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Optimal Stopping of Controlled Jump Diffusion Processes: A Viscosity Solution Approach*

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Abstract

This paper concerns the optimal stopping time problem in a finite horizon of a controlled jump diffusion process. We prove that the value function is continuous and is a viscosity solution of the integrodifferential variational inequality arising from the associated dynamic programming. We also establish comparison principles, which yield uniqueness results. Moreover, the viscosity solution approach allows us to extend maximum principles for linear parabolic integrodifferential operators in $\mathcal{C}^0([0,T] \times I\!R^n)$ and to obtain $\mathcal{C}^{1,2}([0,T) \times I\!R^n)$ existence result for the associated Cauchy problem in the nondegenerate case.

Key words: stochastic control, viscosity solutions, jump-diffusion processes

AMS Subject Classifications: 93E20, 49L25, 60J75

1 Introduction

In this paper, we investigate the optimal stopping time problem of a controlled jump diffusion process and the associated Bellman variational inequality. Let us briefly recall the stochastic background for this problem. On a probability space $(\Omega, \mathcal{F}, P)$ with a filtration $I\!F = (\mathcal{F}_t)_{0 \leq t \leq T}$ satisfying usual assumptions, are defined two processes (W, v) adapted to $I\!F$ where W is a standard d-Brownian motion and v is a homogeneous Poisson random measure with intensity measure $q(dt, dz) = dt \times m(dz)$. m is the Lévy measure on $I\!R^n$ of v and $\tilde{v}(dt, dz) = (v - q)(dt, dz)$ is called the compensated jump martingale random measure of v. Readers are referred

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to Gihman-Skorohod (1972 [12]) or Jacod (1979 [17]) for more precise definitions and properties of random measures. The state of the system is described by the $I\!R^n$-valued process X_t solution of the following stochastic differential equation:

$$\begin{aligned} dX_t &= b(t, X_{t^-}, \alpha_t)dt + \sigma(t, X_{t^-}, \alpha_t)dW_t \\ &\quad + \int_{I\!R^n} \gamma(t, X_{t^-}, \alpha_t, z)\tilde{\nu}(dt, dz) \end{aligned} \tag{1.1}$$

where α_t, the control, belongs to $\mathcal{U}$, the set of all progressively measurable processes with values in a compact metric separable space U, and σ, b, γ are matrix and vectors valued functions satisfying assumptions detailed in Section 2.

We consider the problem in a finite time horizon T of maximizing with control and optimal stopping a running gain f and a terminal reward g with discount rate c, and we introduce therefore the value function:

$$\begin{aligned} v(t,x) &= \sup_{\substack{\tau \in \mathcal{T}_{t,T} \\ \alpha. \in \mathcal{U}}} E_{tx} \left[\int_t^\tau e^{-\int_t^s c(u, X_u)du} f(s, X_s, \alpha_s)ds \right. \\ &\qquad \left. + e^{-\int_t^\tau c(u, X_u)du} g(X_\tau) \right] \end{aligned} \tag{1.2}$$

for all $(t, x) \in [0, T] \times I\!R^n$. E_{tx} is the conditional expectation under P given that $X_t = x$ and $\mathcal{T}_{t,T}$ is the set of all stopping times between t and T. c, f and g are real valued functions satisfying conditions detailed in Section 2. This stochastic control problem applies in finance theory for the American option valuation and the consumption/investment portfolio choice.

The Hamilton-Jacobi-Bellman (HJB in short) equation associated with this problem is a variational inequality involving, at least heuristically, a nonlinear second order parabolic integrodifferential equation (see Bensoussan–J.L. Lions (1982) [3] for example):

$$\min \left\{ c(t,x)v - \frac{\partial v}{\partial t} + \min_{\alpha \in U} \left(-A_t^\alpha v - B_t^\alpha v - f(t,x,\alpha) \right); v - g(x) \right\} = 0 \tag{1.3}$$

in $[0, T) \times I\!R^n$ with the terminal data

$$v(T, x) = g(x) \qquad \forall x \in I\!R^n \tag{1.4}$$

where A_t^α is the linear second-order differential operator:

$$A_t^\alpha v(t,x) = tr\left(\frac{1}{2}\sigma\sigma'(t,x,\alpha)D_x^2 v(t,x)\right) + b(t,x,\alpha).D_x v(t,x)$$

and B_t^α is the integrodifferential operator:

$$B_t^\alpha v(t,x) = \int_{\mathbb{R}^n} [v(t,x+\gamma(t,x,\alpha,z)) - v(t,x) - \gamma(t,x,\alpha,z).D_x v(t,x)]\, m(dz)$$

As it is well-known, there is not in general a smooth solution of the equation (1.3), especially when the diffusion coefficient is degenerate. One is forced to use a notion of weak solutions such as viscosity solutions introduced by Crandall–P.L. Lions (1983 [7]) in the deterministic first order case and by P.L. Lions (1983 [19]) in the second order case for diffusion processes. Soner (1986a-b [23] [24]) has extended the viscosity approach to piecewise-deterministic processes with jumps, but restricts to bounded coefficients. Sayah (1991 [22]) studied also first order Hamilton-Jacobi equations with integral term, under less restrictive assumptions. She obtains existence results via Perron's method. In this paper, we use the dynamic programming approach and viscosity solutions to study the finite horizon problem of mixed optimal stopping and stochastic control of jump diffusion processes. In proving comparison principles, we adopt classical viscosity solution techniques for second order equations (see Crandall–Ishii–P.L. Lions 1992 [6] for a general overview of the theory), by taking into account the non local integral term B. Applied to a problem without stochastic control, the viscosity solution approach provides maximum principles for linear second order parabolic integrodifferential operators in the space of continuous functions $\mathcal{C}^0([0,T] \times \mathbb{R}^n)$, extending then results obtained in Sobolev spaces by Bony 1967 [4], Bensoussan–J.L. Lions [3] or Gimbert–P.L. Lions 1984 [13]. We give a simpler proof adapted to the linear case, which does not use the general viscosity solution method and do not require any Lipschitz condition. Uniqueness results for viscosity solutions and standard regularity results for second order parabolic operators (see e.g. Friedman 1964 [10]) also yield $\mathcal{C}^{1,2}([0,T) \times \mathbb{R}^n)$ existence result for the associated Cauchy problem in the nondegenerate case.

The outline of the paper is as follows. Assumptions and equivalent definitions of viscosity solutions of second order integrodifferential operators are given in the next section, with a careful attention on the integral term. In accordance with financial applications, we allow the coefficients of the state process and of the value function to be unbounded. In Section 3, using the dynamic programming principle and stating some careful estimates, we prove that the value function is continuous and is a viscosity solution of (1.3). Section 4 is devoted to uniqueness, and finally in Section 5, we apply the viscosity solution method to the case of linear integrodifferential operators.

2 Assumptions and Definitions

We assume that the functions $b, \sigma, \gamma, f, g, c$ are continuous with respect to (t, x, α). We also assume also $\gamma(t, x, \alpha, .)$ is bounded uniformly in $\alpha \in U$ in a neighbourhood of $z = 0$ (for example $|z| < 1$). The Lévy measure m is a positive σ-finite measure on $I\!R^n$, eventually with a singularity in 0, such that:

$$\int_{|z| \geq 1} m(dz) \quad < \quad +\infty. \tag{2.1}$$

Let us point out that this standard assumption is satisfied for all stable processes. Furthermore, we shall make the following assumptions: there exist $K > 0, c_0 \in I\!R$, $\rho : I\!R^n \mapsto I\!R_+$, with $\int_{I\!R^n} \rho^2(z) m(dz) < +\infty$, such that for all $t, s \in [0, T]$, $x, y \in I\!R^n$ and $\alpha \in U$,

$$\begin{aligned} |b(t,x,\alpha) - b(t,y,\alpha)| + |\sigma(t,x,\alpha) - \sigma(t,y,\alpha)| &\leq K|x-y| && (2.2)\\ |\gamma(t,x,\alpha,z) - \gamma(t,y,\alpha,z)| &\leq \rho(z)|x-y| && (2.3)\\ |\gamma(t,x,\alpha,z)| &\leq \rho(z)(1+|x|) && (2.4) \end{aligned}$$

$$\begin{aligned} |f(t,x,\alpha) - f(s,y,\alpha)| + |g(x) - g(y)| &\leq K\left[|t-s| + |x-y|\right] && (2.5)\\ c(t,x) &\geq c_0. && (2.6) \end{aligned}$$

Notice that the global Lipschitz conditions (2.2), (2.5) and the continuity of b, σ, f with respect to (t, α) yield the global linear growth conditions:

$$\begin{aligned} |b(t,x,\alpha)| + |\sigma(t,x,\alpha)| &\leq K(1+|x|), && (2.7)\\ |f(t,x,\alpha)| + |g(x)| &\leq K(1+|x|). && (2.8) \end{aligned}$$

Assumptions on b, σ, γ ensure that, for each admissible control $\alpha. \in \mathcal{U}$, there exists a unique strong solution to (1.1) with an initial condition (see Gihman–Skorohod [12]). Let us give two examples for which these technical conditions, especially on the jump component, are satisfied.

Example 1
m is a finite measure on $I\!R$: $m(dz) = \lambda h(z)dz$, and h is a probability density which admits second order moment. $\gamma(t, x, \alpha, z) = xz$ so that assumptions (2.3)–(2.4) are satisfied with $\rho(z) = z$. When $b = \sigma = 0$, (X_t) is a standard jump-Markov process with occurrence in state changes determined by a Poisson process with parameter λ and with transition probability given by $\pi(x, ,) = \frac{1}{x}\int_{,} h(\frac{y}{x})dy$, for any Borelian , of $I\!R$.

Example 2
m is the Lévy measure of a symmetric stable process of order $\beta \in]0,2[$, i.e. $m(dz) = |z|^{-1-\beta}dz$, and $\gamma(t,x,\alpha,z) = z$ so that (2.3)–(2.4) are satisfied with $\rho(z) = 0$.

For $q \geq 0$, we define:

$$\mathcal{C}_q([0,T]\times I\!R^n) \quad = \quad \left\{ \phi \in \mathcal{C}^0([0,T]\times I\!R^n) \ / \ \sup_{[0,T]\times I\!R^n} \frac{|\phi(t,x)|}{1+|x|^q} < +\infty \right\}.$$

Let us define (by abuse of notation) for $\alpha \in U$, $t \in [0,T], x \in I\!R^n, p \in I\!R^n, M \in S^n$ (where S^n is the space of symmetric $n \times n$ matrices) the operator:

$$A^\alpha(t,x,p,M) \quad = \quad tr\left(\frac{1}{2}\sigma\sigma'(t,x,\alpha)M\right) + b(t,x,\alpha).p.$$

For $\eta \in (0,1)$ and $\phi \in \mathcal{C}^2([0,T]\times I\!R^n)$, we can define:

$$B^\alpha_\eta(t,x,\phi) \quad = \quad \int_{|z|\leq\eta} [\phi(t,x+\gamma(t,x,\alpha,z)) - \phi(t,x) - \gamma(t,x,\alpha,z).D_x\phi(t,x)]\, m(dz).$$

Indeed this integral term can be written also:

$$B^\alpha_\eta(t,x,\phi) = \int_{|z|\leq\eta}\int_0^1 (1-y)tr\left(D^2_x\phi(t,x+y\gamma(t,x,\alpha,z))\gamma\gamma'(t,x,\alpha,z)\right) dy m(dz).$$

whose integrand is bounded by: $C_{t,x,\eta}\rho^2(z)$, for all $|z| \leq \eta$, since $\gamma(t,x,\alpha,.)$ is bounded uniformly in $\alpha \in U$ for $|z| < 1$. Therefore, $B^\alpha_\eta(t,x,\phi)$ is convergent and bounded uniformly in $\alpha \in U$ and

$$\lim_{\eta\to 0^+}\sup_{\alpha\in U} B^\alpha_\eta(t,x,\phi) = 0. \tag{2.9}$$

We define also for $\phi \in \mathcal{C}_2([0,T]\times I\!R^n)$:

$$B^{\alpha,\eta}(t,x,p,\phi) \quad = \quad \int_{|z|\geq\eta} [\phi(t,x+\gamma(t,x,\alpha,z)) - \phi(t,x) - \gamma(t,x,\alpha,z).p]\, m(dz).$$

The integrand of $B^{\alpha,\eta}(t,x,p,\phi)$ is bounded by: $C_{p,x}(1+|\gamma(t,x,\alpha,z)|^2)$ and then this integral term is convergent and bounded uniformly in $\alpha \in U$ from assumptions (2.1) on m and (2.4) on γ. Finally, we can define for all

$\phi \in \mathcal{C}^2([0,T] \times I\!R^n) \cap \mathcal{C}_2([0,T] \times I\!R^n)$, the integrodifferential operator:

$$\begin{aligned} B^\alpha(t,x,\phi) &= \int_{I\!R^n} [\phi(t,x+\gamma(t,x,\alpha,z)) - \phi(t,x) \\ &\qquad -\gamma(t,x,\alpha,z).D_x\phi(t,x)]\, m(dz) \\ &= B^\alpha_\eta(t,x,\phi) + B^{\alpha,\eta}(t,x,D_x\phi(t,x),\phi). \end{aligned} \tag{2.10}$$

Therefore, the Bellman equation (1.3) is well defined for all functions $v \in \mathcal{C}^2([0,T] \times I\!R^n) \cap \mathcal{C}_2([0,T] \times I\!R^n)$, and is written as:

$$\begin{aligned} &\min\Big\{ c(t,x)v(t,x) - \frac{\partial v}{\partial t}(t,x) \\ &+ \min_{\alpha\in U}\left(-f(t,x,\alpha) - A^\alpha(t,x,D_xv(t,x),D_x^2v(t,x)) - B^\alpha(t,x,v)\right) \quad ; \\ &v(t,x) - g(x)\Big\} \;=\; 0. \end{aligned} \tag{2.11}$$

But as it is well-known, the value function v defined in (1.2) is not smooth and equation (2.11) should be interpreted in a weaker sense. Adapting the notion of viscosity solutions introduced by Crandall–P.L. Lions [7] and then by Soner [23] and Sayah [22] for first order integrodifferential operators, we define:

Definition 2.1 *(i) Any $v \in \mathcal{C}^0([0,T] \times I\!R^n)$ is a viscosity supersolution (subsolution) of* (2.11) *if*

$$\begin{aligned} &\min\Big\{ c(t,x)v(t,x) - \frac{\partial \psi}{\partial t}(t,x) \\ &+ \min_{\alpha\in U}\left(-f(t,x,\alpha) - A^\alpha(t,x,D_x\psi(t,x),D_x^2\psi(t,x)) - B^\alpha(t,x,\psi)\right) \quad ; \\ &v(t,x) - g(x)\Big\} \geq 0 \end{aligned} \tag{2.12}$$

$(\,\leq 0)$ whenever $\psi \in \mathcal{C}^2([0,T] \times I\!R^n) \cap \mathcal{C}_2([0,T] \times I\!R^n)$ and $v - \psi$ has a global minimum (maximum) at $(t,x) \in [0,T) \times I\!R^n$.

(ii) u is a viscosity solution of (2.11) *if it is both super and subsolution.*

Repeating arguments of Soner ([23] Lemma 2.1) or Sayah ([22] Proposition 2.1), we easily obtain an equivalent formulation for viscosity solutions in $\mathcal{C}_2([0,T] \times I\!R^n)$.

Lemma 2.1 *Let $v \in \mathcal{C}_2([0,T] \times I\!R^n)$. Then v is a viscosity supersolution (subsolution) of* (2.11) *if and only if: $\forall \eta \in (0,1)$,*

$$\begin{aligned} &\min\Big\{ c(t,x)v(t,x) - \frac{\partial \psi}{\partial t}(t,x) \\ &+ \min_{\alpha\in U}\big(-f(t,x,\alpha) - A^\alpha(t,x,D_x\psi(t,x),D_x^2\psi(t,x)) - B^\alpha_\eta(t,x,\psi) \\ &-B^{\alpha,\eta}(t,x,D_x\psi(t,x),v)\big)\,;\, v(t,x) - g(x)\Big\} \geq 0 \end{aligned} \tag{2.13}$$

(≤ 0) whenever $\psi \in \mathcal{C}^2([0,T] \times I\!R^n)$ *and* $v - \psi$ *has a global minimum (maximum) at* $(t,x) \in [0,T) \times I\!R^n$.

Remark
As justified above, the two integral terms

$$B^\alpha_\eta(t,x,\psi)$$

and

$$B^{\alpha,\eta}(t,x,D_x\psi(t,x),v)$$

in the inequality (2.13) are well-defined and bounded uniformly in α, whenever $\psi \in \mathcal{C}^2([0,T] \times I\!R^n)$ and $v \in \mathcal{C}_2([0,T] \times I\!R^n)$.

In proving the uniqueness result for viscosity solutions of second order equations, it is convenient to give an intrinsic characterization of viscosity solutions. First, let us recall the notion of parabolic semijets as introduced in P.L. Lions [19]. Given $v \in \mathcal{C}^0([0,T] \times I\!R^n)$ and $(t,x) \in [0,T) \times I\!R^n$, we define the parabolic superjet:

$$\begin{aligned} \mathcal{P}^{2,+}v(t,x) \quad = \quad & \{(p_0,p,M) \in I\!R \times I\!R^n \times S^n \;/\; v(s,y) \leq v(t,x) + p_0(s-t) \\ & +p.(y-x) + \frac{1}{2}(y-x).M(y-x) + o(|s-t| + |y-x|^2) \\ & \text{as } \; (s,y) \to (t,x)\} \end{aligned}$$

and its closure:

$$\begin{aligned} \bar{\mathcal{P}}^{2,+}v(t,x) \quad = \quad & \Big\{(p_0,p,M) = \lim_{n\to+\infty}(p_{0,n},p_n,M_n) \\ & \quad \text{with } \; (p_{0,n},p_n,M_n) \in \mathcal{P}^{2,+}v(t_n,x_n) \\ & \quad \text{and } \quad \lim_{n\to+\infty}(t_n,x_n,v(t_n,x_n)) = (t,x,v(t,x))\Big\}. \end{aligned}$$

Similarly, we consider the parabolic subjet $\mathcal{P}^{2,-}v(t,x) = -\mathcal{P}^{2,+}(-v)(t,x)$ and its closure $\bar{\mathcal{P}}^{2,-}v(t,x) = -\bar{\mathcal{P}}^{2,+}(-v)(t,x)$. It is proved in P.L. Lions [19] that

$$\begin{aligned} \mathcal{P}^{2,+(-)}v(t,x) \quad = \quad & \Big\{\Big(\frac{\partial\psi}{\partial t}(t,x), D_x\psi(t,x), D_x^2\psi(t,x)\Big), \psi \in \mathcal{C}^2([0,T] \times I\!R^n) \\ & \text{and } v - \psi \text{ has a global maximum (minimum) at } (t,x)\} \end{aligned}$$

Using the above definitions, Lemma 2.1 and continuity of the HJB operator, we have then an intrinsic formulation of viscosity solutions in $\mathcal{C}_2([0,T] \times I\!R^n)$.

Lemma 2.2 *Let* $v \in \mathcal{C}_2([0,T] \times I\!R^n)$ *be a viscosity supersolution (resp. subsolution) of* (2.11). *Then, for all* $\eta \in (0,1)$, $\forall (t,x) \in [0,T) \times I\!R^n$,

$\forall (p_0, p, M) \in \bar{\mathcal{P}}^{2,-}v(t,x)$ (resp. $\bar{\mathcal{P}}^{2,+}v(t,x)$), there exists $\psi \in \mathcal{C}^2([0,T] \times I\!R^n)$ such that

$$\min\Big\{c(t,x)v(t,x) - p_0 + \min_{\alpha\in U}\big(-f(t,x,\alpha) - A^\alpha(t,x,p,M)$$

$$-B^\alpha_\eta(t,x,\psi) - B^{\alpha,\eta}(t,x,p,v)\big)\,;v(t,x) - g(x)\Big\} \;\geq\; 0 \qquad (2.14)$$

(resp. ≤ 0).

Remarks
1. The test function ψ of the above lemma is such that $v - \psi$ has a global maximum (minimum) at (t_n, x_n) with $(t_n, x_n) \to (t, x)$.
2. While Definition 2.1 of viscosity solutions is convenient for establishing that the value function is a viscosity solution of the HJB equation thanks to the dynamic programming principle, this last equivalent formulation (2.14) of viscosity solutions in $\mathcal{C}_2([0,T] \times I\!R^n)$ will be particularly useful for proving comparison principles for (2.11).

3 Dynamic Programming and Viscosity Solutions

In this section, we focus on proving the continuity and the viscosity properties of the value function, as consequences of the dynamic programming principle. To simplify notations, we assume $c(t,x) = c$ (positive constant), which is not a restriction from assumption (2.6) and by considering the function $e^{-c_0' t}v(t,x)$ with $c_0' < c_0$. Therefore, the value function defined in (1.2) can be written as

$$v(t,x) = \sup_{\substack{\tau\,\in\,\mathcal{T}_{T-t}\\ \alpha.\,\in\,\mathcal{U}}} E\left[\int_0^\tau e^{-cs} f(s+t, X_s^{t,x}, \alpha_s)ds + e^{-c\tau} g(X_\tau^{t,x})\right] \quad (3.1)$$

where $\mathcal{T}_{T-t}$ denotes the set of all stopping times between 0 and $T-t$ and $X_s^{t,x}$ is the solution of the stochastic differential equation

$$\begin{aligned} dX_s &= b(s+t, X_{s-}, \alpha_s)ds + \sigma(s+t, X_{s-}, \alpha_s)dW_s \\ &\qquad + \int_{I\!R^n} \gamma(s+t, X_{s-}, \alpha_s, z)\tilde{\nu}(ds, dz), \\ X_0 &= x. \end{aligned}$$

First, we need some preliminary estimates on the moments of the jump diffusion state process. The proof of the following lemma is rejected in appendix.

Lemma 3.1 *Let (2.2), (2.3) and (2.4) hold. For any* $k \in [0,2]$*, there exists* $C = C(k,K,T) > 0$ *such that for all* $h, t \in [0,T]$, $x, y \in I\!R^n$, $\alpha \in \mathcal{U}$, $\tau \in \mathcal{T}_h$*:*

$$E\left|X_\tau^{t,x}\right|^k \leq C(1+|x|^k) \tag{3.2}$$

$$E\left|X_\tau^{t,x} - x\right|^k \leq C(1+|x|^k)h^{\frac{k}{2}} \tag{3.3}$$

$$E\left[\sup_{0\leq s\leq h}\left|X_s^{t,x} - x\right|\right]^k \leq C(1+|x|^k)h^{\frac{k}{2}} \tag{3.4}$$

$$E\left|X_\tau^{t,x} - X_\tau^{t,y}\right|^k \leq C|x-y|^2 \tag{3.5}$$

Remark
Estimates of the moments for stochastic differential equations are generally proved for deterministic time τ (see e.g. Krylov 1980 [18], Gihman-Skorohod [12]). Actually, these results can be generalized for any stopping times, essentially thanks to the optional sampling theorem. Note also that in the diffusion case, estimates of Lemma 3.1 are valid for all orders k, while it is generally not true in the jump diffusion case without any other assumption on γ.

We deduce easily from linear growth condition (2.8) on f, g and from estimate (3.2), with k=1, that v satisfies also a global linear growth condition:

$$v(t,x) \leq C(1+|x|). \tag{3.6}$$

Moreover, from Lipschitz condition (2.5) on f, g, and from estimate (3.5), we deduce that the value function is also Lipschitz in x uniformly in t:

$$|v(t,x) - v(t,y)| \leq C|x-y|. \tag{3.7}$$

As it is well-known, the dynamic programming principle yields that the value function is a viscosity solution of the corresponding HJB equation. The mathematical formulation of Bellman's principle for the optimal stopping of a controlled process is the following:

Proposition 3.1 *For all* $(t,x) \in [0,T] \times I\!R^n$, $h \in \mathcal{T}_{T-t}$*, we have*

$$v(t,x) = \sup_{\substack{\tau \in \mathcal{T}_{T-t} \\ \alpha. \in \mathcal{U}}} E\left[\int_0^{\tau\wedge h} e^{-cs} f(s+t, X_s^{t,x}, \alpha_s)ds + \mathbf{1}_{[\tau<h]} e^{-c\tau} g(X_\tau^{t,x}) + \mathbf{1}_{[h\leq\tau]} e^{-ch} v(t+h, X_h^{t,x})\right].$$

Actually, a stronger statement of Bellman's principle, which is more convenient for deriving the HJB equation, is:

Proposition 3.2 *Let $\epsilon > 0$. For all $(t,x) \in [0,T] \times I\!R^n$ and for each admissible control $\alpha. \in \mathcal{U}$, define the stopping time*

$$\tau^{\epsilon}_{t,x,\alpha} = \inf\left\{0 \leq s \leq T-t,\ v(s+t, X^{t,x}_s) \leq g(X^{t,x}_s) + \epsilon\right\}.$$

Therefore, if $\tau_\alpha \leq \tau^{\epsilon}_{t,x,\alpha}$ for all $\alpha \in \mathcal{U}$, we have:

$$v(t,x) = \sup_{\alpha.\in\mathcal{U}} E\left[\int_0^\tau e^{-cs} f(s+t, X^{t,x}_s, \alpha_s)ds + e^{-c\tau} v(t+\tau, X^{t,x}_\tau)\right]$$

(we omit the dependence of τ, τ^ϵ in t, x, α.)

Remarks

1. Proposition 3.1 is a consequence of Proposition 3.2 as observed in Krylov ([18] p.135).

2. When the control set $\mathcal{U}$ is reduced to a point α_0, i.e. there is no control on the process X, we have the well known result that for $\epsilon = 0$, τ^0 is an optimal stopping time for the problem and that $\{\int_0^s e^{-cu} f(u+t, X^{t,x}_u, \alpha_0)du + e^{-cs} v(s+t, X^{t,x}_s), 0 \leq s \leq \tau^0\}$ is a martingale.

Proposition 3.2 is proved by Krylov [18] for diffusion state process. Let us mention how it may be generalized in the jump diffusion case. We first prove Bellman's principle for the optimal control problem without stopping. It can be studied by approximating an arbitrary control strategy with the aid of step strategies, as in Krylov [18], or by using a semi-group approach as in Bensoussan–J.L. Lions [3] and Nisio (1976 [20]). In both methods, the main point is the continuity of the value function with respect to the space variable x, which is actually the present case (see (3.7)). To generalize the dynamic programming principle for the optimal control and stopping problem, we use the technique of randomized stopping developed by Krylov [18]. This method consists in approximating in a reasonable sense the value function of the optimal control problem with stopping rule, by value functions of a control problem without stopping.

Let us now prove the continuity of the value function.

Proposition 3.3 *Under assumptions (2.2)–(2.5), the value function $v \in \mathcal{C}^0([0,T] \times I\!R^n)$. More precisely, there exists a constant $C > 0$ such that for all $t, s \in [0,T]$, $x, y \in I\!R^n$,*

$$|v(t,x) - v(s,y)| \leq C\left[(1+|x|)|t-s|^{\frac{1}{2}} + |x-y|\right]. \qquad (3.8)$$

Proof: We have already seen that v is Lipschitz in x, uniformly in t (see (3.7)). To prove continuity property in time t, we use the dynamic programming principle. Let $0 \leq t < s \leq T$. Applying Proposition 3.1 to $v(t,x)$ with $h = s-t$, we deduce

$$\begin{aligned}
0 &\leq v(t,x) - v(s,x) \\
&= \sup_{\substack{\tau \in \mathcal{T}_{T-t} \\ \alpha . \in \mathcal{U}}} E\left[\int_0^{\tau\wedge(s-t)} e^{-cu} f(u+t, X_u^{t,x}, \alpha_u)du \right. \\
&\quad + \mathbf{1}_{[\tau<s-t]} e^{-c\tau}\left(g(X_\tau^{t,x}) - g(x)\right) + \mathbf{1}_{[\tau<s-t]} e^{-c\tau}\left(g(x) - v(s,x)\right) \\
&\quad + \mathbf{1}_{[s-t\leq\tau]} e^{-c(s-t)}\left(v(s, X_{s-t}^{t,x}) - v(s,x)\right) \\
&\quad \left. + \mathbf{1}_{[s-t\leq\tau]}\left(e^{-c(s-t)} - 1\right) v(s,x) + \mathbf{1}_{[\tau<s-t]}\left(e^{-c\tau} - 1\right) v(s,x)\right].
\end{aligned}$$

Now, thanks to linear growth condition (2.8) on f, Lipschitz assumption (2.5) on g, relation (3.7) and noting that $g(x) \leq v(s,x)$, $0 \leq 1 - e^{-ch} \leq ch$ and v satisfies (3.6), we deduce that:

$$\begin{aligned}
|v(t,x) - v(s,x)| &\leq C\left\{\int_0^{s-t} (1 + E|X_u^{t,x}|)du + (1+|x|)|s-t| \right. \\
&\quad \left. + \sup_{\substack{\tau \in \mathcal{T}_{s-t} \\ \alpha . \in \mathcal{U}}} E|X_\tau^{t,x} - x| + \sup_{\alpha . \in \mathcal{U}} E|X_{s-t}^{t,x} - x| \right\}.
\end{aligned}$$

We conclude with the estimates of Lemma 3.1. □

Remarks

1. The above proposition shows that v is in $W^1([0,T] \times \mathbb{R}^n)$, the set of continuous functions in $[0,T] \times \mathbb{R}^n$, Lipschitz in x, uniformly in t, and more generally in $UC_x([0,T] \times \mathbb{R}^n)$, the set of continuous functions in $[0,T] \times \mathbb{R}^n$, uniformly continuous in x, uniformly in t.

2. If the linear growth condition (2.8) on f, g is strengthen to $f, g \in \mathcal{C}_\mu([0,T] \times \mathbb{R}^n)$, for $\mu \in [0,1]$, the preceding arguments show that v is also in $\mathcal{C}_\mu([0,T] \times \mathbb{R}^n)$ and (3.8) becomes:

$$|v(t,x) - v(s,y)| \leq C\left[(1 + |x|^\mu)|t-s|^{\frac{1}{2}} + |x-y|\right].$$

The following theorem relates the value function defined by (1.2) (or (3.1)) to the Bellman equation (2.11). We essentially adapt arguments of P.L. Lions [19] to an optimal control and stopping problem of a jump diffusion process.

Theorem 3.1 *Under assumptions* (2.1)–(2.5)*, the value function v is a viscosity solution of* (2.11).

Proof: We already know that $v \in \mathcal{C}^0([0,T] \times I\!R^n)$ (Proposition 3.3). We first prove that v is a supersolution of (2.11). Let $(t,x) \in [0,T) \times I\!R^n$ and $\psi \in \mathcal{C}^2([0,T] \times I\!R^n) \cap \mathcal{C}_2([0,T] \times I\!R^n)$ such that without loss of generality

$$0 = (v - \psi)(t,x) = \min_{[0,T]\times I\!R^n} (v - \psi). \tag{3.9}$$

Apply Proposition 3.1 with $0 < h < T - t$:

$$v(t,x) \quad \geq \quad \sup_{\alpha.\in\mathcal{U}} E\left[\int_0^h e^{-cs} f(s+t, X_s^{t,x}, \alpha_s)ds + e^{-ch} v(t+h, X_h^{t,x})\right].$$

From (3.9), it yields

$$0 \quad \geq \quad \sup_{\alpha.\in\mathcal{U}} E\left[\int_0^h e^{-cs} f(s+t, X_s^{t,x}, \alpha_s)ds + e^{-ch}\psi(t+h, X_h^{t,x}) - \psi(t,x)\right]$$

By applying Itô's formula to $e^{-cs}\psi(t+s, X_s^{t,x})$, we obtain from assumptions (2.1)–(2.5) and estimates of Lemma 3.1:

$$\begin{aligned}\sup_{\alpha\in\mathcal{U}} \frac{1}{h} E\Bigg[\int_0^h \Big(-c\psi(t,x) + \frac{\partial\psi}{\partial t}(t,x) \quad &+ \quad A_t^{\alpha_s}\psi(t,x) \; + \; B_t^{\alpha_s}\psi(t,x) \\ &+ \quad f(t,x,\alpha_s)\Big)\, ds\Bigg] \;\leq\; \epsilon(h).\end{aligned}$$

Choosing $\alpha_s = \alpha \in U$, we find by sending $h \rightarrow 0^+$

$$c\psi(t,x) - \frac{\partial\psi}{\partial t}(t,x) - A_t^{\alpha}\psi(t,x) - B_t^{\alpha}\psi(t,x) - f(t,x,\alpha) \geq 0,$$

which provides supersolution inequality (2.12) since from definition of the value function, $v(t,x) \geq g(x)$. To prove subsolution property, we use Proposition 3.2. Let $(t,x) \in [0,T) \times I\!R^n$ and $\psi \in \mathcal{C}^2([0,T] \times I\!R^n) \cap \mathcal{C}_2([0,T] \times I\!R^n)$ such that without loss of generality

$$0 = (v - \psi)(t,x) = \max_{[0,T]\times I\!R^n} (v - \psi). \tag{3.10}$$

We already know that $v(t,x) \geq g(x)$. If $v(t,x) = g(x)$, the inequality of subsolution is obviously satisfied. Assume therefore that $v(t,x) > g(x)$ and define

$$\epsilon = \frac{v(t,x) - g(x)}{2} > 0.$$

For each control $\alpha \in \mathcal{U}$, define as in Proposition 3.2, the stopping time

$$\tau_{t,x,\alpha}^{\epsilon} = \inf\left\{0 \leq s \leq T - t,\; v(s+t, X_s^{t,x}) \leq g(X_s^{t,x}) + \epsilon\right\}.$$

Thus, we have for all $h > 0$, since $h \wedge \tau^\epsilon \leq \tau^\epsilon$:

$$\begin{aligned} v(t,x) &= \sup_{\alpha.\in\mathcal{U}} E\left[\int_0^{h\wedge\tau^\epsilon} e^{-cs} f(s+t, X_s^{t,x}, \alpha_s)ds \right. \\ &\qquad \left. + \, e^{-c(h\wedge\tau^\epsilon)} v(t+(h\wedge\tau^\epsilon), X_{h\wedge\tau^\epsilon}^{t,x})\right] \end{aligned}$$

and then from (3.10):

$$\begin{aligned} 0 &\leq \sup_{\alpha.\in\mathcal{U}} \frac{1}{h} E\left[\int_0^{h\wedge\tau^\epsilon} e^{-cs} f(s+t, X_s^{t,x}, \alpha_s)ds \right. \\ &\qquad \left. + e^{-ch\wedge\tau^\epsilon} \psi(t+h\wedge\tau^\epsilon, X_{h\wedge\tau^\epsilon}^{t,x}) - \psi(t,x) \ \right]. \end{aligned}$$

Apply Itô's formula to $e^{-cs}\psi(t+s, X_s^{t,x})$ and use as above assumptions (2.1)–(2.5) with estimates of Lemma 3.1 to obtain:

$$\begin{aligned} \epsilon(h) &\leq \sup_{\alpha.\in\mathcal{U}} \frac{1}{h} E\left[\int_0^{h\wedge\tau^\epsilon} \left(-c\psi(t,x) + \frac{\partial\psi}{\partial t}(t,x) + A_t^{\alpha_s}\psi(t,x) \right.\right. \\ &\qquad \left.\left. + \, B_t^{\alpha_s}\psi(t,x) + f(t,x,\alpha_s)\right) ds\right] \\ &\leq \sup_{\alpha\in U} \left\{-c\psi(t,x) + \frac{\partial\psi}{\partial t}(t,x) + A_t^\alpha\psi(t,x) + B_t^\alpha\psi(t,x) + f(t,x,\alpha)\right\} \\ &\qquad . \sup_{\alpha\in\mathcal{U}} E\left[\frac{h\wedge\tau^\epsilon}{h}\right]. \end{aligned} \tag{3.11}$$

Consider the function $\tilde{v}(s,x) = v(s+t,x) - g(x)$. Then $\tilde{v}$ satisfies the same continuity relation (3.8) as v, and for each control $\alpha \in \mathcal{U}$:

$$\begin{aligned} P[\tau_{t,x,\alpha}^\epsilon \leq h] &\leq P\left[\sup_{0\leq s\leq h} \left|\tilde{v}(s, X_s^{t,x}) - \tilde{v}(0,x)\right| \geq \epsilon\right] \\ &\leq \frac{1}{\epsilon^2} E\left[\sup_{0\leq s\leq h} \left|\tilde{v}(s, X_s^{t,x}) - \tilde{v}(0,x)\right|\right]^2 \\ &\leq \frac{C}{\epsilon^2} E\left[(1+|x|)h^{\frac{1}{2}} + \sup_{0\leq s\leq h} |X_s^{t,x} - x|\right]^2 \\ &\leq C'h \end{aligned}$$

where the last inequality is derived from estimate (3.4) and C' is a constant independent of $\alpha \in \mathcal{U}$. Sending therefore $h \to 0^+$ in the inequality (3.11), we have:

$$0 \leq -c\psi(t,x) + \frac{\partial\psi}{\partial t}(t,x) + \sup_{\alpha\in U} \{A_t^\alpha\psi(t,x) + B_t^\alpha\psi(t,x) + f(t,x,\alpha)\}$$

and finally the subsolution inequality (2.12). □

4 Uniqueness

Uniqueness proofs for viscosity solutions of first-order integrodifferential operators were given in Soner [24] and Sayah [22]. The second order case introduces other difficulties which may be overcome thanks to Ishii's lemma (1989 [15]). In this section, we will use an intrinsic version of this lemma, proved in Crandall–Ishii (1990 [5] Theorem 9). We state comparison principles following, for the most part, classical viscosity solution techniques with modifications arising from the integral term.

We shall work in the space of functions $UC_x([0,T]\times I\!R^n)$, even though, under our assumptions, the value function v is more precisely in $W^1([0,T]\times I\!R^n)$. As a matter of fact, this slight extension is straightforward. Since gain and reward functions f and g are generally unbounded in financial applications, we do not restrict to bounded viscosity solutions (recall that $UC_x([0,T]\times I\!R^n)\subset \mathcal{C}_1([0,T]\times I\!R^n)$). Sayah [22] proved an uniqueness result for unbounded viscosity solutions of first order integrodifferential operators. However her method does not apply in our context, since her structure conditions ([22] (5.3)–(5.4) p.1081–1082) are clearly not satisfied here, unless the coefficients b, σ, γ are bounded. To avoid such restrictive assumptions, we shall adapt arguments of Ishii [14] (see also Barles–Buckdahn–Pardoux 1994 [1]) and prove the following comparison principle.

Theorem 4.1 *Assume* (2.1)–(2.5). *Let* u *(resp.* v*)* $\in UC_x([0,T]\times I\!R^n)$ *be a viscosity subsolution (resp. supersolution) of* (2.11). *If* $u(T,x)\leq v(T,x)$ *for all* $x\in I\!R^n$, *then*

$$u(t,x)\leq v(t,x) \qquad \forall (t,x)\in[0,T]\times I\!R^n. \tag{4.1}$$

Proof: First, observe that it suffices to prove comparison inequality (4.1) for all $(t,x)\in(0,T]\times I\!R^n$ by continuity of u and v in $t=0$. For $\beta,\epsilon,\delta,\lambda>0$, let us define the function Φ in $(0,T]\times I\!R^n$:

$$\Phi(t,x,y) = u(t,x) - v(t,y) - \frac{\beta}{t} - \frac{1}{2\epsilon}|x-y|^2 - \delta e^{\lambda(T-t)}\left(|x|^2+|y|^2\right) \tag{4.2}$$

Since $u,v\in\mathcal{C}_1([0,T]\times I\!R^n)$, Φ admits a maximum at $(\bar{t},\bar{x},\bar{y})\in(0,T]\times I\!R^n\times I\!R^n$ (we omit the dependance on β, ϵ, δ and λ to alleviate notations). Writing that $2\Phi(\bar{t},\bar{x},\bar{y})\geq\Phi(\bar{t},\bar{x},\bar{x})+\Phi(\bar{t},\bar{y},\bar{y})$ and using uniform continuity of u and v, we easily check (see e.g. Ishii 1984 [14] Theorem 1.1) that:

$$\frac{1}{\epsilon}|\bar{x}-\bar{y}|^2 \quad\leq\quad \omega(C\epsilon^{\frac{1}{2}}) \tag{4.3}$$

where C is a positive constant independent of β, ϵ, δ, λ, and ω is a modulus of continuity for u and v. From the inequality $\Phi(T,0,0)\leq\Phi(\bar{t},\bar{x},\bar{y})$ and since $u,v\in\mathcal{C}_1([0,T]\times I\!R^n)$, we have:

$$\delta(|\bar{x}|^2+|\bar{y}|^2) \quad\leq\quad C(1+|\bar{x}|+|\bar{y}|).$$

Using Young's inequality, we deduce that there exists a constant C_δ, depending on δ but not on ϵ, such that:

$$|\bar{x}|,\ |\bar{y}| \quad \leq \quad C_\delta. \tag{4.4}$$

It follows from (4.3)–(4.4) that, along a subsequence, $(\bar{t}, \bar{x}, \bar{y})$ converges to $(t_0, x_0, x_0) \in [0,T] \times I\!R^n \times I\!R^n$, as $\epsilon \to 0^+$.

If $\bar{t} = T$ then writing that $\Phi(t,x,x) \leq \Phi(T, \bar{x}, \bar{y})$, we have

$$\begin{aligned} u(t,x) - v(t,x) - \frac{\beta}{t} - 2\delta e^{\lambda(T-t)}|x|^2 \quad &\leq \quad u(T,\bar{x}) - v(T,\bar{x}) \\ &\qquad + v(T,\bar{x}) - v(T,\bar{y}) \\ &\leq \quad \omega(|\bar{x} - \bar{y}|) \end{aligned}$$

where the second inequality follows from uniform continuity of v and since by assumption $u(T,x) \leq v(T,x)$. Sending $\beta, \epsilon, \delta \to 0^+$ and using estimate (4.3), we have: $u(t,x) \leq v(t,x)$. Assume therefore that $\bar{t} < T$. Applying Theorem 9 of Crandall–Ishii [5] to the function $\Phi(t,x,y)$ at point $(\bar{t}, \bar{x}, \bar{y}) \in (0,T) \times I\!R^n \times I\!R^n$, we can find $p_0 \in I\!R$, $M, N \in S^n$ such that

$$\begin{aligned} &\Bigg(p_0 - \frac{\beta}{\bar{t}^2} - \lambda\delta e^{\lambda(T-\bar{t})}\left(|\bar{x}|^2 + |\bar{y}|^2\right), \\ &\frac{1}{\epsilon}(\bar{x} - \bar{y}) + 2\delta e^{\lambda(T-\bar{t})}\bar{x}, M + 2\delta e^{\lambda(T-\bar{t})} I_n \Bigg) \quad \in \quad \bar{\mathcal{P}}^{2,+} u(\bar{t}, \bar{x}) \end{aligned}$$

$$\left(p_0, \frac{1}{\epsilon}(\bar{x} - \bar{y}) - 2\delta e^{\lambda(T-\bar{t})}\bar{y}, N - 2\delta e^{\lambda(T-\bar{t})} I_n \right) \ \in \ \bar{\mathcal{P}}^{2,-} v(\bar{t}, \bar{y})$$

and under Lipschitz condition (2.2) on σ,

$$tr\left(\frac{1}{2}\sigma\sigma'(\bar{t}, \bar{x}, \alpha) M\right) - tr\left(\frac{1}{2}\sigma\sigma'(\bar{t}, \bar{y}, \alpha) N\right) \quad \leq \quad \frac{C}{\epsilon}|\bar{x} - \bar{y}|^2. \tag{4.5}$$

The fact that u and v are respectively viscosity subsolution and supersolution in $\mathcal{C}_2([0,T] \times I\!R^n)$ of (2.11) yields (thanks to Lemma 2.2): $\forall \eta \in (0,1)$,

$$\begin{aligned} \min\Bigg\{ &cu(\bar{t}, \bar{x}) - p_0 + \frac{\beta}{\bar{t}^2} + \lambda\delta e^{\lambda(T-\bar{t})}\left(|\bar{x}|^2 + |\bar{y}|^2\right) + \min_{\alpha \in U}\Big(-f(\bar{t}, \bar{x}, \alpha) \\ &- A^\alpha(\bar{t}, \bar{x}, \frac{1}{\epsilon}(\bar{x} - \bar{y}) + 2\delta e^{\lambda(T-\bar{t})}\bar{x}, M + 2\delta e^{\lambda(T-\bar{t})} I_n) \\ &- B_\eta^\alpha(\bar{t}, \bar{x}, \psi_1) - B^{\alpha,\eta}(\bar{t}, \bar{x}, \frac{1}{\epsilon}(\bar{x} - \bar{y}) + 2\delta e^{\lambda(T-\bar{t})}\bar{x}, u)\Big); u(\bar{t}, \bar{x}) - g(\bar{x}) \Bigg\} \ \leq \ 0 \end{aligned}$$

and

$$\begin{aligned} &\min\{cv(\bar{t}, \bar{y}) - p_0 \\ &+ \min_{\alpha \in U}\Big(-f(\bar{t}, \bar{y}, \alpha) - A^\alpha(\bar{t}, \bar{y}, \frac{1}{\epsilon}(\bar{x} - \bar{y}) - 2\delta e^{\lambda(T-\bar{t})}\bar{y}, N - 2\delta e^{\lambda(T-\bar{t})} I_n) \end{aligned}$$

$$-B_\eta^\alpha(\bar t,\bar y,\psi_2) - B^{\alpha,\eta}(\bar t,\bar y,\frac{1}{\epsilon}(\bar x-\bar y) - 2\delta e^{\lambda(T-\bar t)}\bar y, v)\Big) ; v(\bar t,\bar y) - g(\bar y)\Big\} \;\geq\; 0$$

for some $\psi_1,\psi_2 \in \mathcal{C}^2([0,T]\times I\!R^n)$. Subtracting these two inequalities and remarking that $\min(a,b)-\min(d,e)\leq 0$ implies either $a-d\leq 0$ or $b-e\leq 0$, we divide our consideration into two cases:

(i) the case

$$\begin{aligned}
& c\left[u(\bar t,\bar x) - v(\bar t,\bar y)\right] + \frac{\beta}{\bar t^2} + \lambda\delta e^{\lambda(T-\bar t)}\left(|\bar x|^2+|\bar y|^2\right) \\
\leq\;& \sup_\alpha \left\{ f(\bar t,\bar x,\alpha) - f(\bar t,\bar y,\alpha)\right\} \\
& + \sup_\alpha \Big\{ A^\alpha\left(\bar t,\bar x,\frac{1}{\epsilon}(\bar x-\bar y) + 2\delta e^{\lambda(T-\bar t)}\bar x, M + 2\delta e^{\lambda(T-\bar t)} I_n\right) \\
& \qquad - A^\alpha\left(\bar t,\bar y,\frac{1}{\epsilon}(\bar x-\bar y) - 2\delta e^{\lambda(T-\bar t)}\bar y, N - 2\delta e^{\lambda(T-\bar t)} I_n\right)\Big\} \\
& + \sup_\alpha \left\{ B_\eta^\alpha(\bar t,\bar x,\psi_1) - B_\eta^\alpha(\bar t,\bar y,\psi_2)\right\} \\
& + \sup_\alpha \Big\{ B^{\alpha,\eta}(\bar t,\bar x,\frac{1}{\epsilon}(\bar x-\bar y) + 2\delta e^{\lambda(T-\bar t)}\bar x, u) \\
& \qquad - B^{\alpha,\eta}(\bar t,\bar y,\frac{1}{\epsilon}(\bar x-\bar y) - 2\delta e^{\lambda(T-\bar t)}\bar y, v)\Big\} \\
\equiv\;& (I_1) + (I_2) + (I_{3,\eta}) + (I_3^\eta)
\end{aligned} \tag{4.6}$$

In view of conditions (2.2), (2.5), (2.7) on b,σ,f and from (4.5), we have the classical estimates of (I_1) and (I_2):

$$\begin{aligned}
(I_1) &\leq C|\bar x-\bar y| \\
(I_2) &\leq C\left[\frac{1}{\epsilon}|\bar x-\bar y|^2 + \delta e^{\lambda(T-\bar t)}(1+|\bar x|^2+|\bar y|^2)\right].
\end{aligned}$$

As justified in Section 2, the two integral terms of $(I_{3,\eta})$ are convergent and bounded uniformly in α since $\psi_1,\psi_2\in\mathcal{C}^2([0,T]\times I\!R^n)$ and we have from (2.9):

$$\limsup_{\eta\to 0^+}\,(I_{3,\eta}) \;\leq\; 0$$

As also explained in Section 2, the two integral terms of (I_3^η) are convergent and bounded uniformly in α since $u,v\in\mathcal{C}_2([0,T]\times I\!R^n)$. Moreover, from the definition (4.2) of Φ, the differences of these two integrands is given by:

$$\begin{aligned}
& \left[u(\bar t,\bar x+\gamma(\bar t,\bar x,\alpha,z)) - u(\bar t,\bar x) - \gamma(\bar t,\bar x,\alpha,z).\left(\frac{1}{\epsilon}(\bar x-\bar y) + 2\delta e^{\lambda(T-\bar t)}\bar x\right)\right] \\
-\;& \left[v(\bar t,\bar y+\gamma(\bar t,\bar y,\alpha,z)) - v(\bar t,\bar y) - \gamma(\bar t,\bar y,\alpha,z).\left(\frac{1}{\epsilon}(\bar x-\bar y) - 2\delta e^{\lambda(T-\bar t)}\bar y\right)\right] \\
=\;& \Phi(\bar t,\bar x+\gamma(\bar t,\bar x,\alpha,z),\bar y+\gamma(\bar t,\bar y,\alpha,z)) - \Phi(\bar t,\bar x,\bar y)
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{2\epsilon}|\gamma(\bar t,\bar x,\alpha,z)-\gamma(\bar t,\bar y,\alpha,z)|^2 \\
&+ \delta e^{\lambda(T-\bar t)}\left[|\gamma(\bar t,\bar x,\alpha,z)|^2+|\gamma(\bar t,\bar y,\alpha,z)|^2\right].
\end{aligned}
$$

Since $(\bar t,\bar x,\bar y)$ is a maximum point of Φ in $(0,T]\times I\!R^n\times I\!R^n$: $\Phi(\bar t,\bar x+\gamma(\bar t,\bar x,\alpha,z),\bar y+\gamma(\bar t,\bar y,\alpha,z))-\Phi(\bar t,\bar x,\bar y)\le 0$, and we have, therefore, by assumptions (2.3)–(2.4) on γ: for every $\eta\in(0,1)$

$$
(I_3^\eta) \quad \le \quad C\left[\frac{1}{2\epsilon}|\bar x-\bar y|^2+\delta e^{\lambda(T-\bar t)}(1+|\bar x|^2+|\bar y|^2)\right].
$$

Writing that $\Phi(t,x,x)\le\Phi(\bar t,\bar x,\bar x)$ and using inequality (4.6), we have (recall that $c,\beta>0$):

$$
\begin{aligned}
& u(t,x)-v(t,x)-\frac{\beta}{t}-2\delta e^{\lambda(T-t)}|x|^2 \\
\le\ & \frac{1}{c}\left[(I_1)+(I_2)+(I_{3,\eta})+(I_3^\eta)\right]-\frac{\lambda}{c}\delta e^{\lambda(T-\bar t)}\left(|\bar x|^2+|\bar y|^2\right).
\end{aligned}
$$

Sending $\epsilon,\eta\to 0^+$, with the above estimates of (I_1)-(I_2)-$(I_{3,\eta})$-(I_3^η), we obtain:

$$
u(t,x)-v(t,x)-\frac{\beta}{t}-2\delta e^{\lambda(T-t)}|x|^2\le\frac{2\delta}{c}e^{\lambda(T-t_0)}\left[C(1+2|x_0|^2)-\lambda|x_0|^2\right] \tag{4.7}
$$

Choose λ sufficiently large positive ($\lambda\ge 2C$) and send $\beta,\delta\to 0^+$ to conclude that $u(t,x)\le v(t,x)$.

(ii) the second case occurs if

$$
u(\bar t,\bar x)-v(\bar t,\bar y)+g(\bar y)-g(\bar x)\le 0.
$$

Using Lipschitz condition (2.5) on g and estimate (4.3), we obtain that $\limsup_{\epsilon\to 0^+}(u(\bar t,\bar x)-v(\bar t,\bar y))\le 0$ and finally that

$$
u(t,x)\le v(t,x).
$$

□

5 Applications of Viscosity Solutions to the Case of Linear Integrodifferential Operators

In this section, we apply the viscosity solution approach to a problem without control and stopping time, and we consider therefore the linear parabolic integrodifferential operator

$$
\mathcal{L}v \quad = \quad -\frac{\partial v}{\partial t}-A_t v-B_t v+c(t,x)v.
$$

A and B are the operators defined in the introduction and are associated to a jump diffusion process X_t solution of (1.1), where the dependence upon control α is suppressed.

Given $v \in \mathcal{C}^0([0,T] \times I\!R^n)$, we say that $\mathcal{L}v \geq 0$ (resp. ≤ 0) in the viscosity sense if v is a viscosity supersolution (resp. subsolution), in the sense of Definition 2.1, of $\mathcal{L}v = 0$. We also say that $\mathcal{L}v > 0$ in the viscosity sense if the supersolution inequality is strict. Thanks to the linearity of the operator $\mathcal{L}$, it is easy to check that if $\mathcal{L}v \geq 0$ in the viscosity sense and if $w \in \mathcal{C}^2([0,T] \times I\!R^n) \cap \mathcal{C}_2([0,T] \times I\!R^n)$ such that $\mathcal{L}w \geq 0$, then $\mathcal{L}(v+w) \geq 0$ in the viscosity sense.

5.1 Maximum principles in $\mathcal{C}^0([0,T] \times I\!R^n)$

The results of Section 4 extend in particular maximum principles for the operator $\mathcal{L}$ in Sobolev spaces, proved in Bony [4] or Bensoussan–J.L. Lions [3], to the set $\mathcal{C}_2([0,T] \times I\!R^n)$. We give in the linear case a simpler proof under weaker assumptions and that do not use the classical viscosity solution techniques. We first consider the case of bounded domains. The definition of viscosity solutions in an open set Q of $[0,T] \times I\!R^n$ is straightforward. Let us just note that from Definition 2.1, where the nonlocal integral term bears only on the test function ψ, we don't need to define v outside Q, contrarily to other notions of solutions. To simplify the notations, we take Q of the form $Q = (0,T) \times \mathcal{O}$ where $\mathcal{O}$ is an open bounded domain in $I\!R^n$. Let us define

$$\begin{array}{rl} \bar{Q} \; : \; \text{closure of } Q & \partial Q \; : \; \text{boundary of } Q \\ Q_0 = Q \cup (\{0\} \times \mathcal{O}) & \partial_0 Q = \partial Q \setminus (\{0\} \times \mathcal{O}) \,. \end{array}$$

We have the following result.

Proposition 5.1 *Assume* (2.6). *Let $v \in \mathcal{C}^0(\bar{Q})$. If $\mathcal{L}v \geq 0$ in Q_0, in the viscosity sense, and $v \geq 0$ on $\partial_0 Q$, then*

$$v(t,x) \geq 0 \qquad \forall (t,x) \in \bar{Q}.$$

Moreover, if $\mathcal{L}v > 0$ in Q_0, in the viscosity sense, then

$$v(t,x) > 0 \qquad \forall (t,x) \in Q_0.$$

Proof: As usual, by considering the function $e^{-c_0 t} v$, we amount to $c(t,x) \geq 0$. If the first assertion is false, then for ϵ sufficiently small positive, $v - \epsilon(t-T)$ attains in $\bar{Q}$ a negative minimum at $(t_0,x_0) \in Q_0$ and $v(t_0,x_0) \leq 0$. By application of Definition 2.1 with the test function $\psi = \epsilon(t-T)$, we have

$$-\epsilon + c(t_0,x_0)v(t_0,x_0) \geq 0$$

and then $v(t_0, x_0) > 0$, which is a contradiction. If the second assertion is false, then v attains a nil minimum at $(\bar{t}, \bar{x}) \in Q_0$. Writing the strict supersolution inequality with the test function $\psi = 0$, we have now

$$c(\bar{t}, \bar{x}) v(\bar{t}, \bar{x}) > 0$$

hence, $v(\bar{t}, \bar{x}) > 0$, which is a contradiction. □

Remark
The preceding proof shows that maximum principles in bounded domains are valid more generally for operators $\mathcal{L}$ of the form:

$$\mathcal{L}v \quad = \quad -v_t + c(t,x)v + F(t, x, v, D_x v, D_x^2 v, Bv)$$

where c is a continuous function, $c(t,x) \geq c_0$, and F is a continuous function in $[0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times S^n \times \mathbb{R}$ such that

$$F(t, x, v, 0, 0, 0) \; = \; 0$$

for all $(t, x, v) \in [0, T) \times \mathbb{R}^n \times \mathbb{R}$. For a consistent definition with classical solutions, we also assume that F is elliptic and nonincreasing with respect to its last argument.

Let us now turn out to comparison principles in the whole domain $[0,T] \times \mathbb{R}^n$. We make the usual linear growth assumptions on the coefficients b, σ, γ:

$$|b(t,x)|^2 + |\sigma(t,x)|^2 + \int_{\mathbb{R}^n} |\gamma(t,x,z)|^2 m(dz) \quad \leq \quad K(1 + |x|^2) \tag{5.1}$$

(the growth condition on γ is slightly weakened compared to (2.4)) and we prove the following result.

Proposition 5.2 *Assume* (2.6) *and* (5.1). *Let* $v \in \mathcal{C}^0([0,T] \times \mathbb{R}^n)$ *such that* $\mathcal{L}v \geq 0$ *in* $[0,T) \times \mathbb{R}^n$, *in the viscosity sense, and*

$$v(t,x) \geq -C(1 + |x|^q) \qquad \text{in } [0,T) \times \mathbb{R}^n \tag{5.2}$$

for some $C > 0$ *and* $q \in [0, 2)$. *If* $v(T, x) \geq 0$ *for all* $x \in \mathbb{R}^n$, *then*

$$v(t,x) \geq 0 \qquad \forall (t,x) \in [0,T] \times \mathbb{R}^n$$

Moreover, if $\mathcal{L}v > 0$ *in* $[0,T) \times \mathbb{R}^n$, *in the viscosity sense, then*

$$v(t,x) > 0 \qquad \forall (t,x) \in [0,T) \times \mathbb{R}^n$$

Proof: For $\lambda > 0$, consider the (smooth) function in $\mathcal{C}_2([0,T]\times I\!\!R^n)$:

$$w(t,x) \quad = \quad (1+|x|^2)e^{\lambda(T-t)}.$$

An easy calculation yields

$$B_t w(t,x) \quad = \quad e^{\lambda(T-t)}\int_{I\!\!R^n} |\gamma(t,x,z)|^2 m(dz)$$

hence,

$$\begin{aligned}\frac{\mathcal{L}w}{w} \quad &= \quad \lambda + c(t,x) - \frac{1}{1+|x|^2} tr\,(\sigma\sigma'(t,x)) - \frac{2}{1+|x|^2} b(t,x).x \\ &\quad -\frac{1}{1+|x|^2}\int_{I\!\!R^n} |\gamma(t,x,z)|^2 m(dz).\end{aligned}$$

From assumptions (2.6) and (5.1), we can choose λ sufficiently large positive, such that $\mathcal{L}w \geq 0$. For $\epsilon > 0$, let us define the continuous function $w_\epsilon = v + \epsilon w$. It follows that $\mathcal{L}w_\epsilon \geq 0$ in $[0,T)\times I\!\!R^n$, in the viscosity sense. In view of the asymptotic condition (5.2) on v, we have $\liminf_{|x|\to+\infty} w_\epsilon(t,x) \geq 0$, and there exists therefore $R(\epsilon) > 0$ such that

$$w_\epsilon(t,x) \geq 0 \qquad \forall t \in [0,T),\ |x| \geq R(\epsilon).$$

We also have $w_\epsilon(T,x) \geq 0$ since $v(T,x) \geq 0$. Applying Proposition 5.1 to $w_\epsilon(t,x)$ in the bounded domain $Q = (0,T)\times B(R(\epsilon))$ (where $B(r)$ is the open ball in $I\!\!R^n$ of radius r centered in 0), we deduce that

$$w_\epsilon(t,x) \geq 0 \qquad \forall (t,x) \in [0,T]\times I\!\!R^n,$$

and thus the first asserted result by sending $\epsilon \to 0^+$. If the second statement is false, then $e^{-c_0 t}v$ attains a zero minimum at $(\bar{t},\bar{x}) \in [0,T)\times I\!\!R^n$. Writing the strict supersolution inequality with the test function $\psi = 0$, we have

$$(c(\bar{t},\bar{x}) - c_0)\, v(\bar{t},\bar{x}) > 0,$$

which is a contradiction. □

Remarks

1. In the linear case, a positivity result is proved under weaker assumptions than in the nonlinear case, since no Lipschitz condition on the coefficients is required.

2. As for the second order parabolic case, we can derive maximum principles for functions satisfying exponential growth conditions, by assuming that the coefficients b, σ are bounded. About the jump component, we shall assume either that the function

$$(t,x) \quad \longmapsto \quad \int_{I\!\!R^n} exp\,(\delta|\gamma(t,x,z)|)\, m(dz)$$

is bounded uniformly in (t,x) for all $\delta > 0$, which implies in particular that the measure m is bounded, or to avoid such an assumption that the functions

$$\begin{aligned}
(t,x) &\longmapsto \int_{|z|\geq 1} exp\left(\delta|\gamma(t,x,z)|\right) m(dz) \\
(t,x) &\longmapsto \int_{|z|\leq 1} |\gamma(t,x,z)|^2 m(dz) \\
(t,x) &\longmapsto \mathbf{1}_{|z|\leq 1}|\gamma(t,x,z)|
\end{aligned}$$

are bounded uniformly in (t,x) for all $\delta > 0$. Such results are proved by the same way as in the proof of Proposition 5.2, by considering the function $w(t,x) = exp\left(k\sqrt{1+|x|^2}\ +\lambda(T-t)\right)$, and in the case of an unbounded measure m, by breaking the integral term Bw into two integrals as in (2.10).

5.2 $\mathcal{C}^{1,2}$ existence result for the Cauchy problem

This paragraph is devoted to the existence of a smooth solution $\mathcal{C}^{1,2}$ ($\mathcal{C}^1$ in t and $\mathcal{C}^2$ in x) of the Cauchy problem:

$$\mathcal{L}v = f(t,x) \qquad \forall (t,x) \in [0,T) \times I\!R^n \tag{5.3}$$

$$v(T,x) = g(x) \qquad \forall x \in I\!R^n \tag{5.4}$$

Gihman–Skorohod [12] have proved a $\mathcal{C}^{1,2}$ existence result for (5.3)–(5.4) under strong smoothness conditions on the coefficients $b, \sigma, \gamma, c, f, g$. See also Bensoussan–J.L. Lions [3]. We want to indicate here how the viscosity solution approach allows to reach simply $\mathcal{C}^{1,2}$ existence result for the Cauchy problem under weaker conditions on these coefficients, analog to the ones of Friedman (1975 [11]) for linear second order equations.

Provided that $\gamma(t,x,.) \in L^1(m)$, we denote:

$$\tilde{b}(t,x) \ = \ b(t,x) - \int_{I\!R^n} \gamma(t,x,z) m(dz).$$

Observe therefore that equation (5.3) can be also written, at least formally, as:

$$\tilde{\mathcal{L}}v = f_v(t,x) \qquad \forall (t,x) \in [0,T) \times I\!R^n \tag{5.5}$$

where

$$\begin{aligned}
\tilde{\mathcal{L}}v &= -\frac{\partial v}{\partial t} - tr\left(\frac{1}{2}\sigma\sigma'(t,x)D_x^2 v\right) - \tilde{b}(t,x).D_x v + c(t,x)v \\
f_v(t,x) &= f(t,x) - \int_{I\!R^n} \left[v(t,x+\gamma(t,x,z)) - v(t,x)\right] m(dz).
\end{aligned}$$

Note that the function f_v is well-defined whenever v is Lipschitz in x, and that $\tilde{\mathcal{L}}$ is a linear second order parabolic operator. Remark also that we can send $\eta \to 0^+$ in the super(sub)-solution inequality (2.13) since the limit integral terms are convergent whenever $v \in W^1([0,T] \times I\!R^n)$ and $\gamma \in L^1(m)$. It is, therefore, easily checked thanks to the definition–Lemma 2.1, with $\eta = 0$, that if v is a viscosity solution in $W^1([0,T] \times I\!R^n)$ of the second order integrodifferential equation (5.3), then it is also a viscosity solution of the second order equation (5.5). The idea is thus to characterize the solution v of (5.3) thanks to viscosity notion for parabolic integrodifferential operators developed in the preceding sections and to consider according to the above remark, equation (5.5), for which standard regularity theory for second order parabolic equations can be applied. Uniqueness of viscosity solutions will thus yield that v is smooth.

To apply both viscosity results and smoothness results for linear second order equations (see Friedman [11]), we shall assume that the functions $b, \sigma, \gamma, c, f, g$ are all continuous and:

(H0) There exists $\beta > 0$ such that for all $t \in [0,T]$, $x, \zeta \in I\!R^n$:

$$\zeta' \sigma\sigma'(t,x)\zeta \;\geq\; \beta|\zeta|^2.$$

(H1) b, σ are bounded and locally Lipschitz in (t,x).

(H2) There exists a function $\rho : I\!R^n \mapsto I\!R_+$ with $\int_{I\!R^n} \rho^2(z)m(dz) < +\infty$ such that :

$$\begin{aligned} |\gamma(t,x,z)| &\leq \rho(z) & \forall (t,x) \in [0,T] \times I\!R^n \\ |\gamma(t,x,z) - \gamma(t,y,z)| &\leq \rho(z)|x-y| & \forall (t,x,y) \in [0,T] \times I\!R^n \times I\!R^n \end{aligned}$$

$$(t,x) \longmapsto \int_{I\!R^n} \gamma(t,x,z)m(dz) \quad \text{is locally Lipschitz in } (t,x).$$

(H3) There exists $K > 0$ such that for all $t \in [0,T]$, $x, y \in I\!R^n$:

$$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \quad \leq \quad K|x-y|.$$

(H4) There exists $K > 0$ such that for all $t, s \in [0,T]$, $x, y \in I\!R^n$:

$$|f(t,x) - f(s,y)| + |g(x) - g(y)| \quad \leq \quad K\left[|t-s| + |x-y|\right].$$

(H5) c is bounded and locally Hölder continuous in (t,x).

(H6) m is a bounded measure.

This last assumption (H6) is the more restrictive and is required here to ensure that the function f_v is Lipschitz in x, uniformly in t. We have then the following result.

Proposition 5.3 *Under assumptions (H0)–(H6), there exists a unique solution* $v \in \mathcal{C}^{1,2}([0,T) \times I\!R^n) \cap \mathcal{C}^0([0,T] \times I\!R^n)$ *of the Cauchy problem* (5.3)–(5.4) *that satisfies*

$$|v(t,x)| \quad \leq \quad C(1+|x|^q) \qquad \forall (t,x) \in [0,T] \times I\!R^n$$

for some $q \in [0,2)$*. This solution is given by:*

$$v(t,x) \quad = \quad E_{tx}\left[\int_t^T e^{-\int_t^s c(u,X_u)du} f(s,X_s)ds \right.$$
$$\left. + e^{-\int_t^T c(u,X_u)du} g(X_T)\right]. \tag{5.6}$$

Proof: Let v given by (5.6). Assumptions of Proposition 3.3 and Theorem 3.1 are satisfied and it yields that v is a viscosity solution in $W^1([0,T] \times I\!R^n)$ of (5.3) and hence a viscosity solution of (5.5). From (H2), (H6) and since $v \in W^1([0,T] \times I\!R^n)$, we deduce by the dominated convergence theorem and with the continuity of f that f_v is also continuous in $[0,T] \times I\!R^n$. In view of (H3)–(H4), and since $v \in W^1([0,T] \times I\!R^n)$, we see that f_v is also Lipschitz in x, uniformly in t, whenever m is a bounded measure. Moreover, (H0) means that $\tilde{\mathcal{L}}$ is an uniformly parabolic operator, and by applying Theorem 5.3 of Friedman [11], whose other conditions are satisfied by (H1)–(H5), it yields that the Cauchy problem (5.5)–(5.4) admits a smooth solution $u_v \in \mathcal{C}^{1,2}([0,T) \times I\!R^n) \cap \mathcal{C}_1([0,T] \times I\!R^n)$, which is in particular a viscosity solution of (5.5). According to uniqueness results for viscosity solutions in $\mathcal{C}_1([0,T] \times I\!R^n)$ of second order equations (see e.g. Ishii [15]), we deduce that v coincides with u_v and is therefore smooth. Uniqueness of the Cauchy problem (5.3)–(5.4) is a direct consequence of Proposition 5.2. □

Appendix: Proof of Lemma 3.1

According to Hölder inequality, it suffices to prove estimates (3.2)–(3.5) for $k = 2$. For notational simplicity, hereafter, the C denotes a generic constant in different places.

1) By the optional sampling theorem and from growth conditions (2.4), (2.7) on b, σ, γ, we have for all $\tau \in \mathcal{T}_h$:

$$E\left|X_\tau^{t,x}\right|^2 \quad \leq \quad C\left\{|x|^2 + E\int_0^\tau \left|b(u+t, X_u^{t,x}, \alpha_u)\right|^2 du\right.$$

$$
\begin{aligned}
&+E\int_0^\tau \left|\sigma(u+t,X_u^{t,x},\alpha_u)\right|^2 du \\
&+ \quad E\left[\int_0^\tau \int_{I\!R^n} \left|\gamma(u+t,X_u^{t,x},\alpha_u,z)\right|^2 m(dz)du\right]\Bigg\} \\
\leq \quad & C\left\{1+|x|^2+E\int_0^\tau \left|X_u^{t,x}\right|^2 du\right\}. \qquad (A.1)
\end{aligned}
$$

In particular, for any deterministic time $\tau = s$, this last inequality yields by Fubini's theorem and by Gronwall's lemma:

$$
E\left|X_s^{t,x}\right|^2 \quad \leq \quad C(1+|x|^2). \qquad (A.2)
$$

We obtain then estimate (3.2) by injecting (A.2) into (A.1) and noting that $E\int_0^\tau |X_u^{t,x}|^2 du \leq \int_0^h E|X_u^{t,x}|^2 du$ for any stopping time $\tau \in \mathcal{T}_h$.

2) Similar arguments as above and (A.2) imply that for all $\tau \in \mathcal{T}_h$,

$$
\begin{aligned}
E\left|X_\tau^{t,x}-x\right|^2 \quad & \leq \quad C\int_0^h \left(1+E\left|X_u^{t,x}\right|^2\right) du \\
& \leq \quad C(1+|x|^2)h.
\end{aligned}
$$

3) From Doob's inequality for martingales, conditions (2.4), (2.7) and estimate (A.2), we have

$$
\begin{aligned}
E\left[\sup_{0\leq s\leq h}|X_s^{t,x}-x|\right]^2 \quad & \leq \quad C\Bigg\{E\int_0^h \left|b(u+t,X_u^{t,x},\alpha_u)\right|^2 \\
& \quad +E\int_0^h \left|\sigma(u+t,X_u^{t,x},\alpha_u)\right|^2 du \\
& \quad +E\left[\int_0^h \int_{I\!R^n} \left|\gamma(u+t,X_u^{t,x},\alpha_u,z)\right|^2 m(dz)du\right]\Bigg\} \\
& \leq \quad C(1+|x|^2)h.
\end{aligned}
$$

4) Let us define the process $Z_s = X_s^{t,x_1} - X_s^{t,x_2}$. Applying Itô's formula, we have by the optional sampling theorem: $\forall \tau \in \mathcal{T}_h$,

$$
\begin{aligned}
E|Z_\tau|^2 \quad = \quad & |x_1-x_2|^2 + E\Bigg\{\int_0^\tau \Big[2Z_u'\bar{b}(u+t,X_u^{t,x_1},X_u^{t,x_2},\alpha_u) \\
& +tr\left(\bar{\sigma}\bar{\sigma}'(u+t,X_u^{t,x_1},X_u^{t,x_2},\alpha_u)\right) \\
& +\int_{I\!R^n} \left|\bar{\gamma}(u+t,X_u^{t,x_1},X_u^{t,x_2},\alpha_u,z)\right|^2 m(dz)\Big] ds\Bigg\}
\end{aligned}
$$

where $\bar{b}(s,x,y,\alpha) = b(s,x,\alpha) - b(s,y,\alpha)$,

$$
\bar{\sigma}(s,x,y,\alpha) = \sigma(s,x,\alpha) - \sigma(s,y,\alpha),
$$

$$\bar{\gamma}(s,x,y,\alpha,z) = \gamma(s,x,\alpha,z) - \gamma(s,y,\alpha,z).$$

We obtain then from Lipschitz assumptions (2.2)–(2.3) on b, σ and γ:

$$E|Z_\tau|^2 \leq |x_1 - x_2|^2 + CE\left[\int_0^\tau |Z_u|^2 du\right]. \tag{A.3}$$

This inequality being true in particular for any deterministic time u, we deduce, thanks to Fubini's theorem and Gronwall's lemma, that $E|Z_u|^2 \leq e^{Cu}|x_1 - x_2|^2$. Injecting this last inequality into (A.3), we obtain:

$$E|Z_\tau|^2 \leq |x_1 - x_2|^2 + C|x_1 - x_2|^2 \int_0^h e^{Cu} du$$

which ends the proof. □

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