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Optimality Conditions for Dirichlet Boundary Control Problems of Parabolic Type*

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Abstract

This paper is devoted to the study of finite horizon optimal control problems with Dirichlet boundary control. The main difficulty to overcome is the discontinuity of the trajectories. We prove necessary and sufficient conditions for optimality of trajectory–control pairs. We formulate the necessary condition in terms of an Hamiltonian system for which we show an existence and uniqueness result. This yields a sufficient condition for optimality.

Key words: Boundary control, parabolic equations, Dirichlet boundary conditions, optimal controls, necessary and sufficient optimality conditions

AMS Subject Classifications: 49K20, 49J20, 49L10, 35K20

1 Introduction

In this paper we study the minimization of the functional

$$J(t_0, x_0; \gamma) = \int_{t_0}^{T} L(s, x(s; t_0, x_0, \gamma), \gamma(s))ds + \phi(x(T; t_0, x_0, \gamma)) \ , \qquad (1.1)$$

overall trajectory–control pairs $\{x, \gamma\}$, which are mild solutions of the infinite dimensional controlled system

$$\begin{cases} x'(t) + Ax(t) + F(x(t)) = A^{\beta} B\gamma(t) \\ x(t_0) = x_0 \end{cases} . \qquad (1.2)$$

The control space U and the state space X are two real Hilbert spaces. Here L and ϕ are real–valued smooth function, $A : D(A) \subset X \to X$ is a self–adjoint accretive operator, A^{β} is the β–fractional power of A, $\frac{3}{4} < \beta < 1$,

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B is a bounded linear operator, $F : X \to X$ is a Lipschitz continuous map and $\gamma : [t_0, T] \to U$ is a measurable function. In addition if we consider $B = \mathbf{D}_\beta = A^{1-\beta}\mathbf{D}$, where $\mathbf{D}$ is the Dirichlet map, then system (1.2) is the abstract version of a parabolic equation that is controlled through a Dirichlet type boundary condition, see Section 2 for further details.

The main aim of the present paper is to state necessary and sufficient optimality conditions for boundary control problems (1.1)–(1.2). In the later years, boundary control problems have been studied by many authors. The Linear Quadratic problem has been extensively treated, see for instance [5], [19], [20]. As for nonlinear boundary control problems one of the first case to be studied was the convex problem, see [1], [18], where it is considered a linear state equation and a convex running cost. As for general nonlinear boundary control problems, most of the results that are available in the literature are concerned with necessary optimality conditions, see e.g. [13], [14] and [25]. The Dynamic Programming approach to nonlinear boundary control problems is more recent and uses viscosity solutions, see [9], [10] and [11]. We refer to [17] for second order sufficient conditions for boundary control problems.

In this paper, as it is done in [12] for the Neumann boundary control problem, the running cost L can be unbounded if we assume a coercivity condition of the form

$$\exists \lambda_0 > 0, \lambda_1 \in \mathbb{R} \ : \ L(t,x,\gamma) \geq \lambda_0 |\gamma|^2 + \lambda_1, \forall t \in [0,T], \gamma \in U \ .$$

We consider the value function associated to the optimal control problem (1.1)–(1.2)

$$v(t_0,x_0) = \inf_{\gamma(t) \in U} \left\{ \int_{t_0}^{T} L(s, x(s;t_0,x_0,\gamma), \gamma(s)) ds + \phi(x(T;t_0,x_0,\gamma)) \right\} \ . \tag{1.3}$$

A control $\overline{\gamma}(\cdot)$ is said to be optimal if the infimum in the above equation is attained at $\overline{\gamma}(\cdot)$. The presence of the unbounded operator A^β acting on $B\gamma$ in the state equation, causes, in general, the discontinuity of the trajectories. In order to avoid this difficulty we follow the reasoning of [12] to prove a result of existence and boundedness of optimal controls. We prove more precisely that an optimal control $\overline{\gamma}$ satisfies:

$$|\overline{\gamma}(t)| \leq \frac{C}{(T-t)^{\frac{1}{2}-\varepsilon}} \tag{1.4}$$

for a suitable constant $C > 0$ and for $\varepsilon > 0$ small (see Proposition 2.5). This allows us to consider continuous mild solutions of equation (1.2) for $t < T$ and to prove that the value function enjoys the following regularity result (see [7] and Proposition 2.7). For every $R > 0, \alpha \in [0,1)$ there exists a constant $C_{R,\alpha} > 0$ such that

$$|v(t,x)-v(t,y)| \le C_{R,\alpha}|A^{-\alpha}(x-y)| \qquad \forall |x|,|y| \le R,\ t \in [0,T-\frac{1}{R}]. \quad (1.5)$$

Going back to our goal we use the previous results to state necessary conditions for optimality both in the classical version of the Pontryagin Maximum Principle, see Theorem 3.2, and in the Hamiltonian formulation, see Theorem 3.6. In the proof of the Maximum Principle we adapt the approach of [3] and [15]. Their results do not apply to problem (1.2)–(1.1) since they do not deal with the presence of the unbounded operator A^β. As in [6] for distributed control systems and as in [12] for Neumann boundary control problems, we derive that the superdifferential of the value function v along the optimal trajectory $\overline{x}(\cdot)$ includes the co–state associated to the optimal pair $(\overline{x}(\cdot), \overline{\gamma}(\cdot))$.

We obtain sufficient conditions for optimality adapting the techniques contained in [12] showing that the Hamiltonian system

$$\begin{cases} x'(t) = -Ax(t) - F(x(t)) - A^\beta D_p H(t, x(t), A^\beta p(t)) \\ \\ p'(t) = Ap(t) + [DF(x(t))]^* p(t) + D_x H(t, x(t), A^\beta p(t)) \end{cases}$$

with the initial–terminal condition

$$\begin{cases} x(t_0) = x_0 \\ p(T) = D\phi(x(T)) \end{cases} \quad (1.6)$$

has a solution which is an optimal trajectory. In the case when $F = 0$, substituting the initial–terminal condition above with particular initial–initial condition, we are able to prove that this solution is unique. Therefore, a stronger sufficient condition holds.

We briefly outline the paper. In §2 we recall the main assumptions on the data and the basic material on boundary control problems. In this Section we state some properties of the value function of problem (1.1)–(1.2). In §3 we derive necessary conditions for optimality through the Pontryagin Maximum Principle, see Theorem 3.2. Then we formulate its Hamiltonian version. In §4 we show an existence and uniqueness result for the Hamiltonian system (1.6) which is a sufficient condition for optimality, see Theorems 4.1 and 4.2.

2 Preliminaries

We begin by giving some notations. Let Y be a real Hilbert space. For $a < b \in \mathbb{R}$ we denote by $L^2(a,b;Y)$ the space of all square integrable functions $\gamma : [a,b] \to Y$. If Ω is a subset of another Hilbert space Z, then $C(\Omega;Y)$ will denote the set of all continuous functions $f:\Omega\to Y$. For $p\in[1,+\infty]$, $L^p(\Omega,Y)$ will denote the set of all functions $f:\Omega\to Y$ such that $\|f\|_Y^p$ is integrable on Ω. If $Y=\mathbb{R}$ we will write simply $C(\Omega)$ and $L^p(\Omega)$. Finally $\mathcal{L}(Z;Y)$ will denote the space of all bounded linear operators $T:Z\to Y$.

Let X be a real Hilbert space with norm $|\cdot|$ and scalar product $<\cdot,\cdot>$ and let U be another real Hilbert space. Let $x_0\in X$, $T>0$, $t_0\in[0,T)$, $\gamma\in L^2(t_0,T;U)$ and consider the infinite dimensional controlled system

$$\begin{cases} x'(t)+Ax(t)+F(x(t))=A^\beta B\gamma(t) \\ x(t_0)=x_0 \end{cases} . \tag{2.1}$$

In (2.1), A^β denotes the fractional powers of the operator A, see [23]. In the sequel we assume

$$\begin{array}{ll} (i) & A:D(A)\subset X\to X \text{ is a closed linear operator} \\ & \text{such that } A=A^* \text{ and } <Ax,x>\geq\omega|x|^2 \\ & \text{for some } \omega>0 \text{ and all } x\in D(A); \\ (ii) & \text{the inclusion } D(A)\subset X \text{ is dense and compact ;} \\ (iii) & F:X\to X\ ,\ |F(x)-F(y)|\leq K_F|x-y|\ ,\ \forall x,y\in X \\ & \text{for some } K_F>0; \\ (iv) & \beta\in\left(\frac{3}{4},1\right); \\ (v) & B\in\mathcal{L}(U;D(A^\rho)) \text{ for some } \rho>0. \end{array} \tag{2.2}$$

Remark 2.1

(i) We note that (i) and (ii) imply that $-A$ is the infinitesimal generator of an analytic semigroup satisfying $||e^{-tA}||\leq e^{-\omega t}$ for some $\omega>0$ and all $t\geq 0$.

(ii) Assumption (iii) allows us to treat the case of linear continuous perturbations of A and the case of Nemitski operators associated to Lipschitz continuous functions.

(iii) Assumption (iv) is necessary in order to consider Dirichlet parabolic boundary control problems. In fact it could be enough to take $\beta<1$.

(iv) Hypothesis (v) can be replaced by the weaker one:

$$(v)\text{-bis} \qquad B\in\mathcal{L}(U;X).$$

All the results stated in this paper remain true with simple modifications. Hypothesis (v) allows us to simplify the exposition. Moreover it is verified in our motivating example, as we are going to see.

We recall two useful estimates related to the analyticity of the semigroup e^{-tA}. For every $\theta \in [0,1]$ there exists a constant $M_\theta > 0$ such that

$$|A^\theta e^{-tA}x| \leq \frac{M_\theta}{t^\theta}|x|, \quad \forall t > 0, \forall x \in X. \tag{2.3}$$

Moreover, let $\eta \in (0,1]$ and $\alpha \in (0,\eta)$. Then, a well–known interpolation inequality, see e.g. [23], states that for every $\sigma > 0$ there exists $C_\sigma > 0$ such that

$$|A^\alpha x| \leq \sigma |A^\eta x| + C_\sigma |x|, \quad \forall x \in D(A^\eta). \tag{2.4}$$

System (2.1) is important in applications since it can be seen as the abstract formulation of the following. parabolic partial differential equation controlled by a Dirichlet datum at the boundary

$$\begin{cases} \dfrac{\partial x}{\partial t}(t,\xi) = \Delta_\xi x(t,\xi) + f(x(t,\xi)) & \text{in } (t_0,T)\times\Omega \\ x(t_0,\xi) = x_0(\xi) & \text{on } \Omega \\ x(t,\xi) = \gamma(t,\xi) & \text{on } (t_0,T)\times\partial\Omega \end{cases} \tag{2.5}$$

where $\Omega \subset \mathbb{R}^n$ is open and bounded with a smooth boundary $\partial\Omega$, $T > 0$, $t_0 \in [0,T]$. Moreover $\Delta_\xi = \sum_{j=1}^N \frac{\partial^2 x}{\partial^2 \xi_j}$ is the Laplace operator, $x_0 \in L^2(\Omega)$, $\gamma \in L^2(t_0,T;L^2(\partial\Omega))$ and $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous. See [5] for further details.

Going back to equation (2.1), we observe that it makes sense only in integral form as follows

$$x(t) = e^{-(t-t_0)A}x_0 - \int_{t_0}^t e^{-(t-s)A}F(x(s))ds + A^\beta \int_{t_0}^t e^{-(t-s)A}B\gamma(s)ds\ . \tag{2.6}$$

We say that x is a mild solution of (2.1) if $x \in L^2(t_0,T;X)$ and it is a solution of the above integral equation. We denote such a solution by $x(\cdot;t_0,x_0,\gamma)$.

The following proposition studies the regularity properties of the solution of equation (2.6).

Proposition 2.2 *Assume that* (2.2) *holds. Fix* $0 \leq t_0 < T$ *and let* $\gamma : [t_0,T] \to U$. *Then for any* $x_0 \in X$ *there exists a unique solution*

$$x \in L^2(t_0,T;D(A^{1-\beta+\rho})) \ \text{ where } \rho \text{ is given by (2.2)-(v)} \tag{2.7}$$

such that

$$A^{\frac{1}{2}-\beta}x \in C(t_0,T;X)), \tag{2.8}$$

and

$$|A^{\frac{1}{2}-\beta}x(t)| \le C_0[|x_0| + |F(0)| + ||\gamma||_{L^2(t_0,T;U)}] \qquad \forall t \in [t_0,T] \tag{2.9}$$

for some $C_0 > 0$.

Moreover, if $\gamma(\cdot)$ *is bounded*

$$x \in C([t_0,T];X)). \tag{2.10}$$

Finally, if $x_0 \in D(A^{1-\beta})$ *and* $\gamma(\cdot)$ *is bounded, then*

$$x \in C([t_0,T];D(A^{1-\beta})). \tag{2.11}$$

Proof: We sketch the proof for the reader's convenience. First we focus our attention on the term

$$g(t) = A^{\beta}\int_{t_0}^{t} e^{-(t-s)A}B\gamma(s)ds.$$

By standard arguments (see e.g. [5]) we can show that:

$$\gamma \in L^2(t_0,T;U) \Longrightarrow g \in L^2(t_0,T;D(A^{1-\beta+\rho})) \tag{2.12}$$

and

$$\gamma \in L^{\infty}(t_0,T;U) \Longrightarrow g \in C([t_0,T];D(A^{1-\beta+\rho-\varepsilon})) \tag{2.13}$$

for small $\varepsilon > 0$.

Let $(t_0,x_0) \in [0,T]\times X$ and consider the map $\Lambda : L^2(t_0,T;X) \to L^2(t_0,T;X)$ defined as follows

$$\Lambda x(t) = e^{-(t-t_0)A}x_0 - \int_{t_0}^{t} e^{-(t-s)A}F(x(s))ds + A^{\beta}\int_{t_0}^{t} e^{-(t-s)A}B\gamma(s)ds\ . \tag{2.14}$$

Recalling that the map $t \to e^{-(t-t_0)A}x_0$ belongs to $C([t_0,T];X)$ and to $L^2(t_0,T;D(A^{\frac{1}{2}}))$, see [22], by (2.12) and (2.13) we can see that, for small ρ

$$\begin{array}{l} \Lambda : L^2(t_0,T;D(A^{1-\beta+\rho})) \to L^2(t_0,T;D(A^{1-\beta+\rho})) \\ \Lambda : C([t_0,T];X) \to C([t_0,T];X)) \end{array} \tag{2.15}$$

and, if $x_0 \in D(A^{1-\beta})$, taking $\varepsilon < \rho$ in (2.13)

$$\Lambda : C([t_0,T];D(A^{1-\beta})) \to C([t_0,T];D(A^{1-\beta})). \tag{2.16}$$

The claims (2.7) (2.10) and (2.11) follow by (2.15), (2.16) and Contraction Mapping Principle. claim (2.8) follows by setting $z(t) = A^{\frac{1}{2}-\beta}x(t)$ and applying Contraction Mapping Principle to the equation for z. Finally estimate (2.9) follows by observing that for some $C_1 > 0$

$$|A^{\frac{1}{2}-\beta}x(t)| \leq C_1[|x_0| + |F(0)|] + K_F \int_{t_0}^{t} |x(s)|ds + |A^{\frac{1}{2}-\beta}g(t)|$$

and that, by a standard application of Gronwall inequality

$$\int_{t_0}^{t} |x(s)|ds \leq C_2[|x_0| + |F(0)|] + C_3||\gamma||_{L^2(t_0,T;U)}$$

for some $C_2, C_3 > 0$. ∎

Now let us consider the problem of minimizing the functional

$$J(t_0, x_0; \gamma) = \int_{t_0}^{T} L(t, x(t; t_0, x_0, \gamma), \gamma(t))dt + \phi(x(T; t_0, x_0, \gamma)) \qquad (2.17)$$

over all functions $\gamma \in L^2(t_0, T; U)$ (usually called controls), where the function $x(\cdot; t_0, x_0, \gamma)$ is the mild solution of (2.1). Here $L : [0,T] \times X \times U \to \mathbb{R}$ and $\phi : X \to \mathbb{R}$ are assumed to satisfy the following

$$\begin{array}{ll} (i) & L \in C([0,T] \times X \times U); \\ (ii) & \text{For some constant } C_L > 0 : \\ & |L(t,x,\gamma) - L(t,y,\gamma)| \leq C_L(1 + |x| + |y|)|x-y|, \\ & \forall t \in [0,T], \gamma \in U, |x|, |y| \in X; \\ (iii) & L(t,x,\cdot) \text{ is strictly convex;} \\ & \exists \lambda_0 > 0, \lambda_1 \in \mathbb{R} \ : \ L(t,x,\gamma) \geq \lambda_0|\gamma|^2 + \lambda_1 \\ & \forall (t,x) \in [0,T] \times X, \ \gamma \in U \\ & \text{and } L(t,x,\gamma) - L(t,x,0) \geq \lambda_0|\gamma|^2 + \lambda_1, ; \\ (iv) & \phi \text{ is bounded from below and } \ \forall R > 0 \ \exists C_{\phi,R} > 0 : \\ & |\phi(x) - \phi(y)| \leq C_{\phi,R}|A^{\frac{1}{2}-\beta}(x-y)|, \\ & \forall x, y \in X \text{ such that } |A^{\frac{1}{2}-\beta}x|, |A^{\frac{1}{2}-\beta}y| \leq R. \end{array} \qquad (2.18)$$

Remark 2.3

(i) Assumption (ii) allows us to treat the case of quadratic growth with respect to x of the running cost L. In particular the linear–quadratic case is included in the above framework.

(ii) The strict convexity and the first inequality in hypothesis (2.18)–(iii) guarantee the existence and uniqueness of the optimal control, see

Remark 2.4. The second inequality hypothesis (2.18)–(iii) guarantee an estimate of optimal controls, see Proposition 2.5 which will be crucial in the sequel. Assumption (iii) is needed since $\gamma \in L^2(t_0, T; U)$. If we take γ bounded then we can avoid this hypothesis.

(iii) Assumption (2.18)–(iv) is useful in order to have a meaningful terminal cost. In fact if ϕ satisfies (2.18)–(iv) then $\phi(x) = \psi(A^{\frac{1}{2}-\beta}x)$ for every $x \in X$ and a suitable function ψ Lipschitz continuous on bounded subsets of X. Note also that (2.18)-(iv) implies that at every point x where ϕ is differentiable we have (see [7])

$$D\phi(x) \in D(A^{\beta-\frac{1}{2}}).$$

We define the value function of problem (2.17)–(2.1) as

$$v(t_0, x_0) = \inf_{\gamma(t)\in U} \left\{ \int_{t_0}^{T} L(t, x(t; t_0, x_0, \gamma), \gamma(t))dt + \phi(x(T; t_0, x_0, \gamma)) \right\}. \tag{2.19}$$

A control $\gamma^*(t) \in U$ at which the infimum in (2.19) is attained, is said to be *optimal*, in other words if

$$v(t_0, x_0) = \int_{t_0}^{T} L(s, x(s; t_0, x_0, \gamma^*), \gamma^*((s))ds + \phi(x(T; t_0, x_0, \gamma^*)).$$

Remark 2.4 From assumptions (2.2) and (2.18) we derive, for every $t_0 \in [0, T]$ and $x_0 \in X$ the existence and uniqueness of the optimal control for problem (2.1)–(2.19) (see, e.g. [2]). Moreover the following property holds. Let $R > 0$. There exists $C_1(R) > 0$ such that if $t_0 \in [0, T]$, $|x_0| \le R$ and $\overline{\gamma}$ is the optimal control for $J(t_0, x_0; \cdot)$ then

$$\|\overline{\gamma}\|_{L^2(t_0,T;U)} \le C_1(R). \tag{2.20}$$

Indeed, by (2.18)-(iii) it follows

$$\int_{t_0}^{T} \lambda_0 |\overline{\gamma}(s)|^2 ds + \lambda_1(T - t_0) \le \int_{t_0}^{T} L(s, x(s), \overline{\gamma}(s))ds$$

so that

$$\lambda_0 \int_{t_0}^{T} |\overline{\gamma}(s)|^2 ds \le J(t_0, x_0; \overline{\gamma}) - \phi(x(T)) - \lambda_1(T - t_0) \le J(t_0, x_0; 0) + K$$

for a suitable constant K depending on the lower bound of ϕ, see assumption (2.18)-(iv). Finally,

$$J(t_0, x_0; 0) \le \int_{t_0}^{T} L(t, x(t; t_0, x_0, 0), 0)dt + \phi(x(T; t_0, x_0, 0))$$

where, by a simple application of Gronwall inequality we have

$$|x(t;t_0,x_0,0)| \le e^{K_F T}[|x_0| + |F(0)|(T-t)],$$

which yields, by applying (2.18)-(ii) and (iv)

$$J(t_0,x_0;0) \le C_1(|x_0|),$$

which gives the claim.

The following Proposition states the boundedness, on compact subsets of $[t_0,T)$, of the optimal control $\overline{\gamma}(\cdot)$.

Proposition 2.5 *Assume* (2.2) *and* (2.18)*. Then, for any $R > 0$ there exists a constant $M_R > 0$ such that, for any $t_0 \in [0,T]$, $x_0 \in X$, with $|x_0| \le R$ and any control $\gamma \in L^2(t_0,T;U)$, there exists $\overline{\gamma} \in L^2(t_0,T;U)$ satisfying*

$$\begin{aligned} &(i) \quad J(t_0,x_0;\overline{\gamma}) \le J(t_0,x_0;\gamma) \\ &(ii) \quad |\overline{\gamma}(t)| \le \frac{M_R}{(T-t)^{\frac{1}{2}-\rho}} \quad \forall t \in [t_0,T) \end{aligned} \tag{2.21}$$

where ρ is given in (2.2)*-(v).*

Proof: We follow the approach of [12].

Let $R > 0$ and let $t_0 \in [0,T]$, $|x_0| \le R$ and let $\gamma \in L^2(t_0,T;U)$. Due to assumption 2.18-(iii) and Remark 2.4 we can assume, without loss of generality, that $\|\gamma\|_{L^2(t_0,T;U)} \le C_1(R)$. Define, for any $n \in \mathbb{N}$,

$$I_n = \left\{ t \in [t_0,T] \ : \ |\gamma(t)| > \frac{n}{(T-t)^{\frac{1}{2}-\rho}} \right\}$$

and

$$\gamma_n(t) = \begin{cases} \gamma(t) & \text{if } t \notin I_n \\ 0 & \text{if } t \in I_n. \end{cases}.$$

Moreover, let us set

$$x(t) = x(t;t_0,x_0,\gamma)\ , \ \ x_n(t) = x(t;t_0,x_0,\gamma_n).$$

Then, denoting by $|I_n|$ the Lebesgue measure of I_n, we have

$$J(t_0,x_0;\gamma_n) = J(t_0,x_0;\gamma) + \int_{t_0}^{T} [L(t,x_n(t),\gamma_n(t)) - L(t,x_n(t),\gamma(t))]dt$$

$$+ \int_{t_0}^{T} [L(t,x_n(t),\gamma(t)) - L(t,x(t),\gamma(t))]dt + [\phi(x_n(T)) - \phi(x(T))]$$

$$\le J(t_0,x_0;\gamma)+|\lambda_1|\;|I_n|-\lambda_0\int_{I_n}|\gamma(r)|^2dr$$

$$+C_L\int_{t_0}^{T}(1+|x_n(t)|+|x(t)|)|x_n(t)-x(t)|dt+\tilde{C}|A^{\frac{1}{2}-\beta}(x_n(T)-x(T))|,$$

where $\tilde{C}=C_{\phi,\max\{|A^{\frac{1}{2}-\beta}x_n(T)|,|A^{\frac{1}{2}-\beta}x(T)|\}}$. Now we use (2.9), (2.20) to see that $\tilde{C}$ depends only on ϕ,R, so that we can write $\tilde{C}=\tilde{C}_{\phi,R}$. Then by Schwarz inequality we obtain

$$J(t_0,x_0;\gamma_n)-J(t_0,x_0;\gamma)\le|\lambda_1|\;|I_n|-\lambda_0\int_{I_n}|\gamma(r)|^2dr$$

$$+\tilde{C}_{\phi,R}|A^{\frac{1}{2}-\beta}(x_n(T)-x(T))|$$
$$+2C_L\left[\int_{t_0}^{T}(1+|x_n(t)|^2+|x(t)|^2)dt\right]^{\frac{1}{2}}\left[\int_{t_0}^{T}|x_n(t)-x(t)|^2dt\right]^{\frac{1}{2}}. \tag{2.22}$$

Now, recalling (2.6),

$$|x_n(s)-x(s)|$$
$$\le K_F\int_{t_0}^{s}|x_n(r)-x(r)|dr+\left|A^{\beta}\int_{t_0}^{s}e^{-(s-r)A}B\gamma(r)\chi_{I_n}(r)dr\right| \tag{2.23}$$

where χ denotes the characteristic function of the set I_n. Let

$$\eta(t)=\int_{t_0}^{t}|x_n(s)-x(s)|^2ds.$$

Then, taking the square of (2.23) and integrating,

$$\eta(t)\le K\left\{K_F\int_{t_0}^{t}\eta(s)ds+\int_{t_0}^{t}ds\left|\int_{t_0}^{s}\frac{M_{\beta-\rho}|\gamma(r)|}{(s-r)^{\beta-\rho}}\chi_{I_n}(r)dr\right|^2\right\}$$

$$\le KK_F\int_{t_0}^{t}\eta(s)ds+C(T,\beta,\rho)\int_{I_n}|\gamma(r)|^2dr$$

where K is a positive constant. Hence, by Gronwall's inequality,

$$\eta(t)\le e^{KK_FT}C(T,\beta,\rho)\int_{I_n}|\gamma(r)|^2dr=:C_2\int_{I_n}|\gamma(r)|^2dr. \tag{2.24}$$

From (2.6), (2.23), it follows that

$$|A^{\frac{1}{2}-\beta}(x_n(s)-x(s))|$$

$$\le K_F\int_{t_0}^{s}|x_n(r)-x(r)|dr+\left|A^{\frac{1}{2}}\int_{t_0}^{s}e^{-(s-r)A}B\gamma(r)\chi_{I_n}(r)dr\right|$$

so that, by estimating $\int_{t_0}^{s} |x_n(r) - x(r)| dr$ in the same way of η we obtain

$$|A^{\frac{1}{2}-\beta}(x_n(s)-x(s))| \leq C_3 \int_{I_n} |\gamma(r)| dr + C_4 \int_{I_n} \frac{|\gamma(r)|}{(s-r)^{\frac{1}{2}-\rho}} \chi_{I_n}(r) dr \quad (2.25)$$

for suitable constants $C_3, C_4 > 0$.

Now we estimate the state x. The same estimate will hold for x_n. By (2.6) it follows

$$|x(t)| \leq [|x_0| + |F(0)|(T - t_0)] + K_F \int_{t_0}^{t} |x(s)| ds + \left| A^{\beta} \int_{t_0}^{t} e^{-(t-s)A} B\gamma(s) ds \right|$$

Set $\delta(t) = \int_{t_0}^{t} |x(s)|^2 ds$. Applying the same technique used to estimate $\eta(t)$ we obtain

$$|\delta(t)| \leq C_5 \left[|x_0|^2 + |F(0)|^2 + \int_{t_0}^{t} |\gamma(s)|^2 ds \right] \leq C_6(R). \quad (2.26)$$

Putting (2.24), (2.25) (2.26) in (2.22) and recalling that $||\gamma||_{L^1(I_n)} \leq c||\gamma||_{L^2(I_n)}$ for some positive constant c, we obtain,

$$J(t_0, x_0; \gamma_n) - J(t_0, x_0; \gamma)$$

$$\leq |\lambda_1| \, |I_n| - \lambda_0 \int_{I_n} |\gamma(r)|^2 dr + C_7 \int_{I_n} \frac{|\gamma(r)|}{(T-r)^{\beta}} dr + C_8 \left[\int_{I_n} |\gamma(r)|^2 dr \right]^{\frac{1}{2}}. \quad (2.27)$$

Finally, we claim that the right–hand side of (2.27) is negative for sufficiently large n, which will yield the conclusion of the proof. Indeed,

$$|\lambda_1| \, |I_n| - \frac{1}{3}\lambda_0 \int_{I_n} |\gamma(r)|^2 dr \leq |\lambda_1| \, |I_n| - \frac{n^2 \lambda_0}{3T^{2\beta}} |I_n| < 0$$

provided n is large enough, say $n \geq n_1$. Furthermore,

$$C_7 \int_{I_n} \frac{|\gamma(r)|}{(T-r)^{\beta}} dr - \frac{1}{3}\lambda_0 \int_{I_n} |\gamma(r)|^2 dr$$

$$\leq C_7 \int_{I_n} \frac{|\gamma(r)|}{(T-r)^{\beta}} dr - \frac{n\lambda_0}{3} \int_{I_n} \frac{|\gamma(r)|}{(T-r)^{\beta}} dr < 0$$

if $n \geq n_2$. Finally

$$C_8 \left[\int_{I_n} |\gamma(r)|^2 dr \right]^{\frac{1}{2}} - \frac{1}{3}\lambda_0 \int_{I_n} |\gamma(r)|^2 dr$$

$$\leq \left(C_8 - \frac{n\lambda_0}{3T^\beta}\right)\left[\int_{I_n} |\gamma(r)|^2 dr\right]^{\frac{1}{2}} < 0.$$

The claim follows and the proof is complete. ∎

Remark 2.6 By Proposition 2.5 it follows that, if $\overline{\gamma}$ is optimal at $(t_0, x_0) \in [0,T] \times X$, then, for every $(s,\zeta) \in [t_0,T] \times X$ we have

$$x(t; s, \zeta, \overline{\gamma}) \in C([0,T]; X)$$

and, if $(s,\zeta) \in [t_0,T] \times D(A^{1-\beta})$,

$$x(t; s, \zeta, \overline{\gamma}) \in C([0,T]; D(A^{1-\beta})).$$

More generally, if $\gamma \in L^2(t_0,T;U)$ and $(s,\zeta) \in [t_0,T] \times X$, then, at every $t \in (s,T]$, t Lebesgue point of γ we have

$$x(t; s, \zeta, \gamma) \in D(A^{1-\beta}).$$

Moreover, $A^{1-\beta}x$ is continuous in t and, in particular, t is a Lebesgue point for x and $A^{1-\beta}x$. Indeed, if $t \in (s,T]$ is a Lebesgue point of γ we have that γ is bounded on a neighborhood of t so that

$$\left| A \int_s^t e^{-(t-r)A} B\gamma(r) dr \right|$$

$$\leq \left| A \int_s^{t-\varepsilon} e^{-(t-r)A} B\gamma(r) dr \right| + \left| A \int_{t-\varepsilon}^t e^{-(t-r)A} B\gamma(r) dr \right|$$

$$\leq \frac{M_1 - \rho}{\varepsilon^{1-\rho}} \|\gamma\|_{L^2(t_0,T;U)} + \int_{t-\varepsilon}^t \frac{M_1 - \rho}{(t-r)^{1-\rho}} \mathrm{esssup}_{r \in [t-\varepsilon, t]} |\gamma(r)| dr$$

from which the claim follows by standard arguments.

Let us now recall the definition of some generalized gradients which will be used in the sequel. Let O be an open subset of X. The superdifferential of a function $w : O \to \mathbb{R}$ at a point $x_0 \in R$ is the (possibly empty) set

$$D^+w(x_0) = \left\{ p \in X \ : \ \limsup_{x \to x_0} \frac{w(x) - w(x_0) - < p, x - x_0 >}{|x - x_0|} \leq 0 \right\}. \tag{2.28}$$

We denote by $D^*w(x_0)$ the set of all vectors $p \in X$ for which there exists a sequence $\{x_n\}$ of points of O such that

$$\begin{cases} (i) & x_n \to x_0 \text{ as } n \to +\infty \\ (ii) & w \text{ is Fréchet differentiable at } x_n, \forall n \\ (iii) & Dw(x_n) \rightharpoonup p \text{ as } n \to +\infty \end{cases} \tag{2.29}$$

If the function w is Lipschitz continuous in a neighborhood R_0 of x_0, then w is Fréchet differentiable on a dense subset of R_0 (see [24]). Hence, $D^*w(x_0) \neq \emptyset$.

Assuming (2.2) and (2.18) the value function v is Lipschitz continuous with respect to the negative fractional powers of A. This fact yields a crucial property of the superdifferential of v (see [7], [9], [11]), which is stated in the following proposition.

Proposition 2.7 *Assume* (2.2), (2.18)*. Then, the value function v defined in* (2.19) *is continuous in $[0,T] \times X$. Moreover, for every $R > \frac{1}{T}$ and $\theta \in [0,1)$ there exists a constant $C_{\theta R} > 0$ such that*

$$|v(t,x) - v(t,y)| \leq C_{\theta R}|A^{-\theta}(x-y)| \quad \forall t \in [0, T - \frac{1}{R}], \;\; |x|, |y| \leq R. \tag{2.30}$$

In particular v is sequentially weakly continuous in $[0,T) \times X$ and

$$D_x^+ v(t,x) \subset D(A^\theta) \qquad \forall \theta \in [0,1) \quad \textit{and for all } \;|[0,T) \times X. \tag{2.31}$$

Proof: Let $x, y \in X$, $t \in [0,T]$ and let $\overline{\gamma}$ be optimal for (t,x). We set $x(\cdot) = x(\cdot; t, x, \overline{\gamma})$ and $y(\cdot) = y(\cdot; t, y, \overline{\gamma})$. Then

$$\begin{aligned}
&v(t,x) - v(t,y)\\
&\leq \int_t^T [L(s, x(s), \overline{\gamma}(s)) - L(s(y(s), \overline{\gamma}(s)] + \phi(x(T)) - \phi(y(T))\\
&\leq \int_t^T C_L(1 + |x(s)| + |y(s)|)|x(s) - y(s)|ds + \tilde{C}|A^{\frac{1}{2}-\beta}(x(T) - y(T))|
\end{aligned} \tag{2.32}$$

where $\tilde{C} = C_{\phi, \max\{|A^{\frac{1}{2}-\beta}x(T)|, |A^{\frac{1}{2}-\beta}y(T)|\}}$. Now we estimate the state function using boundedness of optimal controls. Indeed, recalling (2.6) and (2.21)

$$\begin{aligned}
|x(s)| \leq |x| + |F(0)|(T-t) + K_F \int_t^s |x(\sigma)|d\sigma\\
+ \left| \int_t^s \frac{M_\beta}{(s-\sigma)^{\beta-\rho}} \frac{M_R}{(T-\sigma)^{\frac{1}{2}-\rho}} d\sigma \right|
\end{aligned}$$

so that, by Gronwall inequality

$$|x(s)| \leq e^{K_F T} \left[|x| + |F(0)|(T-t) + \frac{C_1(R))}{(T-s)^{\frac{1}{2}-\rho}} \right]. \tag{2.33}$$

Clearly, a similar estimate holds true for $y(\cdot)$.

Now writing x and y in mild form and subtracting we get

$$x(s)-y(s)=e^{-(s-t)A}(x-y)-\int_t^s e^{-(s-\sigma)A}[F(x(s))-F(y(s))]ds.$$

By (2.3) and Lipschitz continuity of F

$$|x(s)-y(s)|\leq \frac{M_\alpha}{(s-t)^\alpha}|A^{-\alpha}(x-y)|+K_F\int_t^s|x(\sigma)-y(\sigma)|d\sigma$$

and by applying Gronwall inequality as in the proof of previous proposition,

$$|x(s)-y(s)|\leq [\frac{C_2}{(s-t)^\alpha}+C_3]|A^{-\alpha}(x-y)| \tag{2.34}$$

for constants $C_2, C_3>0$. Putting estimates (2.33) and (2.34) in (2.32) we have, for $|x|,|y|\leq R$,

$$\begin{aligned} &v(t,x)-v(t,y)\\ &\leq C_L|A^{-\alpha}(x-y)|\cdot\\ &\int_t^T\left(1+C_4(R)\left[1+\frac{1}{(T-s)^{\frac{1}{2}-\rho}}\right]\right)\left[\frac{C_2}{(s-t)^\alpha}+C_3\right]ds\\ &+\tilde{C}|A^{\frac{1}{2}-\beta}x(T)-y(T)|. \end{aligned} \tag{2.35}$$

Now we recall that, if $|x|,|y|\leq R$ then, by (2.9) and Remark 2.4 we have

$$|A^{\frac{1}{2}-\beta}x(T)|,|A^{\frac{1}{2}-\beta}y(T)|\leq M(R),$$

so that $\tilde{C}=\tilde{C}_{\phi,R}$ as in the proof of Proposition 2.5. This yields together with (2.34)

$$v(t,x)-v(t,y)\leq C_{1,R}|A^{-\alpha}(x-y)|+C_{2,R}\left[\frac{K_1}{T-t)^\alpha}+K_2\right]|A^{-\alpha}(x-y)| \tag{2.36}$$

which yields (2.30). On the other hand (2.31) can be easily verified arguing as in [7]. ■

Now, by standard arguments, we verify that the value function v satisfies an inequality related to the following Hamilton–Jacobi equation

$$\begin{cases} -\dfrac{\partial v}{\partial t}(t,x)+H(t,x,A^\beta D_x v(t,x)) \\ \qquad\qquad +<A^{1-\beta}x+A^{-\beta}F(x),A^\beta D_x v(t,x)>=0 \\ v(T,x)=\phi(x) \end{cases} \tag{2.37}$$

where

$$H(t,x,p) = \sup_{\gamma\in U}\left[- < B\gamma, p > -L(t,x,\gamma)\right]. \tag{2.38}$$

Theorem 2.8 *Assume that* (2.2) *and* (2.18) *hold true. Then for every* $\varphi \in C([0,T]\times X)$ *we have,*

$$\begin{array}{ll} (i) & -\frac{\partial\varphi}{\partial t}(t,x) + H(t,x,A^{\beta}D_x\varphi(t,x)) \\ & + < A^{1-\beta}x + A^{-\beta}F(x), A^{\beta}D_x\varphi(t,x) > \leq 0 \\ & \textit{for all } (t,x)\in[0,T)\times D(A^{1-\beta}) \textit{ which are maximum points of} \\ & v-\varphi \textit{ at which } \varphi \textit{ is differentiable} \\ & \\ (ii) & \lim_{t\downarrow 0}\sup_{x\in X}[v(T-t,x)-\phi(e^{-tA}x)]^{+} = 0 \\ & \textit{where } a^{+} = \max\{a,0\}. \end{array} \tag{2.39}$$

Proof: Fix $(t_0,x_0)\in[0,T)\times D(A^{1-\beta})$ and a constant control $\gamma(\cdot)=\gamma$ in U. Set $x(t)=x(t;t_0,x_0,\gamma)$. Now suppose that there exists $\varphi\in C([0,T]\times X)$ differentiable in (t_0,x_0) such that

$$v(t_0,x_0)-\varphi(t_0,x_0) = \max\ (v-\varphi) \geq v(t,x(t))-\varphi(t,x(t)). \tag{2.40}$$

By the Dynamic Programming Principle , we have

$$v(t_0,x_0) \leq \int_{t_0}^{t} L(s,x(s),\gamma)ds + v(t,x(t)).$$

Therefore

$$\frac{\varphi(t_0,x_0)-\varphi(t,x(t))}{t-t_0} \leq \frac{v(t_0,x_0)-v(t,x(t))}{t-t_0} \leq \frac{1}{t-t_0}\int_{t_0}^{t} L(s,x(s),\gamma)ds. \tag{2.41}$$

Since φ is differentiable in (t_0,x_0) we have, by (2.40) and (2.31)

$$D_x\varphi(t_0,x_0) \in D_x^{+}v(t_0,x_0) \subset D(A^{\beta}), \tag{2.42}$$

so that

$$\frac{\varphi(t_0,x_0)-\varphi(t,x(t))}{t-t_0}$$

$$= -\frac{\partial\varphi}{\partial t}(t_0,x_0) - < A^{\beta}D_x\varphi(t_0,y_0), \frac{A^{-\beta}(x(t)-x_0)}{t-t_0} > +\omega(t-t_0). \tag{2.43}$$

Here, and in the sequel of the proof, we denote by $\omega(\cdot)$ a function such that $\omega(r) \downarrow 0$ as $r \downarrow 0$. Recalling from Proposition 2.2 that if $x_0 \in D(A^{1-\beta})$ and $\gamma(\cdot) = \gamma$ then $x \in C([0,T], D(A^{1-\beta}))$, we get

$$\frac{A^{-\beta}(x(t) - x_0)}{t - t_0} = -A^{1-\beta}x_0 - A^{-\beta}F(x_0) + B\gamma + \omega(t - t_0).$$

Substituting the above equality in (2.43) and recalling (2.41) we have

$$-\frac{\partial \varphi}{\partial t}(t_0, x_0) - < A^{\beta} D_x \varphi(t_0, x_0), -A^{1-\beta}x_0 - A^{-\beta}F(y_0) + B\gamma >$$
$$\leq L(t_0, x_0, \gamma) + \omega(t - t_0).$$

Therefore 2.39 (i) follows from the definition of the Hamiltonian H, letting $t \to t_0$ in the above estimate. We still have to prove 2.39 (ii). By definition of value function we have

$$v(T - t, x_0) = \inf_{\gamma(t) \in U} \left\{ \int_{T-t}^{T} L(s, x(s), \gamma(s)) ds + \phi(x(T)) \right\},$$

therefore, for any constant control $\gamma(\cdot) = \gamma$

$$v(T-t, x_0) - \phi(e^{-tA}x_0) \leq \int_{T-t}^{T} L(s, x(s), \gamma) ds + C|A^{\frac{1}{2}-\beta}(x(T) - e^{-tA}y_0)|, \tag{2.44}$$

where C is a positive constant depending on $|x_0|$. From (2.6) it follows that

$$x(T) = e^{-tA}x_0 - \int_{T-t}^{T} e^{-(T-s)A} F(x(s)) ds + A^{\beta} \int_{T-t}^{T} e^{-(T-s)A} B\gamma ds. \tag{2.45}$$

Substituting (2.45) in (2.44) we have

$$v(T-t, x_0) - \phi(e^{-tA}x_0) \leq \int_{T-t}^{T} L(s, x(s), \gamma) ds$$
$$+C \left| -A^{\frac{1}{2}-\beta} \int_{T-t}^{T} e^{-(T-s)A} F(x(s)) ds + A^{\frac{1}{2}} \int_{T-t}^{T} e^{-(T-s)A} B\gamma ds \right|.$$

Since L and F are continuous, as $t \to 0$ we conclude that

$$\limsup_{t \downarrow 0} \sup_{x_0 \in X} \left[v(T-t, x_0) - \phi(e^{-tA}x_0) \right]^{+} \leq 0, \tag{2.46}$$

which yields the conclusion. ∎

Remark 2.9 The above theorem can be used to define viscosity subsolutions of (2.37). Similarly we can define viscosity supersolutions. Therefore we can state an existence result for viscosity solutions of equation (2.37). In order to obtain uniqueness results the definition of viscosity solution has to be modified, see [11].

3 Necessary Conditions

In this section we derive necessary conditions for the problem of minimizing $J(t_0, x_0; \gamma)$ overall controls $\gamma \in L^2(t_0, T; U)$. Here J is defined in (2.17). In addition to hypotheses (2.2),(2.18), we will assume that

$$\begin{array}{ll} (i) & F \text{ is continuously Fréchet differentiable;} \\ (ii) & L \text{ is continuously Fréchet differentiable with respect to } x; \\ (iii) & \phi \text{ is continuously Fréchet differentiable and} \\ & D\phi(A^{\beta-\frac{1}{2}}\cdot) \text{ is continuous on X.} \end{array} \tag{3.1}$$

Remark 3.1 Notice that the above assumptions and (2.2)–(2.18) imply that DF is bounded on X and that $|D_x L(t, x, \gamma)| \leq 2C_L(1+|x|)$, $\forall t, x, \gamma \in [0, T] \times X \times U$.

Let $\gamma \in L^2(t_0, T; U), x(\cdot) = x(\cdot; t_0, x_0, \gamma)$ and $p_T = D\phi(x(T))$. We recall that the *co–state* associated to the triplet $\{\gamma, x, p_T\}$ is formally defined as the mild solution to the problem

$$\begin{cases} p'(t) = Ap(t) + [DF(x(t))]^* p(t) - D_x L(t, x(t), \gamma(t)) \ , \quad t \in [t_0, T) \\ p(T) = p_T, \end{cases} \tag{3.2}$$

which is expressed through the following integral equation

$$p(t) = e^{-(T-t)A} p_T + \int_t^T e^{-(T-s)A} [DF(x(s))]^* p(s) ds$$
$$+ \int_t^T e^{-(T-s)A} D_x L(s, x(s), \gamma(s)) ds.$$

We note that assumption (3.1) (iii) is necessary to have a meaningful terminal datum p_T. Now we can state the main result of this section.

Theorem 3.2 *Assume* (2.2), (2.18), (3.1). *Let* $\{\overline{\gamma}, \overline{x}\}$ *be an optimal pair for problem* (2.19) $-$ (2.1), *with starting point* $(t_0, x_0) \in [0, T] \times X$. *Moreover, set* $p_T = D\phi(\overline{x}(T))$ *and let* $\overline{p}$ *be the corresponding co–state. Then it satisfies the co–state inclusion*

$$\overline{p}(t) \in D_x^+ v(t, \overline{x}(t)) \tag{3.3}$$

for every $t \in [t_0, T]$ *and the Maximum Principle*

$$- < B\overline{\gamma}(t), A^{\beta}\overline{p}(t) > -L(t, \overline{x}(t), \overline{\gamma}(t)) = H(t, \overline{x}(t), A^{\beta}\overline{p}(t)) \qquad (3.4)$$

for a.e. $t \in [t_0, T]$ *Lebesgue point of* $\overline{\gamma}$*, where*

$$H(t, x, p) = \sup_{\gamma \in U}[- < B\gamma, p > -L(t, x, \gamma)]. \qquad (3.5)$$

We prove this result using the approach of [4] (see also [3] and [15]). We begin giving some preliminary results.

Lemma 3.3 *Let* $(t_0, x_0) \in [0, T] \times X$ *and consider the control problem* (2.1) *and* (2.19) *starting at* (t_0, x_0)*. Let* $\overline{\gamma}(\cdot)$ *be an optimal control for this problem and define the function* $W : [t_0, T] \times X \to \mathbb{R}$

$$W(s, \zeta) = \int_s^T L(r, x(r; s, \zeta, \overline{\gamma}), \overline{\gamma}(r))dr + \phi(x(T; s, \zeta, \overline{\gamma})). \qquad (3.6)$$

Then

(i) $\forall \zeta \in D(A^{1-\beta})$*,* $W(\cdot, \zeta)$ *is differentiable at every Lebesgue point of* $\overline{\gamma}$ *in* $[t_0, T]$*.*

(ii) $\forall s \in [t_0, T]$*,* $W(s, \cdot)$ *is continuously Fréchet differentiable on* X*.*

Proof: We follow the approach of [15]. Let $(s, \zeta) \in [t_0, T] \times X$ and let $x(t; s, \zeta, \overline{\gamma})$ be the mild solution of

$$x(t) = e^{-(t-s)A}\zeta - \int_s^t e^{-(t-r)A}F(x(r))dr + A^{\beta}\int_s^t e^{-(t-r)A}B\overline{\gamma}(r)dr\ .$$

Then, by the parameter dependent Contraction Mapping Principle (see [22]), applied to equation (2.14) it follows that $x(t; s, \zeta, \overline{\gamma})$ is Fréchet differentiable with respect to ζ. Moreover, see e.g. [7], if we set

$$\Psi(t) =< D_{\zeta}x(t; s, \zeta, \overline{\gamma}), z > \quad \text{for } z \in X. \qquad (3.7)$$

Then Ψ satisfies, in integral form, the following

$$\begin{cases} \Psi'(t) = -[A + DF(x(t; s, \zeta, \overline{\gamma})]\Psi(t) \\ \Psi(s) = z \end{cases} . \qquad (3.8)$$

By classical results, (see [5], [22]), we have that Ψ is the unique mild solution of (3.8) and

$$\Psi \in C([t_0, T]; X) \cap L^2(t_0, T; D(A^{\frac{1}{2}})) \cap L^1(t_0, T; D(A^{\beta})). \qquad (3.9)$$

Moreover, $|\Psi(t)| \leq M_1|z|$ where M_1 depends only on β, T and K_F. Exploiting (3.8) and setting $x(t) = x(t; s, \zeta, \overline{\gamma})$ then we can write, for $z \in X$

$$< D_\zeta W(s,\zeta), z >$$

$$= \int_s^T < D_x L(r, x(r), \overline{\gamma}(r)), \Psi(r) > dr + < D\phi(x(T)), \Psi(T) >. \tag{3.10}$$

Then, by regularity properties of Ψ, L and $D\phi$, (see (3.9), by Hypotheses (2.18), (3.1) and by Remark 3.1 and by estimate (2.33) we obtain

$$| < D_\zeta W(s,\zeta), z > | \leq K \left[\int_s^T C|\Psi(r)|(1 + |x(r)|)dr + |\Psi(T)| \right] \leq K M_1 |z| \tag{3.11}$$

where K does not depend on z. The above estimate yields the Gâteaux differentiability.

We now prove continuous Fréchet differentiability. Let $\zeta_n, \zeta_0 \subset X$ and let $\zeta_n \stackrel{n\to+\infty}{\longrightarrow} \zeta_0$. Then, setting

$$x_n(t) = x(t; s, \zeta_n, \overline{\gamma}), \qquad x_0(t) = x(t; s, \zeta_0, \overline{\gamma}),$$

we obtain

$$[x_n(t) - x_0(t)] = e^{-(t-s)A}[\zeta_n - \zeta_0] - \int_s^t e^{-(t-r)A}[F(x_n(r)) - F(x_0(r))]dr.$$

Applying Gronwall inequality we get

$$|x_n(T) - x_0(T)| \leq M|\zeta_n - \zeta_0|, \tag{3.12}$$

for some positive M. Now, define $\Psi_n(t) =< D_\zeta x_n(t), z >$ and $\Psi_0(t) =< D_\zeta x_0(t), z >$. By equation (3.8) we obtain

$$[\Psi_n(t) - \Psi_0(t)] = \int_s^t e^{-(t-r)A}[DF(x_0(r))\Psi_0(r) - DF(x_n(r))\Psi_n(r)]dr$$

so that, by Gronwall inequality

$$|\Psi_n(t) - \Psi_0(t)| \leq M_1 e^{K_F(t-s)}|z| \int_s^t |DF(x_0(r)) - DF(x_n(r))|dr. \tag{3.13}$$

Then we recall that, by (3.10), we have

$$< D_\zeta W(s,\zeta_n) - D_\zeta W(s,\zeta_0), z >$$

$$\leq \int_s^T [< D_x L(r, x_n(r), \overline{\gamma}(r)), \Psi_n(r) > - < D_x L(r, x_0(r), \overline{\gamma}(r)), \Psi_0(r) >] \, dr$$

$$+ [< D\phi(x_n(T)), \Psi_n(T) > - < D\phi(x_0(T)), \Psi_0(T) >] \,. \tag{3.14}$$

Putting estimates (3.12)–(3.13) in (3.14) we prove continuous Fréchet differentiability on X.

On the other hand, let $\zeta \in D(A^{1-\beta})$ and define

$$y(t;s,\zeta,\overline{\gamma}) = A^{-\beta}x(t;s,\zeta,\overline{\gamma}).$$

Then y satisfies

$$y(t) = e^{-(t-s)A}\zeta - \int_s^t e^{-(t-r)A}F(y(r))dr + A^{\beta}\int_s^t e^{-(t-r)A}B\overline{\gamma}(r)dr \ . \tag{3.15}$$

Then, by the parameter dependent Contraction Mapping Principle it follows that $y(t;s,\zeta,\overline{\gamma})$ is differentiable with respect to s. Setting

$$\Phi(t) = \partial_s y(t;s,\zeta,\overline{\gamma}), \tag{3.16}$$

we find that Φ satisfies in mild form

$$\begin{cases} \Phi'(t) = -[A + DF(x(t))]\Phi(t) \\ \Phi(s) = A^{1-\beta}\zeta + A^{-\beta}F(\zeta) - B\overline{\gamma}(s) = \Phi_s \end{cases} \tag{3.17}$$

where the initial condition of (3.17) is satisfied for a. e. $s \in [t_0, T]$ (i. e. at every Lebesgue point of $\overline{\gamma}$). By classical results we have that Φ the unique mild solution of (3.17) and

$$\Phi \in C([t_0,T];X) \cap L^2(t_0,T;D(A^{\frac{1}{2}})) \cap L^1(t_0,T;D(A^{\beta})).$$

Now we show that the scalar product $< D_xL(r,x(r),\overline{\gamma}(r)), A^{\beta}\Phi(r) >$ is integrable. In fact

$$\int_s^T < D_xL(r,x(r),\overline{\gamma}(r)), A^{\beta}\Phi(r) > dr \leq \int_s^T C(1+|x(r)|)|A^{\beta}\Phi(r)|dr.$$

Now recalling, from estimate (2.33) , that

$$|x(s)| \leq e^{K_F T}\left[|\zeta| + |F(0)|(T-t) + \frac{C_1(|\zeta|)}{(T-s)^{\frac{1}{2}-\rho}}\right]$$

and deriving that, for some positive C,

$$|A^{\beta}\Phi(r)| \leq C\left[\frac{|\Phi_s|}{(r-s)^{\beta}} + 1\right],$$

we can conclude the integrability of the term under consideration. Therefore, from (3.17) and assumption (3.1) it follows that for every $\zeta \in D(A^{1-\beta})$ the function $W(\cdot,\zeta)$ is differentiable at every Lebesgue point of $\overline{\gamma}$ and

$$\begin{aligned} \partial_s W(s,\zeta) = \int_s^T & < D_xL(r,x(r),\overline{\gamma}(r)), A^{\beta}\Phi(r) > dr \\ -L(s,\zeta,\overline{\gamma}(s)) + & < D\phi(x(T)), A^{\beta}\Phi(T) > \end{aligned} \tag{3.18}$$

which yields (i). ∎

The following Corollary is a straightforward consequence of the previous Lemma and it can be proved arguing as in [3] and [15].

Corollary 3.4 *Assume* (2.2), (2.18), (3.1) *and let* $\zeta \in D(A^{1-\beta})$. *Then, for every* $s \in [t_0, T]$ *and for a. e. Lebesgue point* $t \in [s, T]$ *of* $\overline{\gamma}$ *the function* W *is Fréchet differentiable at* $(t, x(t; s, \zeta, \overline{\gamma})) \in [s, T] \times D(A^{1-\beta})$. *Moreover we have*

$$-\partial_t W(t, x(t)) + < A^{\beta} D_x W(t, x(t)), A^{1-\beta} x(t) + A^{-\beta} F(x(t)) - B\overline{\gamma}(t) > \\ -L(t, x(t), \overline{\gamma}(t)) = 0 \tag{3.19}$$

where $x(t) = x(t; s, \zeta, \overline{\gamma})$. *In addition*

$$W(T, x(T; s, \zeta, \overline{\gamma})) = \phi(x(T; s, \zeta, \overline{\gamma})). \tag{3.20}$$

Proof: Recall that, by the definition of W in (3.6) we have, for every $t \in [s, T]$

$$W(t, x(t)) = \int_t^T L(r, x(r), \overline{\gamma}(r)) dr + \phi(x(T))$$

where $x(r) = x(r; t, x(t), \overline{\gamma}) = x(r; s, \zeta, \overline{\gamma})$. Then, for $h \in \mathbb{R}$ sufficiently small

$$W(t+h, x(t+h)) - W(t, x(t)) = -\int_t^{t+h} L(r, x(r), \overline{\gamma}(r)) dr$$

so that, if t is a Lebesgue point of the map $r \to L(r, x(r), \overline{\gamma}(r))$ (which is true for every Lebesgue point of $\overline{\gamma}$, see Remark 2.6) we get

$$\lim_{h \to 0} \frac{1}{h} \left[W(t+h, x(t+h)) - W(t, x(t)) \right] = L(t, x(t), \overline{\gamma}(t)). \tag{3.21}$$

At this point observe that due to the optimality of $\overline{\gamma}$ we have $v(t_0, x_0) = W(t_0, x_0)$ and also, by Dynamic Programming Principle (see [2]) $v(t, \overline{x}(t)) = W(t, \overline{x}(t))$ for every $t \in [t_0, T]$. Since $v(s, \zeta) < W(s, \zeta)$ for every $(s, \zeta) \in [t_0, T] \times X$ and $W(t, \cdot)$ is Fréchet differentiable for every $t \in [t_0, T]$ (see Lemma 3.3 (ii)), then we have, as in (2.42)

$$D_x W \in D_x^+ v \subset D(A^{\beta}).$$

Moreover, if t is a Lebesgue point for $\overline{\gamma}$ then $x(t) \in D(A^{1-\beta})$ (see Remark 2.6). By Lemma 3.3 (i) and (ii), it follows that $W(\cdot, x(t))$ is differentiable so that

$$\frac{d}{dt} W(t, x(t))$$

$$= \partial_t W(t,x(t)) + < A^\beta D_x W(t,x(t)), -A^{1-\beta}x(t) - A^{-\beta}F(x(t)) + B\overline{\gamma}(t) >$$

which gives the claim, together with (3.21). ∎

Proof of Theorem 3.2: For the reader's convenience we divide the proof in three steps.

Step I Let $(t_0,x_0) \in [0,T] \times D(A^{1-\beta})$ be the starting point and let $\{\overline{\gamma},\overline{x}\}$ be an optimal pair associated to it. We recall that $\overline{x}(t) = x(t;t_0,x_0,\overline{\gamma})$. Let $p \in C([t_0,T];X)$ be the mild solution of the co–state equation (3.2) associated to the optimal pair $\{\overline{\gamma},\overline{x}\}$. We argue as in [15], [3] to show that $p(t) = D_x W(t,\overline{x}(t))$.

Take $z \in D(A)$, $\tau \in [t_0,T]$ and let $\zeta = \overline{x}(\tau) = x(\tau;t_0,x_0,\overline{\gamma})$. Let $\Psi \in C([t_0,T];D(A))$ be the solution of

$$\begin{cases} \Psi'(t) = -[A + DF(x(t))]\Psi(t) \\ \Psi(\tau) = z, \end{cases} \tag{3.22}$$

where, we set $x(t) = x(t;\tau,\zeta,\overline{\gamma})$.

By equation (3.2) we have, for every $\xi \in D(A)$

$$< p'(t),\xi >=< p(t),(A + DF(\overline{x}(t)))\xi > - < D_x L(t,\overline{x}(t),\overline{\gamma}(t)),\xi > .$$

Now, being $\Psi \in C([t_0,T];D(A))$ we get, for every $t \in [t_0,T]$

$$< p'(t),\Psi(t) >$$

$$=< p(t),(A + DF(\overline{x}(t)))\Psi(t) > - < D_x L(t,\overline{x}(t),\overline{\gamma}(t)),\Psi(t) > .$$

Hence, recalling that, by uniqueness, $x(t;\tau,\zeta,\overline{\gamma}) = \overline{x}(t)$, from (3.22) and the previous formula we get

$$\begin{aligned} \partial_t < p(t),\Psi(t) > &=< p'(t),\Psi(t) > + < p(t),\Psi'(t) > \\ &= - < D_x L(t,\overline{x}(t),\overline{\gamma}(t)),\Psi(t) >. \end{aligned} \tag{3.23}$$

Now integrating from τ to T we derive

$$< p(T),\Psi(T) > - < p(\tau),\Psi(\tau) >= -\int_\tau^T < D_x L(t,\overline{x}(t),\overline{\gamma}(t)),\Psi(t) > dt \tag{3.24}$$

which yields

$$< p(\tau),z >= \int_\tau^T < D_x L(t,\overline{x}(t),\overline{\gamma}(t)),\Psi(t) > dt + < p_T,\Psi(T) > . \tag{3.25}$$

On the other hand, by arbitrariness of $z \in D(A)$, by density of $D(A)$ in X and by (3.10), it is easy to see that $p(t) = D_x W(t,\overline{x}(t))$.

Step II Let $\overline{x}(t) = x(t; t_0, x_0, \overline{\gamma})$ be the optimal state. By reasoning as in the proof of the previous Corollary we obtain that $v(t, \overline{x}(t)) = W(t, \overline{x}(t))$ for every $t \in [t_0, T]$ and $v(t, \zeta) \leq W(t, \zeta)$ for every $(t, \zeta) \in [t_0, T] \times X$. Since $x_0 \in D(A^{1-\beta})$ and $\overline{\gamma}(t)$ is bounded on $[0, T)$ then (see Remark 2.6) $\overline{x} \in C([t_0, T); D(A^{1-\beta}))$ and so for every $t \in [t_0, T)$ the pairs $(t, \overline{x}(t))$ belong to $[t_0, T) \times D(A^{1-\beta})$ and are maximum points of $v - W$, which yields (3.3). By Corollary 3.4 we can use W as a test function in inequality (2.39) (i) in Theorem 2.8 since $D_x W \in D_x^+ v \subset D(A^\beta)$. Therefore we derive that for a.e. $t \in [t_0, T]$

$$\begin{aligned} -\partial_t W(t, \overline{x}(t)) + &< A^\beta D_x W(t, \overline{x}(t)), A^{1-\beta}\overline{x}(t) + A^{-\beta} F(\overline{x}(t)) > \\ &+ H(t, \overline{x}(t), A^\beta D_x W(t, \overline{x}(t))) \leq 0. \end{aligned} \tag{3.26}$$

We recall that from (3.19) and from (2.38) for every $s \in [t_0, T]$ and for a. e. t such that $t_0 \leq s \leq t \leq T$, we have

$$\begin{aligned} -\partial_t W(t, \overline{x}(t)) + &< A^\beta D_x W(t, \overline{x}(t)), A^{1-\beta}\overline{x}(t) + A^{-\beta} F(\overline{x}(t)) > \\ &+ H(t, \overline{x}(t), A^\beta D_x W(t, \overline{x}(t))) \geq 0. \end{aligned} \tag{3.27}$$

Comparing inequalities (3.27) and (3.26) we obtain

$$\begin{aligned} -\partial_t W(t, \overline{x}(t)) + &< A^\beta D_x W(t, \overline{x}(t)), A^{1-\beta}\overline{x}(t) + A^{-\beta} F(\overline{x}(t)) > \\ &+ H(t, \overline{x}(t), A^\beta D_x W(t, \overline{x}(t))) = 0. \end{aligned} \tag{3.28}$$

Thus, from (3.19) and (3.28) we find

$$\begin{aligned} &- < A^\beta D_x W(t, \overline{x}(t)), B\overline{\gamma}(t) > - L(t, \overline{x}(t), \overline{\gamma}(t)) \\ &= H(t, \overline{x}(t), A^\beta D_x W(t, \overline{x}(t))) \\ &= \sup_{\gamma \in U} \left\{ - < A^\beta D_x W(t, \overline{x}(t)), B\gamma > - L(t, \overline{x}(t), \gamma) \right\}. \end{aligned} \tag{3.29}$$

Therefore the maximum principle holds if the starting point $(t_0, x_0) \in [0, T] \times D(A^{1-\beta})$.

Step III Now we show that the maximum principle holds if the starting point $(t_0, x_0) \in [0, T] \times X$. Let $\overline{\gamma}(\cdot)$ be an optimal control for problem (2.19)–(2.1) and let $\overline{x}(t) = x(t; t_0, x_0, \overline{\gamma})$ be the corresponding optimal trajectory. Since $\overline{x}(t) \in L^2(t_0, T; D(A^{1-\beta}))$, we can find a sequence $t_n \to t_0$ as $n \to \infty$ such that $\overline{x}(t_n) = x(t_n; t_0, x_0, \overline{\gamma}) \in D(A^{1-\beta})$. Setting $x_n(t) = x(t; t_n, \overline{x}(t_n), \overline{\gamma})$ we have, by Dynamic Programming Principle, that x_n is an optimal trajectory for the problem with the starting point

$(t_n, \overline{x}(t_n)) \in [0,T] \times D(A^{1-\beta})$. Then, if p_n is the mild solution of the problem

$$\begin{cases} -p_n'(t) = -[A + [DF(x_n(t))]^*]p_n(t) + D_x L(t, x_n(t), \overline{\gamma}(t)) \\ \text{for } t \in [t_n, T) \\ \\ p_n(T) = D\phi(x_n(T)) \end{cases} \tag{3.30}$$

by Step I and Step II, p_n satisfies the maximum principle (3.4) for a.e. $t \in [t_n, T]$ and the co–state inclusion (3.3) for every $t \in [t_n, T]$.

Recalling that, by uniqueness, $\overline{x}(t) = x(t; t_n, \overline{x}(t_n), \overline{\gamma}) = x_n(t)$ for every $t \in [t_n, T]$, we obtain that, for every $n \in \mathbb{N}$ and $t \in [t_n, T]$, $p_n(t) = p(t)$, where p is the mild solution of equation (3.2). Then p satisfies the maximum principle (3.4) for a.e. $t \in [t_n, T]$ and the co–state inclusion (3.3) for every $t \in [t_n, T]$. Since $t_n \to t_0$ the result follows for a.e. $t \in [t_0, T]$. ■

Remark 3.5 From assumption (2.2), (2.18) and (3.1) H is Gâteaux differentiable with respect to p. Then by (3.4) and (3.5) for any $v \in D(A^\beta)$ we easily derive

$$< D_p H(t, \overline{x}(t), A^\beta \overline{p}(t)), A^\beta v >= - < B\overline{\gamma}(t), A^\beta v > \ .$$

Then we obtain

$$\overline{\gamma}(t) = -B^* D_p H(t, \overline{x}(t), A^\beta \overline{p}(t)) \tag{3.31}$$

for a.e. $t \in [t_0, T]$. The above equation and (3.3) yield the *feedback law*

$$\overline{\gamma}(t) \in -B^* D_p H(t, \overline{x}(t), A^\beta D_x^+ v(t, \overline{x}(t))) \tag{3.32}$$

for a.e. $t \in [t_0, T]$.

Next, by standard procedure, we reformulate the Pontryagin Maximum Principle in terms of an Hamiltonian system (see e.g.[6]).

Theorem 3.6 *Assume* (2.2), (2.18), (3.1). *Let* $\{\overline{\gamma}, \overline{x}\}$ *be an optimal pair for problem* (2.19)−(2.1), *with starting point* $(t_0, x_0) \in [0,T] \times X$. *Moreover set* $p_T = D\phi(\overline{x}(T))$ *and let* $\overline{p}$ *be the corresponding co–state. Then* H *is Gâteaux differentiable with respect to* (x,p) *at* $(\overline{x}(t), \overline{p}(t))$, *for a. e.* $t \in [t_0, T]$. *Moreover the pair* $(\overline{x}(t), \overline{p}(t))$ *is a mild solution of the Hamiltonian system*

$$\begin{cases} \overline{x}'(t) = -A\overline{x}(t) - F(\overline{x}(t)) - A^\beta D_p H(t, \overline{x}(t), A^\beta p(t)) \\ \\ p'(t) = Ap(t) + [DF(\overline{x}(t)]^* p(t) + D_x H(t, \overline{x}(t), A^\beta p(t)) \end{cases} \tag{3.33}$$

with the initial–terminal condition

$$\begin{cases} \overline{x}(t_0) = x_0 \\ p(T) = D\phi(x(T)) \end{cases} .$$

4 Sufficient Conditions

The next result may be directly derived following the same reasonings contained in [6], Theorem 5.9.

Theorem 4.1 *Assume* (2.2), (2.18), (3.1). *Suppose that for all* $R > 0$,

$$|D_x H(t,x,p) - D_x H(t,y,q)| + |D_p H(t,x,p) - D_p H(t,y,q)|$$
$$\leq C_R[|x-y| + |p-q|]$$

for some constant $C_R > 0$ *and all* $x, y, p, q \in X$ *satisfying* $|x|, |y| \leq R$. *Let* $(t_0, x_0) \in [0,T] \times X$ *and* $p_0 \in D_x^* v(t_0, x_0)$. *Then, the system*

$$\begin{cases} x'(t) = -Ax(t) - F(x(t)) - A^\beta D_p H(t, x(t), A^\beta p(t)) \\ \\ p'(t) = Ap(t) + [DF(x(t))]^* p(t) + D_x H(t, x(t), A^\beta p(t)) \end{cases} \tag{4.1}$$

with the initial–terminal condition

$$\begin{cases} x(t_0) = x_0 \\ p(T) = D\phi(x(T)) \end{cases}$$

has a solution $(\overline{x}, \overline{p})$ *such that* $\overline{x}$ *is an optimal trajectory for problem* (2.19)–(2.1) *corresponding to some control* $\overline{\gamma}$. *Moreover,* $\overline{p}$ *is the co–state associated to* $\overline{\gamma}$ *and satisfies* $\overline{p}(t_0) = p_0$.

The above theorem gives, in some sense, a sufficient condition for optimality. This condition would be more useful if one could guarantee uniqueness of solutions for (4.1). Uniqueness results for problem (4.1) have been obtained in the linear case, see [16] and in [21], [8]. In the next theorem we adapt the reasoning of [12] to the present case to show an existence and uniqueness result for the solution of an Hamiltonian system of kind (4.1). As in [12], we replace the terminal co–state datum with an initial one. We consider the case when $F = 0$. Set $y(t) = A^{-\beta}x(t)$ as in (3.15). Then the Hamiltonian system (4.1) becomes

$$\begin{cases} y'(t) = -Ay(t) - D_p H(t, A^\beta y(t), A^\beta p(t)) \;, \quad y(0) = y_0 = A^{-\beta} x_0 \\ \\ p'(t) = Ap(t) + D_x H(t, A^\beta y(t), A^\beta p(t)) \;, \qquad p(0) = p_0 \end{cases} \tag{4.2}$$

Theorem 4.2 *Assume* (2.2), (2.18) *and* (3.1). *Suppose that*

$$|D_x H(t,x,p) - D_x H(t,y,q)| + |D_p H(t,x,p) - D_p H(t,y,q)|$$
$$\leq L_H[|x-y| + |p-q|]$$

for some constant $L_H > 0$ and all $x, y, p, q \in X$. Let $(t_0, y_0) \in [0,T] \times D(A^{\frac{1}{2}})$ and $p_0 \in D_x^* v(t_0, y_0)$. Then the Hamiltonian system (4.2) has a unique solution $(\overline{y}, p)$ such that $\overline{y}, p \in C([t_0,T]; D(A^{\frac{1}{2}})) \cap L^2(t_0,T; D(A)) \cap W^{1,2}(t_0,T;X)$.

Proof: The existence part is a straightforward consequence of the previous Theorem. Without loss of generality we set $t_0 = 0$. Let $(y_1(t), p_1(t))$ and $(y_2(t), p_2(t))$ be two distinct solutions to system (4.2) and consider $\tilde{y}(t) = y_1(t) - y_2(t)$ and $\tilde{p}(t) = p_1(t) - p_2(t)$. Then $\tilde{y}(t)$ and $\tilde{p}(t)$ satisfy the system

$$\begin{cases} \tilde{y}'(t) = -A\tilde{y}(t) - D_p\tilde{H}(t, A^\beta \tilde{y}(t), A^\beta \tilde{p}(t)) \ , \quad \tilde{y}(0) = 0 \\ \\ \tilde{p}'(t) = A\tilde{p}(t) + D_x\tilde{H}(t, A^\beta \tilde{y}(t), A^\beta \tilde{p}(t)) \ , \qquad \tilde{p}(0) = 0 \end{cases} \tag{4.3}$$

where

$$\begin{aligned} &D_p\tilde{H}(t, A^\beta \tilde{y}(t), A^\beta \tilde{p}(t)) \\ &= D_pH(t, A^\beta y_1(t), A^\beta p_1(t)) - D_pH(t, A^\beta y_2(t), A^\beta p_2(t)), \end{aligned}$$

and

$$\begin{aligned} &D_x\tilde{H}(t, A^\beta \tilde{y}(t), A^\beta \tilde{p}(t)) \\ &= D_xH(t, A^\beta y_1(t), A^\beta p_1(t)) - D_xH(t, A^\beta y_2(t), A^\beta p_2(t)) \end{aligned}$$

Let $\theta \in C^1(\mathbb{R})$ be a function such that

$$\theta(t) = \begin{cases} 1 & 0 \le t \le \dfrac{T}{2} \\ \\ 0 & t = T \end{cases} \qquad \text{and} \quad |\theta'(t)| \le \frac{4}{T}.$$

We set $\overline{y}(t) = \theta(t)\tilde{y}(t)$ and $\overline{p}(t) = \theta(t)\tilde{p}(t)$. Then $\overline{y}(t)$ and $\overline{p}(t)$ satisfy the system

$$\begin{cases} \overline{y}'(t) = -A\overline{y}(t) - D_p\overline{H}(t, A^\beta \overline{y}(t), A^\beta \overline{p}(t)) + g_x(t) \ , \quad \overline{y}(0) = 0 \\ \\ \overline{p}'(t) = A\overline{p}(t) + D_x\overline{H}(t, A^\beta \overline{y}(t), A^\beta \overline{p}(t)) + g_p(t) \ , \qquad \overline{p}(0) = 0 \end{cases} \tag{4.4}$$

where

$$g_x(t) = \theta'(t)\tilde{y}(t) \ \text{ and } \ g_p(t) = \theta'(t)\tilde{p}(t)$$

and

$$\begin{aligned} D_p\overline{H}(t, A^\beta \overline{y}(t), A^\beta \overline{p}(t)) &= \theta(t) D_p\tilde{H}(t, A^\beta \tilde{y}(t), A^\beta \tilde{p}(t)) \\ D_x\overline{H}(t, A^\beta \overline{y}(t), A^\beta \overline{p}(t)) &= \theta(t) D_x\tilde{H}(t, A^\beta \tilde{y}(t), A^\beta \tilde{p}(t)). \end{aligned}$$

Now we set

$$z(t) = e^{\frac{k(t-T)^2}{2}}\overline{y}(t) \ \text{ and } \ q(t) = e^{\frac{k(t-T)^2}{2}}\overline{p}(t),$$

then $z(t)$ and $q(t)$ satisfy the system

$$\begin{cases} z'(t) = -Az(t) + k(t-T)z(t) - D_q H(t, A^\beta z(t), A^\beta q(t)) + f_z(t) \ , \\ z(0) = z(T) = 0 \\ \\ q'(t) = Aq(t) + k(t-T)q(t) + D_z H(t, A^\beta z(t), A^\beta q(t)) + f_q(t) \ , \\ q(0) = q(T) = 0 \end{cases} \tag{4.5}$$

where

$$f_z(t) = e^{\frac{k(t-T)^2}{2}} g_x(t) \quad \text{and} \quad f_q(t) = e^{\frac{k(t-T)^2}{2}} g_p(t)$$

and

$$D_q H(t, A^\beta z(t), A^\beta q(t)) = e^{\frac{k(t-T)^2}{2}} D_p \overline{H}(t, A^\beta \overline{y}(t), A^\beta \overline{p}(t))$$

$$D_z H(t, A^\beta z(t), A^\beta q(t)) = e^{\frac{k(t-T)^2}{2}} D_x \overline{H}(t, A^\beta \overline{y}(t), A^\beta \overline{p}(t)).$$

Then multiplying the first equation of system (4.5) by $z'(t)$ and the second equation by $q'(t)$ we get

$$|z'(t)|^2 = - < Az(t), z'(t) > + < k(t-T)z(t), z'(t) >$$

$$- < D_q H(t, A^\beta z(t), A^\beta q(t)), z'(t) > + < f_z(t), z'(t) >$$

and

$$|q'(t)|^2 = < Aq(t), q'(t) > + < k(t-T)q(t), q'(t) >$$

$$+ < D_z H(t, A^\beta z(t), A^\beta q(t)), q'(t) > + < f_q(t), q'(t) > \ .$$

The above equalities can be rewritten as

$$|z'(t)|^2 = \frac{1}{2}\frac{d}{dt}\{- < Az(t), z(t) > + k(t-T)|z(t)|^2\} - \frac{k}{2}|z(t)|^2$$

$$- < D_q H(t, A^\beta z(t), A^\beta q(t)), z'(t) > + < f_z(t), z'(t) >$$

and

$$|q'(t)|^2 = \frac{1}{2}\frac{d}{dt}\{< Aq(t), q(t) > + k(t-T)|q(t)|^2\} - \frac{k}{2}|q(t)|^2$$

$$+ < D_z H(t, A^\beta z(t), A^\beta q(t)), q'(t) > + < f_q(t), q'(t) > \ .$$

Integrating on $[0, T]$, recalling that z and q vanish at initial and terminal points, we get

$$\int_0^T (|z'(t)|^2 + \frac{k}{2}|z(t)|^2) dt$$

$$\leq \int_0^T |z'(t)|^2 dt + \frac{1}{2}\int_0^T (|D_q H(t, A^\beta z(t), A^\beta q(t))|^2 + |f_z(t)|^2) dt$$

and

$$\int_0^T (|q'(t)|^2 + \frac{k}{2}|q(t)|^2) dt$$

$$\leq \int_0^T |q'(t)|^2 dt + \frac{1}{2}\int_0^T (|D_z H(t, A^\beta z(t), A^\beta q(t))|^2 + |f_q(t)|^2) dt.$$

Therefore these estimates yield

$$k\int_0^T (|z(t)|^2 + |q(t)|^2) dt \leq \int_0^T (|f_z(t)|^2 + |f_q(t)|^2) dt$$
$$+ \int_0^T (|D_q H(t, A^\beta z(t), A^\beta q(t))|^2 + |D_z H(t, A^\beta z(t), A^\beta q(t))|^2) dt\ . \tag{4.6}$$

In [12] the last two terms of the left hand–side of the above inequality are estimated, using the interpolation inequality (2.4), by the quantity

$$C[|A^{\frac{1}{2}} z(t)|^2 + |A^{\frac{1}{2}} q(t)|^2],$$

for some positive constant C. In this case, since $\beta > \frac{1}{2}$, from the interpolation inequality (2.4) it follows for $\beta < \gamma \leq 1$ and for any $\sigma > 0$

$$|D_q H(t, A^\beta z(t), A^\beta q(t))| = e^{\frac{k(t-T)^2}{2}} |D_p \overline{H}(t, A^\beta \overline{y}(t), A^\beta \overline{p}(t))|$$

$$\leq e^{\frac{k(t-T)^2}{2}} \theta(t) |D_p H(t, A^\beta y_1(t), A^\beta p_1(t)) - D_p H(t, A^\beta y_2(t), A^\beta p_2(t))|$$

$$\leq L_H e^{\frac{k(t-T)^2}{2}} \theta(t) \left[|A^\beta(y_1(t) - y_2(t))| + |A^\beta(p_1(t) - p_2(t))|\right]$$

$$= L_H[|A^\beta z(t)| + |A^\beta q(t)|]$$

$$\leq L_H \sigma[|A^\gamma z(t)| + |A^\gamma q(t)|] + L_H C_\sigma[|z(t)| + |q(t)|] \tag{4.7}$$

and

$$|D_z H(t, A^\beta z(t), A^\beta q(t))|$$
$$\leq L_H \sigma[|A^\gamma z(t)| + |A^\gamma q(t)|] + L_H C_\sigma[|z(t)| + |q(t)|] \tag{4.8}$$

where $C_\sigma = \frac{C_1}{\sigma}$, for some positive constant C_1. Notice that from estimates (4.7) and (4.8) it follows

$$|D_q H(t, A^\beta z(t), A^\beta q(t))|^2 + |D_p H(t, A^\beta z(t), A^\beta q(t))|^2$$
$$\leq 2C_2 L_H \sigma[|A^\gamma z(t)|^2 + |A^\gamma q(t)|^2] + 2C_2 L_H C_\sigma[|z(t)|^2 + |q(t)|^2] \tag{4.9}$$

for some positive constant C_2. Let θ be such that $2\beta - 1 < \theta \leq 1$. Then, multiplying by $A^\theta z(t)$ the first equation of (4.5) and integrating, we obtain

$$\int_0^T |A^{\frac{\theta+1}{2}} z(t)|^2 dt \leq kT \int_0^T |A^{\frac{\theta}{2}} z(t)|^2 dt$$

$$+\frac{1}{2}\int_0^T (|D_q H(t, A^\beta z(t), A^\beta q(t))|^2 + |f_z(t)|^2) dt + \int_0^T |A^\theta z(t)|^2 dt.$$

Similarly, multiplying by $-A^\theta q(t)$ the second equation of (4.5) and integrating, we obtain

$$\int_0^T |A^{\frac{\theta+1}{2}} q(t)|^2 dt \leq kT \int_0^T |A^{\frac{\theta}{2}} q(t)|^2 dt$$

$$+\frac{1}{2}\int_0^T (|D_z H(t, A^\beta z(t), A^\beta q(t))|^2 + |f_q(t)|^2) dt + \int_0^T |A^\theta q(t)|^2 dt.$$

Adding the two above inequalities we get

$$\int_0^T (|A^{\frac{\theta+1}{2}} z(t)|^2 + |A^{\frac{\theta+1}{2}} q(t)|^2) dt \leq kT \int_0^T (|A^{\frac{\theta}{2}} z(t)|^2 + |A^{\frac{\theta}{2}} q(t)|^2) dt$$

$$+\frac{1}{2}\int_0^T (|D_q H(t, A^\beta z(t), A^\beta q(t))|^2 + |D_z H(t, A^\beta z(t), A^\beta q(t))|^2) dt$$

$$+\frac{1}{2}\int_0^T (|f_z(t)|^2 + |f_q(t)|^2) dt + \int_0^T (|A^\theta z(t)|^2 + |A^\theta q(t)|^2) dt. \tag{4.10}$$

Choosing $\gamma = \dfrac{\theta+1}{2}$ in (4.9) and applying the interpolation inequality (2.4) to the first and to the last term of the right hand side of (4.10) it follows

$$\int_0^T (|A^\gamma z(t)|^2 + |A^\gamma q(t)|^2) dt \leq +\frac{1}{2}\int_0^T (|f_z(t)|^2 + |f_q(t)|^2) dt$$

$$+\sigma(kT + L_H C + 1)\int_0^T (|A^\gamma z(t)|^2 + |A^\gamma q(t)|^2) dt \tag{4.11}$$

$$+\frac{C_\sigma}{2}(kT + L_H C_2 + 1)\int_0^T (|z(t)|^2 + |q(t)|^2) dt.$$

So setting $\sigma = \dfrac{1}{2(kT + L_H C_2 + 1)}$, recalling $C_\sigma = \dfrac{C_1}{\sigma}$, in the above inequality we obtain

$$\int_0^T (|A^\gamma z(t)|^2 + |A^\gamma q(t)|^2)dt \leq +\frac{1}{2}\int_0^T (|f_z(t)|^2 + |f_q(t)|^2)dt$$
$$+2C_1(kT + L_H C_2 + 1)^2 \int_0^T (|z(t)|^2 + |q(t)|^2)dt. \tag{4.12}$$

Substituting estimate (4.9) for $\sigma = \rho$ in (4.6) and then exploiting (4.12) we derive

$$k\int_0^T (|z(t)|^2 + |q(t)|^2)dt \leq \int_0^T (|f_z(t)|^2) + |f_q(t)|^2)dt$$
$$+C_2 L_H C_\rho \int_0^T (|z(t)|^2 + |q(t)|^2)dt + C_2 L_H \rho \int_0^T (|A^\gamma z(t)|^2 + |A^\gamma q(t)|^2)dt$$
$$\leq C_2[L_H C_\rho + 2C_1 L_H \rho(kT + L_H C_2 + 1)^2] \int_0^T (|z(t)|^2 + |q(t)|^2)dt$$
$$+(C_2 L_H \rho + 1)\int_0^T (|f_z(t)|^2) + |f_q(t)|^2)dt\ . \tag{4.13}$$

We set $\rho = \dfrac{1}{kT + L_H C_2 + 1}$, then $C_\rho \leq C_1(kT + L_H C_2 + 1)$. Therefore, for $T < \dfrac{1}{3L_H C_1 C_2}$ we derive

$$\int_0^T (|z(t)|^2 + |q(t)|^2)dt \leq C(k)\int_0^T (|f_z(t)|^2 + |f_q(t)|^2)dt\ , \tag{4.14}$$

where

$$C(k) = \frac{2}{k(1 - 3L_H C_1 C_2 T) - 3L_H C_1 C_2(1 + L_H C_2)}$$

is positive for k big enough and $C(k) \to 0$ as $k \to \infty$.

From (4.14) directly follows

$$\int_0^T e^{k(t-T)^2}(|\overline{y}(t)|^2 + |\overline{p}(t)|^2)dt \leq C(k)\int_0^T e^{k(t-T)^2}(|g_x(t)|^2 + |g_p(t)|^2)dt\ . \tag{4.15}$$

On the other hand,

$$\int_0^T e^{k(t-T)^2}(|\overline{y}(t)|^2+|\overline{p}(t)|^2)dt \geq \int_0^{\frac{T}{2}} e^{k(t-T)^2}(|\tilde{y}(t)|^2+|\tilde{p}(t)|^2)dt$$

$$\geq e^{k\frac{T^2}{4}} \int_0^{\frac{T}{2}} (|\tilde{y}(t)|^2+|\tilde{p}(t)|^2)dt \tag{4.16}$$

and the following holds

$$\begin{aligned} &\int_0^T e^{k(t-T)^2}(|g_x(t)|^2+|g_p(t)|^2)dt \\ &= \int_0^T e^{k(t-T)^2}|\theta'(t)|^2(|\tilde{y}(t)|^2+|\tilde{p}(t)|^2)dt \\ &\leq \left(\frac{4}{T}\right)^2 \int_{\frac{T}{2}}^T e^{k(t-T)^2}(|\tilde{y}(t)|^2+|\tilde{p}(t)|^2)dt \\ &\leq \left(\frac{4}{T}\right)^2 e^{k\frac{T^2}{4}} \int_{\frac{T}{2}}^T (|\tilde{y}(t)|^2+|\tilde{p}(t)|^2)dt\ . \end{aligned} \tag{4.17}$$

In conclusion, from (4.15), (4.16) and (4.17) we get

$$e^{k\frac{T^2}{4}} \int_0^{\frac{T}{2}} (|\tilde{y}(t)|^2+|\tilde{p}(t)|^2)dt \leq C(k)\left(\frac{4}{T}\right)^2 e^{k\frac{T^2}{4}} \int_{\frac{T}{2}}^T (|\tilde{y}(t)|^2+|\tilde{p}(t)|^2)dt\ .$$

From the above inequality we obtain

$$\int_0^{\frac{T}{2}} (|\tilde{y}(t)|^2+|\tilde{p}(t)|^2)dt \to 0 \quad \text{as} \quad k\to\infty$$

and we conclude that $|\tilde{y}(t)|=|\tilde{p}(t)|=0$ on $[0,\frac{T}{2}]$. Iterating this procedure we obtain the result on [0,T]. ∎

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