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On the Riccati Partial Differential Equation for Nonlinear Bolza and Lagrange Problems*

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Abstract

The recent solution of the output regulation problem for nonlinear control systems gives necessary and sufficient conditions for the local existence of a feedback/feedforward law in terms of the solvability of an "off-line" system of partial differential equations, the "regulator equations." The regulator equations are the nonlinear analogue of the "Sylvester" equations of linear systems theory and in this sense represent an initiation of the study of nonlinear enhancements of the important set of equations arising in linear systems and control which are central in the modern theory, practice and computations of linear systems. While Lyapunov equations are a special case of Sylvester equations which have been used quite a bit in nonlinear control, conspicuous in its absence is the Riccati equation—both in its differential and algebraic forms. The structure of the controller derived in the solution of the regulator problem is, of course, quite reminiscent of that of an LQ or LQR controller, expressed in terms of the "off-line" solution of a Riccati equation. This similarity leads to the question as to whether there exists a corresponding Riccati Partial Differential Equation, which would play a fundamental role not only in optimal control but also in a broader context, for example, in a theory of spectral factorization for nonlinear systems. This now seems to be the case. In this paper, we provide the details for results we announced in 1989 concerning the solution of certain optimal control problems by the use of an off-line Riccati Partial Differential Equation. Independently, Helton and Ben-Artzi discovered a similar Riccati Partial Differential Equation in their investigation into factoring nonlinear systems. Since that time, several other uses of Riccati Partial Differential Equations in finite dimensional nonlinear systems have been discovered.

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Of course, in a subject with such classical origins as optimal control, there are antecedents to virtually every concept. To the best of the author's knowledge, the earliest form of the Riccati PDE was discovered in the application of invariant imbedding methods to the two-point boundary value problems which arise in the application of the Pontryagin Maximum Principle. As cited in the introduction, other forms of Riccati equations have been derived in a variety of other settings. Most notably, the Riccati PDE is also equivalent to the Hamilton-Jacobi-Bellman (HJB) equation, when the value function is sufficiently smooth. One contribution in this paper lies in the development of a geometric existence theory, inspired by our earlier work on the regulator problem, for classical, weak and generalized solutions of the Riccati Partial Differential Equation. For finite and infinite time horizon problems, we also investigate the relationship of such solutions to optimal control laws. Indeed, as corollaries of certain of our existence results, we obtain smoothness results for the value function of the corresponding optimal control problem.

Key words: finite time horizon optimal control, infinite time horizon optimal control, nonlinear control systems, stable manifold, Riccati Partial Differential Equation

1 Introduction

The recent solution (see [1]–[2]) of the output regulation problem for nonlinear control systems gives necessary and sufficient conditions for the local existence of a feedback/feedforward law in terms of the solvability of an "off-line" system of partial differential equations, the "regulator equations." There are several aspects of this approach which have been the starting point for further investigations.

First, in [1] an existence theory is developed for solvability of the regulator equations which is geometric in its nature, reducing to Hautus' existence criterion in terms of transmission zeroes in the linear case and involving zero dynamics in the nonlinear case. However, the key tools for verifying the geometric conditions come from nonlinear dynamics; viz., invariant manifold theory (e.g., the existence and properties of center, stable and unstable manifolds). The systematic development of such geometric conditions for the existence of solutions of partial differential equations, in a control theoretic context, form part of the starting point for this investigation in nonlinear optimal control.

Second, there was some optimism, at least by the authors of [1]-[2], that the "off-line" nature of the regulator equations would facilitate the development of computational methods for nonlinear control. Indeed, the regulator equations are the nonlinear analogue of the "Sylvester" equations of linear systems theory and the development of computational tools for solving such equations would be one of the more important steps in developing a set of computational methods for nonlinear control which would be comparable in scope to those available for linear systems. The paper [3] began an investigation into techniques for solving the regulator equations, expressing the linear equations for the Taylor coefficients of the solutions and giving independent "resonance" criteria for the term by term solution of these linear equations. More recently, Krener [4] has expressed these resonance conditions and the linear equations in terms of the homological equations of Poincaré and has successfully extended his numerical tool box for nonlinear control, also called "Poincaré", to include the regulator equations—solved out to cubic order.

There are, of course, other important equations arising in linear systems and control which are also quite central to the theory, to practice and to the computations. Lyapunov equations are a special case of Sylvester equations and have been extensively used and researched in nonlinear control. However, conspicuous in its absence is the Riccati equation—both in its differential and algebraic forms. The structure of the controller derived in the solution of the regulator problem is, of course, quite reminiscent of that of an LQ or LQR controller, expressed in terms of the "off-line" solution of a Riccati equation. This striking similarity leads to two questions. Can one derive a solution of the regulator equations from an optimal control problem? Is there a corresponding Riccati Partial Differential Equation, which would play a fundamental role not only in optimal control but also in a broader context, for example in spectral factorization?

Concerning the first question, it is important to note that solutions of the regulator equations are often unique up to a choice of a stabilizing state feedback law, despite their derivation via center manifold methods. With this in mind, the first question has been answered in the affirmative in [4], for those solutions of the regulator equations derived from an optimal stabilizing state feedback law.

Concerning the second question question, in 1989 we announced ([5] and more recently, [6]-[7]) the solution of certain optimal control problems by use of an off-line Riccati Partial Differential Equation. Also in 1989, I learned from Bill Helton that he and Ben-Artzi ([8]) had discovered a similar Riccati Partial Differential Equation in their investigation into factoring nonlinear systems. Since that time, several other uses of Riccati Partial Differential Equations in finite dimensional nonlinear systems have been discovered.

Of course, in a subject with such classical origins as optimal control, there are antecedents to virtually every concept—especially in the calculus of variations. Indeed, in 1935, Caratheodory discovered a predecessor of the Riccati PDE in his study of sufficient conditions for the calculus of variations. Referring to this new system of first order partial differential equations as the "fundamental equations of the calculus of variations, " he also derived the well-known necessary conditions of Euler-Lagrange and Weierstrass. To the best of the author's knowledge, the earliest form of the Riccati PDE in modern control was discovered in the application of invariant imbedding methods, pioneered by Bellman, to the two-point boundary value problems arising in the formulation of the transversality conditions in the Pontryagin Maximum Principle (see, for example, the classic text [9] by Melsa). In this context, for example, a form of the "dual" Riccati PDE was also proposed by Nihitala (see, e.g. [10]) for nonlinear filtering.

From one point of view, the Riccati PDE may be regarded as an attempt to eliminate the costate from the expression for the maximizing control in the statement of the Pontryagin Maximum Principle. Since the costate is also constrained to evolve under the adjoint equation, this would require an elimination procedure in the sense of differential algebra. Indeed, a differential algebraic derivation of a higher order system of PDE's for the costate was carried out in this case in [11], using the techniques of differential algebra. However, under suitable rank conditions, the higher order system becomes first order and coincides with the Riccati PDE considered here.

A Riccati Equation, typically realizable by a PDE, is also standard in the theory of optimal control for linear distributed parameter systems and has even been developed for certain classes of nonlinear DPS (see, e.g. [12]–[13]). Finally, the Riccati PDE is also equivalent to the Hamilton-Jacobi-Bellman (HJB) equation, when the value function is sufficiently smooth. One of our contributions lies in the development of a geometric existence theory, inspired by our earlier work on the regulator problem, for classical and non-classical solutions. Indeed, our geometric derivation of the Riccati PDE is independent of the HJB equation and thus, as corollaries of certain of our existence results, we obtain smoothness results for the value function of the corresponding optimal control problem.

For the matrix Riccati equation, the key analytic question is whether or not there exists a finite escape time for solutions with a given initial or final condition. For the Riccati PDE, one has to analyze both the existence of "classical blow-ups"—i.e., finite escape time—where the time derivative becomes infinite and the existence of "shock waves," where the spatial derivative becomes infinite. The general result in LQ theory is that, for those matrix Riccati equations and terminal conditions arising from optimal control problems, finite escape time does not exist. In the nonlinear case, this is closely related to the existence of optimal controls. Indeed, Pontryagin's Maximum Principle rules out the formation of classical blow-ups. There is great evidence (see, e.g., Sections 5, 6 and [16]) that an analogous situation prevails, relating uniqueness of optimal controls to the nonexistence of shock waves, for at least certain classes of nonlinear problems. This relationship, which is suggested by the geometry, provides a link between the existence of shock waves and its finite dimensional "shadow", the existence of bifurcations of optimal controls.

Examples show that generalized solutions to the Riccati PDE can also be used to synthesize optimal controls. This theme has recently been pursued more generally in the forthcoming thesis of N. Caroff [29] which investigates the relationship between set-valued analytic methods, multivalued geometric methods and more analytic approaches to the shock waves, particularly to quantities (such as entropy) preserved throughout the occurrence of shocks.

In this paper, we focus on the development of a geometric existence theory for classical, weak and generalized solutions of the Riccati Partial Differential Equation. For finite and infinite time horizon problems, we also investigate the relationship of such solutions to optimal control laws. In this context, classical solutions are, of course, smooth solutions. Weak solutions are characterized geometrically, but turn out to be continuous everywhere and smooth almost everywhere, leading to the synthesis of continuous optimal feedback control laws. Our development of generalized solutions was inspired by the use of Lagrangian submanifolds as an analogue of "generalized functions" in the geometric theory of nonlinear PDE's (see, e.g., [14], [15]). Thus, generalized solutions are multi-valued but with a smooth, closed, connected Lagrangian submanifold as its graph.

More explicitly, a smooth function determines an exact and hence a closed one form. Geometrically, the graph of a closed one form is a Lagrangian submanifold of the state-costate space, so that Lagrangian submanifolds represent a class of generalized functions—a classical yet powerful point of view. Most recently, in joint work with H. Frankowska (see [16]), we have combined this geometric formalism with techniques from nonsmooth analysis to prove global existence results for solvability of the Riccati Partial Differential Equation. In particular, it is shown, under certain assumptions about the variational problem, that absence of shocks, and hence solvability of the Riccati Partial Differential Equation, are equivalent to uniqueness of optimal trajectories. The paper [16] also includes some new results for the existence of classical solutions, which have as a corollary a nonlinear enhancement of the classical LQ design theory for linear systems.

In Section 2, we set notation and conventions in the course of reviewing preliminaries involving the Pontryagin Maximum Principle and consequent properties for extremals for smooth problems of Bolza type. In particular, we give a concise formulation of the canonical Hamiltonian system and a geometric statement of the transversality conditions.

Starting from this formulation, in Section 3 we derive the Riccati Partial Differential Equation from geometric considerations, quite independent of the Hamilton-Jacobi-Bellman equation. In particular, we give a geometric interpretation of the existence of classical (i.e., smooth) solutions to the Riccati PDE in terms of one parameter family of smooth submanifolds of the state-costate space. That these submanifolds turn out to be Lagrangian is important in our characterization of the existence of classical solutions in terms of higher order smoothness properties of the value function.

In Section 4, we discuss the existence of classical, weak and generalized solutions to the Riccati PDE. Indeed, we begin Section 4 with a discussion of a simple one-dimensional example, consisting of a linear system, quadratic integral performance measure and a non quadratic terminal constraint. For this problem, the Riccati equation reduces to the classical inviscid Burgers' equation, with initial data given by the transversality conditions. This is analyzed for general terminal constraints in Example 7.1, where an optimal control interpretation of Burgers' equation is given, explaining the existence and nonexistence of shock waves in a variational context. In Example 4.1, a case where the terminal constraint is non convex is considered and, as the time horizon is increased, the onset and propagation of shock waves illustrates the concepts of classical, weak and generalized solutions and a corresponding hierarchy of regularity conditions for the value function. It is worth noting that, even as the shock waves propagate, the generalized solutions have an interesting and useful interpretation in terms of the analysis and synthesis of optimal control laws. Following a discussion of Example 4.1, an analysis of the continuity and smoothness properties of weak solutions is given in a series of results which also describe the regularity of the corresponding value function for general classes of Bolza problems. The remainder of Section 4 is devoted to conditions for weak solutions to be classical and the derivation of consequent sufficient conditions, in terms of the system and the cost criteria, for the existence of classical solutions.

Section 5 applies these results on solvability of Riccati PDE's to the construction of optimal control laws in feedback form. From our main theorem, we can deduce that if a global solution of the Riccati PDE exists then there exists a unique optimal control, expressible as a globally defined feedback law. As another corollary we obtain under weaker "local" hypotheses a general local existence result for Bolza problems. From this we are also able to deduce a result previously obtained by Willemstein [17] under much stronger hypotheses; e.g., requiring analyticity as well as positive definiteness of the integral performance measure.

Section 6 addresses the corresponding Lagrange, or infinite time, problem under the same "local" hypotheses. In this case, a formal application of Dynamic Programming would suggest that the one parameter family of generalized solutions, integrated backwards in time, should "converge" to the stable manifold of the canonical Hamiltonian system. In the linear case, the stable manifold is the graph of the (the negative of) the positive semidefinite solution of the algebraic Riccati equation, a relationship which persists, mutatis mutandis, in the nonlinear case. It is worth noting that in [18], Brunovsky announced a sketch based on stable manifold theory of a proof of smoothness of the value function for certain nonlinear optimal stabilization problems. In [19], Lukes gave a complete proof of this fact under rather strong hypotheses requiring positive definiteness of the integral performance measure. For the sake of completeness and for comparison with the finite time horizon case, we derive conditions for the local solvability of the steady-state Riccati Partial Differential Equation and the local optimality of the corresponding control law, for positive semi-definite performance measures.

There are, of course, two extreme cases of Bolza or Lagrange problems which may be considered: A nonlinear system with fairly general nonlinear performance criteria and one very well understood special case, consisting of a linear system with quadratic criteria. In Section 7, we specialize the results obtained in this paper to two intermediate cases which should also be of particular interest; viz., linear systems with more general criteria and nonlinear systems with quadratic criteria. A special case of the former is often referred to as the "simplest problem in the calculus of variations." When our hypotheses apply to a problem of this type, the Riccati Partial Differential Equation reduces to the Euler-Lagrange Equation. This is illustrated in Example 7.1 for a problem with general nonlinear terminal constraints where, as we mentioned above, the Riccati Partial Differential Equation reduces to the inviscid Burgers' equation, with fairly general initial conditions. Moreover, this derivation of Burgers' equation allows for a variational interpretation of the existence and nonexistence of shocks for general initial conditions. In Section 7.2, we specialize the Riccati Partial Differential Equation to the case of nonlinear systems with quadratic integral performance measures, obtaining a PDE which closely resembles the matrix Riccati ODE except for a "nonlinear correction term." It is then shown that for LQ problems linear solutions to the Riccati PDE are, not surprisingly, characterized by the matrix Riccati ODE, from which optimality of the classical solutions can be obtained by the general machinery developed in this paper.

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2 Preliminaries on the Pontryagin Maximum Principle and Extremals for Bolza Problems with Smooth Data

Consider a control system having the form

$$\dot{x} = f(x) + g(x)u(t) \tag{2.1}$$

where $x \in \mathbb{R}^n$, and where for each t, $u(t) \in \mathbb{R}^m$. We assume that the vector fields f, g_i are C^r in x, $r \geq 1$ and that the $u_i(t)$ are piecewise continuous functions. In particular, for each pair $(x(0), u(t))$, the system (2.1) has a unique solution for $t << \infty$. Furthermore, we assume that $f(0) = 0$. For the system (2.1) we shall consider the problem of minimizing the cost functional

$$J^T(x(0), u) = \int_0^T L(x, u)dt + Q(x(T)) \tag{2.2}$$

for both the cases $T < \infty$ and, in Section 6, for $T = \infty$ and $Q(x) \equiv 0$. In Sections 5 and 6 we shall present some explicit solutions, in feedback form, to such optimal control problems, thereby providing constructive existence results. In this section, we review some well-known consequences of the existence of optimal control laws for (2.1)–(2.2) in the case, $T < \infty$.

Our initial assumptions concerning (2.2) are that $Q(x)$ is C^{q+1}, $q \geq 1$ and that $L(x, u)$ is C^{s+1}, $s \geq 1$, and satisfies:

H1: for each fixed x, $\frac{\partial L}{\partial u}(x, \cdot)$ is a diffeomorphism;

H2: for each fixed x, $L(x, u)$ has a minimum at $u = 0$.

In particular, we note that $\frac{\partial^2 L}{\partial u^2}(x, u) > 0$ for all (x, u) and, therefore, that (H1)–(H2) imply that $L(x, u)$ is strictly convex in u.

H3: 0 is a critical point of Q.

In Sections 5 and 6 we shall study the case when Q has a local minimum at 0 in more detail.

Remark 2.1. *Without loss of generality, we shall normalize $L(x, u)$ so that $L(0, 0) = 0$. We shall also assume that $Q(0) = 0$.*

Finally, we set $k = \min(q, r, s)$. Fixing notation, we reinterpret (2.1)–(2.2) as a problem in Mayer form by defining

$$\dot{x}_{n+1} = L(x, u), \quad x_{n+1}(0) = 0$$

and considering an augmented state variable

$$\tilde{x} = \begin{pmatrix} x \\ x_{n+1} \end{pmatrix} \in \mathbb{R}^{n+1}.$$

The corresponding penalty-constraint function

$$\varphi : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \times \mathbb{R}^2 \to \mathbb{R}^{n+4} \tag{2.3}$$

is, therefore, a function of $e = \left(\tilde{x}^0, \tilde{x}^1, t^0, t^1\right)$ where $\tilde{x}^0 = \begin{pmatrix} x^0 \\ x^0_{n+1} \end{pmatrix}$ is an initial state, $\tilde{x}^1 = \begin{pmatrix} x^1 \\ x^1_{n+1} \end{pmatrix}$ a terminal state, t^0 an initial time and t^1 a terminal time. In particular, the Mayer problem corresponding to (2.1)–(2.2) is to minimize

$$\varphi_1(e) = x^1_{n+1} + Q(x^1)$$

subject to the $n+3$ constraints

$$\varphi_2(e) = x^0_{n+1} = 0 \tag{2.4}$$

$$\varphi_{2+i}(e) = x^0_i - x_i(0) = 0, \quad i = 1, \dots, n \tag{2.5}$$

$$\varphi_{n+3}(e) = t^1 - T = 0 \tag{2.6}$$

$$\varphi_{n+4}(e) = t^0 = 0. \tag{2.7}$$

Following the Pontryagin Maximum Principle, we first form an augmented Hamilton function

$$\tilde{H}(\tilde{x}, \tilde{p}, u) = \langle p, f(x) + g(x)u \rangle + p_{n+1} L(x, u)$$

and consider the system on $\mathbb{R}^{2n+2}$

$$\dot{\tilde{x}} = \frac{\partial \tilde{H}}{\partial \tilde{p}} \tag{2.8}$$

$$\dot{\tilde{p}} = \frac{-\partial \tilde{H}}{\partial \tilde{x}}. \tag{2.9}$$

According to the Maximum Principle, for $x(0)$ fixed a necessary condition for $u_*(t)$ to be an optimal control is that there exist an initial condition $\tilde{p}(0)$ such that if $\tilde{x}(0) = \begin{pmatrix} x(0) \\ 0 \end{pmatrix}$, the corresponding solution $(\tilde{x}(t), \tilde{p}(t))$ of (2.4)–(2.5) exists and satisfies the following conditions:

$$\max_{u \in \mathbb{R}^{n+1}} \tilde{H}(\tilde{x}(t), \tilde{p}(t), u) = \tilde{H}\left(x(t), p(t), u_*(t)\right) \quad a.e. \tag{2.10}$$

$$\tilde{p}(t) \not\equiv 0. \tag{2.11}$$

Furthermore, there must also exist a multiplier $\lambda \in \mathbb{R}^{n+4}$ with $\lambda_1 \leq 0$ such that the transversality conditions

$$\tilde{p}(t^0)^T = -\lambda^T \varphi_{\tilde{x}^0}(e) \tag{2.12}$$

$$\tilde{p}(t^1)^T = \lambda^T \varphi_{\tilde{x}^1}(e) \tag{2.13}$$

$$\tilde{H}\left(\tilde{x}(t^0), \tilde{p}(t^0), u_*(t^0)\right) = \lambda^T \varphi_{t^0}(e) \tag{2.14}$$

$$\tilde{H}\left(\tilde{x}(t^1), \tilde{p}(t^1), u_*(t^1)\right) = -\lambda^T \varphi_{t^1}(e) \tag{2.15}$$

are satisfied.

Definition 2.1. *Suppose, for some initial condition $\tilde{x}(0)$, that $u(t)$ is a control for which there exists a solution $\tilde{p}(t)$ of the adjoint system and a "multiplier" $\lambda \in \mathbb{R}^{n+4}$ such that the conditions of the Pontryagin Maximum Principle (2.6)–(2.12) are satisfied. Then, we shall say that $u(\cdot)$ is an extremal control, that the corresponding trajectory $x(\cdot)$ is an extremal trajectory and that the pair $(x(\cdot), p(\cdot))$ is a canonical pair.*

In the remainder of this section, we shall simplify (2.4)–(2.11) in the light of assumptions (H1)–(H2) and the form of (2.1)–(2.2). We first note that this minimization problem is "normal"; i.e. that $p_{n+1} < 0$. Indeed, from (2.10) it follows that

$$\tilde{p}(T)^T = \lambda_1 \left(\nabla Q(x(T)), 1\right) \tag{2.16}$$

so that $p_{n+1} = \lambda_1 \leq 0$. If $p_{n+1} = 0$, then from (2.12) we must have $\tilde{p}(T) = 0$. Since $\tilde{H}$ is linear in p if $p_{n+1} = 0$, (2.5) is linear in p and it would therefore follow that

$$\tilde{p}(t) = 0 \quad 0 \leq t \leq T$$

contrary to (2.7). In particular, we conclude $p_{n+1} < 0$ as claimed. Since the Maximum Principle is invariant under a scaling of $\tilde{H}$ by any positive constant, we may take $p_{n+1} = -1$, leading to the augmented Hamiltonian

$$H(x, p, u) = \langle p, f(x) + g(x)u \rangle - L(x, u).$$

We next note that, by (H1), for each (x, p) there is a unique $u_*(x, p)$, C^k in (x, p), which satisfies

$$0 = \frac{\partial H}{\partial u}|_{u=u_*} = \langle p, g(x) \rangle - \frac{\partial L}{\partial u}(x, u_*).$$

Moreover, in the light of (H2), for each fixed pair (x, p), the value $u_*(x, p)$ in fact maximizes $H(x, p, u)$, since

$$\frac{\partial^2 H}{\partial u^2}|_{u=u_*} = -\frac{\partial^2 L}{\partial u^2}(x, u_*) < 0. \tag{2.17}$$

In particular, if $u_*(t)$ is an optimal control for an initial condition $x(0)$, giving rise to an optimal trajectory $x_*(t)$ and some nontrivial solution $p(t)$ of the adjoint equation satisfying the conditions of the Maximum Principle, we must have

$$u_*(t) = u_*(x_*(t), p(t)). \tag{2.18}$$

There is an important converse to this conclusion. Define a Hamiltonian function, $H_*(x,p)$, via

$$H_*(x,p) = H(x,p,u_*(x,p))$$

where $u_*(x,p)$ is the unique solution of (2.14). From (2.14) one sees

$$\frac{\partial H_*}{\partial x}(x,p) = \frac{\partial H}{\partial x}(x,p,u)|_{u=u_*(x,p)}$$

and

$$\frac{\partial H_*(x,p)}{\partial p} = \frac{\partial H}{\partial p}(x,p,u)|_{u=u_*(x,p)}.$$

Therefore, (2.4)–(2.5) can be written as

$$\dot{x} = \frac{\partial H_*}{\partial p} \tag{2.19}$$

$$\dot{p} = -\frac{\partial H_*}{\partial x} \tag{2.20}$$

together with

$$\dot{x}_{n+1} = L(x,u) \qquad x_{n+1}(0) = 0 \tag{2.21}$$

$$\dot{p}_{n+1} = 0 \qquad p_{n+1}(0) = -1. \tag{2.22}$$

We can now refine the observation implicit in (2.15), which follows from a specialization of the Maximum Principle to the current problem.

Proposition 2.1. *Assume hypotheses (H1)–(H2) hold and consider the "canonical system" (2.16)–(2.17), with the final value condition*

$$p(T) = -\nabla Q(x(T)). \tag{2.23}$$

For any initial condition $x(0)$, consider any extremal control $u(t)$ and a corresponding $p(t)$ for which $(x(t), p(t))$ is a canonical pair. Then, $(x(t), p(t))$ is a solution of the canonical system satisfying the final value condition (2.20). Moreover, $u(t)$ is given by the formula (2.15). Conversely, for any solution of the canonical system satisfying the final value condition (2.20) the control $u(t)$ defined by (2.15) is an extremal control with extremal trajectory $x(t)$ and having $(x(t), p(t))$ as a canonical pair.

Proof: In the light of (2.14)–(2.15), all that remains to be checked is the claim implicit in the converse assertion concerning the existence of a multiplier $\lambda \in \mathbb{R}^{n+4}$, with $\lambda_1 \leq 0$, such that the transversality conditions (2.8)–(2.11) are satisfied. To this end, we define

$$\lambda_1 = \lambda_2 = -1, \tag{2.24}$$

$$(\lambda_3, \ldots, \lambda_{n+2}) = p(0)^T, \tag{2.25}$$

and (2.26)

$$\lambda_{n+3} = -\lambda_{n+4} = H_*(x(T), p(T)) \tag{2.27}$$

□

It is straightforward to check that, with this definition of λ, the transversality conditions are satisfied.

3 Riccati Partial Differential Equations for the Bolza Problem

According to Proposition 2.1, for the Bolza Problem every extremal control $u_*(t)$ for any initial condition $x(0)$ gives rise to a trajectory, or canonical pair, $(x(t), p(t))$ satisfying

$$\dot{x}(t) = \frac{\partial H_*}{\partial p} \tag{3.1}$$

$$\dot{p}(t) = -\frac{\partial H_*}{\partial x} \tag{3.2}$$

and

$$p(T) = -\nabla Q(x(T)). \tag{3.3}$$

Furthermore, $u_*(t)$ may be expressed as

$$u_*(t) = u_*(x(t), p(t)) \tag{3.4}$$

where $u_*(x,p)$ is the unique solution of (2.14). Conversely, any solution $(x(t), p(t))$ of (3.1)–(3.3) generates, via (3.4), an extremal control $u(t)$ with extremal trajectory $x(t)$. In this section, we are interested in developing a closed-loop expression for extremal controls, deferring a discussion of optimality of extremals to Sections 5 and 6. To this end, we develop a nonlinear analogue of the Riccati equation for the "state-costate" equations (3.1)–(3.2) with a nonlinear, but smooth, terminal constraint (3.3). More explicitly, motivated by Dynamic Programming, we shall attempt to find an expression

$$p(t) = -\pi(x(t), t) \tag{3.5}$$

by integrating the smooth constraint (3.3) backwards in time. If this were possible, one could then rewrite (3.4) in feedback form

$$u(t) = u_*(x(t), -\pi(x(t), t)$$

or more simply as

$$u = u_*(x, -\pi(x,t)) \tag{3.6}$$

as in LQ Theory. We shall first impose a further simplifying assumption on (2.1)–(2.2).

H4: The canonical system (3.1)–(3.2) is complete.

Equivalently, we assume that the flow

$$\Phi\left(t, \begin{pmatrix} x(0) \\ p(0) \end{pmatrix}\right) = \begin{pmatrix} x(t) \\ p(t) \end{pmatrix},$$

which is always defined for small t, is in fact defined for all t. In particular, we note that

$$\Phi : \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$$

is jointly C^k and, for each t, the map

$$\Phi_t(\cdot) = \Phi(t, \cdot)$$

is a C^k diffeomorphism

$$\Phi_t : \mathbb{R}^{2n} \to \mathbb{R}^{2n}. \tag{3.7}$$

Now consider the closed, connected C^k submanifold of $\mathbb{R}^{2n}$ defined via

$$M_T = \{(x,p) : p = -\nabla Q(x)\}. \tag{3.8}$$

M_T is of course the submanifold of terminal constraints given by the transversality conditions. We note that for $t \in [0,T]$

$$M_t = \Phi_{t-T}(M_T) \tag{3.9}$$

is a closed, connected C^k submanifold of $\mathbb{R}^{2n}$ consisting of those pairs $(x(t), p(t))$ which satisfy (3.1)–(3.3), with initial time $t_0 = t$. In particular, for $s \in [t,T]$

$$u(s) = u_*(x(s), p(s))$$

is an extremal control for the Bolza problem (2.1)–(2.3) with initial time $t_0 = t$. Moreover, to say

$$M_s = \text{graph}(-\pi(x,s)), \quad t \le s \le T$$

is to say that all extremal controls for $t \leq s \leq T$ can be given in feedback form

$$u(s) = u_*(x(s), -\pi(x(s), s)), \quad t \leq s \leq T.$$

We first derive conditions on $\pi(x,t)$ so that the geometric condition (3.10) will be satisfied.

Theorem 3.1. *Necessary and sufficient conditions for the existence of a C^ℓ function $\pi(x,t)$, $\quad 1 \leq \ell \leq k$, such that*

$$M_t = graph(-\pi(x,t))$$

for $t \in [t_0, T]$, $x \in \mathbb{R}^n$ is that $\pi(x,t)$ satisfy the following "Riccati" partial differential equation, for $(x,t) \epsilon \mathbb{R}^n \times (t_0, T)$

$$\frac{\partial \pi}{\partial t} = \frac{\partial H_*}{\partial x}(x, -\pi) - \frac{\partial \pi}{\partial x}\frac{\partial H_*}{\partial p}(x, -\pi) \tag{3.10}$$

$$\pi(0,t) = 0, \quad \pi(x,T) = \nabla Q(x). \tag{3.11}$$

In particular, the Riccati partial differential equation has a C^k solution if, and only if, it has a C^1 solution.

Proof: Suppose $(x,p) \in M_s$ for $s \in (t_0, T)$ so that $p = -\pi(x,s)$. To say

$$M_t = \text{graph } (-\pi(x,t))$$

with π being C^1 is to say M_t is a regular submanifold, at least C^1, which is transverse to each vertical fiber, $\{x\} \times \mathbb{R}^n \subset \mathbb{R}^{2n}$. By Dynamic Programming, M_t is of course a C^k submanifold. Consider the subset

$$N \subset \mathbb{R}^n \times (t_0, T) \times \mathbb{R}^n$$

defined via

$$N = \{(x,t,p) : (x,p) \, \epsilon M_t\}\,, \tag{3.12}$$

or equivalently

$$N = \{(x,t,p) : p = -\pi\,(x,t)\}\,. \tag{3.13}$$

From (3.14) it follows that N is a closed, connected C^1 submanifold. Since the map

$$proj_2 \;:\; N \to (t_0, T)$$

defined via

$$proj_2\,(x,t,p) = t$$

is a submersion with each preimage being a C^k submanifold (see (3.13)), N is in fact a closed, connected C^k-submanifold. Furthermore, the map

$$P \;:\; N \to \mathbb{R}^n \times (t_0, T)$$

defined via

$$P\,(x, t, p) = (x, t)$$

is a submersion, since π is C^1. Therefore, N is the graph of C^k function, viz, π. In particular, to say that π is C^1 is to say that π is C^k.

We now show that $\pi(x,t)$ must satisfy the Riccati PDE. Choose any final condition $(x(T), p(T)) \in M_T$. The corresponding trajectory $(x(t), p(t)) \in M_t$ is, by hypothesis, of the form $(x(t), -\pi(x(t), t))$ so that $(x(t), -\pi(x(t), t))$ must be a solution of the canonical equations

$$\dot{x}(t) = \frac{\partial H_*}{\partial p}(x(t), -\pi(x(t), t)) \tag{3.14}$$

$$\dot{\pi}(x(t), t) = \frac{\partial H_*}{\partial x}(x(t), -\pi(x(t), t)). \tag{3.15}$$

Since π is C^1 one can also compute the latter equality by the chain rule, using the former equation to obtain

$$\frac{\partial \pi}{\partial t}(x(t), t) = -\frac{\partial \pi}{\partial x}(x(t), t)\frac{\partial H_*}{\partial p}(x(t), -\pi(x(t), t)) \tag{3.16}$$

$$+\frac{\partial H_*}{\partial x}(x(t), -\pi(x(t), t). \tag{3.17}$$

Since for each $t \in [t_0, T]$ M_t is a graph, for each $x \in \mathbb{R}^n$ there exists some p such that $(x, p) \in M_t$. Since all points $(x, p) \in M_t$ have the form $(x(t), p(t))$ for some final condition $(x(T), p(T))$, the pair $(x(t), t)$ can be taken to be an arbitrary pair $(x, t) \in \mathbb{R}^n \times [t_0, T]$. Therefore, π satisfies the Riccati PDE (3.11) on $\mathbb{R}^n \times [t_0, T]$. Since $(0, 0)$ is an equilibrium of (3.1)–(3.2) lying on M_T by (H3), $(0, 0) \in M_t$ for all t. In particular, if $M_t = \text{graph}(-\pi(x, t))$ then $\pi(0, t) = 0$.

Furthermore, since

$$\text{graph}(-\pi(x, T) = M_T = \text{graph}(-\nabla Q(x))$$

one has

$$\pi(x, T) = \nabla Q(x).$$

Conversely, suppose $\pi(x, t)$ is a solution, C^ℓ with $\ell \geq 1$, of the "Riccati PDE" on $\mathbb{R}^n \times (t_0, T)$. We claim

$$(x, -\pi(x, t)) \in M_t.$$

If this were true, then

$$\text{graph}(-\pi(\cdot, t) \subset M_t$$

would be a closed, n-dimensional submanifold of M_t. Moreover, since the map

$$x \mapsto (x, -\pi(x, t))$$

is a submersion of $\mathbb{R}^n$ into M_t, $\text{graph}(\pi(x, t))$ is open in M_t. Since M_t is connected,

$$\text{graph}(-\pi(x, t)) = M_t.$$

We now turn to the proof of the claim. Let $(x_0, s) \in \mathbb{R}^n \times (t_0-, T)$ and consider the trajectory $x_*(t)$ defined, at least for $t \in (s - \varepsilon, s + \varepsilon)$ with $\varepsilon << \infty$, as the solution of

$$\dot{x} = f(x) + g(x)u_*(x, -\pi(x, t)), \quad x(s) = x_0$$

and define

$$p_*(t) = -\pi(x_*(t), t).$$

Lemma 3.1. *$(x_*(t), p_*(t))$ is a trajectory of the canonical system (3.1)–(3.2), for $t \in (s - \varepsilon, s + \varepsilon)$.*

Proof: By construction

$$\dot{p}_*(t) = -\, \dot{\pi}(x_*(t), t) \tag{3.18}$$

$$= -\, \frac{\partial \pi}{\partial t} - \frac{\partial \pi}{\partial x}\frac{\partial H_*}{\partial p}(x_*(t), -\pi(x_*(t), t)) \tag{3.19}$$

$$= -\, \frac{\partial H_*}{\partial x}(x_*(t), -\pi(x_*(t), t)) \tag{3.20}$$

$$-\, \frac{\partial H_*}{\partial x}(x_*(t), p_*(t)). \tag{3.21}$$

Furthermore, for this $p(t)$

$$\dot{x}_*(t) = f(x_*(t) + g(x_*(t))u_*(x_*(t), p_*(t)) \tag{3.22}$$

$$= \frac{\partial H_*}{\partial x}\ (x_*(t), p_*(t))k \tag{3.23}$$

□

Since $(x_0, s) \in \mathbb{R}^n \times (t_0, T)$ is arbitrary and the canonical system is complete, $x_*(t)$ exists for all $t \in (t_0, T)$ and $(x_*(t), -\pi(x_*(t), t))$ is a trajectory of (3.1)–(3.2). By (3.12),

$$(x_*(T), -\pi(x_*(t), T) \in \text{graph}(-\nabla Q) = M_T.$$

Therefore,

$$(x_*(t), -\pi(x_*(t), t)) \in M_t.$$

Since $x_0 \in \mathbb{R}^n$ is arbitrary,

$$\text{graph}(-\pi(\cdot, t)) \subset M_t.$$

□

Remark 3.1. *There are corresponding results which are more local in the spatial variable and which follow from this proof. First suppose there exists an open subset $U_1 \subset \mathbb{R}^n$ containing 0 and that on $U_1 \times [0,T]$ there exists a C^k solution π of the Riccati PDE (3.11), also satisfying the side conditions (3.12). Choose a perhaps smaller neighborhood, U_2,*

$$0 \in U_2 \subset U_1 \subset \mathbb{R}^n,$$

such that trajectories of the system

$$\dot{x} = f(x) + g(x)u_*(x_1 - \pi(x,t))$$

for $t \in [0,T]$ having initial conditions in U_2 remain in U_1. Then, if $proj_1$ is defined on $\mathbb{R}^{2n}$ via

$$proj_1(x,p) = x,$$

then

$$M_t \cap proj_1^{-1}(U_2) \supset gr(-\pi(\cdot, t))$$

if U_2 is taken–as it may–to be connected. Indeed, denoting by $M_t^0(U_2)$ the connected component of $M_t \cap proj_1^{-1}(U_2)$ containing 0, we must have

$$M_t^0 = gr(-\pi(\cdot,t)|_{U_2}) \quad t \in [0,T].$$

Conversely, if there exists an open neighborhood U_1 of 0 in $\mathbb{R}^n$ for which

$$M_t^0 = graph(-\pi(\cdot,t)|_{U_1}), \quad t \in [0,T]$$

then π satisfies the Riccati PDE (3.11) on $U_1 \times [0,T]$. Moreover, choosing U_2 as above, π will also satisfy the side conditions (3.12) on $U_2 \times [0,T]$.

Theorem 3.2. *Suppose $1 \leq \ell \leq k$. For t_0 satisfying $0 \leq t_0 < T$ and $G \subset \mathbb{R}^n$ an open neighborhood of 0, the following statements are equivalent:*

i. There exists a C^ℓ function $\pi(x,t)$, satisfying the Riccati PDE on $G \times (t_0, T)$ and the conditions

$$\pi(0,t) = 0, \quad \pi(x,T) = \nabla Q(x), \quad x \in G, t \in (t_0, T).$$

ii. There exists a $C^{\ell+1}$ function $V(\tilde{x}, t)$ satisfying the Hamilton-Jacobi-Bellman equation for $x \in G$, $t \in (t_0, T)$

$$\frac{\partial V}{\partial t} + \langle \nabla_x V(\tilde{x}),\ f(x) + g(x)u_*(x, -\nabla_x V(\tilde{x}))\rangle \tag{3.24}$$

$$+ L(x, u_*(x, -\nabla_x V(\tilde{x})) = 0, \tag{3.25}$$

and the conditions

$$V(0,t) = 0, \quad V(x,T) = Q(x) + x_{n+1}, \; x \in G.$$

iii. *Condition (i), and hence (ii), holds for $\ell = k$.*

Corollary 3.1. *There exists a C^{k+1} solution of the Hamilton-Jacobi-Bellman equation (3.13) satisfying the side conditions*

$$V(0,t) = 0, \; V(x,T) = Q(x) + x_{n+1}$$

if, and only if, there exists a C^2 solution.

Remark 3.2. *Theorem 3.2 does not assert that $V(x,t)$ satisfies the conditions of the "verification principle" of Dynamic Programming; i.e., that $V(x,t)$ satisfies the Hamilton-Jacobi-Bellman inequality for all admissible controls $u(t)$. This is shown to hold in Section 5 for "weak solutions," which are discussed in Section 4. Nonetheless, according to Theorem 3.2, smoothness results for value functions for Bolza or Lagrange problems would imply the existence of classical solutions to the Riccati PDE. On the other hand, the general existence theory for weak and classical solutions of Riccati Partial Differential Equations yields smoothness results for value functions for Bolza and Lagrange problems (see Sections 4–6). For example, Corollary 3.1 already shows that if one assumes that the value function V of such a problem is C^2, it follows that V is in fact C^{k+1}. This also holds for $k = \infty$ and $k = \omega$. The C^1 case is not similar and is discussed in Section 4. In addition, this general approach provides a geometric framework for an analysis of nonsmooth behavior of value functions and we expect such a framework to provide a more detailed comparison between continuous viscosity solutions of the Hamilton-Jacobi-Bellman equation and the geometry of generalized solutions of the Riccati PDE.*

Proof: That (ii) implies (i) follows from the next two lemmas.

Lemma 3.2. *Suppose $V(\tilde{x},t)$ is a $C^{\ell+1}$ function which satisfies the Hamilton-Jacobi-Bellman equation (3.13). Then,*

$$V(\tilde{x},t) = x_{n+1} + W(x,t) \tag{3.26}$$

for a (necessarily) unique $C^{\ell+1}$ function W which satisfies

$$W(0,t) = 0, \quad W(x,T) = Q(x). \tag{3.27}$$

Proof of Lemma 3.2: To see this, note that to say V satisfies (3.13) is to say that $\dot{V}(\tilde{x}(t),t)$ vanishes along the trajectories of (2.17), (2.19) with $p(t)$ defined via

$$p(t) = -\nabla_x V(\tilde{x},t).$$

As in the proof of Theorem 3.1, one can show that $(x(t), p(t))$ is a canonical pair, since $V(\tilde{x}, t)$ is a solution of (3.13), and therefore that $\tilde{x}(t)$ exists for all time t. In particular,

$$V(\tilde{x}(t), t) = V(\tilde{x}(T), T) = x_{n+1}(T) + Q(x(T)). \tag{3.28}$$

We can now turn to a proof of Lemma 3.2. To say that (3.14) holds is to say

$$\frac{\partial V}{\partial x_{n+1}}(\tilde{x}, t) = 1.$$

Since V is $C^{\ell+1}$, with $\ell \geq 1$, we compute

$$\frac{d}{dt}\left(\frac{\partial V}{\partial x_{n+1}}(\tilde{x}(t), t)\right) = \frac{\partial}{\partial x_{n+1}}\left(\frac{d}{dt}V(\tilde{x}(t), t)\right) = 0$$

along trajectories constructed above. In particular, along any such curve $\tilde{x}(t)$ we have

$$\frac{\partial V}{\partial x_{n+1}}(\tilde{x}(t), t) = \text{constant}$$

for $t \in (t_0, T]$. Moreover, by (3.18) we can evaluate this constant for $t = T$, obtaining (3.19) and hence (3.16).

Lemma 3.3. *$\pi(x, t) = \nabla_x W(x, t)$ is a C^ℓ solution of the Riccati PDE.*

Proof of Lemma 3.3: By the equality of mixed partials, we may compute $\frac{\partial \pi}{\partial t}$ from the Hamilton-Jacobi-Bellman equation, obtaining

$$\frac{\partial \pi}{\partial t} + \frac{\partial \pi}{\partial x}\frac{\partial H_*}{\partial p}(x, -\pi(x, t)) - \frac{\partial H_*}{\partial x}(x, -\pi(x, t)) = 0$$

where

$$\pi(x, t) = \nabla_x Q(x), \quad \pi(0, t) = 0.$$

□

We begin the proof that (i) implies (ii) with an important observation, viz. that M_t is a Lagrangian submanifold, for $t \epsilon [0, T]$. More explicitly, since (3.1)–(3.2) is a Hamiltonian system, with respect to the standard symplectic form

$$\omega = dx \wedge dp,$$

the diffeomorphism (3.7) preserves ω; i.e. Φ_t is a symplectomorphism (see [14]). We note that ω is the standard symplectic form on $\mathbb{R}^{2n}$, thought of as $T^*(\mathbb{R}^n)$ with $x \epsilon \mathbb{R}^n$ and with $p \epsilon T_x(\mathbb{R}^n)$ being regarded as a cotangent vector. With this identification, any Lagrangian submanifold M which is transverse to the "vertical" fiber, $T^*_{x_0}(\mathbb{R}^n)$, is locally (in a neighborhood of x_0) the graph of a closed 1-form, (see e.g. [14]). In particular, as the graph of a gradient (or closed 1-form on $\mathbb{R}^n$), M_T is in fact a Lagrangian submanifold of $(\mathbb{R}^{2n}, w)$.

Since Φ_t is a symplectomorphism and M_T is a Lagrangian submanifold, M_t is also Lagrangian for all t by construction. In particular, if M_t is the graph of a C^ℓ function, $\pi(x,t)$ then M_t is transverse to every vertical fiber and consequently

$$\pi(x,t) = \nabla_x W(x,t) \tag{3.29}$$

for some $C^{\ell+1}$ function $W(x,t)$, by the Poincaré Lemma. We shall normalize W by setting

$$W(0,t) = 0. \tag{3.30}$$

Consequently, since $Q(0) = 0$,

$$W(x,T) = Q(x). \tag{3.31}$$

Now consider the function

$$V\left(\tilde{x},t\right) = x_{n+1} + W\left(x,t\right).$$

By construction,

$$V(0,t) \equiv 0.$$

Defining the quantity

$$F\left(\tilde{x},t\right) = \frac{\partial V}{\partial t} + \langle \nabla_x V,\ f(x) + g(x)u_*\left(x,-\nabla_x V\right)\rangle + L\left(x,u_*\left(x,-\nabla_x V\right)\right)$$

we first note that, for $i = 1,\dots,n$

$$\frac{\partial F}{\partial x_i}\left(\tilde{x},t\right) = 0.$$

Indeed, as in the proof of Lemma 3.3

$$\frac{\partial F}{\partial x} = \frac{\partial \pi}{\partial t} + \frac{\partial \pi}{\partial x}\frac{\partial H_*}{\partial p}\left(x,-\pi(x,t)\right) - \frac{\partial H_*}{\partial x}\left(x,-\pi(x,t)\right)$$

which vanishes identically since π is a solution of the Riccati PDE. Furthermore, since $F\left(\tilde{x},t\right)$ is independent of x_{n+1} we also have

$$\frac{\partial F}{\partial x_{n+1}}\left(\tilde{x},t\right) = 0$$

so that

$$F\left(\tilde{x},t\right) = F(0,t) = L\left(0,u_*(0,0)\right)$$

Since $u_*(x,p)$ is the unique solution of (2.14), we must have

$$u_*(0,0)=0$$

since with $x=p=0$, (2.14) reduces to the critical point equation

$$\frac{\partial L}{\partial u}(0,u)=0.$$

which has $u=0$ as a solution, by hypothesis $(H2)$. Matters being so, the normalization of $L(x,u)$ (cf. Remark 2.1) implies that

$$F\left(\tilde{x},t\right)=L\left(0,u_*(0,0)\right)=0.$$

Therefore, $V\left(\tilde{x},t\right)$ is a C^{r+1} solution of the Hamilton-Jacobi-Bellman equation, satisfying

$$V(0,t)=0, \quad V\left(\tilde{x},T\right)=x_{n+1}+Q(x).$$

Finally, since (i), (ii) are equivalent, (iii) follows from the fact (Theorem 3.1) that if $\pi(x,t)$ is a C^1 solution of the Riccati PDE, then $\pi(x,t)$ is necessarily C^k.

4 Classical, Weak and Generalized Solutions of Riccati Partial Differential Equations

Theorem 3.1 gives a geometric criterion for the existence of a smooth solution of the Riccati PDE, viz

$$M_t=\mathrm{graph}(-\pi(\cdot,t)) \tag{4.1}$$

where M_t is the set of initial conditions $(x(t),p(t))$ for initial time $t=t_0$, which are feasible or compatible with the conditions of the Maximum Principle. On the other hand, Dynamic Programming would suggest that any function π, C^1 or not, satisfying (4.1) would lead to an optimal control law in feedback form

$$u_*(t)=u_*(x(t),-\pi(x(t),t)).$$

Of course, π may fail to C^1 in several ways, among these being either a "classical blow-up" (or finite escape time), where $\frac{\partial\pi}{\partial t}$ becomes infinite, or the existence of "shock waves," where $\frac{\partial\pi}{\partial x}$ becomes infinite. In neither case, would π be considered to be a classical solution of the Riccati PDE. Rather, in this sense π, or even M_t would be a weak solution of the Riccati PDE with considerable importance for the analysis, design or synthesis of optimal control laws. We begin this section by illustrating this point using a simple example.

Example 4.1. Consider the one-dimensional dynamical system

$$\dot{x}_1 = u, \quad x, u \in \mathbb{R}$$

with the following cost function:

$$J(x_1(0), u) = \frac{1}{2}\int_0^T u^2 dt + \frac{1}{2}(x_1^4(T) - x_1^2(T))$$

where $x_1(0)$, $x_1(T)$ are initial and final conditions of the system respectively. Setting the system in Mayer form, we introduce the running cost as a new variable,

$$\dot{x}_2 = \frac{1}{2}u^2 \quad \text{with } x_2(0) = 0.$$

As before the augmented Hamiltonian is

$$H_*(x, p, u) = p_1 u - \frac{1}{2}u^2$$

and Pontryagin Maximum Principle implies that

$$u = p,$$

which must be constant since the canonical system has the form

$$\dot{x} = p \tag{4.2}$$

$$\dot{p} = 0. \tag{4.3}$$

In particular, for this system, the Riccati PDE for $p = -\pi(x, t)$ reduces to the well known inviscid Burgers' equation:

$$\frac{\partial \pi}{\partial t} = \frac{\partial \pi}{\partial x}\pi$$

with the side conditions

$$\pi(T, x) = 2x^3 - x \text{ and } \pi(t, 0) = 0.$$

The fact that this Burgers' equations, with a non-monotone nondecreasing final condition, does not have a solution for all time $t < T$ is classical (see e.g. [20]). Indeed, it is easy to see, either from the method of characteristics or by integrating the canonical equations, that this problem has a classical solution on $[0, T]$ if, and only if, $T < 1$. The weak solution for $T = 1$ has an infinite vertical slope at $x = 0$ corresponding to the existence (or development) of a shock wave; i.e. a point when $\frac{\partial \pi}{\partial x}$ becomes infinite. The simulations in Figure 4.1 illustrate the fact that as we extend the time interval $[0, T]$ for $T > 1$, this shock wave propagates with the vertical tangent bifurcating into two shocks, one traveling to the right and one traveling to the left. These shock waves are depicted for the times: $T =$ .2, .5, .95, 1.5, and 2. We also note that, if $T > 1$, M_t is no longer of the form

$$M_t = \text{graph}(-\pi(x,t)).$$

Indeed, M_t is a "generalized" solution of the Riccati PDE, consisting of the graph of a multivalued function. Nonetheless, M_t remains a smooth curve and the "branches" of this curve as well as the vertical tangents, or shock waves, have an interesting interpretation in terms of the optimal control problem. Briefly, from the Pontryagin Maximum Principle it follows that any optimal control $u_*(t)$ is necessarily constant, $u_*(t) = p$. Integrating the canonical equations, we can explicitly determine the cost functional, as a function of p, yielding

$$J^T(x_0,p) = \frac{p^2T}{2} + \frac{1}{2}(x_0+pT)^4 - \frac{1}{2}(x_0+pT)^2.$$

The extrema of $J^T(x_0,p)$ occur at solutions of

$$T(p + 2(x_0+pT)^3 - (x_0+pT)) = 0$$

which is, of course, just the transversality condition of the Maximum Principle. One can, therefore, compute the Hessian of $J^T(x_0,\cdot)$ obtaining

$$T(1 + 6T(x_0+pT)^2 - T).$$

The zeroes of $\frac{\partial^2 J}{\partial p^2}$ are therefore the points at which M_t, $T-t > 1$, has a vertical tangent. That is, the onset of shock waves for solutions of the Riccati PDE coincides with the

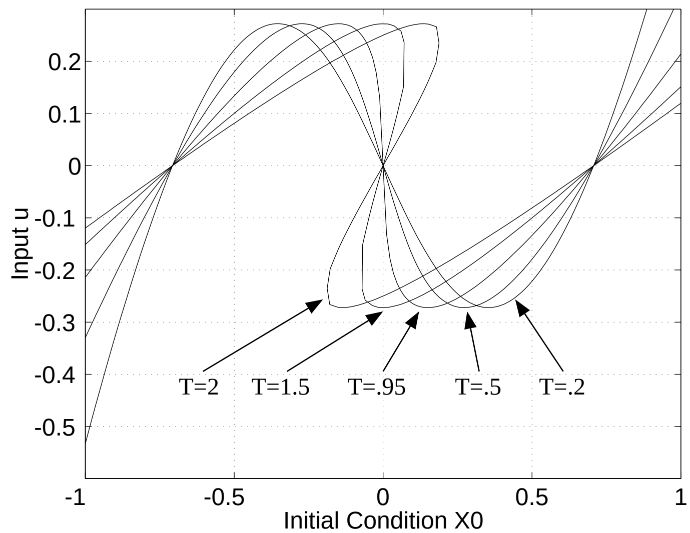

Figure 4.1: U as function of X_0

existence of degenerate critical points of the cost functional. Indeed, explicit calculation, for final times $T > 1$, shows that deleting the two points of vertical tangency from M_t, when $T - t > 1$, leaves 3 connected branches, two semi-infinite branches and one finite branch containing 0. On each of the semi-infinite branches we have $\frac{\partial^2 J}{\partial p^2} > 0$ while on the finite branch we have $\frac{\partial^2 J}{\partial p^2} < 0$, representing a bifurcation of the extremal minimizing controls to a family of 3 extremal controls. One can, of course, develop a synthesis of an optimal control for this problem, using these geometric constructions (see [21]–[22] for further discussion of the example). We emphasize, however, that smoothness of the generalized solution M_t is quite helpful in this analysis.

The existence of shock waves for the Riccati PDE, and the analysis in Example 4.1 in particular, serve to underscore the importance of our first basic result for the existence of weak, or generalized, solutions of the Riccati PDE, a result which in fact was buried in the proof of Theorem 3.2.

Proposition 4.1. *Assume (H1)–(H4) hold.* $\forall t \in \mathbb{R}$, *M_t exists and is uniquely defined as a connected, closed, Lagrangian C^k-submanifold of $\mathbb{R}^{2n}$. Moreover*

$$(0,0) \in M_t, \quad \forall t \in \mathbb{R}$$

and

$$M_T = graph(-\nabla Q).$$

Convention 4.1. *Motivated by Proposition 4.1, we shall refer to the 1-parameter family $(M_t : t \in \mathbb{R})$ of Lagrangian submanifolds as a generalized solution of the Riccati PDE.*

The basic existence and uniqueness result for generalized solutions of the Riccati PDE provided by Proposition 4.1 yields a starting point for the investigation of the existence and uniqueness of weak and classical solutions.

Definition 4.1. *If, for all $t \in [0,T]$ $M_t = graph(-\pi(\cdot,t))$ for some function π–not necessarily C^1, then we shall refer to π as a "weak" solution of the Riccati PDE on $\mathbb{R}^n$. If π is C^k, $k \geq 1$, then we say that π is a classical solution. Suppose, for $N \subset \mathbb{R}^n$ an open neighborhood of 0, we have $(x, \pi(x,t) \in M_t$ for $x \in N$, $t \in [0,T]$. If, in addition, π satisfies the side conditions on N, then we shall say π is a weak solution of the Riccati PDE in $N \times [0,T]$. If, in addition, π is C^k, $k \geq 1$, then we say π is a classical solution.*

Theorem 4.1. *Suppose $M_t = graph(-\pi(\cdot,t))$. Then*

i. *$\pi(x,t)$ is C^k a.e., and continuous everywhere;*

ii. *$\pi(x,t) = -\nabla_x W(x,t)$, for W a C^1 function, which is C^{k+1} a.e.*

Moreover, the set of regular points for π is open and dense.

Proof: Consider the function

$$P : M_t \to \mathbb{R}^n$$

defined via

$$P(x,p) = x \quad \text{for } (x,p) \in M_t.$$

Since M_t is C^k, P is C^k for $k \geq 1$. Since $p = -\pi(x,t)$, P is also a surjection. By Sard's Theorem, almost every value of P is regular. We claim that the set $R \subset \mathbb{R}^n$, of regular values is open and dense. Density follows, of course, from Sard's Theorem. Suppose $x_0 \in \mathbb{R}^n$ is a regular value of P. Note there is a unique p_0,

$$p_0 = -\pi(x_0,t),$$

such that $(x,p) \in M_t$. Since M_t is n-dimensional, the implicit function theorem asserts that π is C^k in x, for x near x_0, and that the set of points

$$\{(x, \pi(x,t))\} \subset M_t$$

is a (relatively) open neighborhood of (x_0, p_0) in M_t.

Therefore, x_0 is an interior point of R, as was to be shown. In particular, $\pi(\cdot,t)$ is a C^k function on $R \subset \mathbb{R}^{2n}$. Moreover, since M_t is closed, $\pi(\cdot,t)$ is continuous everywhere.

We now prove (ii).

Lemma 4.1. *For $x \in R$, $\pi(x,t) = \nabla W(x,t)$ for some globally defined, C^1 function $W(\cdot,t)$ on $\mathbb{R}^n$, which is C^{k+1} on R.*

If π were C^1 everywhere, this would be a consequence of the Poincaré Lemma, as in the proof of Theorem 3.2. Our proof is an adaptation of this classical argument to the case of 1-forms with continuous coefficients, smooth on an open dense set.

Proof of Lemma 4.1: We first define a continuous function of $x \in \mathbb{R}^n$ via

$$W(x,t) = \int_0^1 < \pi(sx,t),\, x > ds.$$

Suppose $x \in \mathbb{R}$, so that $\pi(\cdot,t)$ is C^1 in a neighborhood of x. Computing the Newton quotient,

$$\frac{W(x+\varepsilon v,t) - W(x,t)}{\varepsilon} = \int_0^1 \frac{[\langle \pi(s(x+\varepsilon v),t); x+\varepsilon v\rangle - \langle \pi(sx,t),x\rangle]}{\varepsilon} dt$$

we see that the limit, as $\varepsilon \to 0$, exists for all v and is continuous at v so that $W(\cdot,t)$ is in fact C^1 at x. We claim that

$$D_v W(x,t) = \pi(x,t)\cdot v.$$

By linearity, it suffices to check this for the unit tangent vectors, $v = e_j$. Thus, the claim follows from the explicit calculation

$$\frac{\partial W}{\partial x_j}(x,t) = \sum_{i=1}^n \int_0^1 \frac{\partial \pi_i}{\partial x_j}(sx,t)\cdot x_i\, ds \qquad (4.4)$$

$$+ \int_0^1 \pi_j(sx,t) ds \qquad (4.5)$$

However, since M_t is Lagrangian, at any $x \in R$ we have

$$\frac{\partial \pi_i}{\partial x_j}(x,t) = \frac{\partial \pi_j}{\partial x_i}(x,t)$$

so that

$$\frac{\partial W}{\partial x_j}(x,t) = \int_0^1 \sum_{i=1}^n s\frac{\partial \pi_j}{\partial x_i}(sx,t)\cdot x_i ds + \int_0^1 \pi_j(sx,t) ds \qquad (4.6)$$

$$= \int_0^1 \frac{d}{ds}(s\pi_j(sx,t)) ds = \pi_j(x,t). \qquad \square \qquad (4.7)$$

Remark 4.1. *We have defined $W(x,t)$ as the path integral of the 1 form $\sum_{i=1}^{u} \pi_i(x,t)dx_i$ along the straight line from 0 to x. One could as well have chosen a different base point, say $y \in \mathbb{R}^n$, leading to the function*

$$W_y(x,t) = \int_0^t \langle \pi(s(x-y)+y,t), x-y \rangle ds.$$

A similar calculation would show that, for $x \in R$,

$$\nabla W_y(x,t) = \pi(x,t). \tag{4.8}$$

From (4.2) and Lemma 4.1, it then follows that

$$\nabla (W_y(x,t) - W(x,t)) \equiv 0, \; x \in R. \tag{4.9}$$

Indeed, we claim that for $x \in \mathbb{R}^n$

$$W_y(x,t) = W(x,t) + c \tag{4.10}$$

where c is constant, which can be evaluated at $x = y$ yielding

$$c = W(y,t). \tag{4.11}$$

In particular, we claim that the only effect of the choice of a base point is the addition to W of a constant of integration. This would be trivial if W were differentiable everywhere, a fact we shall derive from (4.4)–(4.5). On the other hand, the claim itself follows from an analysis of the function

$$D(x,t) = W_y(x,t) - W(x,t),$$

which has zero total derivative on R. Therefore,

$$R = \cup_\alpha U_\alpha$$

where $\alpha \in$Range$(D|_R)$ and U_α is the open subset of $\mathbb{R}$

$$U_\alpha = \{x \in R : D(x,t) = \alpha\}.$$

Let $y \in \mathbb{R}^n$ be a boundary point of U_α in $\mathbb{R}^n$, $y \in \delta U_a$. Of course, $D(y,t) = \alpha$ by continuity, so that $y \notin \delta U_\beta$ for any $\beta \neq \alpha$. Therefore, $\bar{U}_\alpha$ is open in $\mathbb{R}^n$ and the decomposition

$$\mathbb{R}^n = \cup_\alpha \bar{U}_\alpha$$

is a decomposition of $\mathbb{R}^n$ into disjoint open subsets. By connectivity,

$$\mathbb{R}^n = U_\alpha$$

for some unique α; i.e.

$$D(x,t) \equiv \alpha \text{ for } x \in \mathbb{R}^n.$$

We can now conclude the proof of Theorem 4.1.

Lemma 4.2. *$W(x,t)$ is C^1 on $\mathbb{R}^n$ with total derivative $\nabla W(x,t) = \pi(x,t)$.*

Proof: From (4.4)–(4.5), we have

$$\frac{W(x+\varepsilon v, t) - W(x,t)}{\varepsilon} = \frac{W_x(x+\varepsilon v, t)}{\varepsilon}.$$

Therefore

$$\lim_{\varepsilon \to 0} \frac{W(x+\varepsilon v, t) - W(x,t)}{\varepsilon} \tag{4.12}$$

$$= \lim_{\varepsilon \to 0} \int_0^1 \frac{\langle \pi(s(\varepsilon v) + x, t), \varepsilon v \rangle}{\varepsilon} ds \tag{4.13}$$

$$= \int_0^1 \langle \pi(x,t), v \rangle ds = \pi(x,t) \cdot v. \qquad \square \tag{4.14}$$

Theorem 4.1 gives information on the regularity properties of a function whose graph coincides with the submanifold M_t of extremal initial conditions for initial time $t_0 = t$. A similar analysis, applied to the submanifold $N \subset \mathbb{R}^{2n+1}$ and the map

$$P : N \to \mathbb{R}^n \times (t_0, T)$$

introduced in the proof of Theorem 3.1, shows that a weak solution exists on $\mathbb{R}^n \times [t_0, T]$ if, and only if,

$$\text{graph } (-\pi(x,t)) = M_t, \quad x \in \mathbb{R}^n, \ t \in [t_0, T]$$

for a continuous function π which is C^k on an open, dense subset $R \subset \mathbb{R}^n \times [t_0, T]$. In fact, $R \cap \mathbb{R}^n \times \{t\}$ is (relatively) open and dense for $t \in [t_0, T]$. We want to complete this circle of ideas by characterizing weak solutions in terms of the Riccati equation and in terms of regularity properties for "candidate" value functions.

Theorem 4.2. *The following statements are equivalent:*

1. *A weak solution of the Riccati equation exists on $\mathbb{R}^n \times [t_0, T]$.*
2. *There exists a continuous function $\pi(x,t)$ defined on $\mathbb{R}^n \times [t_0, T]$, satisfying the side conditions (3.12), and an open dense subset $R \subset \mathbb{R}^n \times [t_0, T]$ on which π is C^1 and satisfied the Riccati PDE.*
3. *There exists a C^1 function $W(x,t)$ defined on $\mathbb{R}^n \times [t_0, T]$, which is C^2 on an open dense subset R, and for which the function $V(\tilde{x}, t) = x_{n+1} + W(x,t)$ satisfies the Hamilton-Jacobi-Bellman (3.15) equation, together with the associated side conditions.*

Proof: As we have remarked, to say (1) is to say

$$M_t = \mathrm{graph}(-\pi(x,t))$$

for $x \in \mathbb{R}^n,\ t \in [t_0, T]$, for a continuous function π which is C^k on an open dense subset, $R \subset \mathbb{R}^n \times [t_0, T]$, of regular points. Sine $R \cap \mathbb{R}^n \times \{t\}$ is open and dense for all $t \in [t_0, T]$,

$$R \cap \mathbb{R}^n \times \{T\}$$

is open and dense in $\mathbb{R}^n$. Choosing final data $(x(T), p(T), T)$ for $(x(T), T) \cdot \in R \cap \mathbb{R}^n \times \{T\}$ and integrating backwards along trajectories of the canonical system, we find, as in the proof of Theorem 3.1, that π satisfies the Riccati PDE on an open dense subset of $\mathbb{R}^n \times [t_0, T]$, so that (2) must be satisfied.

Conversely, if π is C^1 on an open dense subset R of $\mathbb{R}^n \times [t_0, T]$ and satisfies the Riccati PDE on R, then–as in the proof of Theorem 3.1–we know that for $(x,t) \in R,\ (x, -\pi(x,t)) \in M_t$. By continuity of π, we deduce that

$$M_t \supset \text{graph } (-\pi(\cdot, t)), \quad \text{for each } t \in [t_0, T].$$

Finally, by Brouwer's Principle of Invariance of Domain [23], $\mathrm{graph}(-\pi(\cdot, t))$ is an open submanifold of M_t, which is also closed in M_t. Since M_t is connected,

$$M_t = graph(-\pi(\cdot, t)) \quad \text{for } t \in [t_0, T]$$

and therefore π is a weak solution of the Riccati PDE.

That (1) implies (3) follows from the fact that any weak solution is C^k in time, since

$$\pi(x(t), t) = -p(t)$$

where $(x(t), p(t))$ is a trajectory of the canonical system, and that on the subset R of regular points

$$\pi(x,t) = \nabla w(\cdot, t)$$

as in the proof of Lemma 4.1. Therefore, W is C^2 $Rx[t_0, T] \subset \mathbb{R}^n \times [t_0, T]$. Since π satisfies the Riccati equation on $R \times [t_0, T]$, W–or equivalently V–satisfies the Hamilton-Jacobi-Bellman equation, by Theorem 3.2. Conversely, if W is C^1 everywhere and at least C^2 on $R \times [t_0, T]$, the function π defined via

$$\pi(x,t) = \nabla W(x,t)$$

is a continuous function on $\mathbb{R}^n \times [t_0, T]$, at least C^1 on $R \times [t_0, T]$, satisfying the Riccati PDE on $R \times [t_0, T]$ by Theorem 3.2. □

Remark 4.2. *In the hierarchy of classical, weak and generalized solutions of the Riccati PDE, we have shown that there is a corresponding hierarchy* *of regularity for the value function: To say the value function is C^2 is to say it is C^k, which is to say a classical solution of the Riccati PDE exists. To say the value function V is C^1 but not C^2 is to say V is C^k on an open dense subset and that a weak solution of the Riccati PDE exists. To say that V is not C^1 is to say the unique generalized solution of the Riccati PDE is not a weak solution but is instead multi-valued.*

We now turn to sufficient conditions for a weak solution to be classical. Example 4.1 demonstrates the evolution of a classical solution ($T - t_0 < 1$) to a weak solution ($T - t_0 = 1$) to a generalized solution ($T - t_0 > 1$) which is coexistent with the onset and propagation of shock waves for the Riccati equation.

Proposition 4.2. *If M_{t_0} is transverse to the "vertical fiber" $\{x_0\} \times \mathbb{R}^n \subset \mathbb{R}^{2n}$ at the point $(x_0, p_0) \in M_{t_0}$, then there exists $\varepsilon > 0$ such that*

$$M_t \cap B_\varepsilon(x_0, p_0) = graph(-\pi(\cdot, t)) \text{ for } |t - t_0| < \varepsilon$$

for a C^k function $\pi(x,t)$, satisfying the Riccati PDE. Moreover, $\pi(x,t) = \nabla_x W(x,t)$ where

$$V(\tilde{x}, t) = x_{n+1} + W(x,t)$$

is a C^{k+1} solution of the Hamilton-Jacobi-Bellman equation.

Proof of Proposition 4.2: Consider the C^k submanifold $N \subset \mathbb{R}^{2n+1}$ defined via

$$N = \{(x,t,p) : (x,p) \in M_t\}\,.$$

Since N is transverse to the fiber $\{x_0\} \times \{t_0\} \times \mathbb{R}^n$ at the point (x_0, t_0, p_0), N is also transverse to the fibers $\{x\} \times \{t\} \times \mathbb{R}^n$, for (x,t) in a sufficiently small neighborhood of (x_0, t_0), at all points on N in a neighborhood of (x_0, t_0, p_0). Therefore, there exists an $\varepsilon > 0$ such that, for $|t - t_0| < \varepsilon$, M_t is transverse to the vertical fiber $\{x\} \times \mathbb{R}^n$ at all points (x,p) in $M_t \cap B_\varepsilon(x_0, p_0)$. By the implicit function theorem,

$$M_t \cap B_\varepsilon(x_0, p_0) = \mathrm{graph}(-\pi(\cdot, t))$$

for a C^k function $\pi(x,t)$. As in the proof of Theorem 3.1, π is locally a solution of the Riccati PDE. By the Poincaré Lemma

$$\pi(x,t) = \nabla_x W(x,t)$$

for a C^{k+1} function W. Finally, from Theorem 3.2 it follows that

$$V(\tilde{x}, t) = x_{n+1} + W(x,t)$$

is a C^{k+1} solution of the Hamilton-Jacobi-Bellman equation. □

Corollary 4.1. *Suppose* $M_{t_0} = graph(-\pi(\cdot, t_0))$ *and is transverse to the vertical fibers* $\{x\} \times \mathbb{R}^n$*, for all* $x \in \mathbb{R}^n$*. Then,* π *is a* C^k *function, defined in a neighborhood of* $\mathbb{R}^n \times \{t_0\} \subset \mathbb{R}^{n+1}$*. In particular, in this neighborhood,* π *satisfies the Riccati Partial Differential Equation.*

Proof of Corollary 4.1: By the proof of Theorem 4.1, $\pi(\cdot, t)$ is C^k on the set of regular points $R \subset \mathbb{R}^n$, which coincides with $\mathbb{R}^n$ since each vertical fiber is transverse to M_t. In particular, $\pi(\cdot, t)$ is C^k everywhere. According to Proposition 4.2, there exists a neighborhood of $\mathbb{R}^n \times \{t_0\} \subset \mathbb{R}^{n+1}$ on which M_t is transverse to each vertical fiber. Consequently, π is C^k on this neighborhood and, as in the proof of Theorem 3.1, satisfies the Riccati PDE. □

Taking $t_0 = T$ and using a standard compactness argument, we deduce

Corollary 4.2. *Suppose* $K \subset \mathbb{R}^n$ *is a relatively compact open neighborhood of* 0*. There exists a* t_0 *such that for* $(x,t) \in K \times [t_0, T]$ *there exists a* C^k *solution* π *of the Riccati Partial Differential Equation, satisfying*

$$\pi(x, T) = \nabla Q(x), \quad \textit{for } x \in K.$$

Early work by Barbu and Da Prato [12]–[13] establishes a result for certain systems which, in finite dimensions, is an improvement of Corollary 4.2, wherein the relatively compact set K may be replaced by $\mathbb{R}^n$. Explicitly, they consider the system

$$\dot{x} = f(x) + u$$

and the integral performance measure

$$L(x, u, t) = \ell(t, x) + \frac{1}{2}\|u\|^2$$

where the drift vector field is either monotone [12] or is the gradient of a function [13] and derive a Riccati PDE in [12], [13] which, of course, coincides with the one derived here. Essentially, they show that the Riccati equation has, for some $T > 0$, a global solution on $\mathbb{R}^n \times [0, T]$. While these results and Corollary 4.2 is an existence result which is local in time, the next existence result is local in the spatial variable but valid for $t \in [0, T]$.

Theorem 4.3. *Assume (H1)-(H4) hold and suppose* L *and* Q *are nonnegative and at least* C^3*. Then, there exists an* $\varepsilon > 0$ *such that*

$$M_t \cap B_\varepsilon(0,0) = \textit{graph}\ (-\pi(\cdot, t))$$

for $t \in [0, T]$*, where* π *is a* C^k *function satisfying the Riccati Partial Differential Equation. In particular, in a neighborhood of* $x = 0$

$$\pi(x, t) = \nabla_x W(x, t)$$

where

$$V(\tilde{x}, t) = x_{n+1} + W(x, t)$$

is a C^{k+1} solution of the Hamilton-Jacobi-Bellman equation.

Lemma 4.3. *In a neighborhood of $(x, u) = (0, 0)$,*

$$L(x, u) = x^T S x + u^T R u + 0(\|x\|^3 + \|u\|^3)$$

where $S = S^T$ and $R = R^T > 0$.

Proof of Lemma 4.3: First note that, from the fact that $L(0, 0) = 0$ and $L(0, u)$ has a minimum at $u = 0$, we must have

$$L(x, u) = x^T S x + x^T L u + u^T R u + R(x, u) \tag{4.15}$$

where

$$R(x, 0) = 0, \ \frac{\partial R}{\partial u}(x, 0) = 0, \ \frac{\partial^2 R}{\partial u^2}(x, 0) = 0$$

and

$$R(x, u) = 0\left(||x||^3 + ||u||^3\right).$$

By (H2) $u = 0$ is a nondegenerate minimum of $L(x, u)$ in u, so that $R = R^T > 0$. Fixing x, (4.6) also gives an expansion of $L(x, u)$ in u. In particular, since $L(x, \cdot)$ has a critical point at $u = 0$ by (H2) we must have

$$x^T L = 0$$

for all x in a neighborhood of $x = 0$. By equality of mixed partials, $S = S^T$. Recapitulating,

$$L(x, u) = x^T S x + u^T R u + R(x, u)$$

where $R(x, u) = 0(\|x\|^3 + \|u\|^3)$.

We now turn to the corresponding expansion for the reduced Hamiltonian. Recalling that $H_*(0, 0) = 0$ and that, from (2.14), we must have

$$\frac{\partial H_*}{\partial x}(0, 0) = 0, \qquad \frac{\partial H_*}{\partial p}(0, 0) = 0$$

it follows that $H_*(x, p)$ admits an expansion

$$H_*(x, p) = (x^T p^T) H(x, p) + 0(\|x\|^2 + \|p\|^3) \tag{4.16}$$

where H is a $2n \times 2n$ symplectic matrix.

Expanding the critical point equation (2.14) to terms of order 2, we obtain

$$B^T p - R\bar{u} = 0,$$

where $u_*(x,p) = \bar{u} + 0\left(\left\|\begin{pmatrix} x \\ p \end{pmatrix}\right\|^2\right)$. In particular,

$$\bar{u}(x,p) = R^{-1}B^T p. \tag{4.17}$$

Therefore, developing a Taylor expansion for the Hamiltonian H_* we obtain

$$H_*(x,p) = \langle p, Ax + BR^{-1}B^T p\rangle - \frac{1}{2}x^T X x \tag{4.18}$$
$$- \frac{1}{2}p^T BR^{-1}B^T p + 0\left(\left\|\begin{pmatrix} x \\ p \end{pmatrix}\right\|^2\right) \tag{4.19}$$

so that $(x^T,\, p^T)H(x,p)$ is the Hamiltonian of the linear optimal control problem

$$\min_u J(x_0, u)$$

where

$$J(x_0,u) = \frac{1}{2}\int_0^T (x^T S x + u^T R u)dt + x^T \bar{Q} x$$

subject to the dynamic constraint

$$\dot{x} = Ax + Bu, \quad x(0) = x_0 \tag{4.20}$$
$$y = Cs \tag{4.21}$$

and where $Q(x) = x^T Q x$. Classical LQ theory asserts that this problem has a solution $\bar{u}$ given by (4.8) with $p = -P(t)x$ for $P(t)$ the unique solution on $[0,T]$ of the Riccati equation

$$\dot{P} = -A^T P - PA + PBR^{-1}B^T P - S \tag{4.22}$$

with final condition

$$P(T) = \bar{Q}.$$

In particular, for each $t \in [0,T]$

$$P(t,x) = P(t)x$$

satisfies the Riccati PDE to second order. Therefore,

$$T_0(M_t) = \text{graph}(-P)$$

so that M_t is transverse to the vertical fiber, $x = 0$, in $\mathbb{R}^{2n}$. Consequently,

$$\text{proj}_1|_{M_t} : M_t \to \mathbb{R}^n$$

defined via

$$\text{proj}_1(x,p) = x$$

is a submersion at (0,0). Now consider the submanifold N, constructed in (3.13)–(3.14), and the map P defined via

$$P(x,t,p) = (x,t),$$

which was analyzed in the proof of Theorem 3.1. Since proj_1 is a submersion at (0,0), there exists $\delta > 0$ such that P is a submersion at $(0,0,t)$, for all $t \in (-\delta, T+0)$. In particular, there exists a neighborhood U of 0 in $\mathbb{R}^n$ such that

$$M_t \supset \text{gr}(-\pi)$$

for π a C^k function on $U \times [0,T]$. Equivalently, there exists an $\varepsilon > 0$ such that

$$M_t \cap B_\varepsilon(0,0) = \text{gr}(-\pi(\cdot,t)|_{B_\varepsilon(0)}).$$

It now follows (cf. Remark 3.1) that, on a perhaps smaller neighborhood, π is a C^k solution to the Riccati equation.

5 Feedback Synthesis of Optimal Controls for the Bolza Problem

In this section, we return to the basic optimal control problem (2.1)–(2.2) which we analyze under the hypotheses (H1)–(H4). Our goal is to develop explicit solutions to such control problems, in the form of a feedback law which can be determined off-line by solving a Riccati PDE. We first show how weak solutions give rise to optimal control laws.

Theorem 5.1. *Consider an open neighborhood N of 0 in $\mathbb{R}^n$ and a time $t_0 < T$. Suppose a weak solution of the Riccati PDE, satisfying the side conditions, exists on $N \times [t_0, T]$. Then there exists a possibly smaller neighborhood G so that the closed-loop control law*

$$u_*(t) = u_*(x(t), -\pi(x(t),t))$$

satisfies, for any initial condition $x_*(t_0) \in G$

$$\int_{t_0}^{T} L(x_*(t), u_*(t))dt + Q(x_*(T)) \leq \int_{t_0}^{T} L(x(t), u(t))dt + Q(x(T)) \quad (5.1)$$

for any control law $u(t)$ for which the corresponding trajectory $x(t)$ lies in N for $t_0 \leq t \leq T$. Moreover, u_* is unique in the sense that if $u(t) \neq u_*(t)$ for an open interval of time, the inequality is strict for each initial condition in G.

Definition 5.1. *We refer to a control law satisfying the above conditions as the locally optimal control defined on N.*

Proof: To say a weak solution to the Riccati PDE exists on N is to say

$$M_t \cap (N \times \mathbb{R}^n) \supset \operatorname{graph}(\pi(x,t)), \quad x \in N \tag{5.2}$$

where $\pi(\cdot,t)$ is a continuous function, C^k almost everywhere, which necessarily vanishes at $x=0$ and satisfies

$$\pi(x,T) = \nabla Q(x), \qquad x \in N. \tag{5.3}$$

According to Theorem 4.1 we also have, for $x \in N, \quad t \in [t_0,T]$

$$\pi(x,t) = \nabla W(x,t)$$

where $W(\cdot,t)$ is C^1 in x, C^{k+1} almost everywhere. Since for an arbitrary point, $(\bar{x},\bar{t}) \in N \times (t_0,T)$, the pair

$$(\bar{x}(t), -\pi(\bar{x}(t),t) \qquad (t-t_0) << \infty$$

is a trajectory of the Hamiltonian system (3.1)–(3.2) with initial condition $\bar{x}(\bar{t}) = \bar{x},\ p(\bar{t}) = -\pi(\bar{x},\bar{t})$, it follows that $p(t) = -\pi(\bar{x}(t),t)$ is a solution of the adjoint system. In particular, $\pi(x,\cdot)$ and hence $W(x,\cdot)$ is C^1 in t, for each $x \in N$.

Therefore, the function

$$V(\tilde{x},t) = x_{n+1} + W(x,t)$$

is C^1 on $N \times (t_0,T)$. Consider the C^k submanifold $M \subset \mathbb{R}^{2n+1}$

$$M = \{(x,t,p) : (x,p) \in M_t\}$$

and the C^k map

$$P : M \to \mathbb{R}^{n+1}$$

defined via

$$P(x,t,p) = (x,t).$$

Using Sard's Theorem as in the proof of Theorem 4.1, we deduce that $\pi(x,t)$ is in fact C^k on an open dense subset of $N \times (t_0,T)$. Therefore, V is a C^1 function on $N \times (t_0,T)$ which is C^k on an open dense subset. Following the proof of Theorem 3.2, V satisfies the Hamilton-Jacobi-Bellman equation

$$\frac{\partial V}{\partial t} + \langle \frac{\partial V}{\partial x},\ f(x) + g(x)u_*(x, -\pi(x,t))\rangle + L(x,u_*) = 0 \tag{5.4}$$

$$(x,t) \in N \times (t_0,T) \tag{5.5}$$

almost everywhere. However, since V is C^1, we see that, in fact, V is a solution of (5.2) everywhere on $N \times (t_0, T)$. Moreover, it follows from (5.3) that V satisfies the side condition

$$V(x,T) = \nabla Q(x) + x_{n+1}, \qquad x \in N.$$

Now choose a neighborhood $0 \in G \subset N$ so that for any initial data $x_0 \in G$, the trajectory

$$x_*(t), \qquad t_0 \leq t \leq T$$

of the closed loop system

$$\dot{x} = f(x) + g(x)u_*(x,t), \qquad x(0) = x_0$$

remains in N. We shall check that the function V, and the control u_*, satisfy the conditions of the verification principle of Dynamic Programming, viz., that for initial data in G, V is nondecreasing along any trajectory which remains in N and that V is constant along the trajectory $x_*(t)$. In particular, we shall show, that on G, $V(x,t)$ is the value function of the optimal control problem and that, locally, u_* is the unique optimal control in the sense of Definition 5.1.

That V is constant along the trajectories $x_*(t)$ is expressed by the Hamilton-Jacobi-Bellman equation (5.4). Now consider an admissible control $u(t)$. Compute

$$\dot{V}(x(t),t) = \frac{\partial V}{\partial t}(x(t),t) + \langle \frac{\partial V}{\partial x}, (x(t),t)), f(x(t)) + g(x(t))u(t)\rangle \qquad (5.6)$$

$$+ L(x(t),u(t)) \qquad (5.7)$$

in the standard manner and observe that, since

$$\frac{\partial V}{\partial x} = \frac{\partial W}{\partial x} = \pi(x,t)$$

and

$$-\pi(x(t),t) = p(t)$$

where $p(t)$ is a solution of the adjoint equation corresponding via the Maximum Principle to $u_*(t)$, we have

$$-\langle \frac{\partial V}{\partial x}(x(t),t), f(x(t)) + g(x(t))u(t))\rangle + L(x(t),u(t)) \qquad (5.8)$$

$$= H(x_*(t),p(t),u(t)) \leq H(x_*(t),p(t),u_*(x_*(t),t)) \qquad (5.9)$$

$$= -\langle \frac{\partial V}{\partial x}(x(t),t), f(x(t)) + g(x(t))u_*(x(t),t)\rangle \qquad (5.10)$$

$$+ L(x(t),u_*(x(t),t)) \qquad (5.11)$$

for almost all t. Consequently,

$$\dot{V} \geq \frac{\partial V}{\partial t}(x(t),t) + \langle \frac{\partial V}{\partial x}(x(t),t), f(x(t)) + g(x(t))u_*(x(t), -\pi(x(t),t))\rangle \tag{5.12}$$

$$+ L(x(t), u_*(x(t),t)) \tag{5.13}$$

or, by (5.4)

$$\dot{V}(x(t),t) \geq 0. \tag{5.14}$$

Since by hypotheses (H1)–(H2) $u_*(x(t), -\pi(x(t),t))$ is a nondegenerate, unique minimum of $H(x,p,u)$ (see e.g. (2.13)–(2.14) and the ensuing discussion), the inequality in (5.4) is strict whenever

$$u(t) \neq u_*(t)$$

holds on an open subinterval of (t_0, T). Consequently, the inequality (5.5), and hence (5.1), is also strict on the subinterval. □

Remark 5.1. *Example 4.1 shows that this result is sharp, in the following sense. For $T > 1$, M_t is of course no longer the graph of a function; rather, in this case, M_t contains 3 branches, with the unique finite branch containing* 0*. For $T > 1$, this branch is the graph of an extremal so that for each fixed t, $t \in [0,T]$, M_t is a graph in a sufficiently small neighborhood of* 0*. Yet as we have noted, for $T > 1$ this extremal we have noted locally maximizes the cost functional. This is in harmony with Theorem 5.1, as it should be. Indeed, while for each fixed t, there exists an $\varepsilon_t > 0$ such that $M_t \cap B_{\varepsilon_t}(0,0)$ is the graph of a function, there is no fixed $\varepsilon > 0$ such that*

$$M_t \cap B_\varepsilon(0,0) = \textit{ graph } (\pi(\cdot,t)), \quad t \in [1,T]$$

for $\pi(\cdot,t)$ defined on a sufficiently small neighborhood of 0*.*

Remark 5.2. *As discussed in Remark 4.2, the proof of Theorem 5.1 shows that a weak solution of the Riccati PDE exists on $N \times [t_0,T]$ if, and only if, the value function of the Bolza problem is C^1 on $N \times [t_0,T]$; see Example 4.1 for $t_0 = T-1$. In general, the value function is C^2, and hence C^k, a shock wave. At the first onset of a shock until the onset of ∇, V is still C^1 everywhere and C^2, and hence C^k, almost everywhere. If a shock wave bifurcates then a generalized solution appears, corresponding to nondifferentiation of the value function.*

Theorem 5.1 has several corollaries. First, we note that from the definition of G and from hypothesis (H4), we know that if $N = \mathbb{R}^n$, then G may also be taken as $G = \mathbb{R}^n$, yielding global optimal control laws.

Corollary 5.1. *Consider a time $t_0 < T$. Suppose a weak solution $\pi(x,t)$ of the Riccati PDE, satisfying the side conditions, exists on $\mathbb{R}^n \times [t_0, T]$. Then, the control law*

$$u_*(t) = u_*(x(t), -\pi(x(t), t))$$

is, for any initial condition $x(t_0) \in \mathbb{R}^n$, the unique optimal control. In particular, the unique optimal control law can be expressed as a feedback control law:

$$u_*(x) = u_*(x, -\pi(x,t)).$$

We now turn to some local consequences of Theorem 5.1. Just as some of the more local aspects, e.g. the construction of the neighborhood G, of this theorem were unnecessary in the global case, some of the global hypotheses are superfluous in the local case.

For example, since we shall work in a neighborhood of the equilibrium, 0, hypothesis (H4) is superfluous. Indeed ([16]), for any fixed control law $u(t)$ and any final time $T > 0$, there exists a neighborhood N of 0 such that for $x_0 \in N$ the unique solution $x(t)$ of (2.1) exists for $t \in [0, T]$. In addition, in the proof of Lemma 4.3, we have noted that, in the light of (H1), hypothesis (H2) implies that $L(x, \cdot)$ has a nondegenerate minimum at $u = 0$. Conversely, the assumption that $L(0, \cdot)$ has a nondegenerate minimum at $u = 0$ will imply a local version of (H1), viz.

$(H1)'$: for all x in a neighborhood of 0, $\frac{\partial L}{\partial u}(x, \cdot)$ is a diffeomorphism.

More formally, we assume L is C^{s+1}, $s \geq 2$, and note that (H1)$'$ is implied by

$(H2)'$: For all x in a neighborhood of 0, $L(x, \cdot)$ is nonnegative definite and has a nondegenerate minimum at $u = 0$.

As before, we shall normalize L by setting $L(0,0) = 0$, We shall also assume

$(H3)'$: Q is nonnegative definite and C^{q+1}, $q \geq 2$, in a neighborhood of 0, with $Q(0) = 0$.

We continue our assumption that the vector fields f, g_i are at least C^r, for $r \geq 1$, and retain the notation $k = \min\{q, r, s\}$.

Corollary 5.2. *Assume hypotheses (H2)$'$, (H3)$'$ hold. Then there exists a neighborhood N of 0 in $\mathbb{R}^n$ such that a solution of the Riccati PDE exists on $N \times [0, T]$. Furthermore, for initial data in a possibly smaller neighborhood of 0, the control law*

$$u_*(t) = u_*(x(t), -\pi(x(t), t))$$

is the (locally) unique optimal control, in the sense of Definition 5.1.

Proof: According to Theorem 4.3 there exists an open neighborhood N of 0 is $\mathbb{R}^n_*$ so that the solution of the Riccati PDE satisfying the side conditions exists on $N \times [0,T]$. Optimality of the corresponding control law on a possibly smaller neighborhood of 0 follows from Theorem 5.1.

One consequence of Corollary 5.2 is a local result, proved under stronger assumptions by Willemstein [17], who extended the local analysis developed by Lukes [19] for of the steady-state problem (also see Section 6).

Corollary 5.3. *(Willemstein [17]) Suppose that f, g_i, L and Q are analytic functions in a neighborhood of their respective origins and suppose*

$$L(x,u) = x^T sx + u^T Ru + \ell(x,u), \tag{5.15}$$

$$Q(x) = x^T Qx + q(x) \tag{5.16}$$

where $\ell(x,u)$ and $q(x)$ represent the remainder consisting of highest order terms. Then there exists a locally unique optimal feedback control law.

Remark 5.3. *Willemstein's proof rests very heavily on analyticity through the exploitation of power series expansions and does not prove that the control law, which happens to be representable as an analytic state feedback law, is locally optimal among all open loop control laws admissible in the sense of Definition 5.1. Rather, it is shown in [17] that this feedback law is locally optimal among all admissible analytic feedback laws.*

6 The Riccati Partial Differential Equation for Infinite Horizon Lagrange Problems

In this section, we consider a class of infinite time Lagrange problems for control systems (2.1)–(2.2) with criterion

$$\min_u J(x_0,u),$$

where

$$J(x_0,u) = \int_0^\infty L(x,u)dt. \tag{6.1}$$

We show that the infinite time problem can be regarded as a limit of the finite-time problem, in the sense that an optimal control can be shown to be a C^k function $u_*(x)$ having the form

$$u_*(x) = u_*(x,p), \ p = -\pi(x) \tag{6.2}$$

where $\pi(x)$ satisfies the steady-state Riccati PDE:

$$\frac{\partial H_*}{\partial x}(x,-\pi(x)) - \frac{\partial \pi}{\partial x}\frac{\partial H_*}{\partial p}(x,-\pi(x)) = 0. \tag{6.3}$$

We shall also show that π satisfies the constraints

$$\pi(0) = 0 \text{ and } \frac{\partial \pi}{\partial x}(0) = \frac{\partial \pi}{\partial x}(0)^T \geq 0. \tag{6.4}$$

The question of whether an optimal closed-loop law exists locally for such nonlinear problems with analytic data has been extensively researched, beginning with [24], [25], [18], [19]. In particular, the work of Al'brecht [24]–[25] focused on developing series expansions for an optimal state feedback law using Lyapunov functions. At that time, it was unknown whether this series would converge or would even be the Taylor series of some analytic or smooth function. Under the hypothesis that $L(x,u)$ is positive definite and that all vector fields and functions are smooth, Brunovsky [18] gave a sketch of a proof that an analytic optimal closed-loop control law exists. It is worth emphasizing the fact that his proof reposed quite heavily on the existence of a stable manifold for hyperbolic Hamiltonian systems.

In [19] Lukes gave a detailed existence theory, under the same hypotheses as Brunovsky, including a proof of an appropriate version of the stable manifold theorem, implicitly proving that the stable manifold $W^s(0)$ is Lagrangian by demonstrating that

$$W^s(0) = \text{graph}(-\nabla_x W)$$

for a smooth function W. Since that time, the development of LQ theory has provided a complete existence theory under much weaker positive semidefiniteness conditions on $L(x,u)$, so that one expects to be able earlier to improve the pioneering existence results of [18]–[19]. Indeed, following the earlier announcements, [5]–[7], in this section we give a proof of the local existence of an optimal feedback control law of the form

$$u_x(x) = -g(x)^T \pi(x)$$

where $\pi(x)$ is a solution of the steady-state Riccati equation (6.3)–under weaker conditions on $L(x,u)$. Briefly, the Riccati equation (6.3) is the invariance condition for the C^k submanifold

$$M_\infty = \text{graph}(-\pi)$$

and the side condition (6.4) reflects the fact that

$$M_\infty = W^s(0).$$

Thus, our existence theory for the Riccati equation (6.3)–(6.4) is just the stable manifold theorem, as in [18] but under weaker conditions. And, the fact that $W^s(0)$ is Lagrangian allows us to deduce, from the stable manifold theorem, a smoothness result for the value function, as a solution of the Hamilton-Jacobi-Bellman equation.

We now turn to a study of the asymptotic properties of the canonical system. As in the proof of Proposition 4.3, we have an expansion

$$L(x,u) = \frac{1}{2}x^T Sx + \frac{1}{2}u^T Ru + 0(\|x\|^3 + \|u\|^3)$$

where $S = S^T$, $R = R^T > 0$. If L is positive semidefinite we may also factor S as

$$S = C^T C$$

forming an output y for the system (2.1), defined via

$$y = Cx. \tag{6.5}$$

Proposition 6.1. *Assume (H2)′ and suppose L is of class C^{s+1}, for $s \geq 2$. Suppose the system (2.1), (6.5) is locally exponentially stabilizable and locally exponentially detectable. Then, the canonical system has $(x,p) = (0,0)$ as a hyperbolic equilibrium, with an n-dimensional stable manifold having the form*

$$W^s(0) = graph(-\pi)$$

for some C^k function, defined via $p = -\pi(x)$. In particular, π satisfies the "steady-state Riccati PDE" (6.3) and the side constraint (6.4).

Proof: As in the proof of Proposition 4.3, our starting point is the expansion of the reduced Hamiltonian

$$H_*(x,p) = (x^T p^T)H(x,p) + 0(\|x\|^3 + \|p\|^3) \tag{6.6}$$

where H is a $2n \times 2n$ symplectic matrix. Our first claim is the following.

Lemma 6.1. *If (2.1)–(6.5) is locally exponentially stabilizable and detectable, H has no purely imaginary eigenvalues. In particular, the canonical system has a C^k, n-dimensional stable manifold $W^s(0)$ and a C^k, n-dimensional unstable manifold, $W^n(0)$.*

Proof of Lemma 6.1: As in the proof of Proposition 4.3, expanding the Hamiltonian H_* we obtain

$$H_*(x,p) = \langle p, Ax + BR^{-1}B^T p\rangle - \frac{1}{2}x^T C^T Cx \tag{6.7}$$

$$-\frac{1}{2}p^T BR^{-1}B^T p + 0\left(\left\|\begin{pmatrix} x \\ p \end{pmatrix}\right\|^3\right) \tag{6.8}$$

so that $(x^T p^T)H(x,p)$ is the Hamiltonian of the linear optimal control problem,

$$\min_u J(x_0,u)$$

where

$$J(x_0, u) = \frac{1}{2}\int_0^\infty (x^T C^T Cx + u^T Ru)dt$$

subject to the dynamic constraint

$$\dot{x} = Ax + Bu,\ x(0) = x_0 \tag{6.9}$$

$$y = Cx. \tag{6.10}$$

On the other hand, to say (2.1)–(6.5) is locally exponentially stabilizable, resp. locally exponentially detectable, is to say that (A, B) is stabilizable, resp. (A, C) is detectable. Consequently, it is classical that H has no purely imaginary eigenvalues. Since λ is an eigenvalue of H is, and only if, $-\lambda$ is, we see that H has an n-dimensional stable subspace V^s and an n-dimensional unstable subspace V^u. More explicitly, if λ is an eigenvalue of H, denote by V_λ the generalized eigenspace for H with eigenvalue λ. Then, the subspaces

$$V^s = \bigoplus_{\substack{\lambda\in\sigma(H)\\ Re(\lambda)\leq 0}} V_\lambda,$$

$$V^u = \bigoplus_{\substack{\lambda\in\sigma(H)\\ Re(\lambda)> 0}} V_\lambda$$

have dimension n. Since

$$T_0 W^s(0) = V^s \tag{6.11}$$

and

$$T_0 W^u(0) = V^u \tag{6.12}$$

we see that

$$\dim W^s(0) = \dim W^u(0) = n. \qquad \square$$

Returning to the proof of Proposition 6.1, we note that under the same hypotheses on (A, B, C), it is also classically known that

$$V^s = \text{graph}(-P) \tag{6.13}$$

where $p = -Px$ for P the unique positive semidefinite solution of the algebraic Riccati equation

$$A^T P + PA - PBR^{-1}B^T P + C^T C = 0/$$

According to the stable manifold theorem, $W^s(0)$ is locally a C^k submanifold of $\mathbb{R}^{2n}$, which is locally invariant under the canonical system, while from (6.7) and (6.9) it follows that $W^s(0)$ is transverse to the vertical fiber, $x = 0$, in $\mathbb{R}^{2n}$. In particular,

$$\mathrm{proj}_1|_{W^s(0)} : W^s(0) \to \mathbb{R}^n$$

defined via

$$\mathrm{proj}_1(x,p) = x$$

is a submersion so that, by the implicit function theorem, in a neighborhood of 0 we have

$$W^s(0) = \mathrm{graph}(-\pi)$$

for some C^k function, $p = -\pi(x)$. Moreover,

$$\pi(0) = 0 \text{ and } \frac{\partial \pi}{\partial x}(0) = P \geq 0.$$

Finally, we note that to say $W^s(0)$ is locally invariant under the canonical system is to say

$$\nabla(p - \pi(x)) \cdot \begin{bmatrix} \frac{\partial H_*}{\partial x} \\ -\frac{\partial H_*}{\partial p} \end{bmatrix} |_{p=-\pi(x)} = 0 \tag{6.14}$$

(6.10) is, of course, just the steady-state Riccati equation. □

Definition 6.1 (19). *Fix a neighborhood G_1 of 0 in $\mathbb{R}^n$. A family of control laws $u(t;x_0)$ is said to be locally optimal on G_1 if there exists a neighborhood G_2,*

$$0 \in G_1 \subset G_2$$

such that the closed loop trajectories $x_*(t)$ for initial data in G_1 remain in G_2 and, for any initial condition $x_0 \in G_1$ and any control $u(t)$ for which

i. $x(t) \in G_2, \quad t \geq 0$;

ii. $J(x_0, u)$ *is finite;*

iii. $x(t) \to 0$ *as* $t \to \infty$;

we have

$$J(x_0, u_*) \leq J(x_0, u).$$

Theorem 6.1. *Assume (H2)′ and suppose L is of class C^{s+1}, $s \geq 2$. Suppose also that the system (2.1), (6.5) is locally exponentially stabilizable and locally exponentially detectable. Then, there exists an $\varepsilon > 0$ such that the steady state Riccati PDE has a positive semidefinite solution $\pi(x)$ on a neighborhood $B_\varepsilon(0)$ of 0 and for which the feedback control law*

$$u_*(x) = u_*(x, -\pi(x))$$

defined via (6.2) is locally optimal on $B_\varepsilon(0)$.

Proof: As above, the stable manifold theorem and a first-order analysis show that there exists an ε_1 such that for $\|x\| < \epsilon_1$ a stable manifold of the canonical equations exists and has the form

$$W^s(0) = \text{ graph } (-\pi(\cdot), \pi(0) = 0), \text{ and } \frac{\partial \pi}{\partial x}(0) \geq 0,$$

verifying the local existence of a solution of the steady-state Riccati PDE. Moreover, since the stable manifold of a Hamiltonian is isotropic (see [26] for an elegant proof of this fact), $W^s(0)$ is a Lagrangian submanifold and therefore

$$\pi(x) = \nabla V(x), \tag{6.15}$$

where we normalize $V(x)$ by setting $V(0) = 0$. We now show that

$$u_*(x) = u_*(x, -\pi(x)) \tag{6.16}$$

is optimal. We first remark that, from the proof of Proposition 6.1, it follows that the restriction of the canonical system to $W^s(0)$ is given by the closed-loop system

$$\dot{x} = f(x) + g(x)u_*(x). \tag{6.17}$$

In particular, for $x(0)$ sufficiently small, the trajectory $x_*(t)$ of (6.13) exists for all time, $t \geq 0$.

Lemma 6.2. *$H_*(x,p)$ vanishes on $W^s(0)$ is a neighborhood of (0,0).*

Proof: Locally, on $W^s(0)$, all trajectories of the canonical system, in particular trajectories of (6.13), tend exponentially to (0,0). Since

$$H_*(x,p) = \langle p, f(x) + g(x)u_* \rangle - L(x, u_*)$$

is constant along trajectories of the canonical system and H_* vanishes at (0,0), it follows that H_* vanishes on $W^s(0)$. □

If $(x(0), p(0)) \in W^s(0)$, we denote by $(x_*(t), p_*(t))$ the corresponding trajectory. We note that by (6.11) we have

$$p_*(t) = -\pi(x_*, t) = -\nabla V(x_*)(t).$$

Since $H_*|_{W^s(0)} = 0$, we then have

$$-\dot{V}(x_*(t)) = L(x_*(t),\ u_*(t)) \tag{6.18}$$

along the extremal trajectory with initial condition $x_*(0) = x_0$. Integrating (6.14) we obtain

$$\int_0^s L(x_*(t), u_*(t))dt = -\int_0^s \dot{V}(x_*(t))dt \tag{6.19}$$

$$= -V(x_*(s)) + V(x_0). \tag{6.20}$$

Since $x_*(s) \to 0$ as $s \to \infty$ and $V(0) = 0$, we deduce that

$$\int_0^\infty L(x_*(t), u_*(t))dt = \lim_{s\to\infty} \int_0^s L(x_*(t), u_*(t))dt = V(x_0).$$

In particular, the extremal cost is finite for any trajectory on $W^s(0)$.

Summarizing, a stable manifold argument shows that $\pi(x)$ is defined for $x \in B_{\varepsilon_1}(0)$, that the closed-loop system (6.13) is locally asymptotically stable on a possibly smaller neighborhood of 0 and that the cost-to-go for initial data in this domain of a tradition is finite and given by $V(x_0)$. Since (6.13) is, in particular, Lyapunov stable there exists ε, ε_2 such that

$$0 < \varepsilon < \varepsilon_2$$

and

i. π exists on $B_{\varepsilon_2}(0) = G_2$;

ii. trajectories of (6.13) initialized in $B_\varepsilon(0) = G_1$ stay in $B_{\varepsilon_2}(0)$ and tend to 0.

We claim that u_* is locally optimal on G_1. Now suppose $u(t)$ is any control which renders $J(x_0)$ finite and for which the corresponding trajectory tends to 0 and remains in $G_2 = B_{\varepsilon_2}(0)$. Lemma 6.3 $V(x_0) \leq \int_0^\infty (L(x,u))dt$.

Proof: Consider $W_u(x, x_{n+1}) = V(x) + x_{n+1}$ where

$$\dot{x}_{n+1} = L(x,u), \; x_{n+1}(0) = 0.$$

We have noted above that $\dot{W}_{u_*} \equiv 0$. If $u = u_* + v$, then

$$\begin{aligned}
\dot{W}_u &= \langle \nabla V, f(x) + g(x)u \rangle + L(x,u) && (6.21)\\
&= \langle \nabla V, f(x) + g(x)u_* \rangle + \langle \nabla V, g(x)V \rangle + L(x,u) && (6.22)\\
&= L(x, u_* + v) - L(x,u) + \langle \nabla V, g(x)v \rangle && (6.23)\\
&= \frac{\partial L}{\partial u}|_{u=u_*} \cdot v + \langle \nabla V, g(x)v \rangle + v^T \frac{\partial^2 L}{\partial u^2}(\lambda\xi)v && (6.24)\\
&= v^T \frac{\partial^2 L}{\partial u^2}(\lambda\xi)v \geq \dot{W}_{u_*} = 0. && (6.25)
\end{aligned}$$

Therefore, along $(x(t), u(t))$, we must have

$$L(x,u) \geq -\dot{V}(x(t)$$

with strict inequality if $u(t) \neq u_*(t)$ for t is an open interval. Integrating this inequality, we find

$$\int_0^T L(x,(t),u(t))dt \geq -V(x(T)) + V(x(0))$$

and thus, in the limit as $T \to \infty$

$$\int_0^\infty L(x(t),u(t))dt \geq V(x_0),$$

with strict inequality, if $u \neq u_*$. □

Remark 6.1. *These results were announced at the conference Nonlinear Synthesis in Sopron, Hungary in June 1989 (see [5]); at the 2nd Conference on Computation and Control* in Bozeman, Montana in August, 1990 (see [6]); and at the *1st European Control Conference in Grenoble in July 1991 (see [7]). While [5] and [7] treat these results for the standard quadratic functional, [6] sketches a derivation of our main result for infinite time horizon problems under the present hypotheses on $L(x,u)$. At the 1^{st} European Control Conference, we learned from van der Schaft that he had recently independently discovered a similar treatment of the steady-state aspect of our theory, (see [27]).*

7 Examples and Illustrations

7.1 Classical calculus of variations

There are, of course, two extreme cases of Bolza problems which may be considered: A fairly general problem, with a nonlinear system (1.1) and a general nonlinear criteria and one very well understood special case, consisting of a linear system and quadratic criteria. There are, therefore, two intermediate cases which should also be of particular interest; viz., nonlinear systems with quadratic criteria and linear systems with more general nonlinear criteria. One particularly interesting special case of the latter is what is often referred to as the simplest problem in the calculus of variations. More explicitly, consider the system

$$\dot{x} = u \tag{7.1}$$

and the criterion

$$J(x_0,u) = \int_0^T L(x,u)dt + Q(x(T)).$$

This problem, with $Q \equiv 0$, is of course equivalent to the problem of minimizing the functional

$$J(x, \dot{x}) = \int_0^T L(x, \dot{x}) dt.$$

Not all of the simplest problems of the classical calculus of variations satisfy the hypotheses (H1)–(H3); for example, the isoperimetric problem does not satisfy (H1). However, for those problems which do satisfy these hypotheses, the corresponding Riccati PDE is, in fact, equivalent to the Euler-Lagrange equation.

To see this, recall from Section 3 that the Riccati PDE

$$\frac{\partial H_*}{\partial x} = \frac{\partial \pi}{\partial t} + \frac{\partial \pi}{\partial x}\frac{\partial H_*}{\partial p}$$

is equivalent, indeed is derived from, the identity

$$p = -\pi(x, t)$$

and the consequent equation

$$\dot{p} = -\dot{\pi}(x, t) \tag{7.2}$$

computed along solutions $(x(t), p(t))$ of the canonical equation. Moreover, from the Pontryagin Maximum Principle applied to the Hamiltonian of the problem,

$$H(x, p, u) = \langle p, u \rangle - L(x, u)$$

it follows that

$$\frac{\partial L}{\partial u}(x_*, u_*) = p = -\pi(x_*, t) \tag{7.3}$$

and that

$$\dot{p} = \frac{-\partial H}{\partial x} = \frac{\partial L}{\partial x}. \tag{7.4}$$

Substituting (7.3)–(7.4) into (7.2), we deduce the Euler-Lagrange equation

$$\frac{\partial L}{\partial x} = \frac{d}{dt}\left(\frac{\partial L}{\partial u}\right) \tag{7.5}$$

as an equivalent form of the Riccati Partial Differential Equation together with the side conditions

$$\frac{\partial L}{\partial u}(x(0), \dot{x}(0)) = 0 \tag{7.6}$$

$$\frac{\partial L}{\partial u}(x(T), \dot{x}(T)) = -\nabla Q(x(T)). \tag{7.7}$$

For systems of the form (7.1), many approaches to optimal control simplify considerably. For example, in this case the differential-algebraic elimination theoretic techniques of Fliess et al [11] give a first order system of equations identical to the Riccati PDE, from which the Euler-equations (7.5) can also be easily derived [11].

Remark 7.1. *Since the Riccati PDE turns out for such simple problems as (7.1), to be equivalent to the Euler-Lagrange equations, it is therefore not surprising that many of the systems of partial differential equations of mathematical physics can arise as Riccati Partial Differential Equations. One such example is the inviscid Burgers' equation, as we shall now illustrate.*

Example 7.1. For the scalar system

$$\dot{x} = u, \qquad x, u \in \mathbb{R} \tag{7.8}$$

we want to minimize the cost functional

$$J^T(x_0) = \frac{1}{2}\int_0^T u(t)^2 dt + Q(x(T)) \tag{7.9}$$

for some arbitrary but fixed C^k function, $Q(x)$. From the Pontryagin Maximum Principle it follows that any optimal control $u_*(t)$ is necessarily constant,

$$u_*(t) = p.$$

In particular, integrating the canonical equations gives

$$J^T(x_0) = \frac{p^2 T}{2} + Q(x_0 + pT).$$

If $k \geq 1$, as a function of p $J^T(x_0)$ has extrema at solutions of

$$f_T(p, x_0) = p + Q'(x_0 + pT) = 0, \tag{7.10}$$

which of course is the transversality condition arising in the Maximum Principle. Continuing in this elementary approach, it is important to know when (7.8) has a solution p, $p = p(x_0)$. If $k \geq 2$, an implicit function theorem argument will imply existence of a smooth solution provided

$$1 + TQ''(x) \neq 0, \quad \text{for } x \in \mathbb{R}. \tag{7.11}$$

If one assumes, as is natural, that $Q''(x_0) > 0$ for some x_0 then the sufficient condition (5.4) implies of course that $Q(x)$ is strictly convex. However, in order to understand the more general case, say of a convex penalty function

$Q(x)$, one needs to analyze the possible bifurcation of solutions to (7.9) at points x_0 where

$$Q''(x_0) = 0.$$

Indeed, near such bifurcation points there may be multiple optima or even no optimum.

On the other hand, an analysis of (5.1)-(5.4) via the methods described in Sections 3–5 and boil down to the existence of solutions to the Riccati PDE (4.3)–which, in this case, is just the well-known inviscid Burgers' equation

$$\frac{\partial \pi}{\partial t} = \frac{\partial \pi}{\partial x} \cdot \pi \tag{7.12}$$

$$\pi(0,t) = 0, \pi(x,T) = Q'(x(T)). \tag{7.13}$$

Since the canonical equations are linear and hence complete, we need only analyze the onset of shock waves, i.e. points where $\frac{\partial \pi}{\partial x}$ becomes infinite, in the construction of $\pi(x,t)$ by integrating the transversality condition backwards in time. On the other hand, it is well-known (see e.g. [20]) that for this Burgers equations, integrated backwards in time, an initial condition

$$\pi(x,T) = F(x)$$

gives rise to a global solution if, and only if, $F(x)$ is monotone nondecreasing. That is, an analysis of the Riccati PDE shows that global existence of an optimal feedback law for (5.1)-(5.4) is equivalent to convexity of the penalty function $Q(x)$. As a special case of the general results obtained in [16] we can also assert that convexity of the penalty function $Q(x)$ is equivalent to the absence of shocks for the Riccati PDE and also to the uniqueness of optimal control laws.

7.2 Nonlinear quadratic (NLQ) problems

For the sake of comparison with the linear case and as an illustration of the results developed in Sections 3–6, consider the finite time horizon optimal control problem for the system

$$\dot{x} = f(x) + g(x)u, \ y = h(x) \tag{7.14}$$

with performance measure

$$J^T(x_0, u) = \frac{1}{2}\int_0^T (\|y(t)\|^2 + \|u(t)\|^2)dt + Q(x(T)), \ 0 \le t \le T \tag{7.15}$$

Since $L(x,u) = \frac{1}{2}(\|u\|^2 + \|y\|^2)$, it is clear that hypotheses (H1)-(H2) are satisfied. Moreover, the defining equation for $u_*(x,p)$ is simply

$$\langle p, g(x)\rangle - u_* = 0.$$

In this case the Riccati PDE also takes a perhaps more familiar form,

$$\frac{\partial \pi}{\partial t} = -\left(\frac{\partial f}{\partial x}\right)^T \pi(x,t) - \frac{\partial \pi}{\partial x}(x,t)f(x) - \frac{\partial h}{\partial x}(x)^T h(x) \tag{7.16}$$

$$+ \frac{\partial \pi}{\partial x} g(x)g(x)^T \pi(x,t) + \left(\frac{\partial g}{\partial x} \cdot g^T \pi\right)^T \pi(t,x). \tag{7.17}$$

$$\pi(0,t) = 0, \ \pi(x,T) = +\nabla_x Q(x)$$

And, $u_*(x,t) = -g(x)^T \pi(x,t)$ is an optimal control for the unconstrained minimization problem, $\underset{u}{\rightarrow} \min J^T(x_0, u)$. In this case, Theorem 3.1 provides a basic local existence result for the Riccati PDE.

For general NLQ problems, however, the Riccati PDE contains a new "nonlinear correction term",

$$\pi(x,t)^T \frac{\partial g}{\partial x} g(x)^T \pi(x,t),$$

which vanishes in the linear case.

Similar calculations apply to infinite time problems such as (7.10) where the cost to be minimized over u is

$$J(x_0, u) = \int_0^\infty \|y(t)\|^2 + \|u(t)\|^2 dt. \tag{7.18}$$

Again, one computes that an optimal feedback control has the form

$$u_* = g(x)^T p. \tag{7.19}$$

Moreover one can take $p = -\pi(x)$ where $\pi(x)$ is a solution to the steady-state Riccati PDE:

$$\left(\frac{\partial f}{\partial x}\right)^T \pi(x) + \frac{\partial \pi}{\partial x} f(x) - \frac{\partial \pi}{\partial x} g(x)g(x)^T \pi(x) + \frac{\partial h^T}{\partial x} h(x) \tag{7.20}$$

$$+ \left(\frac{\partial g}{\partial x} \cdot g^T \pi\right)^T \pi(x) = 0 \tag{7.21}$$

$$\pi(0) = 0 \tag{7.22}$$

and where π also satisfies the constraint

$$\frac{\partial \pi}{\partial x}(0) = \frac{\partial \pi}{\partial x}(0)^T \geq 0. \tag{7.23}$$

In this case, Proposition 6.1 provides a basic local existence result for this steady-state Riccati PDE and, according to Theorem 6.1, leads to a unique local optimal control u_* which takes the form

$$u_* = -g(x)^T \pi(x). \tag{7.24}$$

Example 7.2. Suppose the system (7.10) is linear, i.e.,

$$\dot{x} = Ax + Bu,\ y = Cx,$$

and in (7.11) the penalty on the terminal state is quadratic

$$Q(x(T)) = \frac{1}{2} x^*(T)^T Q x(T).$$

In this case, the function

$$\pi(x,t) = P(t)x$$

is a solution of the Riccati PDE if, and only if, the equation

$$\dot{P}(t)x = -A^T P(t)x - P(t)Ax - C^T Cx \tag{7.25}$$

$$+ P(t)BB^T P(t)x, \tag{7.26}$$

$$P(T)x = Qx \tag{7.27}$$

is satisfied. After eliminating x in (7.10), this results in the standard Riccati ordinary differential equation of Linear Quadratic Theory. According to Theorem 5.1 (see also Remark 5.2) and the global existence theory for this Riccati Ordinary Differential equation, , the control law

$$u = -B^T P(t)x$$

is the globally unique optimal control, in feedback form.

The infinite horizon problem also has a familiar classical interpretation. Again, the function

$$\pi(x) = Px$$

is a solution of the steady state Riccati PDE if, and only if, the algebraic Riccati equation

$$A^T P + PA - PBB^T P + C^T C = 0$$

has a solution. In this case, the side condition (6.4) boils down to the existence of a positive semidefinite solution P_* which is known to exist provided (A, B) is stabilizable and (A, C) is detectable. In this case, Theorem 6.1 yields the familiar result that

$$u_* = -BB^T P_* x$$

is an optimal control law in feedback form.

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