Journal of Mathematical Systems, Estimation, and Control
Vol. 8, No. 1, 1998, pp. 1–21

Control Theory with Singular State-Space Constraints*

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Abstract

A new concept of partially defined system on a singular set is introduced. The setting is based on sheaf theory, and on a concept of locally integrable vector field on a differentiable space, introduced by K. Spallek. Short introductions to both areas are provided. The results consist of accessibility and observability criteria. They extend classical theorems as well as local observability criterion given recently by Bartosiewicz. Also an overview of integrability conditions for Lie-algebra sheaves is presented.

Key words: partially defined system, accessibility, local observability, singular constraints, sheaf, differentiable space, lie algebra, real radical

AMS Subject Classifications: 93B05, 93B07, 93C10, 32K15, 32B10, 14P05

1 Introduction

In classical theory of nonlinear control systems (see [19, 18, 23]) one studies a differential equation of the form:

$$\dot{x}(t) = f(x(t), u(t)),$$

where $x(t)$ belongs to a differential manifold M (called the state space) and u is a control function with values $u(t)$ in a set $U \subset \mathbb{R}^m$. For every fixed value $\omega \in U$, $f(\cdot, \omega)$ is a vector field on M. A *control* is a function $u : [0, T] \to U$, usually piecewise constant or piecewise analytic. Then for each $x_0 \in M$ there is a trajectory of this equation, corresponding to the

*Received January 12, 1995; received in final form October 16, 1995. Summary appeared in Volume 8, Number 1, 1998.
†Supported in part by a European Union grant and in part by a KBN grant.

control u and the initial condition x_0. Many papers showed that a manifold is a much better state space for nonlinear control systems than just $\mathbb{R}^n$. However it is not the best one.

In physical and technical sciences one often has to impose upon $x(t)$ constraints of the form (see e.g., [1])

$$g_1(x(t)) = 0, \ldots, g_k(x(t)) = 0,$$

where g_i are analytic or smooth functions. Usually it is assumed that the gradients of the functions $g_1, \ldots, g_k$ are linearly independent at each point, so the resulting state space is again a manifold. If this assumption is dropped then our restricted state space is no longer a manifold. Instead, it is an analytic (or a differentiable) space. Our aim is to extend the concept of control system to a system on such a subset, that is to a system with state-space constraints. Though our approach would allow for state spaces which are virtually arbitrary subsets of $\mathbb{R}^n$, one would rather keep in mind the following example.

Example 1.1 Let $M = \mathbb{R}^2$ and consider the control system

$$\begin{aligned} \dot{x}_1 &= 2x_2u_1 \\ \dot{x}_2 &= 3x_1^2u_1 \\ \dot{x}_3 &= u_2. \end{aligned}$$

When we impose the constraint

$$x_1^3 \Leftrightarrow x_2^2 = 0,$$

we can see that the vector fields associated with the system are tangent to the subset defined by the constraint. This set is an algebraic (and analytic) variety, and in particular a differentiable space, but not a differential manifold, since 0 is a singular point. Thus all the considerations can be restricted to this space. Systems on algebraic varieties of this type arise naturally when one studies realizations of polynomial systems [2]. □

Since not all the components of the state $x(t)$ might be available to the operator of the system, one usually introduces the observation $y(t) \in \mathbb{R}^r$ related to the state by an equation of the form

$$y(t) = h(x(t)),$$

where h is called an output or observation map. Each component h_i of h represents some measuring device.

In this paper we allow for infinitely many such devices which leads rather to a family of real functions than just a map h. Moreover we want the observation functions as well as the vector fields constituting the dynamics to be defined on open subsets of the state space. This corresponds to the fact that a measuring device might have a bounded range of operation or that only part of the dynamics can be used at a particular point. The language of the sheaf theory, which lies at the background of differentiable spaces, is especially useful here. Many of the problems we study here are of a local nature. They are best expressed with the aid of germs of vector fields, functions and sets. The language of sheaves allows us to connect those local investigations with the global picture.

A similar attempt was made earlier in [6], where the language of universe spaces was used to deal with partially defined systems. Differentiable spaces are more concrete objects than universe spaces, so our approach is a more down-to-earth one. We also emphasize a local nature of our definitions and results. In particular we do not name our vector fields and observation functions, so there is no natural correspondence between vector fields and observation functions at different points of the state space.

Our main concerns are the classical problems of controllability and observability. We examine the rank condition for accessibility on differentiable spaces, integrability of distributions and decomposition of the system. We also prove the extension of the criterion of local observability obtained recently in [3] to our new setting. The criterion is expressed with the language of real radicals which makes it more algebraic than traditional ones. On the other hand, it is far from a recent popular algebraic approach to observability (see [11, 12, 13, 14]). The algebraic approach, based on differential algebra, excludes the concept of state space which is basic in our theory.

2 A Soft Introduction to Sheaves

Although the sheaf theory is not absolutely necessary to express the ideas of this paper, it forms a natural and beautiful language simplifying and clarifying our concepts. We believe that sooner or later sheaves will find their way to the vocabulary of contemporary control theorists, and now it is the right moment to start. We are going to present here a few definitions and examples, following a nice introduction in [15]. We assume that the reader is familiar with germs of functions and sets ([16] is a good source).

A *sheaf* on a topological space X is a topological space $\mathcal{S}$ together with a local homeomorphism $\pi : \mathcal{S} \to X$ called *projection*. Then the *stalk* $\mathcal{S}_x := \pi^{-1}(x)$ is a discrete subset of $\mathcal{S}$ for every $x \in X$. Although in different sources one can find different definitions of sheaves, the one given here is the simplest one. To clarify the concept we need more definitions.

Let $(\mathcal{S}_1, \pi_1)$ and $(\mathcal{S}_2, \pi_2)$ be sheaves on X. A continuous map $\varphi : \mathcal{S}_1 \to \mathcal{S}_2$ is called a sheaf mapping if $\pi_2 \circ \varphi = \pi_1$ which means that φ *respects* *the stalks*. A subset $\mathcal{S}'$ of a sheaf $\mathcal{S}$ equipped with the relative topology is called a subsheaf of $\mathcal{S}$ if $(\mathcal{S}', \pi_{|\mathcal{S}'})$ is a sheaf on X.

Let Y be a subspace of X. A continuous map $s : Y \to \mathcal{S}$ is called a *section* over Y if $\pi \circ s = \mathrm{id}_Y$. Then the value of s at $x \in Y$, denoted by s_x, belongs to $\mathcal{S}_x$. The set of all sections over Y is denoted by , $(Y, \mathcal{S})$ or $\mathcal{S}(Y)$. Observe that the family $\{s(U) := \bigcup_{s \in U} s_x \;\; : \;\; U \subset X \text{ open }, s \in \mathcal{S}(U)\}$ forms a basis for the topology of $\mathcal{S}$.

Let us associate to every open set U in X some set $S(U)$. Suppose that for a pair of open sets $\emptyset \neq V \subset U$ in X we have a *restriction map* $r_V^U : S(U) \to S(V)$ such that

$$r_U^U = \mathrm{id} \;\; \text{and} \;\; r_W^V \circ r_V^U = r_W^U,$$

whenever $W \subset V \subset U$. Then $S := (S(U), r_V^U)$ is called a *presheaf* over X.

The most important examples of presheaves (at least for us) are presheaves of functions and vector fields. For example, one may assign $S(U)$ to be the set of all continuous (or smooth or analytic) functions on U. Then the restriction map r_V^U is simply the restriction of the domain U of functions in $S(U)$ to a subset $V \subset U$.

Every sheaf $\mathcal{S}$ over X defines the canonical preasheaf

$$, (\mathcal{S}) := (\mathcal{S}(U), r_V^U),$$

where $r_V^U(s)$ is the restriction of s to V. This is the presheaf of sections. On the other hand, given a presheaf, one can construct a corresponding sheaf by "localization." In fact, this is a common way of defining and introducing sheaves (see [15]). In the case of the presheaf of real (continuous) functions on a topological space, the sheaf obtained by "localization" consists of germs of real continuous functions. More precisely, the stalk $\mathcal{S}_x$ consists of germs of functions at point x. (We assume that the reader knows what a germ is). Obviously, the projection π assigns to a germ at x the point x. The concept of sheaf is convenient when we want to consider such germs at different points.

Sheaves and presheaves defined so far are set-valued. We would rather need sheaves of rings, modules or algebras, as functions and vector fields often form such structures. Let us concentrate on the last case, where algebras are over the field of reals. The sheaf $\mathcal{S}$ is a sheaf of algebras if for every $x \in X$ the stalk $\mathcal{S}_x$ is an algebra and all operations in the algebra are continuous. Again, as an example, one can take the sheaf of germs of real (continuous) functions. The germs at point x form an algebra over $\mathbb{R}$: one can add them, multiply them, and multiply them by reals.

If $\mathcal{S}$ is a sheaf of algebras then for every open U in X, $S(U)$ has a natural structure of an algebra. Thus the canonical presheaf , $(\mathcal{S})$ is a presheaf of algebras.

Let X be a C^∞ or C^ω manifold and let $\mathcal{D}^N$ denote the sheaf of all germs of C^N real functions on X ($N = \omega$ or $N = \infty$). The stalk $\mathcal{D}^N_x$ is an algebra over $\mathbb{R}$ of all C^N germs at x. We can safely add two such germs, multiply them, or multiply one of them by a real scalar. A section of $\mathcal{D}^N$ over U is a continuous map which assigns to every $x \in U$ a germ at x of a C^N function. Such a section can be identified with a C^N real function on U (i.e., a *partially defined function* on X). This is due to the fact that the algebra of germs at x is local (see [15]). Thus the canonical presheaf $\Gamma(\mathcal{D}^N)$ consists of C^N functions defined on open subsets of X.

Similarly we can construct the sheaf $\mathcal{V}^N$ of germs of C^N vector fields on a manifold X. Sections can be identified with partially defined vector fields on X. Sections over U form a Lie algebra over $\mathbb{R}$ with respect to the Lie bracket. This algebraic structure is carried down to stalks of the sheaf, so $\mathcal{V}^N$ is a sheaf of Lie algebras.

The sheaves $\mathcal{D}^N$ have different properties for $N = \omega$ and for $N = \infty$. If $N = \infty$ then the sheaf $\mathcal{D}^\infty$ is *soft*. This means that for every closed subset $A \subset X$ and for every section s over A, s can be extended to the entire space X [15]. This property does not hold for $\mathcal{D}^\omega$. On the other hand, if we can extend an analytic section over an open set (or an analytic germ at a point x) to the entire space, this extension is unique.

3 Differentiable Spaces and Locally Integrable Vector Fields

In this section we introduce locally integrable vector fields on subsets of $\mathbb{R}^n$. Though we could do this on arbitrary subsets (arbitrary differentiable spaces), we prefer to use a set of the form

$$C = \{x \in \mathbb{R}^n : g_1(x) = 0, \ldots, g_k(x) = 0\}. \tag{1}$$

If the functions g_i are analytic, C is an analytic set. In the C^∞ case, an arbitrary closed subset of $\mathbb{R}^n$ may be given form (1). If the equalities in (1) are substituted with inequalities, for g_i's analytic, C is a semi-analytic set. A point x of C is *regular* if the dimension of the space spanned by gradients of functions g_i is constant in some neighborhood of x. Otherwise x is singular. If all points of C are regular, then C is a C^∞ or C^ω submanifold of $\mathbb{R}^n$.

Though it is usually assumed that the set C is regular, i.e., it is a manifold, there are cases in which singular points appear naturally. Many such examples can be found in robotics.

Example 3.1 (suggested by K. Tchoń) Consider a two-arm robot, or simply a double pendulum, with arms of the same length equal 1. If θ_1 denotes the angle between the horizontal line and the first arm, and θ_2 is the angle between the directions of the arms, then the end point of the robot (pendulum), (x, y), is described by the equations

$$x = \cos\theta_1 + \cos(\theta_1 + \theta_2), \tag{2}$$

$$y = \sin\theta_1 + \sin(\theta_1 + \theta_2). \tag{3}$$

Suppose now that the end point is to stay on the circle $x^2 \Leftrightarrow x + y^2 = 0$, to perform some job. Though the circle is a perfect manifold, this restriction leads to a complicated curve in θ_1 and θ_2 coordinates. They, together with their derivatives, form the state of the robot; the state space is a cartesian product of a torus and a plane. The restriction on x and y leads to the following equation on the torus

$$\cos\frac{\theta_2}{2}(2\cos\frac{\theta_2}{2} + \cos(\theta_1 + \frac{\theta_2}{2})) = 0.$$

Since the left-hand side is a product of two analytic functions, the set C of the solutions of the equation is a union of two analytic curves on the torus. Although both curves are regular, the points where they intersect are singular points of the entire set. There are two such points: $(0, \pi)$ and (π, π). Hence, the state space C of the restricted robot is not a manifold. □

A real function φ on C is of class C^N if φ has a C^N extension to an open set in $\mathbb{R}^n$, containing C. Such a set C together with the sheaf of germs of functions of class C^N on C will be called a (reduced) *differentiable space* of class C^N. Actually, the class of differentiable spaces is much larger, but we do not want to introduce too much theory here.

Let us point out the main reasons for studying systems on differentiable spaces.

1) Differentiable spaces form a much larger and more flexible class than differential manifolds, but this class is still sufficiently concrete and most of the standard analysis can be done here.

2) If a group G acts on a differential manifold, then in reasonable cases the quotient M/G exists in the category of differentiable spaces. The quotient is a manifold iff G operates fixed-point free. The singular case, when the resulting space is not a manifold, was discussed in [8, 24, 25]. When the quotient space is a differentiable space, we can still push down onto M/G many objects from M, like functions, vector fields or a metric ([17]).

3) The language of sheaves, used in the theory of differentiable spaces, allows for easy manipulation of partially defined functions and vector fields. Such objects appear for example in systems described by rational functions. Partially defined systems arise also as realizations of input/output equations that can be factorized (see [5]). In real life problems, the output

functions, which represent measuring devices, need not be defined on the whole state space. Each device may have a limited and different range of operation. For example, different instruments are used to measure low and high voltages.

To avoid abstract or more complicated constructions we describe only the case of reduced locally compact differentiable spaces in $\mathbb{R}^n$, i.e., the spaces whose underlying sets have the form (1). In the last section we give some remarks on a more general case.

Similarly as functions we define vector fields of class C^N on a set C.

Assume now that f is a C^N vector field on $\mathbb{R}^n$. Then through every point of the set C there passes a local trajectory (integral curve) defined on some interval $(-\epsilon, \epsilon)$. This trajectory does not have to stay in C. However if it does, and this holds for local trajectories through every point of C, we call such a vector field locally integrable on C [28] (ϵ may be different for different points). Later in this section we give some comments on this property.

Let us recall the usual definition of the tangent space to C at a point p. The tangent space T_pC consists of all vectors $v \in \mathbb{R}^n$ such that $d\varphi(p)(v) = 0$ for every differentiable function φ that vanishes near p on C. A vector field f on $\mathbb{R}^n$ is tangent to C at $p \in C$, if $f(p) \in T_pC$.

Remark 3.2

1. If C has no singularities, i.e., C is a manifold, we are in a classical situation. Every (tangent) vector field on C is then locally integrable. Hence we are interested in the case when C does have singularities.

2. A locally integrable vector field on $U \subset C$ is always tangent to C at every point of U.

3. If C is an analytic set then every C^ω or C^∞ vector field tangent to C is locally integrable [28].

4. In general, a vector field on C, tangent to C, need not be locally integrable. Take for example $C = \{x \in \mathbb{R} : x \geq 0\}$ and the vector field $f \equiv 1$. Then f is tangent to C, but it does not have local trajectories passing through $x = 0$, so it is not locally integrable on C. On the other hand, the vector field $f(x) = x$ (or $f(x) = x\frac{\partial}{\partial x}$ in a derivation form) *is* locally integrable. □

Once the vector field is locally integrable on C, we can work with it in exactly the same way as one does on manifolds. Moreover, C may be replaced by any subset of $\mathbb{R}^n$, even highly singular. This is based on results of K. Spallek [27] which we are going to describe now.

The ordinary tangent space is too large for our purposes. Define T_p^iC to be the set of all tangent vectors $v \in T_pC$ for which there exist a neighborhood U of p in C and a locally integrable vector field f on U, such that $f(p) = v$. Observe that for a regular point p of C, $T_pC = T_p^iC$.

Theorem 3.3 [28]

1. *Assume that C is a locally compact subset of $\mathbb{R}^n$, $U \subset C$ is open (in C), and f is a vector field on U. Then f is locally integrable iff for every $q \in U$, $f(q) \in T_q^iC$.*

2. *The sheaf $\mathcal{V}^i(C)$ of germs of locally integrable vector fields on C is a Lie algebra sheaf on C, i.e., $\mathcal{V}^i$ is closed under addition, Lie brackets and multiplication by germs of functions.*

3. *There exists a unique family $\mathcal{F}(C) = \{M_j : j \in J\}$ of sets satisfying:*

   (a) every M_j is a connected, 1:1 immersed submanifold of C,

   (b) $\bigcup M_j = C$,

   (c) for all $j \in J$ and $p \in M_j$, $T_pM_j = T_p^iC$ (so T_p^iC is a vector space).

4. *For a point $p \in C$, $d_p := \dim T_p^iC$ is the largest number s such that in a neighborhood of p, C is diffeomorphic to some $\tilde{C} \times \mathbb{R}^s$, where $\tilde{C}$ is a subset of some $\mathbb{R}^k$.* □

Now we are going to give some comments and examples which should clarify presented results and stress their importance. More details can be found in original papers.

The family $\mathcal{F}(C)$, called the *natural foliation of* C, can often be explicitly found either by geometric arguments (just looking at the set) or by calculations. For example, if $C \subset \mathbb{R}^2$ is the graph of a function that is nowhere smooth, the foliation of C consists of isolated points (C is called then geometrically degenerated).

If C is a cone in $\mathbb{R}^n$ given by the equation

$$x_1^2 + \ldots + x_{n-1}^2 = x_n^2$$

with the restriction $x_n \geq 0$, the foliation consists of the zero point and the rest. This implies that trajectories of locally integrable vector fields cannot pass through 0 (in other words, they are constant).

Consider several sheets (halfplanes) attached to each other along the line — their common boundary (a book). Then the foliation consists of the line and open halfplanes. All vector fields tangent to the book are locally integrable ([28]). In particular, their trajectories passing through points in the line must stay in the line.

The cone and the book are important instances of the general case studied in [28]. They are *integrable spaces*, i.e., spaces on which all tangent vector fields are locally integrable. The Cone Lemma and the Book Lemma, proved in [28], describe large classes of integrable spaces based on the given examples.

To be integrable the book must have at least two sheets. If there is only one sheet (i.e., a closed halfplane), not every tangent vector field is locally integrable. The integrable ones are tangent to the boundary.

If G is a finite linear group acting on $\mathbb{R}^n$, then the foliation of the quotient space $\mathbb{R}^n/G$ given in Theorem 3.3 may be constructed in the following way. First take the connected components of the set of regular points. Then proceed in the same way with the set of singular points (which has its own regular points). We can show that this gives so-called Whitney stratification ([33]). Moreover, locally integrable vector fields on $\mathbb{R}^n/G$ are the images under the quotient map of G-equivariant vector fields on $\mathbb{R}^n$ ([17]).

Note that our foliation of the set C is in general singular. This means that the dimension of the leaves may change. But we can prove the following property: the boundary in C of any leaf is a union of some leaves of the foliation.

4 Systems

Let X be a constrained set of the form (1), where the functions g_i are of the class C^N and N is either ∞ or ω. X will be called the space (this is in fact an example of a differentiable space). All the definitions we are going to give are valid in both cases, but some of the results are true, or are known to be true, only in the analytic case. Let $\mathcal{D}^N$ denote the sheaf of germs of C^N real functions on X, and $\mathcal{V}^i$ the sheaf of germs of locally integrable vector fields on X. Then the stalk $\mathcal{V}^i_x$ is a module over $\mathcal{D}^N$ and a Lie algebra with respect to the Lie bracket (Theorem 3.3). When φ is a germ of a C^N function at $x \in X$, we often use the same symbol φ to denote a representative of this germ, i.e., a C^N function on a neighborhood of x. On the other hand, if φ is a function defined on a neighborhood of x, φ_x means the germ of φ at x. The same applies to germs of vector fields.

By a *dynamics* on X we mean a set-sheaf $\mathcal{V}$ on X such that for every $x \in X$, $\mathcal{V}_x$ is a non-empty subset of $\mathcal{V}^i_x$. This means that at each point x of X we have a family of admissible integrable vector fields, each vector field defined on some open neighborhood of x. This family may be finite, but is never empty. If $\mathcal{V}_x$ is finite, we say that $\mathcal{V}$ is *finite at* x, and if it is finite at every $x \in X$, we call $\mathcal{V}$ *locally finite*. We call $\mathcal{V}_x$ the *local dynamics at* x.

Let $\mathcal{V}$ be a dynamics on X. We say that a vector field V defined on an open set U *belongs to* $\mathcal{V}$, if $V \in$, $(U, \mathcal{V})$, i.e., for every $x \in U$, $V_x \in \mathcal{V}_x$. As representatives of germs from $\mathcal{V}_x$ we always take elements of $\Gamma(U,\mathcal{V})$ where U is some open neighborhood of x.

Let $\mathcal{V}$ be a dynamics on X. By a *trajectory* of $\mathcal{V}$ we mean any continuous and piecewise C^N map $\gamma : I \to X$, where I is an interval, such that $\dot\gamma(t) \in \mathcal{V}(\gamma(t))$ for all $t \in I$ for which the derivative $\dot\gamma(t)$ exists. Observe that $\mathcal{V}(\gamma(t))$ is a subset of $T^i_{\gamma(t)}X$. We say that the trajectory γ *starts at* $x_0 \in X$ if $I = [0,T)$ or $I = [0,T]$, and $\gamma(0) = x_0$.

We say that a trajectory $\gamma : [a,b] \to X$ is *simple* if $a = t_0 < \ldots < t_k = b$ and $\gamma_{|[t_i,t_{i+1}]}$ is a trajectory of V_{i+1} for some $t_1,\ldots,t_{k-1}$ and some vector fields $V_1,\ldots,V_k$ belonging to $\mathcal{V}$. It is easy to see that all trajectories are simple if the dynamics is locally finite. In most cases only simple trajectories will be taken into account.

An *observation structure* on X is a set-sheaf $\mathcal{H}$ on X such that for every $x \in X$, $\mathcal{H}_x$ is a non-empty subset of $\mathcal{D}^N$. This means that at each point x in X we have a set of real functions, each defined on some open neighborhood of x. Such a function represents a measuring device which delivers information about the current position of a trajectory starting from x. Usually such information is not complete, i.e., it does not allow us to establish this position. If $\mathcal{H}_x$ is a finite set, we say that $\mathcal{H}$ is *finite at* x. We call $\mathcal{H}$ *locally finite* if it is finite at all $x \in X$. The set $\mathcal{H}_x$ is called the *local observation structure at* x. A representative of an element from $\mathcal{H}_x$ is always understood as an element of $\Gamma(U,\mathcal{H})$, where U is a neighborhood of x.

A *system* on a space X is a pair $\Sigma = (\mathcal{V},\mathcal{H})$ where $\mathcal{V}$ is a dynamics and $\mathcal{H}$ is an observation structure on X. We say that the system is *of class* C^N if X, $\mathcal{V}$ and $\mathcal{H}$ are of class C^N, and we say that it is *locally finite* (resp. *finite at* x) if both $\mathcal{V}$ and $\mathcal{H}$ are locally finite (resp. finite at x). If Σ is a system on X and $x \in X$, then $\Sigma_x = (\mathcal{V}_x,\mathcal{H}_x)$ is the *local system at* x (associated with Σ).

Let $x \in X$. We say that a dynamics $\mathcal{V}$ (resp. observation structure $\mathcal{H}$) is *locally generated at* x if there is an open neighborhood U of x such that for every $y \in U$, $\mathcal{V}_y = \Gamma(U,\mathcal{V})_y$ (resp. $\mathcal{H}_y = \Gamma(U,\mathcal{H})_y$). Then U is called a *distinguished neighborhood of* x (corresponding to $\mathcal{V}$ or $\mathcal{H}$). A system $(\mathcal{V},\mathcal{H})$ is *locally generated at* x if both $\mathcal{V}$ and $\mathcal{H}$ are locally generated at x.

A system Σ is called *global* if for every $x \in X$, $\mathcal{V}_x = \Gamma(X,\mathcal{V})_x$ and $\mathcal{H}_x = \Gamma(X,\mathcal{H})_x$. Thus one can say that Σ is locally generated at x if Σ is locally global at x, i.e., is global when restricted to a distinguished neighborhood of this point. In particular, every global system is locally generated.

5 Reachability Properties

By a *reachable set* of the dynamics $\mathcal{V}$ at $x_0 \in X$ we mean the set of all points $\gamma(t)$ for all trajectories γ of $\mathcal{V}$ that start at x_0 and all $t \geq 0$ for which γ is defined. We denote this set by $R(\mathcal{V}, x_0)$, or simply $R(x_0)$ if $\mathcal{V}$ is fixed.

We say that $\mathcal{V}$ is *accessible at* x_0 if $R(x_0)$ has the non-empty interior in X. Let U be an open subset of X. By $\mathcal{V}_{|U}$ we denote the dynamics $\mathcal{V}$ restricted to the set U. We say that $\mathcal{V}$ is *locally accessible at* x_0 if for every neighborhood U of x_0, the dynamics $\mathcal{V}_{|U}$ is accessible at x_0, and is *locally reachable at* x_0 if for every neighborhood U of x_0, $R(\mathcal{V}_{|U}, x_0)$ contains a neighborhood of x_0. Observe that the restriction to U means that the trajectories of the dynamics cannot leave U.

In other words, $\mathcal{V}$ is locally accessible at x_0 if for every neighborhood U of x_0, the germ of $R(\mathcal{V}_{|U}, x_0)$ at x_0 has the non-empty interior, and is locally reachable if for every U this germ is full, i.e., it is the germ of a neighborhood of x_0. (The interior of a set-germ A at x is defined as the germ at x of the interior of a representative of A.) Obviously, the latter property implies the former one.

Let $\mathcal{L}(\mathcal{V})$ denote the sheaf of Lie algebras generated by the dynamics. This means that the stalk $\mathcal{L}(\mathcal{V})_x$ is a Lie algebra generated by the germs of the vector fields from $\mathcal{V}_x$. If $\mathcal{V}$ is fixed, we skip it in the above notation. Note that $\mathcal{L}(x)$, i.e., $\mathcal{L}$ evaluated at x, is a subspace of $T^i_x X$ for every $x \in X$.

The following theorem is an immediate generalization of results of Krener [20], and Sussmann and Jurdjevic [32]. Observe however the assumptions about regularity and local generatedness.

Theorem 5.1 *If x_0 is a regular point of X and $\mathcal{L}(x_0) = T_{x_0} X$, then $\mathcal{V}$ is locally accessible at x_0.*

If $\mathcal{V}$ is analytic and locally generated then also the converse is true. □

If the point x_0 is not regular then local accessibility at x_0 cannot be achieved, since locally trajectories are confined to a subset of X. Let U be an open subset of X and let $\mathcal{O}(U, x_0)$ denote the U-orbit at x_0 of the family $\mathcal{V}^i$ of all locally integrable vector fields on X restricted to U. It consists of all those points $x \in U$ that can be joined with x_0 by trajectories of $\mathcal{V}^i$ that do not leave U. The U-orbit is a C^N manifold (see e.g., [28]). X-orbit at x_0 is denoted by $\mathcal{O}(x_0)$ and called just orbit. Obviously, for any dynamics $\mathcal{V}$ on X, $R(\mathcal{V}_{|U}, x_0)$ is a subset of $\mathcal{O}(U, x_0)$.

We say that $\mathcal{V}$ is *structurally accessible at* x_0 if $R(x_0)$ has non-empty interior in $\mathcal{O}(x_0)$, and it is *structurally locally accessible at* x_0 if for every neighborhood U of x_0, the reachable set $R(\mathcal{V}_{|U}, x_0)$ has nonempty interior in $\mathcal{O}(U, x_0)$. Similarly, $\mathcal{V}$ is *structurally locally reachable at* x_0 if for every open neighborhood of x_0, $R(\mathcal{V}_{|U}, x_0)$ contains an open neighborhood U of x_0 in $\mathcal{O}(U, x_0)$.

Theorem 5.2 *If $\mathcal{L}(x_0) = T^i_{x_0}X$ then $\mathcal{V}$ is structurally locally accessible at x_0.*

If $\mathcal{V}$ is analytic and locally generated, then also the converse is true.

Proof: After reducing to the orbit we are in the classical setting of [20] and [31]. The only difference is that the vector fields now are not globally defined. But if we assume that $\mathcal{L}(x_0) = T^i_{x_0}X$ then we have enough vector fields at x_0 to construct a set with non-empty interior in $\mathcal{O}(U, x_0)$. On the other hand, proving the converse, we assume local generatedness, which makes the dynamics locally global. □

Example 5.3 Let X be $\mathbb{R}^2$ and let $\mathcal{V}_x = \{\partial_1, \partial_2\}$ for $x_1 > 0$ and $\mathcal{V}_x = \{\partial_1\}$ for $x_1 \leq 0$, where $\partial_i = \partial/\partial x_i$ and $x = (x_1, x_2)$. Take $x_0 = 0$. Then $\mathcal{V}$ is locally accessible at 0, but $\mathcal{L}(0) = \text{span}\{\partial_1\} \neq T_0X$. Observe that $\mathcal{V}$ is not locally generated at 0. □

6 Integrability of Distributions

When the system is not accessible we want to restrict it to a space (manifold) on which it has this property. In other words, we are interested in restricting our system to the integral manifold of $\mathcal{L}$ passing through a distinguished (initial) point. However such manifolds not always exist. They do if $\mathcal{L}$ is integrable. Integrability means that for every $x_0 \in X$ there is a manifold N immersed in X such that for every $y \in N$, $T_yN = \mathcal{L}(y)$. The definition is the same as for globally defined vector fields, but, as we shall see, there are some differences in their characterizations.

One might ask why we restrict the system to the integral manifold, which is a regular object, and not to the reachable set — a wilder space, just right for our approach. One reason is tradition; we do not want to change everything. Second, we would get something of little interest: every system may be restricted to its reachable set (from a point) on which it is reachable. One might like this, but the real progress could be made rather in analysis of the reachable set. And the third reason is complications: the reachable set need not be locally compact. The theory of locally integrable vector fields on such spaces is not yet fully developed. Reachability (global or local) corresponds to forward trajectories, defined for positive times. This leads to a concept of local half-integrability, the future topic of our research (see the last section). Finally, the integral manifolds are disjoined (or equal). This is not true for reachable sets corresponding to different points.

When $\mathcal{L}$ is integrable, the state space X may be decomposed into disjoint integral manifolds such that the dimension of $\mathcal{L}(x)$ is equal to the dimension of the manifold passing through x. This means that the system restricted to such a manifold is locally accessible at each point. This leads to a natural decomposition of the system on a singular space X into a family of locally accessible systems on manifolds. Thus, the integrability of $\mathcal{L}$ is crucial in this decomposition.

Below we summarize the results on integrability. We do this only for the reason of completeness of our paper, so we do not intend to give here all the details, examples and comments that one would like to see. Those can be found in [7], where the global case was discussed, and in original papers, especially in [29], where many examples and counter examples were presented. We state the results for the sheaf of Lie algebras $\mathcal{L}$ defined by a dynamics $\mathcal{V}$.

Let us first recall finite generatedness conditions that appeared in the literature. Some of them have been defined only for global vector fields, so we extend them to partially defined ones, using the language of sheaves.

The sheaf $\mathcal{L}$ is *locally finitely generated* at $x \in X$ if there is an open neighborhood U of x and vector fields $V_1, \ldots, V_k \in\ , (U, \mathcal{L})$ such that for every $y \in U$, $\mathcal{L}_y$ is contained in the module over $\mathcal{D}^N_y$ generated by the germs of $V_1, \cdots, V_k$. This condition may be expressed in a more global form. Namely $\mathcal{L}$ is locally finitely generated at x iff there is a neighborhood U of x and vector fields $V_1, \ldots, V_k \in\ , (U, \mathcal{L})$ such that $, (U, \mathcal{L})$ is contained in a module over $, (U, \mathcal{D}^N)$ generated by $V_1, \ldots, V_k$ [19]. We say that $\mathcal{L}$ is locally finitely generated if it is locally finitely generated at x for every $x \in X$.

The sheaf $\mathcal{L}$ is *locally of finite type* if for every $x \in X$ there exist germs of vector fields $(V_1)_x, \ldots, (V_k)_x \in \mathcal{L}_x$ such that $\mathrm{span}\{V_1, \ldots, V_k\}(x) = \mathcal{L}(x)$ and for every $V_x \in \mathcal{L}_x$ there is a neighborhood U of x and functions $\varphi_{ij}, i, j = 1, \ldots, k$, in $, (U, \mathcal{D}^N)$ such that

$$[V, V_i](y) = \varphi_{i1}(y)V_1(y) + \cdots + \varphi_{ik}(y)V_k(y)$$

for all $y \in U$ and $i = 1, \ldots, k$ [21].

The sheaf $\mathcal{L}$ is *locally softly of finite type* if for every $x \in X$ there exist $(V_1)_x, \ldots, (V_k)_x \in \mathcal{L}_x$ such that $\mathrm{span}\{V_1, \ldots, V_k\}(x) = \mathcal{L}(x)$ and for every $V_x \in \mathcal{L}_x$ there is $\epsilon > 0$ and C^∞ functions α_{ij} defined on $(\Leftrightarrow\epsilon, \epsilon)$ such that

$$[V, V_i](\gamma^V_t(x)) = \alpha_{i1}(t)V_1(\gamma^V_t(x)) + \cdots + \alpha_{ik}(t)V_k(\gamma^V_t(x))$$

for $t \in (\Leftrightarrow\epsilon, \epsilon)$ and $i = 1, \ldots, k$ [31] (the name of the property is given by us).

The sheaf $\mathcal{L}$ is *pointwise finitely generated* if for every $x \in X$ the module over $\mathcal{D}^N_x$ generated by $\mathcal{L}_x$ is finitely generated. $\mathcal{L}$ is *locally weakly finitely generated* if for every $x \in X$ there exist a neighborhood U of x and vector fields $V_1, \ldots, V_k \in\ , (U, \mathcal{L})$ such that $\mathrm{span}\{V_1, \ldots, V_k\}(y) = \mathcal{L}(y)$ for all $y \in U$ [29].

The sheaf $\mathcal{L}$ is *topologically closed* if $\overline{\mathcal{L}} = \mathcal{L}$, where the bar denotes the closure in the Whitney topology (this topology is defined by uniform convergence of all partial derivatives on compact sets; a vector field is identified with an n-tuple of real functions).

Assume that the system is smooth and consider the following conditions for the sheaf $\mathcal{L}$:
a) $\mathcal{L}$ is pointwise finitely generated and topologically closed;
b) $\mathcal{L}$ is locally finitely generated;
c) $\mathcal{L}$ is pointwise finitely generated and locally weakly finitely generated;
d) $\mathcal{L}$ is integrable;
e) $\mathcal{L}$ is invariant under $\mathcal{L}$;
f) $\mathcal{L}$ is locally softly of finite type;
g) $\mathcal{L}$ is locally of finite type;
h) $\mathcal{L}$ is pointwise finitely generated.
Then the following implications hold:

Theorem 6.1 *a) ⇒b) ⇒c) ⇒d) ⇔ e) ⇒f) and c) ⇒h) ⇒g) ⇒f).*

Proof: The first three implications are proved in [29]. The equivalence of d) and e) is shown in [30, 31]. The implication e) ⇒f) is proved in [31], h) ⇒g) shown in [29] and g) ⇒f) shown in [31]. Some proofs should be translated into the language of sheaves. □

In general, none of the conditions f),g),h) implies d). If the dynamics are analytic, a sheaf version of the Nagano theorem (see [22]) holds [29].

Theorem 6.2 *If $\mathcal{L}$ is analytic and locally generated then it is integrable.* □

Local generatedness is essential as the following example shows.

Example 6.3 Let $X = \mathbb{R}$ and define $\mathcal{V}$ by $\mathcal{V}_x = \{(0\frac{d}{dx})_x\}$ for $x \leq 0$ and $\mathcal{V}_x = \{(0\frac{d}{dx})_x, (\frac{d}{dx})_x\}$ for $x > 0$. Then neither $\mathcal{V}$ nor $\mathcal{L} = \mathrm{span}\mathcal{V}$ is locally generated at 0. Observe that $\mathcal{L}$ is not integrable. □

7 Local Observability

Let $x_0 \in X$ and $\mathcal{V}$ be a dynamics on X. Consider now a finite number of germs of vector fields $V_1, \ldots, V_k \in \mathcal{V}_{x_0}$ and choose their representatives belonging to $\mathcal{V}$ (calling them $V_1, \ldots, V_k$). Let $\gamma_i(t, x)$ denote the trajectory of V_i starting at x at time t. For small nonnegative $t_1, \ldots, t_k$ we can define the following map:

$$\gamma(t_1, \ldots, t_k) = \gamma_k(t_k, \gamma_{k-1}(t_{k-1}, \ldots, \gamma_1(t_1, x_0)\ldots)).$$

The map γ is defined on an intersection of the positive orthant in $\mathbb{R}^k$ and an open neighborhood of 0. We call γ a *k-trajectory of $\mathcal{V}$ starting at x_0*.

Let $\Sigma = (\mathcal{V}, \mathcal{H})$ be a system on X. By the *k-behavior of Σ at x_0* we mean the set $B_k(\Sigma, x_0)$ of germs at 0 of all maps $h \circ \gamma$ where h is a representative of a germ from $\mathcal{H}_{x_0}$ and γ is a k-trajectory of $\mathcal{V}$ starting at x_0. The sequence $(B_k(\Sigma, x_0))_{k \in \mathbb{N}}$ is called the *local behavior of Σ at x_0* and denoted by $B(\Sigma, x_0)$. We skip Σ in the above notation if it is fixed.

Let x_1 and x_2 belong to X and Σ be a system on X. We say that x_1 and x_2 *are locally indistinguishable* if $B(x_1) = B(x_2)$, i.e., $B_k(x_1) = B_k(x_2)$ for $k \in \mathbb{N}$. If this is not the case we say that they are *locally distinguishable*.

The concept of local indistinguishability defined above differs from the classical indistinguishability in two aspects. First, we are satisfied with germs of the maps $h \circ \gamma$, which means that in order to check equality of behaviors, it is enough to check equality of such maps only for small $t_1, \ldots, t_k$. Another feature of our definition is that we do not name or numerate elements in $\mathcal{V}_x$ and $\mathcal{H}_x$. This means that we do not have any correspondence between observation functions used at x_1 and observation functions used at x_2. This also concerns the vector fields of local dynamics at x_1 and x_2. Classically, when all vector fields and observation functions are globally defined on X, we still know locally which germ in $\mathcal{H}_x$ or in $\mathcal{V}_x$ corresponds to which global function or vector field. Thus we have then a natural correspondence between elements of $\mathcal{V}_x$ and $\mathcal{V}_y$, and similarly, between elements of $\mathcal{H}_x$ and $\mathcal{H}_y$. If the global system is analytic, then those correspondences are bijective. Obviously, if the system is global and x_1, x_2 are classically indistinguishable in arbitrarily short time, then they are locally indistinguishable in our sense. The converse is not true.

Example 7.1 Let $X = \mathbb{R}$, $\mathcal{V}_x = \{V_{1x}, V_{2x}\}$, $V_1 = -\frac{d}{dx}$, $V_2 = \frac{d}{dx}$ and $\mathcal{H}_x = \{x^2\}$ at each $x \in X$. Then $(\mathcal{V}, \mathcal{H})$ is a global system on X. Take $x_1 = -1$ and $x_2 = 1$. Let γ_j^i denote the trajectory of V_j starting at x_i. Then $\gamma_j^i(t) = (-1)^i + (-1)^j t$ for all $t > 0$. Denote the only observation function by h. Then $h \circ \gamma_1^1 \neq h \circ \gamma_1^2$ and that is enough for classical distinguishability of x_1 and x_2. On the other hand, $h \circ \gamma_j^i = h \circ \gamma_i^j$ for $i, j = 1, 2$, so $B_1(x_1) = B_1(x_2)$. Similarly one shows that $B_k(x_1) = B_k(x_2)$ for all $k \in \mathbb{N}$, which means that x_1 and x_2 are locally indistinguishable. □

Example 7.2 Let $X = \mathbb{R}$, $\mathcal{V}_x = \{x\frac{d}{dx}\}$ and $\mathcal{H}_x = \{-x, x\}$. Obviously, any two points $x_1, x_2 \in X$ are classically distinguishable. But for $x_1 = -1$ and $x_2 = 1$,

$$B_1(x_1) = \{\epsilon^t, -\epsilon^t\} = \{-\epsilon^t, \epsilon^t\} = B_1(x_2),$$

so x_1, x_2 are locally indistinguishable, since the local behaviors consist only of B_1. □

In both examples above a simple symmetry was used to show that our concept is weaker than the classical one. It is an interesting problem to establish conditions under which the concepts coincide. They coincide at least in the trivial case where the dynamic and the observation structures are just one-element sets. The system is global in this case, and each local behavior contains only one element.

We say that the system Σ is *locally observable at* $x_0 \in X$ if there is an open neighborhood U of x_0, such that for every $x \in U$, x and x_0 are locally distinguishable.

In what follows we are going to give a criterion of local observability which extends that of [3] (see also [4] for the smooth case). The reader can find there examples and comments that show the importance of real radicals used in the criterion.

Let $\mathcal{S}(\Sigma)$ denote a subsheaf of $\mathcal{D}^N$ such that for each $x \in X$, $\mathcal{S}(\Sigma)_x$ is the algebra over $\mathbb{R}$ generated by the germs at x of the form

$$L_{V_k} \dots L_{V_1} h, \tag{4}$$

where $h \in \mathcal{H}_x$, $k \geq 0$, $V_i \in \mathcal{V}_x$, and L_V means the Lie derivative with respect to (the germ of) the vector field V. It is clear that $\mathcal{S}(\Sigma)$, denoted shortly by $\mathcal{S}$, is the smallest algebra-subsheaf of $\mathcal{D}^N$ that contains $\mathcal{H}$ and is closed with respect to the Lie derivatives associated with the vector fields of $\mathcal{V}$.

Now let I_x be the ideal in $\mathcal{D}_x^N$ generated by all those elements of $\mathcal{S}_x$ which vanish at x. Obviously, I_x is contained in the maximal ideal m_x of D_x^N. Let $\sqrt[\mathbb{R}]{I_x}$ denote the real radical of I_x in $\mathcal{D}_x^N$ (see [3, 9, 26]). By definition, $\sqrt[\mathbb{R}]{I_x}$ consists of all elements $a \in \mathcal{D}_x^N$ for which there is $m > 0$, $k \geq 0$ and elements $b_1, \dots, b_k \in \mathcal{D}_x^N$, such that

$$a^{2m} + b_1^2 + \dots + b_k^2 \in I_x.$$

The reader might want to check [3, 4] in order to learn more about computation of real radicals.

Theorem 7.3 *Assume that* Σ *is analytic (on an analytic set* X*), finite and locally generated at* x_0*. Then,* Σ *is locally observable at* x_0 *iff* $\sqrt[\mathbb{R}]{I_{x_0}} = m_{x_0}$.

In order to prove the above theorem we shall first establish a few important lemmas.

Lemma 7.4 *Assume that* Σ *is finite and locally generated at* x_0 *and let* U *be a distinguished neighborhood of* x_0*. Then the following conditions are equivalent:*
a) for every neighborhood W *of* x_0 *there is* $x \in W$ *such that* $B(x) = B(x_0)$*;*
b) for every neighborhood W *of* x_0 *contained in* U *there is* $x \in W$ *such that*

for every $h \in$, $(U,\mathcal{H})$*, every* $k \in \mathbb{N}$*, and every vector fields* $V_1,\ldots,V_k \in$, $(U,\mathcal{V})$*, the germs at* 0 *of* $h \circ \gamma^x$ *and* $h \circ \gamma^{x_0}$ *are equal, where* γ^x *and* γ^{x_0} *denote the* k*-trajectories corresponding to* $V_1,\ldots,V_k$ *and starting at* x *and* x_0 *respectively.*

Proof b) $\Rightarrow$ a) is obvious since behaviors at x and x_0 consist exactly of the elements described in b).
a) $\Rightarrow$ b) Since we have finite numbers of vector fields in local dynamics on U and finite numbers of output functions, k-behaviors $B_k(x)$ are finite for all $x \in U$. Let for $x \in U$

$$B_k(x) = \{h_i \circ \gamma_j^x\}_{i=1,\ldots r, j=1,\ldots m(k)}$$

where k-trajectories and output functions are numbered in some fixed way independent of x. It may happen that some elements in $B_k(x)$ encoded with different indices are equal. We assume a) and want to show that arbitrarily close to x_0 there is x such that $h_i \circ \gamma_j^x = h_i \circ \gamma_j^{x_0}$ for all i, j. Suppose that for some i, j this equality does not hold. Since arbitrarily close to x_0 there is x such that $B(x) = B(x_0)$, and because the behaviors are finite, there is a sequence x_n such that $x_n \to x_0$ when $n \to \infty$, and for all $n \in \mathbb{N}$, $h_i \circ \gamma_j^{x_n} = h_m \circ \gamma_s^{x_0}$ for some m and s. From the continuous dependence of k-trajectories on the initial condition, we get $h_i \circ \gamma_j^{x_0} = h_m \circ \gamma_s^{x_0}$, so also $h_i \circ \gamma_j^{x_n} = h_i \circ \gamma_j^{x_0}$. □

The proof of Lemma 7.4 depends on the finiteness of Σ_{x_0}. It is not clear whether the lemma holds without this assumption. Local generatedness is of course essential for the formulation of the lemma.

Lemma 7.5 *Assume that* Σ *is global when restricted to an open set* U *and* $x_1, x_2 \in U$*. Let* γ^{x_1} *and* γ^{x_2} *denote* k*-trajectories starting from* x_1 *and* x_2*, respectively, and corresponding to the same sequence of vector fields from the dynamics. Then the following implication holds:*

If $(h \circ \gamma^{x_1})_{x_1} = (h \circ \gamma^{x_2})_{x_2}$ *for every observation function* h*, every* k*-trajectory and every* k*, then* $L_{V_s} \ldots L_{V_1} h(x_1) = L_{V_s} \ldots L_{V_1} h(x_2)$ *for every observation function* h *and all vector fields* $V_1,\ldots,V_s$ *in the dynamics.*
Moreover, if Σ *is analytic then the converse is also true.*

Proof: The proof is essentially the same as in [18] and [10]. □

Let X be an analytic set (i.e., the functions that define it are analytic). For an ideal $J \subset \mathcal{D}_x^\omega(X)$, let $Z(J)$ be (the germ at x of) the zero set of J in X. Since J is finitely generated, $Z(J)$ is well defined (see [16]). Let $I(Z(J))$ be the ideal in $\mathcal{D}_x^\omega(X)$ of all germs of analytic functions that vanish on $Z(J)$.

Lemma 7.6 *Assume that Σ is an analytic system defined on an analytic set X. If Σ is locally generated at x_0, then the following are equivalent:*
a) $Z(I_{x_0}) \neq \{x_0\}$;
b) arbitrarily close to x_0 there is x such that $\varphi(x_0) = \varphi(x)$ for every $\varphi \in \mathcal{S}_{x_0}$.

Proof: a) holds iff arbitrarily close to x_0 there is x such that all representatives of germs $\varphi \in I_{x_0}$ (all defined in some neighborhood of x_0) are 0 at x iff all representatives of germs $\varphi \in \mathcal{S}_{x_0}$ take on the same values at x_0 and this x iff b) holds. □

Lemma 7.7 *If X is an analytic set and J is an ideal in $\mathcal{D}^{\omega}_x(X)$ then $I(Z(J)) = \sqrt[\mathbb{R}]{J}$.*

Proof: This is a simple extension of the Risler result [26]. □

Proof of Theorem 7.3:
Suppose that Σ is not locally observable at x_0. This is equivalent to the fact that arbitrarily close to x_0 there is an x such that $B(x) = B(x_0)$. By Lemmas 7.4, 7.5 and 7.6 this is equivalent to the fact that $Z(I_{x_0})$ is different from $\{x_0\}$. And this, by Lemma 7.7, is equivalent to $I(Z(I_{x_0})) = m_{x_0}$. □

Let us stress the importance of the proved result. Suppose that we have a standard global system on a manifold M. Let us construct the "localization" of this system at every point $x \in M$ by taking germs at x of vector fields that form the dynamics, and output functions. Thus we get a system in our sense. The ideal I_x and its real radical are the same for the original global system and its localization. Theorem 7.3 is a "generalization" of an earlier result in [3], where the same condition on the radical was proved to be equivalent to standard local observability. This means that under the assumptions of Theorem 7.3 both concepts of local observability coincide. This is quite surprising, for the definitions of indistinguishability for both cases are different and not equivalent.

8 Remarks on Future Work

There are two things we plan to work on. The first concerns spaces that are not locally compact. Such spaces may appear as reachable sets.

Example 8.1 Let X be a subset of $\mathbb{R}^2$ that consists of the line $x_2 = 0$ and the open half-plane $x_1 > 0$. Observe that X is not locally compact. Let $V = (-1, 0)$ be a vector field on X. Although local trajectories through each point of X do exist, they do not define a local flow in the neighborhood of 0. We call this phenomenon *weak local integrability* of the vector field V. (On locally compact spaces local integrability always implies existence of the local flow.) □

The second thing is half local integrability. This property is related to existence of local forward trajectories, i.e., defined for small positive times. We have three different propositions for such a concept and it is not clear at the moment which one will appear the most fruitful. The forward integrability is essential in control theory, where the direction of time is one of the basic things to take into account. We do not know yet which properties of locally integrable vector fields can be transferred to local half integrability.

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Communicated by Michel Fliess