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SUMMARY
Optimality Conditions and Synthesis for the Minimum Time Problem*

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1 Introduction

In this paper we are concerned with the *minimum time optimal control* problem for the system

$$\begin{cases} y'(t) = f(y(t), u(t)), & t \geq 0, \\ y(0) = x, \end{cases} \tag{1.1}$$

where $x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \times U \to \mathbb{R}^n$ are given, U being a complete separable metric space. We assume that f is continuous and

$$\begin{cases} i) & \forall\, R > 0,\ \exists\, c_R > 0 \text{ such that} \\ & \forall\, u \in U,\ f(\cdot, u) \text{ is } c_R-\text{Lipschitz on } B_R(0) \\ ii) & \exists\, k > 0 \text{ such that} \\ & \forall\, x \in \mathbb{R}^n,\ \sup_{u \in U} |f(x,u)| \leq k(1 + |x|). \end{cases} \tag{1.2}$$

A measurable function $u : [0, +\infty[\to U$ is called a *control* and the corresponding solution of the state equation (1.1) is denoted by $y(\cdot; x, u)$.

Given a nonempty closed set $K \subset \mathbb{R}^n$ (called the *target*) and a point $x \in K^c$, we are interested in minimizing, over all controls u, the time taken for the solution $y(\cdot; x, u)$ to reach K. The value function, denoted by

$$T(x) = \inf_u \{t \geq 0 : y(t; x, u) \in K\} \tag{1.3}$$

as x ranges over R^n, is called the *Minimum Time* function of the problem.

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It is well known that T may be discontinuous even for very smooth data. However, if in addition

$$\forall\, x \in \mathbb{R}^n, \quad f(x,U) \quad \text{are closed and convex,} \tag{1.4}$$

then T is lower semicontinuous.

In this paper we focus our attention on necessary and sufficient optimality conditions. In particular, we are interested in extending to time optimal control the results of the first two authors ([2]) for the Mayer problem.

2 Optimality Conditions

For any non–empty set $S \subset \mathbb{R}^n$, we denote by S^c its complement, by S^- its (negative) polar by $T_S(x)$ the contingent cone to S at a point $x \in \bar{S}$, and by $\Pi_S(x)$ the set of perpendiculars to S at x.

Let $\varphi : \mathbb{R}^n \mapsto \mathbb{R} \cup \{\pm\infty\}$ be an extended function, and let $x_0 \in \mathbb{R}^n$ be such that $\varphi(x_0) \neq \pm\infty$. We set

$$D^+\varphi(x_0) = \left\{ p \in \mathbb{R}^n \mid \limsup_{x \to x_0} \frac{\varphi(x) - \varphi(x_0) - \langle p, x - x_0 \rangle}{|x - x_0|} \ \le\ 0 \right\} \ ;$$

and, for any vector $v \in \mathbb{R}^n$,

$$D_\downarrow \varphi(x_0)(v) \ = \ \limsup_{h \to 0+,\ v' \to v} \frac{\varphi(x_0 + hv') - \varphi(x_0)}{h}.$$

Finally, for any Lipschitz arc $z : [a,b] \to \mathbb{R}^n$ and any $t \in [a, b[$ we set

$$Dz(t) \ := \ \operatorname{Limsup}_{s \downarrow t} \frac{z(s) - z(t)}{s - t}$$

where Limsup denotes the Painlevé–Kuratowski upper limit [1].

We begin with a refined version of Pontryagin's Maximum Principle. Assume (1.2) and suppose that f is differentiable with respect to x. Let $\bar{u}$ be an optimal control for problem (1.3) at some point x_0 and set $T_0 = T(x_0)$, $\bar{y}(t) = y(t; x_0, \bar{u})$. Then, for every $p_0 \in T_{K^c}(\bar{y}(T_0))^-$ we show that the solution $p : [0, T_0] \to \mathbb{R}^n$ of the adjoint system

$$-\,p'(t) \ = \ \left(\frac{\partial f}{\partial x}(\bar{y}(t), \bar{u}(t)) \right)^\star p(t), \quad p(T_0) \ = \ -p_0 \tag{2.5}$$

satisfies the minimum principle

$$\langle p(t), v \rangle \ = \ \min_{u \in U} \langle p(t), f(\bar{y}(t), u) \rangle \tag{2.6}$$

for all $t \in [0, T_0]$ and all vectors $v \in\ \operatorname{Limsup}_{s \to t} \frac{\bar{y}(s) - \bar{y}(t)}{s - t}$.

We also derive a co–state inclusion of the type obtained in [3] and [2] for the (Lipschitz) value function of Bolza and Mayer problems. We recall that the Hamiltonian H associated to control system (1.1) is given by

$$H(x,p) = \sup_{u\in U} \langle p,\ f(x,u)\rangle \ , \quad \forall\, x,\, p \in \mathbb{R}^n \, . \tag{2.7}$$

Let (y_0, u_0) be an optimal pair at a point $x_0 \in K^c$ and set $T_0 = T(x_0)$. Suppose that $\nu \in \Pi_{K^c}(y_0(T_0))$ is such that

$$H(y_0(T_0), \nu) > 0. \tag{2.8}$$

Then, we prove that the solution p of problem (2.5) with

$$p_0 = H(y_0(T_0), \nu)^{-1}\nu$$

satisfies

$$p(t) \in D^+T(y_0(t)) \ , \quad \forall t \in [0, T_0[\, . \tag{2.9}$$

Furthermore, since $\Pi_{K^c}(y_0(T_0)) \subset T_{K^c}(y_0(T_0))^-$, we conclude that, if $\nu \in \Pi_{K^c}(y_0(T_0))$ and (2.8) holds true, then the solution of the adjoint system (2.5) with $p_0 = H(y_0(T_0), \nu)^{-1}\nu$ satisfies both the minimum principle (2.6) and the co–state inclusion (2.9).

From the above necessary conditions we derive the following necessary and sufficient optimality result.

Theorem 2.1 *Assume* (1.2), (1.4)*, and suppose that f is differentiable with respect to x. Let $x_0 \in K^c, T_0 > 0$, and let $u_0(\cdot)$ be a fixed control such that the corresponding trajectory $y_0(\cdot) = y(\cdot; x_0, u_0)$ satisfies $y_0(t) \notin K$ for all $t \in [0, T_0[,\ y_0(T_0) \in K$, and* (2.8) *for some $\nu \in \Pi_{K^c}(y_0(T_0))$. Then, u_0 is time optimal if and only if*

$$D_{\downarrow}T(y_0(t))(v) = -1,\ \forall v \in Dy_0(t),\ \forall t \in [0, T_0[.$$

3 Time Optimal Synthesis

Let us assume that f is differentiable with respect to x and consider the set–valued maps

$$U(x) = \begin{cases} \{u \in U \mid \langle p, f(x,u)\rangle = -1 \, , \ \forall p \in D^+T(x)\} & \text{if } D^+T(x) \neq \emptyset \\ \emptyset & \text{otherwise} \end{cases}$$

and $F(x) = f(x, U(x))$. It is not difficult to verify that if (1.4) holds true, then F has closed convex, possibly empty, images.

Suppose that y is an optimal trajectory at a point $x \in K^c$, set $T_0 = T(x)$, and let $H(y(T_0), \nu) > 0$ for some vector $\nu \in \Pi_{K^c}(y(T_0))$. Then, we show that y is a solution of the differential inclusion

$$\begin{cases} y'(t) & \in & F(y(t))\, , \quad \text{a.e. } t \in [0, T_0] \\ y(0) & = & x \end{cases} \tag{3.10}$$

and $\min_{p \in D^+T(y(t))} |p| \leq M,\ \forall t \in [0, T_0[$. for some constant $M > 0$.

Conversely, we obtain the following result.

Theorem 3.1 *Assume* (1.2), (1.4) *and let* $x \in K^c, T_0 > 0$. *Suppose that* $y(\cdot)$ *is a solution of* (3.10) *satisfying* $y(t) \notin K$ *for all* $t \in [0, T_0[$, $y(T_0) \in K$,

$$\min_{p \in D^+T(y(t))} |p| \ \leq \ M \ , \quad \forall\, t \in [0, T_0[\tag{3.11}$$

for some constant $M > 0$. *Then* y *is time optimal.*

Moreover, we can drop assumption (3.11) if, instead of an absolutely continuous solution of (3.10), we consider a contingent solution of it, i.e. a continuous arc y such that

$$Dy(t) \cap F(y(t)) \neq \emptyset \,, \quad \forall\, t \in [0, T_0[\,.$$

Finally, we consider another time optimal feedback in the form

$$G(x) = \left\{ f(x,u) \mid u \in U, \ \exists\, p \in D^+T(x) \ : \ \langle p, f(x,u) \rangle = -1 \right\} \,,$$

and the differential inclusion

$$\begin{cases} y'(t) & \in & G(y(t)) \\ y(0) & = & x \,. \end{cases} \tag{3.12}$$

Theorem 3.2 *Assume* (1.2), (1.4), *and suppose that* f *is differentiable with respect to* x. *Let* $x \in K^c, T_0 > 0$, *and let* $u(\cdot)$ *be a fixed control. Suppose that trajectory* $y(\cdot) = y(\cdot; x, u)$ *satisfies* $y(t) \notin K$ *for all* $t \in [0, T_0[$, $y(T_0) \in K$, *and* (2.8) *for some* $\nu \in \Pi_{K^c}(y(T_0))$. *Then,* $u(\cdot)$ *is time optimal if and only if* y *is a contingent solution of* (3.12) *in* $[0, T_0]$.

References

[1] J.-P. Aubin and H. Frankowska. SET-VALUED ANALYSIS. Basel: Birkhäuser, 1990.

[2] P. Cannarsa and H. Frankowska. Some characterizations of optimal trajectories in control theory, *SIAM J. on Control and Optimization*, **29** (1991), 1322–1347.

[3] F.H. Clarke and R.B. Vinter. The relationship between the maximum principle and dynamic programming, *SIAM J. Control Optim.*, **25** (1987), 1291–1311.

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