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On Models of Gaussian Reciprocal Processes and the Reconstruction of Periodic Jacobi Matrices*

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Abstract

We consider algorithms to reconstruct models of scalar Gaussian reciprocal processes from covariance information. These methods exploit the fact that the covariance matrices of these processes have structured inverses, either periodic Jacobi or Jacobi matrices. We develop these relationships and show how to pass back and forth between the covariance and its inverse both directly and by way of some knowledge of the eigenstructure. In particular, we show how to use eigenpairs to reconstruct the matrices. This approach is markedly different from existing algorithms which use all of the eigenvalues of the full matrix and its largest principal submatrix.

1 Introduction

We will consider various algorithms for identifying the model of a scalar Gaussian reciprocal process from some knowledge of the covariance. These algorithms are developed by exploiting the special structure exhibited by the inverses of the covariance matrices. In this sense the paper concentrates most heavily on linear algebra and its methods. The paper is organized in the following way. We begin by giving a brief introduction to models of Gaussian reciprocal processes, including a discussion of the structure of the various matrices we will be interested in examining. Next we give some results on the relationship between Jacobi and periodic Jacobi matrices and their inverses. In this section we will derive the basic relationships in a direct and constructive manner. Next we show how a very few eigenpairs

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(generally two) can be used to reconstruct the inverse of a covariance matrix for such a process. We give some results that show when this can be done. We emphasize that the methods herein are developed only for scalar processes. Finally, we look very briefly at one particular special case, the stationary case, which is particularly interesting since the eigenvectors are known *a priori*.

2 Models of Scalar Gaussian Reciprocal Processes

A stochastic process $x(k)$ defined on $[1, N]$ is *reciprocal* if for any subinterval $[l, m]$ of $[1, N]$ the process in the interior of $[l, m]$ is conditionally independent of the process in $[1, N] - [l, m]$ given $x(l)$ and $x(m)$. For a more rigorous definition see [11]. Reciprocal processes generalize Markov processes since a Markov process is reciprocal while the converse, in general, is not true, see [11] for an example of a process which is reciprocal and not Markov.

It is known [12, 13, 6] that a discrete time Gaussian reciprocal process $x(k)$, **under the assumption that the covariance of**

$$[\ x(k-1) \quad x(k+1)\ ]^T$$

is full rank, satisfies a nearest neighbor model like the following

$$m_0(k)x(k) - m_-(k)x(k-1) - m_+(k)x(k+1) = \xi(k) \tag{2.1}$$

where $\xi(k)$ is a zero mean Gaussian process with covariance

$$
\begin{aligned}
E[\xi(k)\xi(k)] &= m_0(k) && (2.2)\\
E[\xi(k)\xi(k+1)] &= -m_-(k+1) = -m_+(k) && (2.3)\\
E[\xi(1)\xi(n)] &= -m_-(1) = -m_+(N) && (2.4)\\
E[\xi(k)\xi(k+l)] &= 0 \quad \text{otherwise.} && (2.5)
\end{aligned}
$$

In matrix form the model (2.1) can be written as

$$\Lambda \mathbf{x} = \xi \tag{2.6}$$

where

$$\mathbf{x}^T = [\ x(1) \quad x(2) \quad \cdots \quad x(N)\ ], \tag{2.7}$$

$$\xi^T = [\ \xi(1) \quad \xi(2) \quad \cdots \quad \xi(N)\ ], \tag{2.8}$$

and Λ is the following periodic Jacobi matrix

$$\Lambda = \begin{bmatrix} m_0(1) & -m_+(1) & & & & -m_+(n) \\ -m_+(1) & m_0(2) & -m_+(2) & & & \\ & -m_+(2) & & & \ddots & \\ & & & \ddots & & -m_+(n-1) \\ -m_+(n) & & & & -m_+(n-1) & m_0(n) \end{bmatrix}$$

Note that all omitted entries in this matrix are zeros. The covariance structure (2.3) of the noise process corresponds to

$$E[\xi \xi^T] = \Lambda. \tag{2.9}$$

From (2.6) and (2.9) we see that

$$E[\mathbf{x} \xi^T] = \mathrm{I} \tag{2.10}$$

and, therefore, that

$$\Lambda \mathbf{R} = \mathrm{I} \tag{2.11}$$

where $\mathbf{R} = E[\mathbf{x}\mathbf{x}^T]$. Thus, **the noise process ξ is the conjugate process of x,** the matrix Λ characterizing the model of the reciprocal process x is a periodic Jacobi matrix (i.e., a symmetric tridiagonal with entries in the Northeast and Southwest corners, these are discussed in detail in Section 6), and the covariance $\mathbf{R}$ of $\mathbf{x}$ is its inverse. Moreover, by virtue of this relationship, these matrices have a related eigenstructure. If $(\lambda_k, \mathbf{u}_k)$ are the eigenpairs of Λ then $(1/\lambda_k, \mathbf{u}_k)$ are the eigenpairs of $\mathbf{R}$.

This leads us to consider the possibility of identifying the reciprocal model (2.1) of such a process starting from the eigenvectors of its covariance. This is equivalent to reconstructing the matrix Λ from its eigenstructure which is a well known problem in the literature see, for example, [9], [10] and references therein.

We propose an algorithm to reconstruct Λ from its two extremal eigenpairs $(\lambda_1, \mathbf{u}_1)$ and $(\lambda_n, \mathbf{u}_n)$. We show that the algorithm is well posed. The extremal eigenpairs also have the advantage that they can be easily computed from the covariance $\mathbf{R}$ using Krylov sequence methods like the Lanczos algorithm or power and inverse iteration [8].

The algorithm also serves to identify Markov models. Note first that the Markov processes are a subclass of the reciprocal processes, and the model corresponds to a Jacobi matrix (a subclass of the periodic Jacobi matrices). In [13], it was shown that a Markov process $x(k)$ satisfying the model

$$\Omega \mathbf{x} = B \mathbf{w} \tag{2.12}$$

where

$$\Omega = \begin{bmatrix} 1 & & & & \\ -a(1) & 1 & & & \\ & -a(2) & \ddots & & \\ & & \ddots & & \\ & & & -a(n-1) & 1 \end{bmatrix} \tag{2.13}$$

$$B = \mathrm{diag}(b(k)) \tag{2.14}$$

and

$$\mathbf{w} = \begin{bmatrix} w(0) & w(1) & \cdots & w(N-1) \end{bmatrix} \tag{2.15}$$

where $w(k)$ are Gaussian, zero mean, independent random variables with unitary variance, also satisfies a reciprocal model like (2.6) where

$$\Lambda = \Omega^T Q^{-1} \Omega \tag{2.16}$$

with $Q = BB^T$ and

$$\xi = \Omega^T Q^{-1} \mathbf{w}. \tag{2.17}$$

Therefore, Ω and B can be obtained from Λ by performing a Cholesky factorization.

In practice, the covariance $\mathbf{R}$ will be corrupted by noise; we will then need to work with the covariance $\mathbf{R}_y$ of the observations

$$\mathbf{y} = C\mathbf{x} + \mathbf{v} \tag{2.18}$$

where $C = \mathrm{diag}_{N \times N}(c)$. If $v(k)$ are independent Gaussian random variables identically distributed with zero mean and variance v, then $\mathbf{R}_y$ is related to $\mathbf{R}$ by

$$\mathbf{R}_y = C\mathbf{R}C^T + vI. \tag{2.19}$$

One can show that as long as $v \neq 0$ the covariance $\mathbf{R}_y$ does not have a Jacobi inverse. We can refine the algorithm so that given $\mathbf{R}_y$ it estimates v and Λ such that

$$\Lambda(\mathbf{R}_y - vI) = \mathrm{I}. \tag{2.20}$$

3 Jacobi and Green's Matrices

A real symmetric tridiagonal matrix

$$J = \begin{bmatrix} \alpha_1 & \beta_1 & & & \\ \beta_1 & \alpha_2 & \beta_2 & & \\ & \beta_2 & & \ddots & \\ & & \ddots & & \beta_{n-1} \\ & & & \beta_{n-1} & \alpha_n \end{bmatrix} \tag{3.21}$$

will be called a *Jacobi* matrix. If, furthermore, $\beta_i^2 > 0$ for $i = 1, 2, ..., n - 1$, then we will call the matrix a *simple Jacobi* matrix. We note that a Jacobi matrix can be written as a direct sum of simple Jacobi and diagonal matrices. Jacobi matrices occur in many applications.

Given $\mathbf{u}, \mathbf{v} \in \Re^n$, the real symmetric matrix G given by

$$G_{i,j} = \begin{cases} u_i v_j & \text{if } i \leq j \\ u_j v_i & \text{if } i > j \end{cases}$$

is called a Green's matrix (or one-pair matrix) with defining pair $(\mathbf{u}, \mathbf{v})$. Note that G depends on only $2n - 1$ independent parameters, the same as a Jacobi matrix, since we can arbitrarily normalize either $\mathbf{u}$ or $\mathbf{v}$.

For example, the 6×6 Green's matrix looks like

$$\begin{bmatrix} u_1v_1 & u_1v_2 & u_1v_3 & u_1v_4 & u_1v_5 & u_1v_6 \\ u_1v_2 & u_2v_2 & u_2v_3 & u_2v_4 & u_2v_5 & u_2v_6 \\ u_1v_3 & u_2v_3 & u_3v_3 & u_3v_4 & u_3v_5 & u_3v_6 \\ u_1v_4 & u_2v_4 & u_3v_4 & u_4v_4 & u_4v_5 & u_4v_6 \\ u_1v_5 & u_2v_5 & u_3v_5 & u_4v_5 & u_5v_5 & u_5v_6 \\ u_1v_6 & u_2v_6 & u_3v_6 & u_4v_6 & u_5v_6 & u_6v_6 \end{bmatrix},$$

which can be thought of as a symmetrization of the outer-product $\mathbf{u}\mathbf{v}^T$.

Although Green's matrices are usually represented by a defining pair they may also be represented by the entries from the diagonal and first super-diagonal (or first sub-diagonal). This manner of characterizing them is consistent with that for Jacobi matrices. The remainder of the entries can be found by applying the relationship

$$G_{i,j} G_{i+1,j+1} = G_{i,j+1} G_{i+1,j}, \tag{3.22}$$

which holds for all i, j such that $|i - j| \geq 1$. It is important to note that for a second-order scalar discrete Markov process only the variances at each step, $E[x(i)x(i)]$, and the covariances of adjacent steps, $E[x(i)x(i+1)]$, are necessary for determining all other second-order information since these processes also satisfy

$$E[x(i)x(j)]E[x(i+1)x(j+1)] = E[x(i)x(j+1)]E[x(i+1)x(j)] \quad (3.23)$$

for all i, j such that $|i-j| \geq 1$. This can be seen from (2.12) that for $j < i$ implies

$$E[x(i{+}1)x(j)] = a(i)E[x(i)x(j)], \quad E[x(i{+}1)x(j{+}1)] = a(i)E[x(i)x(j{+}1)]$$

while for $j > i$

$$E[x(i)x(j{+}1)] = a(j)E[x(i)x(j)], \quad E[x(i{+}1)x(j{+}1)] = a(j)E[x(i{+}1)x(j)].$$

It follows that the covariance matrix of such a process is a positive definite Green's matrix (positive definiteness comes directly from the fact that it is a covariance).

We introduce the following well-known fact.

Theorem 3.1 *Let J be an $n \times n$ nonsingular simple Jacobi matrix, then J^{-1} is a Green's matrix.*

Although the proof of this classical result is widely known, it uses determinant relationships and is not particularly constructive. We give a direct proof of this fact that yields a number of interesting and useful relationships. Throughout, we denote by $\mathbf{e}_i$ the i'th axis vector (that is, a vector with a 1 in the i'th position and zeros elsewhere), and for convenience we use $\mathbf{e}_L$ to denote the last axis vector (1 in the last position). The sizes of the various axis vectors should be taken from context. and let

Proof: Since J is non-singular choose $\mathbf{u}$ to be the solution of $J\mathbf{u} = \mathbf{e}_n$ and $\mathbf{v}$ to be the solution to $J\mathbf{v} = (1/u_1)\mathbf{e}_1$ (note that since J is simple it must be true that $u_1 \neq 0$).

We will prove the claim by showing that $J\mathbf{g}_k = \mathbf{e}_k$ where $\mathbf{g}_k$ is the k'th column of that G defined by $(\mathbf{u}, \mathbf{v})$. We begin by showing this for $1 \leq k < n$.

For an arbitrary index $1 \leq k < n$ write

$$J = \begin{bmatrix} J_1 & \beta_k \mathbf{e}_L \mathbf{e}_1^T \\ \beta_k \mathbf{e}_1 \mathbf{e}_L^T & J_2 \end{bmatrix},$$

and let

$$\mathbf{v} = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix} \text{ and } \mathbf{u} = \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix}$$

where $\mathbf{v}_1, \mathbf{u}_1 \in \Re^k$, all other indices are apparent from the context. Now

$$J\mathbf{v} = \begin{bmatrix} J_1\mathbf{v}_1 + \beta_k \mathbf{e}_L \mathbf{e}_1^T \mathbf{v}_2 \\ J_2\mathbf{v}_2 + \beta_k \mathbf{e}_1 \mathbf{e}_L^T \mathbf{v}_1 \end{bmatrix} = \begin{bmatrix} J_1\mathbf{v}_1 + \beta_k v_{k+1}\mathbf{e}_L \\ J_2\mathbf{v}_2 + \beta_k v_k \mathbf{e}_1 \end{bmatrix}.$$

So that

$$\begin{aligned} J_1\mathbf{v}_1 + \beta_k v_{k+1}\mathbf{e}_L &= \frac{1}{u_1}\mathbf{e}_1, & (3.24) \\ J_2\mathbf{v}_2 + \beta_k v_k\mathbf{e}_1 &= \mathbf{0}. & (3.25) \end{aligned}$$

And, by a similar process

$$\begin{aligned} J_1\mathbf{u}_1 + \beta_k u_{k+1}\mathbf{e}_L &= \mathbf{0}, & (3.26) \\ J_2\mathbf{u}_2 + \beta_k u_k\mathbf{e}_1 &= \mathbf{e}_L. & (3.27) \end{aligned}$$

Next, we note that by virtue of the structure of G we can write

$$\mathbf{g}_k = \begin{bmatrix} v_k\mathbf{u}_1 \\ u_k\mathbf{v}_2 \end{bmatrix}.$$

Therefore

$$\begin{aligned} J\mathbf{g}_k &= \begin{bmatrix} v_kJ_1\mathbf{u}_1 + \beta_k u_k\mathbf{e}_L\mathbf{e}_1^T\mathbf{v}_2 \\ u_kJ_2\mathbf{v}_2 + \beta_k v_k\mathbf{e}_1\mathbf{e}_L^T\mathbf{u}_1 \end{bmatrix} \\ &= \begin{bmatrix} v_kJ_1\mathbf{u}_1 + \beta_k u_k v_{k+1}\mathbf{e}_L \\ u_kJ_2\mathbf{v}_2 + \beta_k v_k u_k\mathbf{e}_1 \end{bmatrix}. \end{aligned}$$

Substituting from equations 3.25 and 3.26 gives

$$\begin{aligned} J\mathbf{g}_k &= \begin{bmatrix} -\beta_k v_k u_{k+1}\mathbf{e}_L + \beta_k u_k v_{k+1}\mathbf{e}_L \\ -\beta_k u_k v_k\mathbf{e}_1 + \beta_k v_k u_k\mathbf{e}_1 \end{bmatrix} \\ &= \beta_k(u_kv_{k+1} - v_ku_{k+1})\mathbf{e}_k. \end{aligned}$$

Hence JG is diagonal and we only need to show that $\beta_k(u_kv_{k+1} - v_ku_{k+1}) = 1$ for $1 \le k < n$ to complete the proof. First, if $k = 1$ then, by the definition of $\mathbf{u}$ and $\mathbf{v}$, we have

$$\begin{aligned} \alpha_1u_1 + \beta_1u_2 &= 0, & (3.28) \\ \alpha_1v_1 + \beta_1v_2 &= 1/u_1. & (3.29) \end{aligned}$$

Scaling these equations by v_1 and u_1, respectively, and then subtracting the first from the second to eliminate the terms involving α_1 gives

$$\beta_1(u_1v_2 - v_1u_2) = 1.$$

Next, if $1 \le k < n$ we have

$$\begin{aligned} \beta_{k-1}u_{k-1} + \alpha_ku_k + \beta_ku_{k+1} &= 0, & (3.30) \\ \beta_{k-1}v_{k-1} + \alpha_kv_k + \beta_kv_{k+1} &= 0, & (3.31) \\ & & (3.32) \end{aligned}$$

which leads to

$$\beta_{k-1}(v_k u_{k-1} - u_k v_{k-1}) = \beta_k(u_k v_{k+1} - v_k u_{k+1}).$$

Hence, by induction $\beta_k(u_k v_{k+1} - v_k u_{k+1}) = 1$ for $1 \leq k < n$.

Finally, we note that $\mathbf{g}_n = v_n \mathbf{u}$ which implies that $J\mathbf{g}_n = v_n \mathbf{e}_n$. To complete the proof we need to show that $v_n = 1$. We know that

$$\begin{aligned} \beta_{n-1}u_{n-1} + \alpha_n u_n &= 1, \\ \beta_{n-1}v_{n-1} + \alpha_n v_n &= 0. \end{aligned}$$

Whence

$$\beta_{n-1}(v_n u_{n-1} - u_n v_{n-1}) = v_n.$$

But we also know from before that

$$\beta_{n-1}(v_n u_{n-1} - v_{n-1} u_n) = 1.$$

Hence $v_n = 1$ and the proof is complete. ■

4 Inverting a Green's Matrix

Just as the inverse of a simple Jacobi matrix is a Green's matrix, the inverse of a non-singular Greens matrix is a simple Jacobi matrix. To construct the inverse we compute the β_k using the relationship $\beta_k(u_k v_{k+1} - v_k u_{k+1}) = 1$ that was developed in the proof above. In particular

$$\beta_k = \frac{1}{u_k v_{k+1} - v_k u_{k+1}}.$$

To find the α_k we note that for $1 < k < n$ we have the following two equations:

$$\begin{aligned} \beta_{k-1}u_{k-1} + \alpha_k u_k + \beta_k u_{k+1} &= 0, \\ \beta_{k-1}v_{k-1} + \alpha_k v_k + \beta_k v_{k+1} &= 0, \end{aligned}$$

which leads to

$$\begin{aligned} \alpha_k &= -\beta_{k-1}\frac{u_{k-1}}{u_k} - \beta_k \frac{u_{k+1}}{u_k}, \\ \alpha_k &= -\beta_{k-1}\frac{v_{k-1}}{v_k} - \beta_k \frac{v_{k+1}}{v_k}. \end{aligned}$$

We note that at least one of these must be well-defined for each $1 < k < n$, since setting $u_k = v_k = 0$ for any k would give a zero row (and column) in G which contradicts the non-singular assumption. We get α_1 by noting that

$$\alpha_1 u_1 + \beta_1 u_2 = 0.$$

Whence

$$\alpha_1 = -\beta_1 \frac{u_2}{u_1},$$

And similarly

$$\alpha_n = -\beta_{n-1} \frac{v_{n-1}}{v_n}.$$

Both of which must be well defined, again because either $u_1 = 0$ or $v_n = 0$ would imply that G is singular.

We also note that the following relationships can be derived for the ratios between successive u_i and v_i. Define

$$\gamma_k = \frac{u_k}{u_{k+1}} = \frac{G_{k,k+1}}{G_{k+1,k+1}}, \tag{4.33}$$

$$\rho_k = \frac{v_{k+1}}{v_k} = \frac{G_{k,k+1}}{G_{k,k}}. \tag{4.34}$$

Then

$$\gamma_k = \frac{-\beta_k}{\alpha_k - \beta_{k-1}\gamma_{k-1}}, \tag{4.35}$$

$$\rho_k = \frac{-\beta_k}{\alpha_{k+1} - \beta_{k+1}\rho_{k+1}}, \tag{4.36}$$

where

$$\gamma_1 = \frac{-\beta_1}{\alpha_1}, \tag{4.37}$$

$$\rho_{n-1} = \frac{-\beta_{n-1}}{\alpha_n}. \tag{4.38}$$

5 The Cholesky Factorization of a Green's Matrix

It is often useful to find the Cholesky factorization of a covariance matrix. We now give two direct algorithms for computing the Cholesky factorization of a positive definite Green's matrix $G \in \Re^{n\times n}$. We wish to find $L \in \Re^{n\times n}$ such that $LL^T = G$. First, we derive an algorithm that uses the defining pair to compute the Cholesky factorization. Consider the following block partitioning where $\hat{\mathbf{v}}^T = \begin{bmatrix} v_2 & v_3 & \dots & v_n \end{bmatrix}$ and $\hat{G}$ is the $n-1 \times n-1$ submatrix that remains after deleting the first row and column of G. Then, the block form of the Cholesky factorization is

$$\begin{bmatrix} u_1 v_1 & u_1 \hat{\mathbf{v}}^T \\ u_1 \hat{\mathbf{v}} & \hat{G} \end{bmatrix} = \begin{bmatrix} \gamma & \mathbf{0}^T \\ \mathbf{a} & \hat{L} \end{bmatrix} \begin{bmatrix} \gamma & \mathbf{a}^T \\ \mathbf{0} & \hat{L}^T \end{bmatrix}.$$

We are led to the following equations

$$\begin{aligned}
\gamma^2 &= u_1 v_1, \\
\mathbf{a} &= \frac{1}{\gamma} u_1 \hat{\mathbf{v}}, \\
\hat{L}\hat{L}^T &= \hat{G} - \frac{u_1^2}{u_1 v_1} \hat{\mathbf{v}}\hat{\mathbf{v}}^T, \\
&= \hat{G} - \frac{u_1}{v_1} \hat{\mathbf{v}}\hat{\mathbf{v}}^T.
\end{aligned}$$

Notice that the Schur complement is itself a Green's matrix with defining pair $(\hat{\mathbf{u}} - \frac{u_1}{v_1}\hat{\mathbf{v}}, \hat{\mathbf{v}})$. We are led to the following algorithm

for $j = 1$ to $n-1$
 $L_{j,j} = \sqrt{u_j v_j}$
 for $i = j+1$ to n
  $L_{i,j} = \sqrt{\frac{u_j}{v_j}} v_i$
  $u_i = u_i - \frac{u_j}{v_j} v_i$
 endfor
endfor
$L_{n,n} = \sqrt{u_n v_n}$

If we use the fact that the elements of a Green's matrix satisfy equation 3.22 we can develop an algorithm that computes the Cholesky factorization using only the entries from the diagonal and first sub-diagonal (super-diagonal). We omit the derivation as it is quite tedious, but we note that it follows from applying the relation 3.22 to the algorithm above.

for $j = 1$ to n
 if $j == 1$
  $L_{j,j} = \sqrt{G_{1,1}}$
 else
  $L_{j,j} = \sqrt{G_{j,j} - \frac{G_{j,j-1}^2}{G_{j-1,j-1}}}$
 endif
 for $i = j+1$ to n
  $L_{i,j} = L_{j,j} \prod_{k=j}^{i-1} \rho_k$
 endfor
endfor

where the ρ_k are as defined in 4.34

It is interesting to note that one step of Cholesky LR on a Green's matrix preserves the structure. Hence, we can use this algorithm to derive a number of quotient-difference like algorithms, ala Rutishauser, for finding the eigenvalues of a Green's matrix without prior reduction to tridiagonal form [3]. These processes are analogous to those used for Jacobi matrices but operate on the inverse. This relationship mirrors the one between the divide-and-conquer algorithms for the symmetric eigenproblem based on extension (arrow matrices) and those based on modification (diagonal plus rank-one) as these structures are also inverses.

6 Periodic Jacobi Matrices

A real symmetric matrix with non-zero entries only on the diagonal, first super-diagonal, first sub-diagonal, and northeast and southwest corners, e.g.,

$$K = \begin{bmatrix} \alpha_1 & \beta_1 & & & \gamma \\ \beta_1 & \alpha_2 & \beta_2 & & \\ & \beta_2 & & \ddots & \\ & & \ddots & & \beta_{n-1} \\ \gamma & & & \beta_{n-1} & \alpha_n \end{bmatrix} \tag{6.39}$$

is called a *periodic Jacobi* matrix. We will assume that $\beta_i^2 > 0$ for $i = 1, 2, ..., n-1$.

It is known that the inverse of the covariance of a scalar discrete reciprocal process is a periodic Jacobi matrix. It is not surprising then that the periodic Jacobi matrices include the Jacobi matrices as a sub-class in just the same way that the reciprocal processes include the Markov processes as a sub-class.

It is a useful fact that a periodic Jacobi matrix is a rank-one modification of a Jacobi matrix. If we let $\mathbf{s} = \mathbf{e}_1 + \mathbf{e}_n$ then the periodic Jacobi matrix K above can be written in the form $K = J + \gamma \mathbf{s}\mathbf{s}^T$ where

$$J = \begin{bmatrix} \alpha_1 - \gamma & \beta_1 & & & \\ \beta_1 & \alpha_2 & \beta_2 & & \\ & \beta_2 & & \ddots & \\ & & \ddots & & \beta_{n-1} \\ & & & \beta_{n-1} & \alpha_n - \gamma \end{bmatrix}.$$

Note that J is the model for a Markov process with the same reciprocal dynamics as the process represented by K, **i.e., the two processes have the same conjugate process $\xi(i)$ for $1 < i < n$.**

To investigate the inverse of a periodic Jacobi we recall the Sherman-Morrison formula

$$(A + \mathbf{u}\mathbf{v}^T)^{-1} = A^{-1} - \frac{A^{-1}\mathbf{u}\mathbf{v}^T A^{-1}}{1 + \mathbf{v}^T A^{-1}\mathbf{u}}.$$

For the case under consideration we can write

$$(J+\gamma \mathbf{s}\mathbf{s}^T)^{-1} = J^{-1} - \frac{\gamma}{1+\gamma \mathbf{s}^T J^{-1}\mathbf{s}}(J^{-1}\mathbf{s})(J^{-1}\mathbf{s})^T.$$

Since we know that J^{-1} is a Greens' matrix, we see that the inverse of a periodic Jacobi is a Greens' matrix plus a symmetric outer-product. It is clear from the formula above that

$$K^{-1}\mathbf{s} = \tau J^{-1}\mathbf{s}$$

where

$$\begin{aligned}\tau &= 1 - \frac{\gamma \mathbf{s}^T J^{-1}\mathbf{s}}{1+\gamma \mathbf{s}^T J^{-1}\mathbf{s}} \\ &= \frac{1}{1+\gamma \mathbf{s}^T J^{-1}\mathbf{s}}\end{aligned}$$

from which we can see the following:

Theorem 6.1 *Let R be the covariance of a reciprocal process $u(t)$ indexed on $t = 1, 2, ..., n$ and let $\mathbf{w} = R\mathbf{s}$. There exists a unique scalar σ such that $R + \sigma \mathbf{w}\mathbf{w}^T$ is the covariance of a Markov process with the same reciprocal dynamics as the process $u(.)$. Moreover, if $1 \le i < n$ and $1 \le j < n$ are integers such that $|i-j| \ge 1$ then*

$$\sigma = \frac{-\det(\hat{R})}{\begin{bmatrix} \mathbf{w}_{i+1} & -\mathbf{w}_i \end{bmatrix} \hat{R} \begin{bmatrix} \mathbf{w}_{j+1} \\ -\mathbf{w}_j \end{bmatrix}}$$

where

$$\hat{R} = \begin{bmatrix} R_{i,j} & R_{i,j+1} \\ R_{i+1,j} & R_{i+1,j+1} \end{bmatrix}.$$

The theorem allows us to find the Green's matrix that is the inverse of J, which is the model of a Markov process with the same reciprocal dynamics as those of the reciprocal process in question. In order to invert the covariance matrix R we will need to invert this Green's matrix (this was discussed in an earlier section) and we need to find γ. If we define $\phi := \mathbf{s}^T R\mathbf{s}$ then we can show that

$$\tau = \frac{1}{1+\sigma\phi}$$

and further, since $\gamma = \sigma\tau$ we see that

$$\gamma = \frac{\sigma}{1+\sigma\phi}.$$

7 Reconstructing a Jacobi Matrix

We wish to develop an algorithm to reconstruct J from the knowledge of two of its eigenpairs $(\lambda, \mathbf{u})$ and $(\mu, \mathbf{v})$. The eigenvector recurrence for Jacobi matrices is

$$\beta_{i-1}u_{i-1} + \alpha_i u_i + \beta_i u_{i+1} = \lambda u_i \tag{7.40}$$

where $(\lambda, \mathbf{u})$ is any eigenpair of J, u_i is the ith element of $\mathbf{u}$, and $\beta_0 = \beta_n = 0$. Applying this relation to both eigenpairs gives

$$\begin{aligned} \beta_{i-1}u_{i-1} + \alpha_i u_i + \beta_i u_{i+1} &= \lambda u_i, \\ \beta_{i-1}v_{i-1} + \alpha_i v_i + \beta_i v_{i+1} &= \mu v_i. \end{aligned}$$

Combining these two equations and eliminating α_i gives

$$\beta_{i-1}(v_i u_{i-1} - u_i v_{i-1}) + \beta_i(u_{i+1}v_i - v_{i+1}u_i) = (\lambda - \mu)u_i v_i. \tag{7.41}$$

Define $\delta_i := u_{i+1}v_i - v_{i+1}u_i$ for $i = 1, 2, ..., n-1$ then we may rewrite 7.42 in the following form:

$$\beta_i\delta_i - \beta_{i-1}\delta_{i-1} = (\lambda - \mu)u_i v_i. \tag{7.42}$$

Since $\beta_0 = \beta_n = 0$ we get the following initial and terminal conditions

$$\beta_1\delta_1 = (\lambda - \mu)u_1 v_1, \tag{7.43}$$

$$\beta_{n-1}\delta_{n-1} = (\mu - \lambda)u_n v_n. \tag{7.44}$$

Provided that $\delta_i \neq 0$ for $i = 1, 2, ..., n-1$, we can combine (7.42) with (7.43) and get a special case of the Christoffel-Darboux identity,

$$\beta_i\delta_i = (\lambda - \mu)\sigma_i, \tag{7.45}$$

for $i = 1, 2, ..., n-1$ where we define

$$\sigma_i := \sum_{k=1}^{i} u_k v_k.$$

Note that since $\mathbf{u}$ and $\mathbf{v}$ are orthogonal it must be true that

$$-\sigma_i = \sum_{k=i+1}^{n} u_k v_k.$$

so there is also a backward formula,

$$\beta_i\delta_i = -(\lambda - \mu)\sum_{k=i+1}^{n} u_k v_k. \tag{7.46}$$

There are several ways to determine the α_i. It is possible to obtain an equation for the α_i using the β_i and a single eigenpair by solving the scalar equation 7.40 to get

$$\alpha_i = \lambda - \beta_{i-1}\frac{u_{i-1}}{u_i} - \beta_i\frac{u_{i+1}}{u_i}. \tag{7.47}$$

If we do not use the extremal eigenpairs then this formula can break down since it is possible that $u_i = 0$. In this case we can use the second eigenpair provided that $v_i \neq 0$. Unfortunately, it is also possible that both u_i and v_i are zero. Then we must use one of the two formulas that follow, at least one of which must be well-defined for any two eigenpairs.

$$\alpha_i\delta_i = \mu u_{i+1}v_i - \lambda v_{i+1}u_i - \beta_{i-1}\Delta_i \tag{7.48}$$

$$\alpha_i\delta_{i-1} = \lambda u_i v_{i-1} - \mu v_i u_{i-1} - \beta_i\Delta_i \tag{7.49}$$

where we define

$$\Delta_i := u_{i+1}v_{i-1} - v_{i+1}u_{i-1},$$

for $i = 2, 3, 4, ..., n-1$. We can use equation 7.48 to get α_1 and equation 7.49 to get α_n provided we assume that $\beta_0 = \beta_n = 0$.

We can use these equations to reconstruct the original matrix from the two eigenpairs provided that $\delta_i \neq 0$ for any $i = 1, 2, ..., n-1$. Using (7.47), (7.45), and (7.46) we can reconstruct the original matrix in $13n - 12$ flops.

In order to determine when these formulas can be applied, we need some additional results. We introduce the following fact from [14].

Theorem 7.1 *Let $J \in \Re^{n\times n}$ be a simple Jacobi matrix with non-negative off-diagonal elements and assume that the eigenvalues are ordered so that $\lambda_1 > \lambda_2 > ... > \lambda_n$. Then the number of sign changes between consecutive elements of the kth eigenvector of J, denoted s_k, is $k-1$.*

We refer the reader to [14] for a proof but note that it can be derived from the Sturm sequence property for the characteristic polynomials of the principal sub-matrices. With this fact in hand we can prove the following theorem.

Theorem 7.2 *If $J \in \Re^{n\times n}$ is a simple Jacobi matrix with non-negative off-diagonal elements and if $(\lambda, \mathbf{u})$ and $(\mu, \mathbf{v})$ are the extremal eigenpairs of J, that is $\lambda = \lambda_1$ and $\mu = \lambda_n$, then $v_i u_{i+1} - u_i v_{i+1} \neq 0$ for any $i = 1, 2, ..., n-1$.*

Proof: The proof follows trivially by noting that the strict interlacing property for simple Jacobi matrices (see [16] p. 300) guarantees that none of the numbers $u_i, u_{i+1}, v_i, v_{i+1}$ can be zero. And, since u_i and u_{i+1} must have the same sign and v_i and v_{i+1} must have opposite signs (from fact 1), it follows that both terms in $u_i v_{i+1} - v_i u_{i+1}$ have opposite signs and are nonzero so this difference is really a sum of two strictly positive (negative) numbers and hence is not zero. ■

Hence, if we choose the two extremal eigenpairs of a simple Jacobi matrix with non-negative off-diagonal entries we can always reconstruct the original matrix using the formulas above. Notice that the δ_i are computed without cancellation in this case because of the sign pattern. Moreover, if we use the smallest (largest) eigenpair in (7.47) to get the α_i, then these can be reconstructed from the derived β_i and the data without further cancellation if the matrix is positive (negative) definite. If the matrix is indefinite then there is only one additional cancellation for each of the α_i. If the matrix is singular then choosing the eigenvector associated with the zero eigenvalue prevents further cancellation.

Note that any Jacobi matrix has exactly $2n-1$ real degrees of freedom and that two eigenpairs contain $2n+2$ numbers but, in fact, also have $2n-1$ real degrees of freedom since there are two arbitrary scaling parameters for the eigenvectors and a single orthogonality condition. The eigenpairs contain precisely the right amount of information.

8 Breakdown of the Jacobi Reconstruction

The algorithm can breakdown if one does not use the extremal eigenpairs since some of the δ_i may be zero. For an example of this consider the matrix

$$\begin{bmatrix} 6 & 2 & 0 & 0 \\ 2 & 4 & 5 & 0 \\ 0 & 5 & 4 & 2 \\ 0 & 0 & 2 & 6 \end{bmatrix}. \tag{8.50}$$

This matrix is a simple Jacobi matrix with non-negative off-diagonal entries and shares no eigenvalues with its principal sub-matrices. The eigenvalues are 10, $(5+\sqrt{65})/2$, 5, $(5-\sqrt{65})/2$ and the eigenvectors associated with 10 and 5 are $[1\ 2\ 2\ 1]^T$ and $[-2\ 1\ 1\ -2]^T$, respectively. Using these two eigenpairs the algorithm breaks down in computing β_2. Some manipulation of the scalar equations shows that the two eigenpairs in question are, in fact, eigenpairs of any matrix of the form

$$\begin{bmatrix} 6 & 2 & 0 & 0 \\ 2 & 9+\ell & -\ell & 0 \\ 0 & -\ell & 9+\ell & 2 \\ 0 & 0 & 2 & 6 \end{bmatrix}, \tag{8.51}$$

which can be written as a sum in the following way:

$$\begin{bmatrix} 6 & 2 & 0 & 0 \\ 2 & 9 & 0 & 0 \\ 0 & 0 & 9 & 2 \\ 0 & 0 & 2 & 6 \end{bmatrix} + \ell \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{8.52}$$

In general, the reconstruction algorithm will break down at any $1 \leq k \leq n-1$ such that $\delta_k = 0$, that is, $u_{k+1}v_k - v_{k+1}u_k = 0$. Notice that if $\delta_k = 0$, then both $\delta_{k-1} \neq 0$ and $\delta_{k+1} \neq 0$ since if either was also zero it would imply that two distinct eigenvalues share the same eigenvector. When a breakdown occurs at k it implies that the given input data does not restrict the value of β_k, we can assign β_k arbitrarily and proceed with the reconstruction using equation 7.42. Furthermore, if $\delta_k = 0$ let $H^{(k)}$ be

$$H^{(k)} = 0_{k-1} \oplus \begin{bmatrix} \frac{u_{k+1}}{u_k} & -1 \\ -1 & \frac{u_k}{u_{k+1}} \end{bmatrix} \oplus 0_{n-k-1}.$$

Then $\mathbf{u}$ and $\mathbf{v}$ are both eigenvectors of the symmetric tridiagonal matrix $H^{(k)}$ associated with the eigenvalue 0. Hence, if J is any symmetric tridiagonal with eigenpairs $(\lambda, \mathbf{u})$ and $(\mu, \mathbf{v})$ then so is $J + \ell H^{(k)}$. There will be one such matrix for each δ_k that is zero.

We can modify the algorithm so that it deals gracefully with breakdowns by proceeding in two steps. First, we build a reducible Jacobi matrix by running the recurrence and setting $\beta_k = 0$ any time $\delta_k = 0$. The following fragment implements this

```
s = 0
for i = 1 to n − 1
    if δ_i == 0
       β_i = 0
       s = 0
    else
       s = s + u_i v_i
       β_i = (λ−μ)s / δ_i
    endif
endfor
```

We will call the output, J, of this step the particular solution. Next, for each k such that $\delta_k = 0$ we construct a matrix $H^{(k)}$ as described above, we will call these homogeneous solutions. Finally, the most general solution to the problem is given by

$$J + \sum_{k \ni \delta_k = 0} \ell_k H^{(k)},$$

where the ℓ_k may take on any values.

9 Reconstructing a Periodic Jacobi Matrix

Reconstructing a periodic Jacobi matrix $K \in \Re^{n\times n}$ is not much different from reconstructing the Jacobi matrix. If we take $(\lambda, \mathbf{u})$ to be any eigenpair of K then the eigenvector recurrences are

$$\beta_{i-1}u_{i-1} + \alpha_i u_i + \beta_i u_{i+1} = \lambda u_i \tag{9.53}$$

for $i = 2, 3, ..., n-1$ and

$$\begin{aligned} \alpha_1 u_1 + \beta_1 u_2 + \gamma u_n &= \lambda u_1, && (9.54)\\ \alpha_n u_n + \beta_{n-1} u_{n-1} + \gamma u_1 &= \lambda u_n. && (9.55) \end{aligned}$$

Proceeding as before we get the following relationship

$$\beta_i \delta_i - \beta_{i-1}\delta_{i-1} = (\lambda - \mu) u_i v_i \tag{9.56}$$

for $i = 2, 3, ..., n-1$.

The initial and terminal conditions are more difficult in the periodic case. We have only two equations in the three unknowns

$$\begin{aligned} \beta_1 \delta_1 &= (\lambda - \mu) u_1 v_1 + \gamma(u_1 v_n - v_1 u_n), && (9.57)\\ \beta_{n-1}\delta_{n-1} &= (\mu - \lambda) u_n v_n + \gamma(u_1 v_n - v_1 u_n). && (9.58) \end{aligned}$$

Extend the definition of δ_i to include $\delta_0 := u_1 v_n - v_1 u_n =: \delta_n$. Then combining (9.56) with (9.57) gives a special case of the Christoffel-Darboux identity,

$$\beta_i \delta_i = (\lambda - \mu)\sigma_i + \gamma \delta_0$$

for $i = 1, 2, ..., n-1$. There is a backward formula which follows by orthogonality as before.

The α_i satisfy equations 7.48 and 7.49 for $i = 2, 3, ..., n-1$. If we extend the definition of Δ_i to include

$$\begin{aligned} \Delta_1 &:= u_2 v_n - v_2 u_n,\\ \Delta_n &:= u_1 v_{n-1} - v_1 u_{n-1}. \end{aligned}$$

then the following equations hold for α_1 and α_n

$$\begin{aligned} \alpha_1 \delta_1 &= \mu u_2 v_1 - \lambda v_2 u_1 - \gamma \Delta_1,\\ \alpha_n \delta_{n-1} &= \lambda u_n v_{n-1} - \mu v_n u_{n-1} - \gamma \Delta_n. \end{aligned}$$

9.1 Parameterized form for the periodic Jacobi

In reconstructing the periodic Jacobi we have left the corner element γ as a free parameter. This is unavoidable with the given data since we do not have enough information to completely determine the $2n$ unknowns. As a result, if the reconstruction proceeds without a breakdown, we will have a parameterized family of solutions. We can see that the periodic Jacobi matrix with eigenpairs $(\lambda, \mathbf{u})$ and $(\mu, \mathbf{v})$ can be written as the sum of a Jacobi matrix and an arbitrary multiple of a periodic Jacobi matrix in the form

$$K = J + \gamma \hat{K},$$

where J is the Jacobi matrix that is found using the reconstruction algorithm described earlier, and $\hat{K}$ is a periodic Jacobi matrix with ones in the NE and SW corners and whose remaining elements are given by

$$\begin{aligned} \hat{\beta}_i &= \frac{\delta_0}{\delta_i}, \\ \hat{\alpha}_i &= -\frac{\delta_0 \Delta_i}{\delta_{i-1}\delta_i}. \end{aligned}$$

We can save $2n-1$ multiplies at the cost of one additional division by letting $\hat{K}$ be the periodic Jacobi matrix with $1/\delta_0$ in the NE and SW corners and whose remaining elements are given by

$$\begin{aligned} \hat{\beta}_i &= \frac{1}{\delta_i}, \\ \hat{\alpha}_i &= \frac{-\Delta_i}{\delta_{i-1}\delta_i}. \end{aligned}$$

9.2 Feasible regions

We have seen that two eigenpairs do not contain enough information to completely reconstruct a periodic Jacobi matrix. With this limited information we can only recover a parameterized form such as the one above. Fortunately, the parameterization is of such a form that with additional information it is often possible to derive simple equations that will yield the unknown parameter γ. For instance, if any of the non-zero elements of K is known then γ can be solved for immediately.

Even less direct information can be used in an effort to determine or restrict the value of γ. For example, recall that our motivation for developing this algorithm is model identification of reciprocal stochastic processes. In this case, we are reconstructing the inverse of a covariance matrix and so there is certain additional information that is at our disposal. In particular, since covariance matrices are positive definite, so are their inverses. It is well-known that the diagonal elements of a positive definite matrix must be positive. We can use this fact to limit the possible values that γ may assume by imposing this constraint. That is, we can determine a *feasible* region for γ by finding the open interval (l, r) such that the diagonal of $J + \gamma\hat{K}$ has positive diagonal elements for all $\gamma \in (l, r)$. In particular, for each diagonal element of $J+\gamma\hat{K}$ to be positive we need to impose the linear constraints

$$\alpha_i + \gamma\hat{\alpha}_i > 0.$$

It follows that

$$\gamma > -\frac{\alpha_i}{\hat{\alpha}_i} \quad \text{for all } i \text{ such that } \hat{\alpha}_i > 0$$

and

$$\gamma < -\frac{\alpha_i}{\hat{\alpha}_i} \quad \text{for all } i \text{ such that } \hat{\alpha}_i < 0.$$

We note that this is a rather minimal *necessary* condition and is not sufficient to guarantee that $J + \gamma\hat{K}$ will be positive definite. We have presented it because it leads to a simple algorithm and experience shows that the interval determined in this manner is often quite small so that with no further information it may be possible to get a reasonable estimate for the inverse of the covariance.

Of course, a variety of methods can be used to determine γ given additional information. These techniques will depend on the available data.

10 Identification of Stationary Processes

It is known that when a scalar Gaussian reciprocal process is stationary then the inverse of its covariance must be a circulant periodic Jacobi matrix, that is

$$K = \begin{bmatrix} \alpha & \beta & & & \beta \\ \beta & \alpha & \beta & & \\ & \beta & & \ddots & \\ & & \ddots & & \beta \\ \beta & & & \beta & \alpha \end{bmatrix}. \tag{10.59}$$

Circulant matrices have a very special structure and are intimately related to the discrete Fourier transform (DFT). It is well known that if $A \in \Re^{n\times n}$ is a circulant matrix then the $n \times n$ matrix $U_{i,j} = \omega^{(i-1)(j-1)}$, where ω is the generator for the n'th roots of unity[1], diagonalizes A. It is not difficult to show that the eigenvalues of K are

$$\begin{aligned} \lambda_k &= \alpha + \beta(\omega^k + \bar{\omega_k}) \\ &= \alpha + 2\beta\cos\frac{2\pi k}{n} \end{aligned}$$

[1]This matrix is often called the DFT matrix of order n.

where $k = 0, 1, 2, ..., n-1$. Furthermore, the eigenvector associated with the eigenvalue λ_k is

$$\mathbf{v}_k = \begin{bmatrix} 1 \\ \omega^k \\ \omega^{2k} \\ \vdots \\ \omega^{(n-1)k} \end{bmatrix}.$$

In particular, we see that $\alpha + 2\beta$ is an eigenvalue of K whose associated eigenvector is the ones vector. Furthermore, if n is even then $\alpha - 2\beta$ is an eigenvalue, whose associated eigenvector is the sign-permuted ones vector. Moreover, the remaining eigenvalues each occur with algebraic multiplicity 2. This fact can lead to breakdown since the reconstruction requires that the eigenvalues be distinct.

Fortunately, this structure leads to a very simple procedure for identifying the model for such a process. In particular, we know that if we are given the covariance R of such a process, then

$$R\mathbf{v}_k = \frac{1}{\alpha + 2\beta\cos\frac{2\pi k}{n}}\mathbf{v}_k$$

from which we can derive a number of equations for α and β. As an example, assume we have a scalar stationary Gaussian reciprocal process defined on $t = 1, 2, ..., n$ where n is even, and assume that its covariance matrix, R, is known (or has been estimated). Then

$$\begin{aligned} R\mathbf{w} &= \frac{1}{\alpha + 2\beta}\mathbf{w}, \\ R\mathbf{z} &= \frac{1}{\alpha - 2\beta}\mathbf{z}, \end{aligned}$$

where $\mathbf{w}$ is the ones vector, and $\mathbf{z}$ is the sign permuted ones vector ($\mathbf{z} = \begin{bmatrix} 1 & -1 & 1 & ... \end{bmatrix}^T$). By performing two matrix vector multiplies (for this case we need only $n-1$ additions and 1 subtraction) we can compute the left-hand sides of the following two equations

$$\begin{aligned} (R\mathbf{w})_1 &= \frac{1}{\alpha + 2\beta}, \\ (R\mathbf{z})_1 &= \frac{1}{\alpha - 2\beta}, \end{aligned}$$

and we see that

$$\begin{aligned} \alpha &= \frac{1}{2}\left(\frac{1}{(R\mathbf{w})_1} + \frac{1}{(R\mathbf{z})_1}\right), \\ \beta &= \frac{1}{4}\left(\frac{1}{(R\mathbf{w})_1} - \frac{1}{(R\mathbf{z})_1}\right). \end{aligned}$$

By exploiting the structure of the covariance estimation algorithm it should be possible to make an estimate of this type directly from observed trajectories.

11 Conclusions

We have developed some methods for identifying models of scalar Gaussian reciprocal processes. The algorithms we developed exploit the fact that covariances for these processes have structured inverses, in particular periodic Jacobi matrices. We point out that although we have not pursued it here many of these algorithms can be extended to the non-scalar case.

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