computations are justified:

$$\frac{d}{dt}(Px(t), x(t))_X = 2Re(Ax(t) + Bu_n(t), Px(t))_X, \quad \forall t \in [0,T] \tag{5.2}$$

recalling (1.1.1). Since P satisfies the LOI (1.2.16) integrating (5.2) from 0 to T yields

$$(Px(T), x(T))_X + \int_0^T F(x(t), u_n(t))dt \geq (P\alpha, \alpha)_X. \tag{5.3}$$

Letting $n \longrightarrow \infty$ we obtain

$$(Px(T), x(T))_X + \int_0^T F(x(t), u(t))dt \geq (P\alpha, \alpha)_X. \tag{5.4}$$

Since $x(\cdot\,) \in L_2(0,\infty;X)$, then there exists a sequence of positive reals $\{T_p\}$ such that $x(T_p) \longrightarrow 0$ as $T_p \longrightarrow \infty$. Then, (5.1) follows by letting $T_p \longrightarrow \infty$ in (5.4). □

5.1 Definition of the operator P_n and its properties

We first introduce the cost functional

$$J_n(x,u) = \int_0^\infty F_n(x(\tau), u(\tau))d\tau \tag{5.1.1}$$

where

$$F_n = (F_1x, x)_X + 2Re(F_2x, u)_U + (F_{3,n}u, u)_U + \epsilon_n \|u\|_U \quad (5.1.2)$$
$$F_{3,n} = F_3 + \epsilon_n I \quad (5.1.3)$$

and $\{\epsilon_n\}$ is a decreasing sequence of positive reals converging to 0 as $n \longrightarrow \infty$.

Proposition 5.1.1 *The map* $\alpha \longrightarrow (x_n^0(\cdot\ ;\alpha); u_n^0(\cdot\ ;\alpha))$ *from* $X \longrightarrow L_2(0,\infty;X) \times L_2(0,\infty;U)$ *is linear and continuous. Thus, the optimal cost*

$$\inf_{(x,u)\in\mathcal{M}_\alpha} J_n(x,u) = J_n^0(\alpha) = V_n^0(\alpha) = (P_n\alpha,\alpha)_X \quad \alpha \in X \quad (5.1.4)$$

is a continuous Hermitian form on X.

Proof: We use the fact that the quadratic form $F(x,u) \geq 0$ since the number $\rho \geq 0$ (by assumption) Hence, the form $F_n(x,u)$ is coercive, i.e., $F_n(x,u) \geq \epsilon_n\|u\|_U$. Thus, Theorem 3.1 applies. Hence, if we denote the optimal solution to the problem

$$\inf_{(x,u)\in\mathcal{M}_\alpha} J_n(x,u) = \inf_{(x,u)\in\mathcal{M}_\alpha} \int_0^\infty F_n(x(\tau),u(\tau))d\tau \quad (5.1.5)$$

by $\{x_n^0(t;\alpha); u_n^0(t;\alpha)\}$, then we have that, for each $n \geq 0$, there exists a self-adjoint operator, $P_n \in \mathcal{L}(X)$, $P_n \geq 0$ such that

$$\inf_{(x,u)\in\mathcal{M}_\alpha} J_n(x,u) = J_n^0(\alpha) = V_n^0(\alpha) = (P_n\alpha,\alpha)_X \quad \alpha \in X\Box \quad (5.1.6)$$

Proposition 5.1.2 *Assume (H.1) – (H.4). Moreover, assume that the number* ρ *defined in Eq. (1.2.2) is nonnegative. Then, the operator* P_n*, guaranteed by Proposition 5.1.1 satisfies the following, for each* n *fixed:*

1. $B^*P_n \in \mathcal{L}(D(A);U)$.

2. *The operator* P_n *is the satisfies the following (nonstandard) algebraic Riccati equation*

$$(Ax, P_ny) + (P_nx, Ay) - ([(P_nB + F_2^*)F_{3,n}^{-1}(B^*P_n + F_2) - F_1]x, y) = 0, \quad (5.1.7)$$

$\forall x, y \in D(A)$. *Equivalently,*

$$2Re(Ax+Bu, P_nx) + F_n(x,u) = \|F_{3,n}^{1/2}[u - F_{3,n}^{-1}(B^*P_n + F_2)x]\|_U, \quad (5.1.8)$$

$\forall (x,u) \in D(A) \times U$ *(see Eq. (4.2)).*

3. P_n *is the unique solution to Eq. (5.1.7) within the class of operators* $\{Q\ :\ Q \in \mathcal{L}(X)\}$.

Proof: Since the form $F_n(x,u)$ is coercive, Theorem 3.1 applies. Proposition 5.1.1 guarantees that there exists a bounded self-adjoint operator $P_n \geq 0$. As in the proof of Lemma 3.5.1, we can show that $B^*P_n \in \mathcal{L}(D(A);U)$. Using the regularity of B^*P_n and Theorem 3.1, we obtain, for each n fixed,

$$(Ax, P_n y) + (P_n x, Ay) - ([(P_n B + F_2^*)F_{3,n}^{-1}(B^*P_n + F_2) - F_1]x, y) = 0, \quad (5.1.9)$$

$\forall x, y \in D(A)$. Equivalently,

$$2Re(Ax + Bu, P_n x) + F_n(x,u) = \|F_{3,n}^{1/2}[u - F_{3,n}^{-1}(B^*P_n + F_2)x]\|_U, \quad (5.1.10)$$

$\forall (x,u) \in D(A) \times U$ (see Eq. (4.2)). □

5.2 Convergence of the sequence $\{P_n\}$ to P

Theorem 5.2.1 *Assume hypotheses (H.1) – (H.4). Then, the sequence of operators $\{P_n\}$ (defined in Eq. (5.1.4)) converges to an operator $P \in \mathcal{L}(X)$ as $n \longrightarrow \infty$. Hence, P satisfies the linear operator inequality (1.2.16) in a weak sense.*

Proof: By Theorem 3.1 there exists a self-adjoint operator $P_n \in \mathcal{L}(X)$ such that

$$\inf_{\{x(\cdot\,;\alpha),u(\cdot\,;\alpha)\}\in\mathcal{M}_\alpha} J_n(x(\cdot\,;\alpha), u(\cdot\,;\alpha)) = (P_n\alpha;\alpha)_X, \quad \alpha \in X \quad (5.2.1)$$

with J_n given by Eq. (5.1.1) Moreover, P_n satisfies

$$2Re(Ax + Bu, P_n x)_X + F_n(x,u) = \|F_3^{1/2}(u - h_n x)\|_U^2 \quad \forall (x,u) \in D(A) \times U \quad (5.2.2)$$

with

$$h_n = -BF_{3,n}^{-1}(B^*P + F_2) \quad (5.2.3)$$

(see Eq. (5.1.8)). Thus,

$$2Re(Ax + Bu, P_n x)_X + F_n(x,u) \geq 0 \quad \forall (x,u) \in D(A) \times U. \quad (5.2.4)$$

Now, we note that since $F_{n+1} \leq F_n$ for all n, the sequence $\{P_n\}$ is decreasing in the norm of $\mathcal{L}(X)$. Moreover, $\{P_n\}$ is uniformly bounded in the norm of $\mathcal{L}(X)$:

$$\rho\|\alpha\|_X^2 \leq \inf_u J(\alpha) \leq (P_n\alpha, \alpha)_X \leq (P_0\alpha, \alpha)_X, \quad \alpha \in X. \quad (5.2.5)$$

Thus, by the Principle of Uniform Boundedness, there exists an operator P_0 such that $P_n \longrightarrow P_0$ strongly to P_0 in $\mathcal{L}(X)$. Thus, P_0 satisfies the LOI (5.2.4) in a weak sense as each of the P_n satisfy the LOI. □

Remark 5.2.1 We are not authorized to pass through the limit on the sequence $\{P_n\}$ in the LOI as the regularity of the limit operator P_0 is unclear, in particular, we do not know if B^*P_0 is well-defined on a dense set.

Proof of Theorem 1.2.3

(i) follows from Theorem 5.2.1;

(ii) follows from Lemma 5.1. □

6 Applications of the Theory

6.1 Damped Euler-Bernoulli plate equation with Dirichlet control

Let Ω be a smooth bounded domain in $\mathbb{R}^n$, $n \leq 3$. We consider the following equation

$$
\begin{aligned}
z_{tt} + \Delta^2 z + c^2 z_t &= 0 && in\ (0,T] \times \Omega = Q && (6.1.1a)\\
z(0,\cdot\,) = z_0;\quad z_t(0,\cdot\,) &= z_1 && in\ \Omega && (6.1.1b)\\
z|_{\Sigma} &\equiv 0 && in\ (0,T] \times ,\ \equiv \Sigma && (6.1.1c)\\
\Delta z|_{\Sigma} &\equiv u && in\ \Sigma && (6.1.1d)
\end{aligned}
$$

$c \neq 0$, with boundary control $u \in L_2(\Sigma)$. Consistently with optimal regularity results [28], [16], the cost functional which we wish to consider is

$$
J(u,z) = \int_0^\infty \{\|z(t)\|^2_{H_0^1(\Omega)} + \|z_t(t)\|^2_{H^{-1}(\Omega)}\}dt \tag{6.1.2}
$$

with initial data $\{z_0, z_1\} \in H_0^1(\Omega) \times H^{-1}(\Omega)$.

Abstract setting

Let $x = [z, z_t]$, $X = H_0^1(\Omega) \times H^{-1}(\Omega)$, $U = L_2(,\ )$. Then, to put (7.1.1) – (7.1.2) into the abstract model (1.1.1), (1.1.3), we introduce the operators

$$
\mathcal{A}h \equiv \Delta^2 h, \quad D(\mathcal{A}) = \{h \in H^4(\Omega);\ h|_{,} = \Delta h|_{,} = 0\} \tag{6.1.3}
$$

$$
A = \begin{bmatrix} 0 & I \\ -\mathcal{A} & -c^2 I \end{bmatrix}, \quad Bu = \begin{bmatrix} 0 \\ \mathcal{A}\mathcal{G}u \end{bmatrix}, \quad F_1 = I, \quad F_2 = F_3 = 0 \tag{6.1.4}
$$

where $\mathcal{G}$ is the Green maps defined by

$$
\begin{aligned}
h = \mathcal{G}v &\Longleftrightarrow \{\Delta^2 h = 0;\ h|_{,} = 0;\ \Delta h|_{,} = v\} && (6.1.5a)\\
\mathcal{G}\ &:\ continuous\ L_2(,\ ) \longrightarrow H^{5/2}(\Omega). && (6.1.5b)
\end{aligned}
$$

The operator A defined in (7.1.4) is uniformly stable on $D(A^{1/4})\times[D(A^{1/4})]'$ and hence on X by the equivalence of the two norms. By [13], $(-A)^{-1}B \in L(U;X)$. We can show that [15],

$$B^* e^{A^* t}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \frac{\partial \Delta\phi(t)}{\partial\nu}|_{\Gamma_0} \quad z \in X \tag{6.1.6a}$$

where $\phi(t) = \phi(t;\phi_0,\phi_1)$ solves the corresponding homogeneous problem for $[\eta_1,\eta_2] \in X$

$$\begin{aligned} \phi_{tt} + \Delta^2\phi + c^2\phi_t &= 0 \quad in\ (0,T]\times\Omega = Q && (6.1.7a)\\ \phi(0,\cdot\,) = \phi_0;\ \ \phi_t(0,\cdot\,) &= \phi_1 \quad in\ \Omega && (6.1.7b)\\ \phi|_\Sigma &\equiv 0 \quad in\ (0,T]\times\Gamma \equiv \Sigma && (6.1.7c)\\ \Delta\phi|_\Sigma &\equiv 0 \quad in\ (0,T]\times\Gamma \equiv \Sigma && (6.1.7d) \end{aligned}$$

with

$$\phi_0 \equiv -\mathcal{A}^{-1/2}\eta_2 \in D(\mathcal{A}^{3/4}) = M; \quad \phi_1 \equiv \eta_1 \in D(\mathcal{A}^{1/4}) = H_0^1(\Omega) \tag{6.1.8}$$

Hence, an equivalent formulation of assumption (H.2) is the inequality

$$\int_\Sigma |\frac{\partial\Delta\phi}{\partial\nu}|^2 d\Sigma \le C_T \|[\phi_0,\phi_1]\|^2_{M\times H_0^1(\Omega)} \tag{6.1.9}$$

which is an independent regularity result which indeed holds true for general smooth Ω [15]. Thus, assumptions (H.1) – (H.4) are satisfied [13].

We now verify that the number $\rho \ge 0$. To do this, we consider the case when the initial condition $\alpha = 0$. Then,

$$\begin{aligned} J(x,u) &= \|Lu\|^2_{L_2(0,\infty;U)} \\ &\ge 0. \end{aligned} \tag{6.1.10}$$

Thus, the number $\rho \ge 0$. Hence, we have shown that Theorem 1.2.3 applies to problem (7.1.1) – (7.1.2).

6.2 Damped Kirchoff plate with boundary control in the bending moment

Let Ω be a smooth bounded domain in $R^n, n \le 3$. We consider the Kirchoff plate equation

$$\begin{aligned} z_{tt} + \Delta^2 z - \rho\Delta z_{tt} + c^2 z_t &= 0 \quad in\ (0,T]\times\Omega = Q && (6.2.1a)\\ z(0,\cdot\,) = z_0;\ \ z_t(0,\cdot\,) &= z_1 \quad in\ \Omega && (6.2.1b)\\ z|_\Sigma \equiv 0,\ \ \Delta z|_\Sigma &\equiv u \quad in\ (0,T]\times\Gamma \equiv \Sigma && (6.2.1c) \end{aligned}$$

$c \neq 0$, $\rho > 0$, with boundary control $u \in L_2(\Sigma)$, and initial data $\{z_0, z_1\} \in H^2(\Omega) \times H_0^1(\Omega)$. Consistently with optimal regularity theory [29], we take the following cost functional

$$J(u,z) \equiv \int_0^\infty \{\|z(t)\|^2_{H^2(\Omega)} + \|z_t(t)\|^2_{H_0^1(\Omega)}\}dt. \tag{6.2.2}$$

Abstract setting

To put problem (7.2.1) – (7.2.2) into the abstract model (1.1.1), (1.1.3), we introduce the positive self-adjoint operators

$$\mathcal{A} = \Delta^2 h; \quad D(\mathcal{A}) = \{h \in H^4(\Omega) : h|_, = \Delta h|_, = 0\} \tag{6.2.3}$$

$$\mathcal{A}^{1/2} = -\Delta h; \quad D(\mathcal{A}^{1/2}) = H^2(\Omega) \times H_0^1(\Omega) \tag{6.2.4}$$

and define the operators

$$A \equiv \begin{bmatrix} 0 & I \\ -\mathsf{A} & -c^2 I \end{bmatrix}; \qquad Bu \equiv \begin{bmatrix} 0 \\ \mathcal{A}\mathcal{G}u \end{bmatrix}; \tag{6.2.5a}$$

$$\mathsf{A} = (I + \rho\mathcal{A}^{1/2})^{-1}\mathcal{A}, \tag{6.2.5b}$$

$\mathcal{G}$ is the Green map defined in (7.1.5) and $\mathcal{G} = -\mathcal{A}^{-1/2}D_0$, with D_0 defined by

$$h = Dg \Longleftrightarrow (\Delta + c^2)h = 0 \ in \ \Omega; \quad h|_, = g. \tag{6.2.6}$$

By elliptic theory [8] and [27]

$$D \ : \ continuous \ L_2(,) \longrightarrow H^{1/2}(\Omega) \subset H^{1/2-2\epsilon}(\Omega) \equiv D(A_D^{1/4-\epsilon}), \quad \forall \epsilon > 0 \tag{6.2.7}$$

where the operator A_D is defined by

$$A_D h = -\Delta h, \quad D(A_D) \equiv H^2(\Omega) \cap H_0^1(\Omega). \tag{6.2.8}$$

$$F_1 = I, \quad F_2 = F_3 = 0. \tag{6.2.9}$$

Let $x = [z, z_t]$ and define the spaces

$$X = [H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega) = D(\mathcal{A}^{1/2}) \times D(\mathcal{A}^{1/4}), \quad U = L_2(,). \tag{6.2.10}$$

We can show [13] that for $[\eta_1, \eta_2] \in X$,

$$B^* e^{A^* t} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \frac{\partial \Delta\phi(t)}{\partial\nu}|_, , \quad z \in X \tag{6.2.11}$$

where $\phi(t) = \phi(t; \phi_0, \phi_1)$ solves the corresponding homogeneous problem

$$\phi_{tt} - \rho\Delta\phi_{tt} + \Delta^2\phi + c^2\phi_t = 0 \quad in \ (0,T] \times \Omega = Q \tag{6.2.12a}$$

$$\phi(0,\cdot) = \phi_0; \quad \phi_t(0,\cdot) = \phi_1 \quad in \ \Omega \tag{6.2.12b}$$

$$\phi|_\Sigma \equiv 0 \quad in \ (0,T] \times , \equiv \Sigma \tag{6.2.12c}$$

$$\Delta\phi|_\Sigma \equiv 0 \quad in \ (0,T] \times , \equiv \Sigma \tag{6.2.12d}$$

with

$$\phi_0 \equiv (I+\rho\mathcal{A}^{1/2})^{-1}\eta_2 \in D(\mathcal{A}^{3/4}); \quad \phi_1 \equiv (I+\rho\mathcal{A}^{1/2})^{-1}\mathcal{A}^{1/2}\eta_1 \in D(\mathcal{A}^{1/2}) \tag{6.2.13}$$

Thus, assumption (H.3) can be rewritten by the inequality

$$\int_\Sigma |\frac{\partial \Delta\phi}{\partial\nu}|^2 d\Sigma \leq C_T \|[\phi_0, \phi_1]\|^2_{D(\mathcal{A}^{3/4})\times D(\mathcal{A}^{1/2})} \tag{6.2.14}$$

which indeed holds true [13]. By [13], hypotheses (H.1) – (H.4) also hold true.

We now verify that the number $\rho \geq 0$. To do this, we consider the case when the initial condition $\alpha = 0$. Then,

$$J(x,u) \quad = \|Lu\|^2_{L_2(0,\infty;U)} \tag{6.2.15}$$

$$\geq 0. \tag{6.2.16}$$

Thus, the number $\rho \geq 0$. Hence, we have shown that Theorem 1.2.3 applies to problem (7.2.1) – (7.2.2).

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