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A Superposition Theorem for Solutions of the Riccati Difference Equation*

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Abstract

We show that, given any pair of solutions of the matrix Riccati difference equation, it is possible to construct a whole family of solutions via projective superposition laws. This extends known results for the discrete-time algebraic Riccati equation.

Key words: Riccati difference equation, families of solutions, projective superposition laws

AMS Subject Classifications: 49B99, 93C55, 93B40

1 Introduction

In this paper, we study the matrix Riccati difference equation in the form which arises in the least squares optimal estimation problem for linear, time-varying, stochastic systems, namely

$$\begin{aligned} &X(t+1) = A(t)X(t)A^*(t) \\ &-A(t)X(t)\hat{C}^*(t)(R(t)+\hat{C}(t)X(t)\hat{C}^*(t))^{-1}\hat{C}(t)X(t)A^*(t)+Q(t). \end{aligned}$$

Here $A(t)$, $Q(t)=Q^*(t)\geq 0$, of dimensions $n\times n$, $\hat{C}(t)$, of dimensions $q\times n$, and $R(t)=R^*(t)>0$, of dimensions $q\times q$, are complex, matrix-valued functions, defined on an interval $[t_0, t_1-1]$ of the integers $\mathbb{Z}$. In view of the positive definiteness of $R(t)$, the previous equation can be equivalently rewritten as

$$\begin{aligned} &X(t+1) = A(t)X(t)A^*(t) \\ &-A(t)X(t)C^*(t)(I+C(t)X(t)C^*(t))^{-1}C(t)X(t)A^*(t)+Q(t), \end{aligned} \tag{1.1}$$

where we have set $C(t)=R^{-\frac{1}{2}}(t)\hat{C}(t)$. Hence, in what follows, we shall deal with the Riccati difference equation (RDE) in the form (1.1).

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The matrix Riccati difference equation possesses a large literature due to its crucial role in optimal filtering and control of linear systems, see e.g. [1]. Of particular interest here are the results on the parametrization of the equilibrium solutions of the discrete time algebraic Riccati equation ($DARE$)

$$X - AXA^* + AXC^*(I + CXC^*)^{-1}CXA^* - Q = 0. \qquad (1.2)$$

G. Ruckebusch [11], [12], H.K. Wimmer [15] - [17], A.C.M. Ran and H.L. Trentelman [10] have established, under system theoretic assumptions, a one–to–one correspondence between solutions of the discrete-time algebraic Riccati equation and the invariant subspaces of a certain *feedback* matrix. These results also allow to express each solution of the $DARE$ as a certain projective combination of two particular solutions (see Theorem 1.1 of [16] and Theorem 3.3 of [10]). This represents the discrete-time counterpart of the parametrization of solutions of the continuous-time algebraic Riccati equation due to J.C. Willems [14], W.A. Coppel [3] and M.A. Shayman [13].

In this paper, we prove that there exists a time-varying counterpart of some of these results for the RDE. In particular, our main result, Theorem 3.2 below, shows that an appropriate projective combination of any pair of solutions is still a solution of the RDE. These results are the discrete-time counterpart of those presented in [8] for the Riccati differential equation. As for several other aspects of the Riccati equations, the discrete- time case is more involved, and, consequently, the derivation is nontrivial.

2 Preliminary Results

We collect in this section, which is largely expository, some elementary results on Riccati difference equations that will be needed in the rest of the paper.

A Hermitian, $n \times n$, complex, matrix-valued function $X(t) = X^*(t)$, defined on $[t_0, t_1]$, will be referred to as a *solution* of the RDE if, on the same time interval, $S_X(t) := I + C(t)X(t)C^*(t)$ is nonsingular, and $X(t)$ satisfies the RDE. We associate to each solution $X(t)$ of the RDE on $[t_0, t_1]$ the *feedback matrix* $A_X(t)$ defined as

$$\begin{aligned} A_X(t) \quad := \quad & A(t) - A(t)X(t)C^*(t)(I + C(t)X(t)C^*(t))^{-1}C(t) \\ & = A(t)(I + X(t)C^*(t)C(t))^{-1}. \end{aligned} \qquad (2.1)$$

Notice that $A_X(t)$ is nonsingular on $[t_0, t_1 - 1]$ if and only if $A(t)$ is. Let $X(t)$, $X_i(t)$, $i = 1, 2$, be solutions of the RDE, and let $A_X(t)$, $A_{X_i}(t)$, $i = 1, 2$, be the corresponding feedback matrices. We then have the following useful identities.

Lemma 2.1 *Given two arbitrary solutions $X_1(t)$ and $X_2(t)$ of the RDE on $[t_0, t_1]$, let $\Delta_{21}(t) := X_2(t) - X_1(t)$. Then, the corresponding feedback matrices satisfy the following identities:*

$$A_{X_2}(t) = A_{X_1}(t)\big(I - \Delta_{21}(t)C^*(t)S^{-1}_{X_2}(t)C(t)\big), \tag{2.2}$$

$$A_{X_2}(t) = A_{X_1}(t)\left(I + \Delta_{21}(t)C^*(t)S^{-1}_{X_1}(t)C(t)\right)^{-1}. \tag{2.3}$$

Proof: Formula (2.2) is known for the $DARE$ [16, p.125], and it can be derived for the RDE in the same fashion. To establish (2.3), write (2.2) as

$$A_{X_2}(t) = A_{X_1}(t)\left(I - \Delta_{21}(t)C^*(t)\left(S_{X_1}(t) + C(t)\Delta_{21}(t)C^*(t)\right)^{-1} C(t)\right),$$

and then apply the well-known matrix identity

$$I - F(D + GF)^{-1}G = (I + FD^{-1}G)^{-1}. \tag{2.4}$$

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Given two arbitrary solutions of the RDE on $[t_0, t_1]$, $X_1(t), X_2(t)$, we define $\Delta_{21}(t) := X_2(t) - X_1(t)$.

Proposition 2.2 *[6, 15] Let $X_1(t), X_2(t)$ be two arbitrary solutions of the RDE on $[t_0, t_1]$. Then, $\Delta_{21}(t) := X_2(t) - X_1(t)$ is a solution of the following homogeneous matrix Riccati equation:*

$$\begin{aligned} \Delta_{21}(t+1) &= A_{X_1}(t)\Delta_{21}(t)A^*_{X_1}(t) \\ &-A_{X_1}(t)\Delta_{21}(t)C^*(t)(I + C(t)X_2(t)C^*(t))^{-1}C(t)\Delta_{21}(t)A^*_{X_1}(t). \end{aligned} \tag{2.5}$$

Using (2.2) in (2.4), we obtain

$$\Delta_{21}(t+1) = A_{X_1}(t)\Delta_{21}(t)A^*_{X_2}(t) = A_{X_2}(t)\Delta_{21}(t)A^*_{X_1}(t), \quad t \in [t_0, t_1 - 1]. \tag{2.6}$$

This relation generalizes a similar formula for the $DARE$, cf. [9, p.197].

Let us introduce the transition matrix $\Psi(t, s)$ associated to the matrix function $A_X(t)$, $t \in [t_0, t_1 - 1]$ defined by

$$\begin{aligned} &\Psi(t+1, s) := A_X(t)\Psi(t, s), \qquad t_0 \le s < t \le t_1 - 1, \\ &\Psi(s, s) := I, \qquad t_0 \le s \le t_1. \end{aligned}$$

Proposition 2.3 *Consider two arbitrary solutions $X_1(t), X_2(t)$ of the RDE on $[t_0, t_1]$, and let $\Delta_{21}(t) := X_2(t) - X_1(t)$. Let $\Psi_i(t, s), i = 1, 2$ be the transition matrices of the corresponding feedback matrices $A_{X_i}(t), i = 1, 2$. Then, for $t_0 \le s \le t \le t_1$, we have*

$$\Delta_{21}(t) = \Psi_1(t, s)\Delta_{21}(s)\Psi^*_2(t, s) = \Psi_2(t, s)\Delta_{21}(s)\Psi^*_1(t, s). \tag{2.7}$$

Proof: Relation (2.6) follows at once from (2.5) and the definition of $\Psi_i(t,s)$. •

This result has some immediate consequences.

Corollary 2.4 *If $A(t)$ is nonsingular on $[t_0, t_1 - 1]$, the difference of two solutions of the RDE has constant rank on $[t_0, t_1]$.*

Corollary 2.5 *Let $A(t)$ be nonsingular on $[t_0, t_1 - 1]$, and consider three arbitrary solutions $X_1(t), X_2(t), X(t)$ of the RDE. Define $\Delta_{21}(t) := X_2(t) - X_1(t)$ and $\Delta(t) := X(t) - X_1(t)$. If $Ker\Delta_{21}(s) \subseteq Ker\Delta(s)$, for some $s \in [t_0, t_1]$, then $Ker\Delta_{21}(t) \subseteq Ker\Delta(t)$, for all t, $s \leq t \leq t_1$.*

Proof: Let x be such that $\Delta_{21}(t)x = 0$. It follows from (2.6) that

$$\Psi_2(t,s)\Delta_{21}(s)\Psi_1^*(t,s)x = 0.$$

Using the invertibility of $\Psi_2(t,s)$ and the hypothesis on $Ker\Delta(s)$, we get

$$\Psi(t,s)\Delta(s)\Psi_1^*(t,s)x = 0.$$

Using (2.6) again, we get $\Delta(t)x = 0$. •

3 Families of Solutions of the RDE

Consider two arbitrary (Hermitian) solutions $X_1(t), X_2(t)$ of the RDE on $[t_0, t_1]$. Let $\Delta_{21}(t) := X_2(t) - X_1(t)$, let $A_{X_1}(t), A_{X_2}(t)$ be the corresponding feedback matrices, and let $\Psi_1(t,s)$ and $\Psi_2(t,s)$ the associated transition matrices.

Lemma 3.1 *Let M_0 and N_0 be subspaces of $\mathbb{C}^n$ such that*

$$M_0 \dot{+} N_0 = \mathbb{C}^n,$$

where $\dot{+}$ denotes direct sum. Suppose that the sequences of subspaces $\{M(t)\}$, $\{N(t)\}$, $t \in [t_0, t_1]$, satisfy for $t_0 \leq s \leq t \leq t_1$

$$M(t_0) = M_0, \quad N(t_0) = N_0, \tag{3.1}$$

$$M(t) \dot{+} N(t) = \mathbb{C}^n, \tag{3.2}$$

$$\Psi_1(t,s)M(s) \subseteq M(t), \tag{3.3}$$

$$\Psi_2(t,s)N(s) \subseteq N(t). \tag{3.4}$$

Let $\Pi(t)$, $t \in [t_0, t_1]$, be the matrix representing the oblique projection onto $M(t)$ along $N(t)$. Let us define

$$X(t) = (I - \Pi(t))X_1(t) + \Pi(t)X_2(t), \quad t \in [t_0, t_1]. \tag{3.5}$$

The matrix valued function (3.5) is Hermitian on $[t_0, t_1]$ if and only if

$$\Delta_{21}(t_0)M_0^\perp \subseteq N_0. \tag{3.6}$$

Proof: Clearly, $X(t)$ is Hermitian if and only if $\Delta(t) := X(t) - X_1(t) = \Pi(t)\Delta_{21}(t)$ is. Moreover, $\Delta(t)$ is Hermitian on $[t_0, t_1]$ if and only if

$$\Pi(t)\Delta_{21}(t)(I - \Pi^*(t)) = 0, \quad t \in [t_0, t_1].$$

This is equivalent to

$$\Delta_{21}(t)M^\perp(t) \subseteq N(t), \quad t \in [t_0, t_1],$$

which, for $t = t_0$, gives (3.6). Conversely, suppose that (3.6) holds. Using (2.6), (3.3), (3.4), we obtain

$$\begin{aligned} \Delta_{21}(t)M^\perp(t) &= \Psi_2(t,t_0)\Delta_{21}(t_0)\Psi_1^*(t,t_0)M^\perp(t) \\ &\subseteq \Psi_2(t,t_0)\Delta_{21}(t_0)M_0^\perp \subseteq \Psi_2(t,t_0)N_0 \subseteq N(t), \end{aligned} \tag{3.7}$$

where we have used the fact that (3.3) implies that $\Psi_1^*(t,t_0)M^\perp(t) \subseteq M_0^\perp$.
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Notice that in the special case of $A(t)$ nonsingular for all t in $[t_0, t_1 - 1]$, equality must hold in the inclusions (3.3) and (3.4).

We are now ready to establish our main result.

Theorem 3.2 *Under the assumptions of Lemma 3.1, and in the same notation, let $X(t)$ be defined by (3.5). If $S_X(t) := (I + C(t)X(t)C^*(t))$ is nonsingular for all t in $[t_0, t_1 - 1]$, then $X(t)$ is a solution of the RDE on $[t_0, t_1]$.*

Proof: In view of Proposition 2.2, it suffices to show that $\Delta(t) = \Pi(t)\Delta_{21}(t)$ satisfies equation (2.4). First of all, notice that (3.3) and (3.4) give

$$\begin{aligned} (I - \Pi(t))\Psi_1(t,s)\Pi(s) &= 0, \quad t_0 \le s \le t \le t_1, \\ \Pi(t)\Psi_2(t,s)(I - \Pi(s)) &= 0, \quad t_0 \le s \le t \le t_1. \end{aligned}$$

These yield

$$\Pi(t+1)A_{X_1}(t)\Pi(t) = A_{X_1}(t)\Pi(t), \quad t_0 \le t \le t_1 - 1, \tag{3.8}$$

$$\Pi(t+1)A_{X_2}(t) = \Pi(t+1)A_{X_2}(t)\Pi(t), \quad t_0 \le t \le t_1 - 1. \tag{3.9}$$

In the remaining calculations, the omitted time argument is always t. Let us multiply both terms of (2.5) on the left by $\Pi(t+1)$. Using (3.9), we get

$$\Pi(t+1)\Delta_{21}(t+1) = \Pi(t+1)A_{X_2}\Pi\Delta_{21}A_{X_1}^*.$$

Due to identity (2.2), the latter gives

$$\Pi(t+1)\Delta_{21}(t+1) = \Pi(t+1)\left(A_{X_1} - A_{X_1}\Delta_{21}C^*S_{X_2}^{-1}C\right)\Pi\Delta_{21}A_{X_1}^*,$$

and, from (3.8), it follows that

$$\begin{aligned}\Pi(t+1)\Delta_{21}(t+1) &= A_{X_1}\Pi\Delta_{21}A_{X_1}^* - \\ &\quad \Pi(t+1)A_{X_1}\Delta_{21}C^*S_{X_2}^{-1}C\ \Pi\Delta_{21}A_{X_1}^*. \quad (3.10)\end{aligned}$$

We now claim that

$$\Pi(t+1)A_{X_1}\Delta_{21}C^*S_{X_2}^{-1} = A_{X_1}\Pi\Delta_{21}C^*S_X^{-1},$$

or, equivalently,

$$\begin{aligned}&\Pi(t+1)A_{X_1}\Delta_{21}C^*(S_{X_1}+C\Delta_{21}C^*)^{-1} = \\ &\quad A_{X_1}\Pi\Delta_{21}C^*(S_{X_1}+C\Pi\Delta_{21}C^*)^{-1}. \quad (3.11)\end{aligned}$$

Using both (3.8) and (3.9), we get

$$\begin{aligned}A_{X_1}\Pi\Delta_{21}C^* &= \Pi(t+1)A_{X_1}\Pi\Delta_{21}C^* + \Pi(t+1)A_{X_2}(I-\Pi)\Delta_{21}C^* \\ &\quad +\Pi(t+1)A_{X_1}\Delta_{21}C^* \\ &\quad -\Pi(t+1)A_{X_1}\Delta_{21}C^*.\end{aligned}$$

Rearranging the terms, we obtain

$$A_{X_1}\Pi\Delta_{21}C^* = \Pi(t+1)A_{X_1}\Delta_{21}C^* - \Pi(t+1)(A_{X_1}-A_{X_2})(I-\Pi)\Delta_{21}C^*.$$

Using (2.2), the latter turns into

$$\begin{aligned}&A_{X_1}\Pi\Delta_{21}C^* = \Pi(t+1)A_{X_1}\Delta_{21}C^* \\ &\quad -\Pi(t+1)\left(A_{X_1}\Delta_{21}C^*(S_{X_1}+C\Delta_{21}C^*)^{-1}C\right)(I-\Pi)\Delta_{21}C^*,\end{aligned}$$

which can be rewritten as follows

$$\begin{aligned}&A_{X_1}\Pi\Delta_{21}C^* = \quad (3.12) \\ &\quad \Pi(t+1)A_{X_1}\Delta_{21}C^*\left(I-(S_{X_1}+C\Delta_{21}C^*)^{-1}C(I-\Pi)\Delta_{21}C^*\right).\end{aligned}$$

A straightforward calculation also shows that

$$\begin{aligned}&I-(S_{X_1}+C\Delta_{21}C^*)^{-1}C(I-\Pi)\Delta_{21}C^* = \\ &\quad (S_{X_1}+C\Delta_{21}C^*)^{-1}(S_{X_1}+C\Pi\Delta_{21}C^*). \quad (3.13)\end{aligned}$$

Plugging (3.13) into (3.12), we get (3.11). Plugging in turn (3.11) into (3.10), we finally get

$$\Pi(t+1)\Delta_{21}(t+1) = A_{X_1}\Pi\Delta_{21}A_{X_1}^* - A_{X_1}\Pi\Delta_{21}C^*(I+CXC^*)^{-1}C\Pi\Delta_{21}A_{X_1}^*.$$

Thus, $\Delta(t)=\Pi(t)\Delta_{21}(t)$ satisfies (2.4), and the proof is complete. •

Notice that the previous proof used neither the symmetry of the RDE nor the definiteness of Q. In fact, it is possible to extend the previous result to general, possibly asymmetric, Riccati difference equations [5]. How large is a family of solutions generated through (3.5) depends crucially on the pair $(X_1(t), X_2(t))$. For instance, if $X_1(t) = X_2(t)$, by (3.5) we cannot generate any new solution.

There are two assumptions in the previous theorem. First, the sequences of subspaces $\{M(t)\}$, $\{N(t)\}$, satisfying (3.1)-(3.4) must exist. Second, for each $t \in [t_0, t_1 - 1]$, the matrix $S_X(t)$, must be nonsingular. The phenomenon of having $S_X(t)$ singular, at a certain $t \in [t_0, t_1 - 1]$, is the discrete time analogous of the occurrence of a finite escape time for the Riccati differential equation [7], [4]. In the special case when $A(t)$ nonsingular for all t in $[t_0, t_1 - 1]$, the previous two hypotheses are equivalent. Indeed, we have the following result.

Proposition 3.3 *Assume that the matrix-valued function $A(t)$ is nonsingular on $[t_0, t_1 - 1]$, and consider subspaces M_0 and N_0 such that $M_0 \dot{+} N_0 = \mathbb{C}^n$. Consider the sequences of subspaces $M(t) = \Psi_1(t, t_0)M_0$ and $N(t) = \Psi_2(t, t_0)N_0$, $t \in [t_0, t_1]$. Define $X(t) = (I - \Pi(t))X_1(t) + \Pi(t)X_2(t)$ as in Theorem 3.2. Then*

$$det S_X(t) \neq 0, \quad \forall t \in [t_0, t_1 - 1],$$

if and only if (3.2) holds for all $t \in [t_0, t_1]$.

Proof: For any $t \in [t_0, t_1 - 1]$, we calculate explicitly $det S_X(t)$. Denote by $U_{M^\perp}(t)$ and $U_{N^\perp}(t)$ a basis of $M^\perp(t)$ and $N^\perp(t)$, respectively. We have, omitting the argument t,

$$\begin{aligned} det S_X &:= det(I + CXC^*) = det\,(I + C^*C(X_1(I - \Pi^*) + X_2\Pi^*)) \\ &= det\left(I + C^*C\,(X_1(U_{M^\perp}|0) + X_2(0|U_{N^\perp}))\,(U_{M^\perp}U_{N^\perp})^{-1}\right) \\ &= det\left(A^*A^{-*}\,((I + C^*CX_1)U_{M^\perp}|(I + C^*CX_2)U_{N^\perp})\,(U_{M^\perp}|U_{N^\perp})^{-1}\right). \end{aligned}$$

From this, using (3.3), (3.4) and (2.1), we have

$$det S_X(t) = det A^*(t) det(U_{M^\perp}(t+1)|U_{N^\perp}(t+1)) det(U_{M^\perp}(t)|U_{N^\perp}(t))^{-1}.$$

Since $A(t)$ is assumed to be nonsingular, $det S_X(t) \neq 0$ if and only if $M^\perp \dot{+} N^\perp = \mathbb{C}^n$ for all t. It remains to observe that the latter condition is equivalent to (3.2). •

A simple case where $S_X(t)$ is nonsingular, when $X(t)$ is a solution of the form (3.5), is described by the following proposition.

Proposition 3.4 *Consider two solutions of the RDE, $X_1(t)$ and $X_2(t)$ on $[t_0, t_1]$, with $X_2(t_0) \geq X_1(t_0) \geq 0$. Let $X(t)$ be a Hermitian solution of* *the RDE of the form (3.5). Then,* $X(t)$ *is such that* $detS_X(t) > 0$, *for all* $t \in [t_0, t_1]$.

Proof: It is straightforward to verify [2], using (2.4), that, under the present assumptions, $\Delta_{21}(t) \geq 0$, for each $t \in [t_0, t_1]$. From this also $\Pi(t)\Delta_{21}(t) = \Delta_{21}(t)\Pi^*(t) \geq 0$ follows. Hence,

$$\begin{aligned} &detS_X(t) := det(I + C(t)X(t)C^*(t)) \\ &= det(I + C(t)X_1(t)C^*(t) + C(t)\Pi(t)\Delta_{21}(t)C^*(t)) > 0. \end{aligned}$$

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4 Equilibrium Solutions

Suppose the Riccati difference equation RDE (1.1) has constant coefficients.

Theorem 4.1 *Let* X_1 *and* X_2 *be equilibrium solutions of the RDE (1.1), and let* $\Delta_{21} := X_2 - X_1$. *Let* M *and* N *be subspaces such that* $M \dot{+} N = \mathbb{C}^n$ *and* $\Delta_{21} M^{\perp} \subseteq N$. *Define* Π *to be the matrix which projects onto* M *along* N. *If* M, N *are invariant subspaces of* A_{X_1}, A_{X_2}, *respectively, and if* $S_X := I + CXC^*$ *is nonsingular,* $X = (I - \Pi)X_1 + \Pi X_2$ *is a solution of the DARE (1.2).*

Proof: For $t \in [t_0, t_1]$, define the sequences of subspaces $\{M(t) := M\}$ and $\{N(t) = N\}$. These satisfy all the assumptions of Theorem 3.2. Hence, X is a solution of the RDE (1.1). Since it is time invariant, it solves the $DARE$ (1.2). •

For the purpose of immediate comparison we state below, in our context, a result of [10], which provides a parametrization of all the solutions of the $DARE$. This result establishes in particular that, under suitable hypotheses, two extreme solutions of the $DARE$ exist. Choosing these as reference solutions, we obtain *all* the solutions of the $DARE$.

Theorem 4.2 *Consider the* $DARE$ *with* A *nonsingular and define the function*

$$\psi(z) \doteq C(Iz^{-1} - A)^{-1}Q(Iz - A^T)^{-1}C^T + I.$$

Suppose that $\psi(\eta) > 0$, *for a certain* η *on the unit circle, and that* (A, C) *is an observable pair. Then there exist maximal and minimal solutions of* $DARE$, X_1 *and* X_2, *such that*

$$X_2 \leq X \leq X_1,$$

for each solution X. *Furthermore all the solutions* X *of the* $DARE$ *may be obtained by*

$$X = (I - \Pi)X_1 + \Pi X_2,$$

where M is a A_{X_1}-invariant subspace, $(\Delta_{21} M^{\perp})$ is invariant with respect to A_{X_2}, and Π projects onto M along $(\Delta_{21} M^{\perp})$.

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References

[1] B.D.O. Anderson and J.B. Moore. *Optimal Filtering.* Englewood Cliffs, NJ: Prentice Hall, 1979.

[2] R.R. Bitmead and M. Gevers. Riccati difference and differential equations: convergence, monotonicity and stability, in *The Riccati Equation*, (S.Bittanti, A. Laub, J.C. Willems, eds.). Berlin: Springer Verlag, 1991, 263-291.

[3] W.A. Coppel. Matrix quadratic equations, *Bull. Austral. Math. Soc.,* **10** (1974), 377-401.

[4] P. Crouch and M. Pavon. On the existence of solutions of the Riccati differential equation, *Systems Control Lett.,* **9** (1987), 203-206.

[5] D. D'Alessandro. Geometric aspects of the Riccati difference equation in the nonsymmetric case, *Linear Algebra and its Applications*, to appear.

[6] C.E. de Souza, Monotonicity and stabilizability results for the solutions of the Riccati difference equation, *Proc. Workshop on the Riccati Equation in Control, Systems and Signals*, S. Bittanti, ed.), Como, Italy, (1989), 38-41.

[7] C. Martin, Finite escape time for Riccati differential equations, *Systems Control Lett.* **1** (1981), 127-131.

[8] M. Pavon and D. D'Alessandro. Families of solutions of matrix Riccati Differential equations, *SIAM Journal on Control and Optimization*, to appear.

[9] G. Picci. *Elementi di Elaborazione Statistica del Segnale*, Padova: CLEUP Editore, 1986.

[10] A.C.M. Ran and H.L. Trentelman. Linear quadratic problems with indefinite cost for discrete time systems, *SIAM J. Matrix Anal. Appl.,* **14** (1993), 776-797.

[11] G. Ruckebusch. Représentations Markoviennes de processus Gaussien stationnaires, Thèse 3ème cycle, Univ. of Paris VI, May 1975.

[12] G. Ruckebusch. Représentations markoviennes de processus gaussiens stationaires et applications statistiques, in *Journées de Statistique de Processus Stochastiques*, 115-139 (D.Dacunha-Castelle and B.Van Cutsem, eds.), Lecture Notes in Mathematics. 636 Berlin: Springer, 1978.

[13] M.A. Shayman. Geometry of the algebraic Riccati equation, Part 1, *SIAM J. Control and Optimization,* **21** (1983), 375-394.

[14] J.C. Willems. Least-squares stationary optimal control and the algebraic Riccati equation, *IEEE Trans. Automat. Control,* **16** (1971), 621-634.

[15] H.K. Wimmer. Stabilizing and unmixed solutions of the discrete time algebraic Riccati equation, *Proc. Workshop on the Riccati Equation in Control, Systems and Signals*, (S. Bittanti, ed.), Como, Italy, (1989), 95-98.

[16] H.K. Wimmer. Geometry of the discrete-time algebraic Riccati equation, *J. Math. Syst. Estim. Control,* **2** (1992), 123-132.

[17] H.K. Wimmer. Unmixed solutions of the discrete-time algebraic Riccati equation, *SIAM J. Control and Optimization,* **30** (1992), 867-878.

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